Introduction to Spatial Filtering / Array Processing

Stephan Weiss

Centre for Signal & Image Processing
Department of Electronic & Electrical Engineering
University of Strathclyde, Glasgow, Scotland, UK

UDRC-EURASIP Summer School, Edinburgh, 22 June 2023

Introduction to Spatial Filtering / Array Processing — Overview

Contents:

1.1 Intuitive Beamforming

- A farfield wavefront arrives at a linear uniformly spaced sensor array:

- due to the direction of arrival (DOA) and finite propagation speed, the wavefront will arrive at different sensors with a delay $\Delta\tau$;
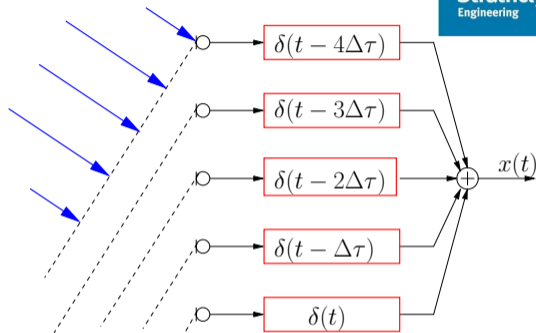
- with appropriate processing (beamforming), the sensor signals can be aligned to create constructive interference at the output $x(t)$;

- the above is a simple delay-and-sum beamformer [16, 42, 64, 43].

1.2 Spatial Sampling

▶ For unambiguous spatial sampling, we need to take at least two samples per wavelength of the highest frequency component in the array signals [42];

▶ analogy from temporal sampling (Nyquist): take at least two samples per period (relating to the highest frequency component);

▶ Wavelength λ and frequency f are related by the propagation speed c in the medium: $\lambda = \frac{c}{f}$;
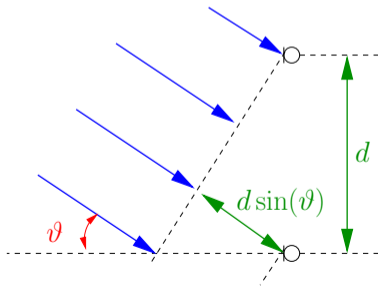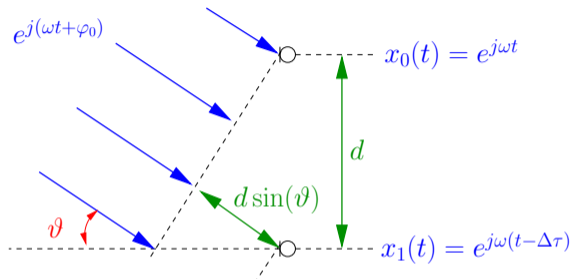

▶ maximum sensor distance

$$d = \frac{\lambda_{\max}}{2} = \frac{c}{2f_{\max}} \ .$$

▶ time delay between sensors

$$\Delta\tau = \frac{d\sin(\vartheta)}{c} = \frac{\sin(\vartheta)}{2f_{\max}} \ .$$

Spatial and Temporal Sampling

▶ Consider the array signals $x_0(t)$ and $x_1(t)$ due to a source $e^{j(\omega t + \varphi_0)}$:

▶ sampling with $t = nT_s$ leads to

$$x_0[n] = e^{j\omega nT_s} \qquad \text{and} \qquad x_1[n] = e^{j\omega(nT_s - \Delta\tau)}$$

▶ with $f_{max} = \frac{f_s}{2} = \frac{1}{2T_s}$ and normalised angular frequency $\Omega = \omega T_s$,

$$x_0[n] = e^{j\Omega n} \qquad \text{and} \qquad x_1[n] = e^{j\Omega n} \cdot e^{-j\Omega \sin(\vartheta)} .$$

1.3 Steering Vector

▶ A narrowband source with normalised angular frequency Ω illuminates an M-element linear array of equi-spaced sensors:

$$\mathbf{x}[n] = \begin{bmatrix} x_0[n] \\ x_1[n] \\ \vdots \\ x_{M-1}[n] \end{bmatrix} = e^{j\Omega n} \cdot \begin{bmatrix} 1 \\ e^{-j\Omega \sin(\vartheta)} \\ \vdots \\ e^{-j(M-1)\Omega \sin(\vartheta)} \end{bmatrix} = e^{j\Omega n} \cdot \mathbf{s}_{\Omega,\vartheta}$$

▶ the vector $\mathbf{s}_{\Omega,\vartheta}$ characterises the phase shifts of waveform with frequency Ω and DOA ϑ measured at the array sensors;

▶ since a narrowband signal $e^{j\Omega n}$ only causes phase shifts rather than delays, constructive interference can be accomplished by a set of complex multipliers rather than processors $\delta(t - m\Delta\tau)$, $m = 0, 1, \ldots (M-1)$;

▶ beamforming problem: how to select the set of complex coefficients?

