

DOES MY VOTE COUNT?

A Glimpse into the SMC's Multiplicative Reasoning Project

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The SMC is currently engaged in the writing of material which will support pupils' multiplicative reasoning whilst also promoting principles of equity and democracy. Here we offer a glimpse into a small part of the project.






Fair allocation of seats via simple proportions

In a democracy, we might expect that a vote cast by one person has the same weight, the same effect, as anyone else's vote. At the very least we should expect that our elections would include such a design principle. In practice, such fair representation proves to be illusory and impracticable, the mathematician's best efforts falling short in the face of competing philosophies and systems, and in some cases, in the struggle between political parties to gain an edge, occasionally even by shady manoeuvres.






Imagine that we have five political parties. We won't give them traditional names because to do so might give the impression of some bias towards a particular party when we simply want to highlight the mathematics. Let's represent them by five mathematical shapes, Circle, Square, Rectangle, Triangle and Hexagon.

Example 1

In this first example of (perfectly) fair representation, let's keep the numbers small, for convenience. In a tiny ward (voting area) a Council of 20 people is being elected by 100 voters. The number of votes cast for each of the parties is shown in the table below. How many Council members should each of the parties get?






Party						Total
Votes	40	25	5	10	20	100

To get from 100 to 20, we need to scale down by a factor of 5, which gives 8, 5, 1, 2, 4. These numbers sum to 20, so the calculations are correct. So we have:






Party						Total
Votes	40	25	5	10	20	100
Members	8	5	1	2	4	20

Example 2

In another ward, a Council of 25 is being elected in a poll for which the turnout was 400. The votes cast were as follows:

Party						Total
Votes	32	88	40	172	68	400






Now $400/25 = 16$, so it takes 16 votes to elect a member. Following Example 1, we need to scale by a factor of $1/16$ which gives:

Party						Total
Votes	32	88	40	172	68	400
Members	2	$5\frac{1}{2}$	$2\frac{1}{2}$	$10\frac{3}{4}$	$4\frac{1}{4}$	25

The Circle Party can be fairly represented (by having two members) but this is not possible for the other four parties because the number of members for each of them must be a whole number. In this particular case, the Squares and the Rectangles have many interests (or even properties) in common and they agree to work together. This is also true of the Triangles and the Hexagons. Furthermore, the election rules allow them to decide what to do about the fractions. The Squares and Rectangles agree to have 5 members and 3 members respectively, but the Triangles demand 11 members and the Hexagons eventually agree. The final arrangement sees the Squares and the Hexagons slightly under-represented but it's not a bad compromise.

Example 3

For a third ward, there is a convention that parties cannot enter into the sort of arrangements seen in Example 2. Instead, simple rounding to the nearest whole number is used. There are 377 votes cast to determine the makeup of a 23-member Council. So we need to scale the number of votes cast for each party by $23/377$ and then round accordingly.

Party						Total
Votes	42	31	67	100	137	377
Members	2.6	1.9	4.1	6.1	8.4	23.1
	3	2	4	6	8	23






Constituencies, and First Past The Post

In the previous examples the main issue in allocating seats fairly is to distribute fractional remainders. A perfectly fair remedy would be to chop individual politicians into fractional pieces. However, while this may be a popular solution it is not entirely legal. Now we look at another aspect of the division of votes in a representative democracy. In most legislatures, the allocation of seats to parties is done at a fine-grained level. That is, the electoral area is partitioned into smaller regions. These pieces are given many names such as constituencies, wards or precincts. Votes are counted at the constituency level and a certain number of members are sent to the legislature based on the results. Some balance is attempted when partitioning voters so that each constituency represents a coherent community, and that the population of each partition is almost uniform. There are many reasons for dividing an electoral area in this way; in particular it means that voters can contact a representative who is aware of the local issues affecting the constituency.






The rules for creating constituencies differ from country to country and also depend on the type of legislature within a country. In our examples we will simplify matters by assuming that each constituency has an identical population. The number of members each constituency elects also varies in practice. For example, for the elections to the Westminster parliament each constituency provides a single MP while in the Glasgow City Council elections, each ward is represented by either three or four councillors.

Example 4

A council is elected over 5 wards each electing 4 members. The votes cast are as follows.






Party						Total
Ward 1	50	25	0	25	0	100
Ward 2	20	25	40	5	10	100
Ward 3	5	25	50	0	20	100
Ward 4	50	25	0	20	5	100
Ward 5	25	25	10	0	40	100
Totals	150	125	100	50	75	500

A reasonable allocation of members is given by dividing the votes in each ward by 25 and rounding. Notice that in this case the total number of seats allocated to each party is perfectly proportional to the number of votes they received, namely 25 votes per seat, and these totals match exactly a fair allocation of 20 seats to the 500 total votes.

Party					
Ward 1	2	1	0	1	0
Ward 2	1	1	2	0	0
Ward 3	0	1	2	0	1
Ward 4	2	1	0	1	0
Ward 5	1	1	0	0	2

Example 5

Consider the situation when the results in Example 4 are repeated but that each ward allocates a single member. This member will be from the party which receives the most votes in a ward. The allocation is then as follows.

Party					
Ward 1	1	0	0	0	0
Ward 2	0	0	1	0	0
Ward 3	0	0	1	0	0
Ward 4	1	0	0	0	0
Ward 5	0	0	0	0	1

Is this fair? We might imagine that the Squares and their supporters would be very disappointed. Despite receiving the second most votes overall they won nothing. On the other hand, the strong local support for the Hexagons in Ward 5 means that they will be represented in the legislature. Furthermore, the Hexagon's councillor will have a casting vote in resolving disputes between the Circles and the Rectangles which in effect makes their single representative the most powerful of all.

There are strong arguments in support of the system characterised in Example 5, known as **First Past the Post**. For example, it ensures that parties with strong local support are represented (like the Hexagons) and it can reduce the chances of extremist politicians getting enough support to gain representation. Nevertheless, it can lead to results which, at least mathematically, appear unfair and there are frequent calls to replace the system where it still exists. In general, the more seats are allocated per constituency the fairer the final allocation in terms of overall votes, but there are plenty of exceptions to this statement.

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