1.4 Data Independent Beamformer
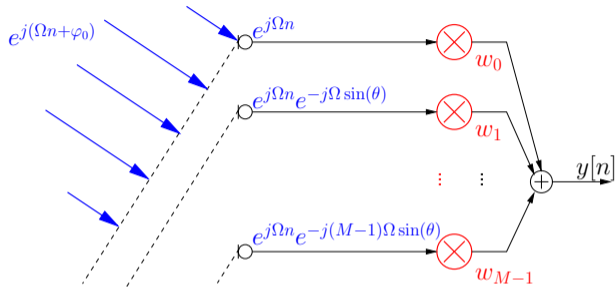
▶ Challenge:
  find a set of complex multipliers
  $w_m, \; m = 0, 1, \ldots (M-1)$:

▶ to steer the array characteristic towards this source, the output

$$y[n] = [w_0 \; w_1 \; \ldots \; w_{M-1}] \mathrm{e}^{j\Omega n} \begin{bmatrix} 1 \\ \mathrm{e}^{-\mathrm{j}\Omega\sin(\vartheta)} \\ \vdots \\ \mathrm{e}^{-\mathrm{j}(M-1)\Omega\sin(\vartheta)} \end{bmatrix} = \mathrm{e}^{\mathrm{j}\Omega n}\mathbf{w}^{\mathrm{H}}\mathbf{s}_{\Omega,\vartheta}$$

should satisfy $y[n] = \mathrm{e}^{\mathrm{j}\Omega n}$, leading to $\mathbf{w}^{\mathrm{H}}\mathbf{s}_{\Omega,\vartheta} = 1$.

Coefficient Vector

▶ For later convenience and compatibility, the Hermitian transpose operator $\{\cdot\}^{\mathrm{H}}$ is used to denote the coefficient vector

$$\mathbf{w}^{\mathrm{H}} = [w_0 \ w_1 \ \ldots \ w_{M-1}] \ ;$$

▶ as a result, the vector $\mathbf{w}$ hold the complex conjugates of the coefficients,

$$\mathbf{w} = \begin{bmatrix} w_0^* \\ w_1^* \\ \vdots \\ w_{M-1}^* \end{bmatrix} \ ;$$

▶ to access the actual unconjugated coefficients, the beamforming vector $\mathbf{w}^*$ has to be considered

▶ note that

$$\mathbf{w}^{\mathrm{H}} \mathbf{s}_{\Omega,\vartheta} = 1 \qquad \longrightarrow \qquad \mathbf{s}_{\Omega,\vartheta}^{\mathrm{H}} \mathbf{w} = 1 \ .$$

Narrowband Beamforming — Single Source

▶ The expression $\mathbf{s}_{\Omega,\vartheta}^{\mathrm{H}}\mathbf{w} = 1$ forms a system with one equation and M unknowns

$$\boxed{\mathbf{s}_{\Omega,\vartheta}^{\mathrm{H}}} \boxed{\mathbf{w}} = \boxed{1}$$

▶ general solution to an underdetermined system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is the right pseudo-inverse $\mathbf{A}^{\dagger}$ [14],

$$\mathbf{x} = \mathbf{A}^{\dagger}\mathbf{b} = \mathbf{A}^{\mathrm{H}}(\mathbf{A}\mathbf{A}^{\mathrm{H}})^{-1}\mathbf{b} \; ;$$

▶ here:

$$\mathbf{w} = (\mathbf{s}_{\Omega,\vartheta}^{\mathrm{H}})^{\dagger} \cdot 1 = \mathbf{s}_{\Omega,\vartheta} \cdot (\mathbf{s}_{\Omega,\vartheta}^{\mathrm{H}}\mathbf{s}_{\Omega,\vartheta})^{-1} \cdot 1 = \frac{\mathbf{s}_{\Omega,\vartheta}}{\|\mathbf{s}_{\Omega,\vartheta}\|_2^2} = \frac{1}{M}\mathbf{s}_{\Omega,\vartheta} \; ;$$

▶ the complex conjugation for $\mathbf{w}^*$ inverts and therefore compensates the phase of the steering vector, which could have been foreseen

▶ the formulation via the pseudo-inverse will be powerful for more complicated cases.

Narrowband Beamformer Example

▶ Source parameters: $\Omega = \frac{\pi}{2}$ and $\vartheta = 30°$; array parameter: $M = 5$;

▶ steering vector (with $\Omega \sin(\vartheta) = \frac{1}{4}\pi$):

$$\mathbf{s}_{\Omega,\vartheta}^{\mathrm{T}} = [1 \ \ e^{-j\frac{1}{4}\pi} \ \ \dots \ \ e^{-j\frac{4}{4}\pi}]$$

▶ coefficient vector is given by $\mathbf{w} = (\mathbf{s}_{\Omega,\vartheta}^{\mathrm{H}})^{\dagger}$;

▶ numerical solution in Matlab;

```
Omega=1/4; theta = pi/6; M=5;
s = exp(-sqrt(-1)*Omega*sin(theta)*(0:(M-1)'));
w = pinv(s');
```

▶ `angle([s conj(w)])/pi` yields:

-0.00000	0.00000
-0.25000	0.25000
-0.50000	0.50000
-0.75000	0.75000
-1.00000	1.00000

Beam Pattern I

- ▶ The beamformer has a unit gain towards a source with frequency Ω and DoA θ; what is its gain response towards other angles of arrival?
- ▶ the beam pattern measures the response of a beamformer by sweeping the angle ψ of a source with frequency Ω

$$g(\Omega, \psi) = \mathbf{w}^{\mathrm{H}} \mathbf{s}_{\Omega,\psi}$$

- ▶ beam pattern for the previous example:
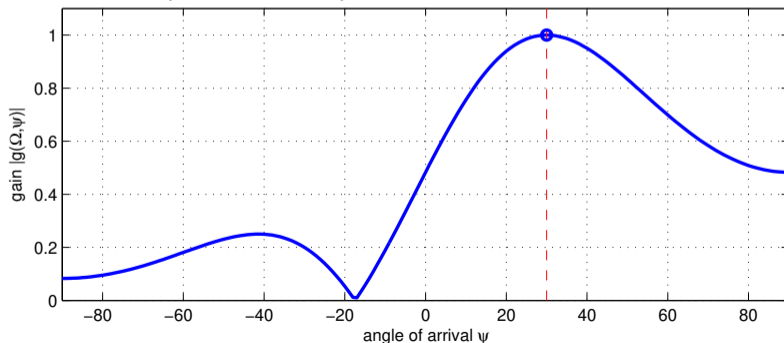
Beam Pattern II

▶ Beam patterns for $\Omega = \frac{\pi}{2}$ and $\vartheta = 30°$ with variable M:
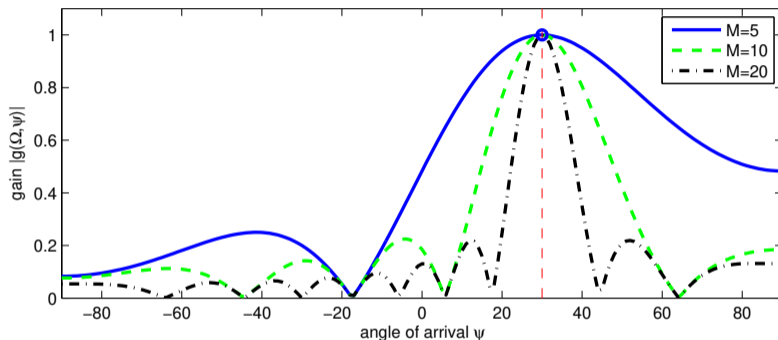
▶ increasing the sensor number M narrows the main beam, and increases the number of spatial zeros;

▶ analogous characteristic in the time domain: increased temporal support leads to higher frequency resolution.

Interference

- Many scenarios contain a source of interest and a number of interferers:
  signal of interest:
  $\{\Omega_0, \vartheta_0\}$
  two interferers:
  $\{\Omega_1, \vartheta_1\}$, $\{\Omega_2, \vartheta_2\}$

- we would like to control the beampattern to place spatial nulls in the directions of the interfering sources;

- Problem formulation and solution :
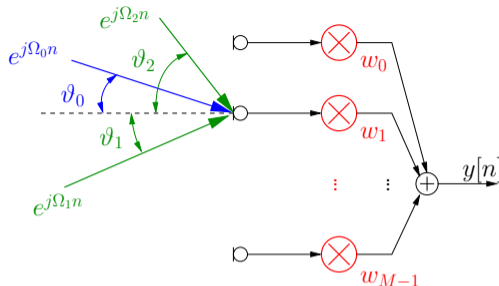
$$\begin{bmatrix} \mathbf{s}_{\Omega_0,\vartheta_0}^{\mathrm{H}} \\ \mathbf{s}_{\Omega_1,\vartheta_1}^{\mathrm{H}} \\ \mathbf{s}_{\Omega_2,\vartheta_2}^{\mathrm{H}} \end{bmatrix} \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \longrightarrow \quad \mathbf{w} = \begin{bmatrix} \mathbf{s}_{\Omega_0,\vartheta_0}^{\mathrm{H}} \\ \mathbf{s}_{\Omega_1,\vartheta_1}^{\mathrm{H}} \\ \mathbf{s}_{\Omega_2,\vartheta_2}^{\mathrm{H}} \end{bmatrix}^{\dagger} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} .$$

Narrowband BF Example — Multiple Sources

▶ The signal of interest illuminates an $M = 5$ element array at a frequency $\Omega_0 = \frac{\pi}{2}$ with a DoA $\vartheta_0 = 30°$

▶ two interferers at $\Omega_1 = \Omega_2 = \Omega_0$ are present with DoA $\vartheta_1 = -45°$ and $\vartheta_2 = 60°$

▶ results via right pseudo-inverse of steering vectors

$\angle \mathbf{s}_{\Omega_0,\vartheta_0}$	$\angle \mathbf{s}_{\Omega_1,\vartheta_1}$	$\angle \mathbf{s}_{\Omega_2,\vartheta_2}$	$\angle \mathbf{w}^*$	$|\mathbf{w}|$
0.00	0.00	0.00	-42.81	0.3172
45.00	63.64	-77.94	-105.01	0.3004
90.00	127.28	-155.89	-90.00	0.2343
135.00	-169.08	126.17	-74.99	0.3004
180.00	-105.44	48.23	-137.19	0.3172

▶ the angle of $\mathbf{w}$ is no longer intuitive; also note that the coefficients in $\mathbf{w}$ no longer have the same modulus

▶ amongst a manifold of solutions, the right pseudo-inverse provides the minimum norm solution.

Multiple Source Example — Beampattern

▶ Beam pattern one source of interest and two interferers:
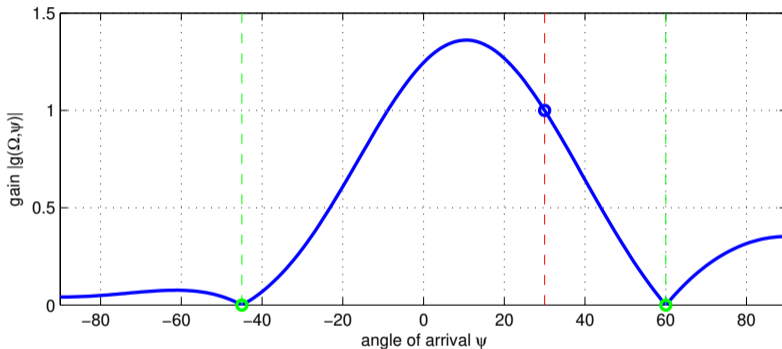
▶ the pseudo-inverse is the minimum norm solution, keeping the general gain response as low as possible;

▶ the minimum norm property protects against spatially white noise.
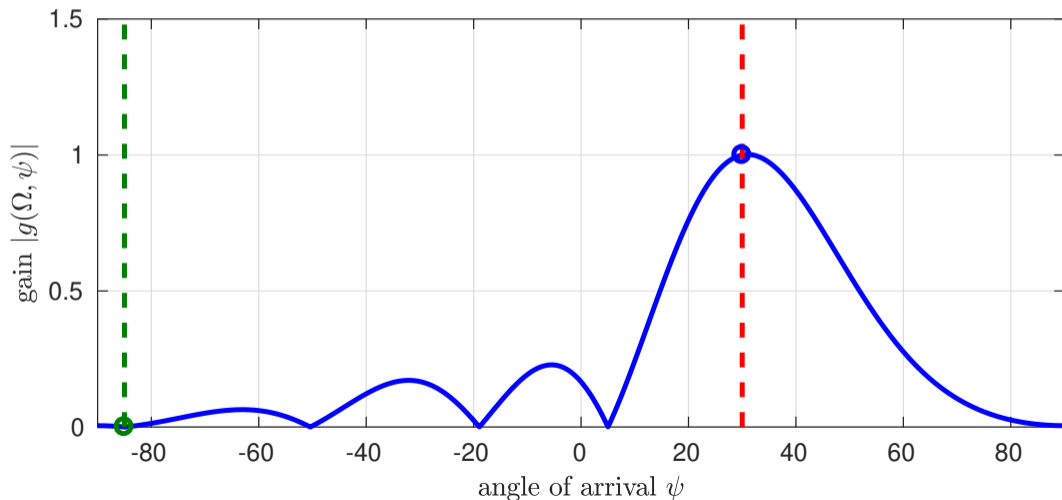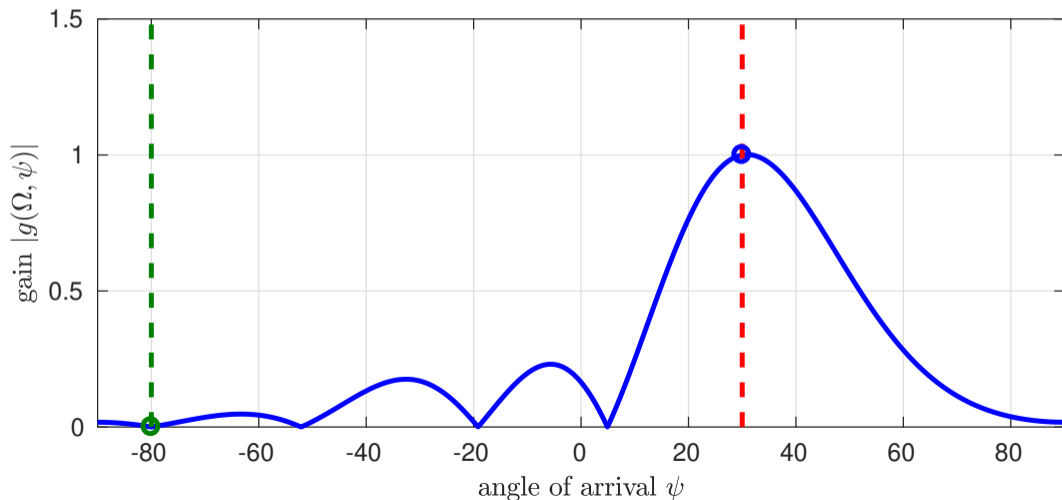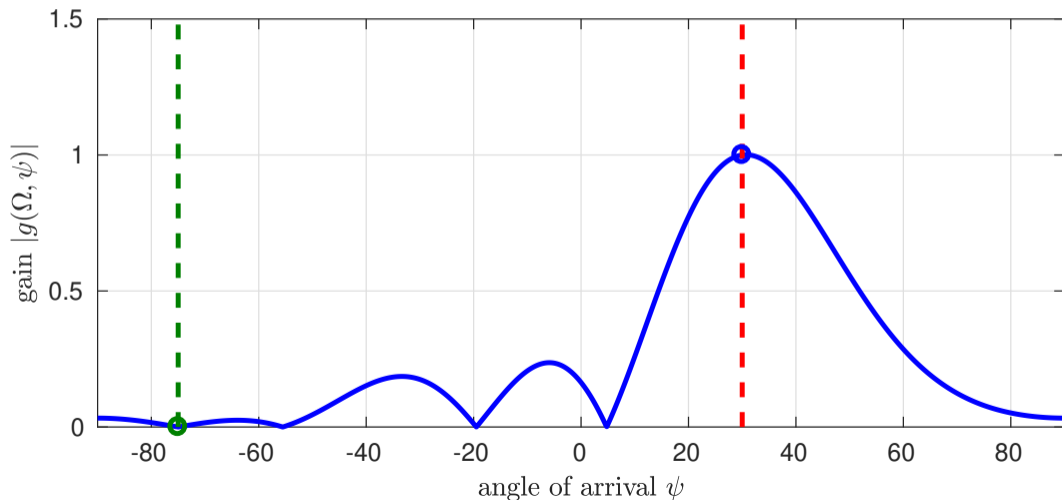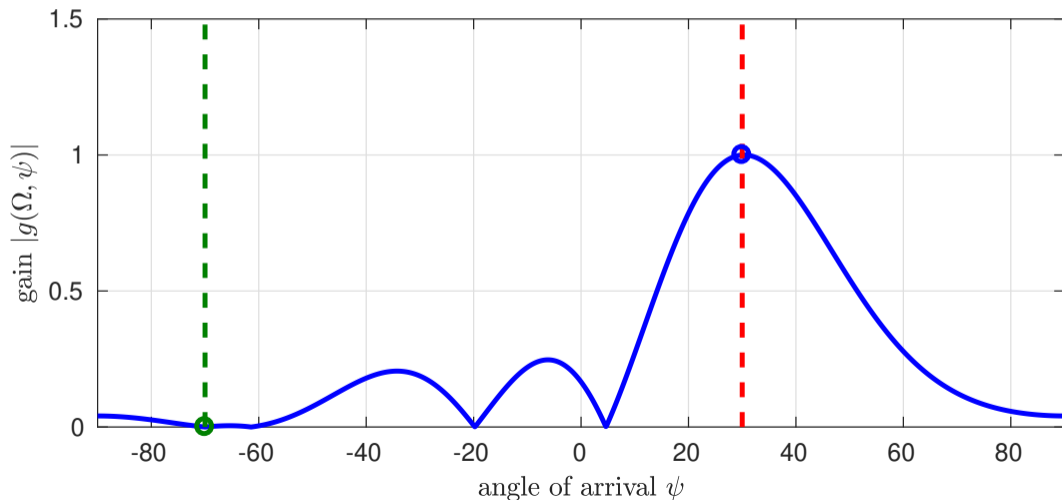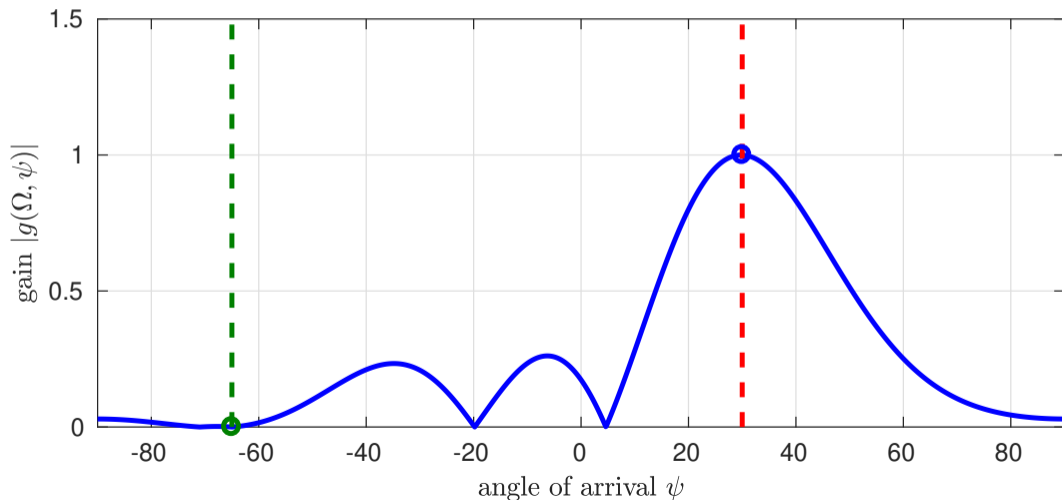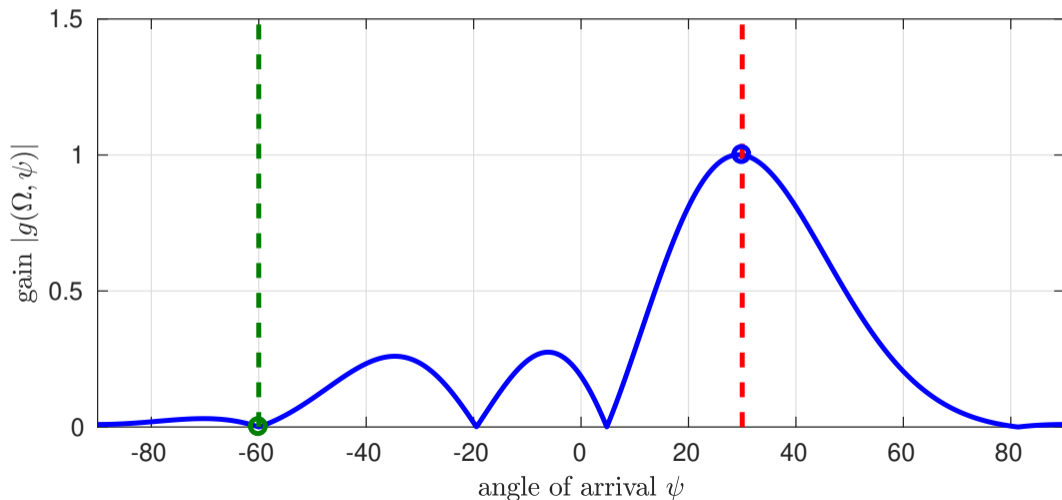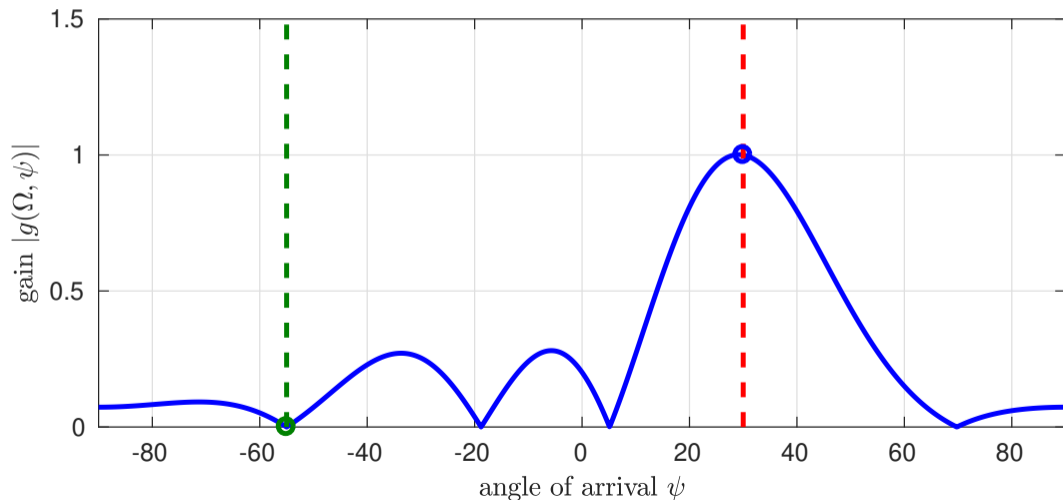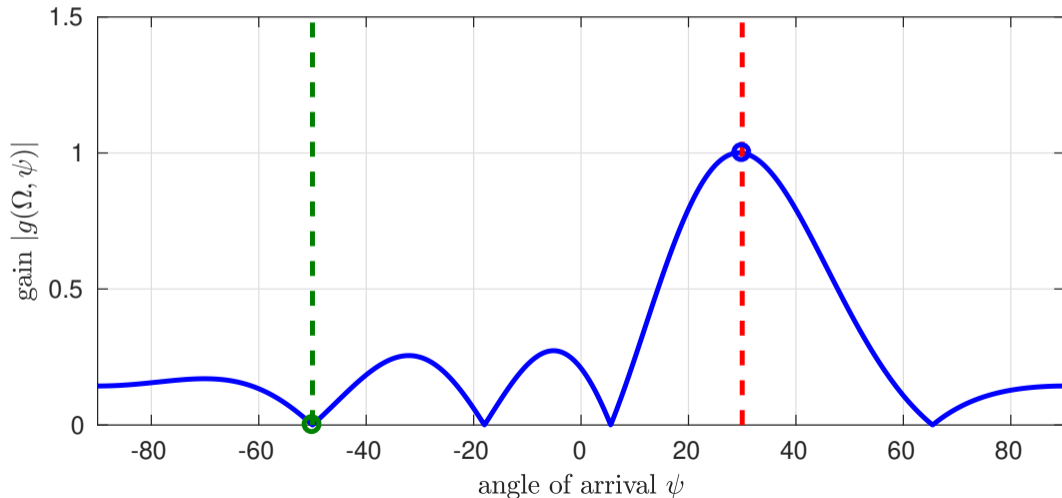
Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

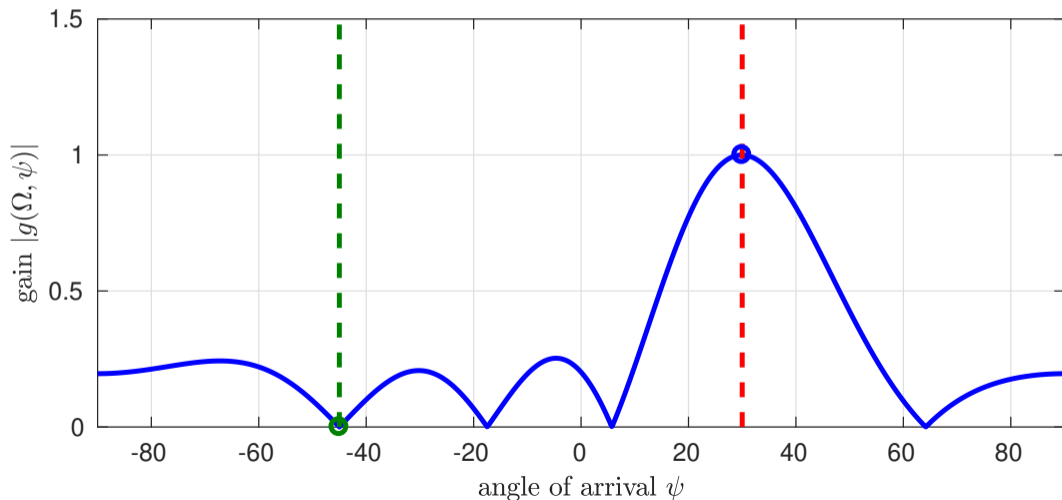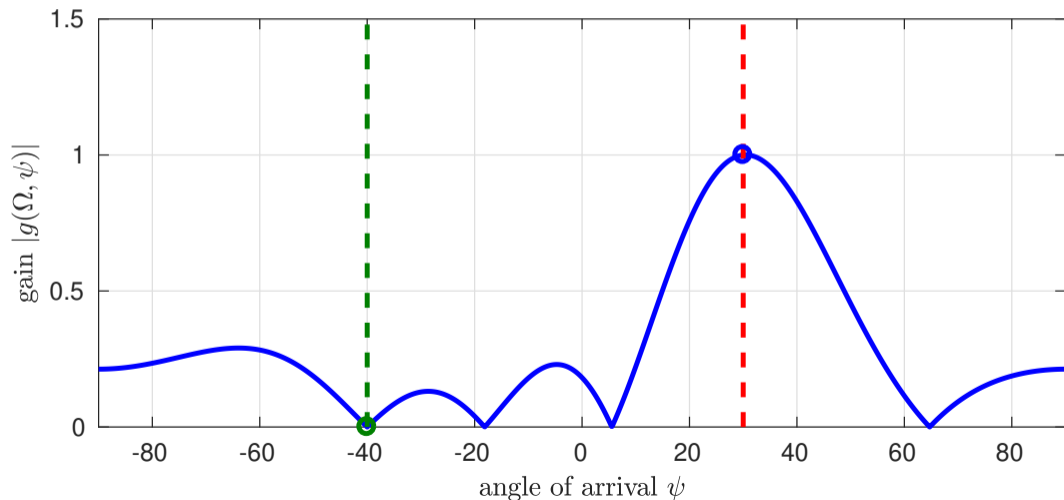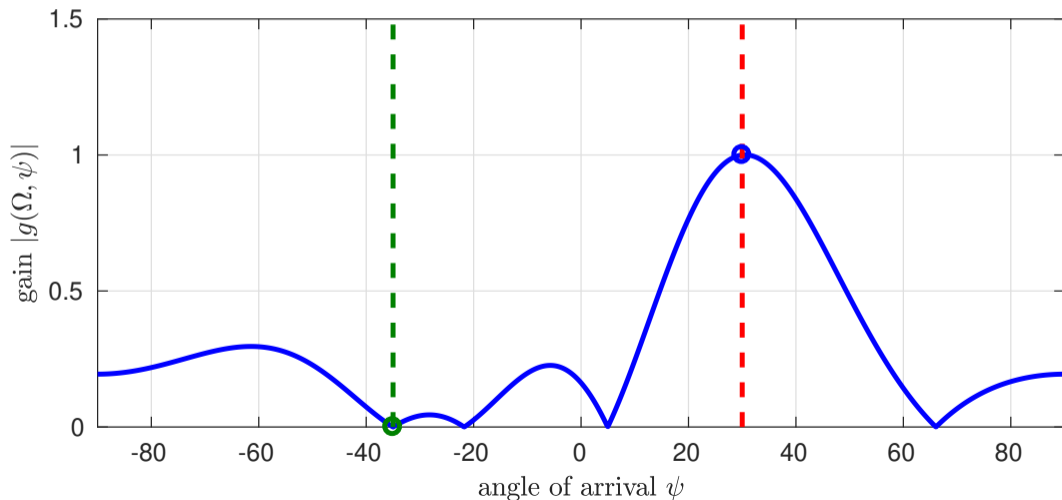▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:
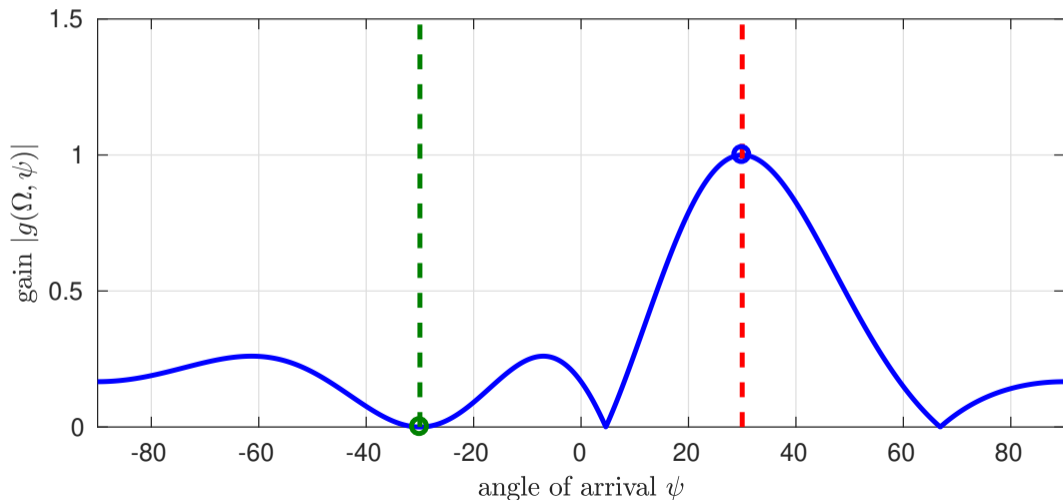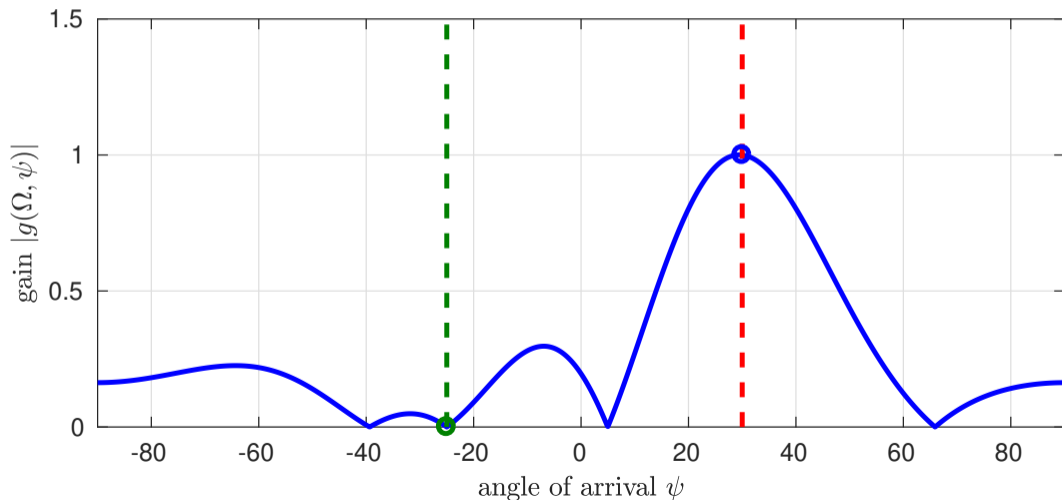
Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

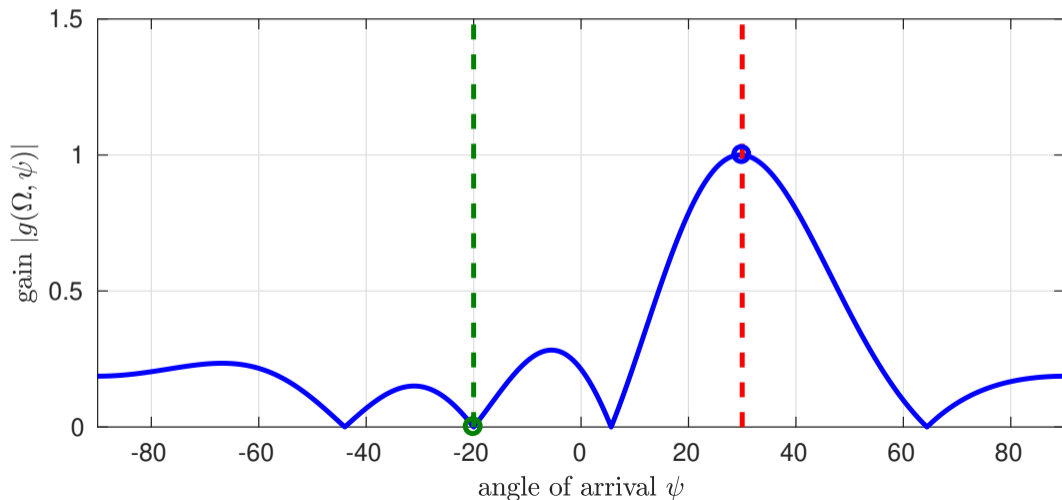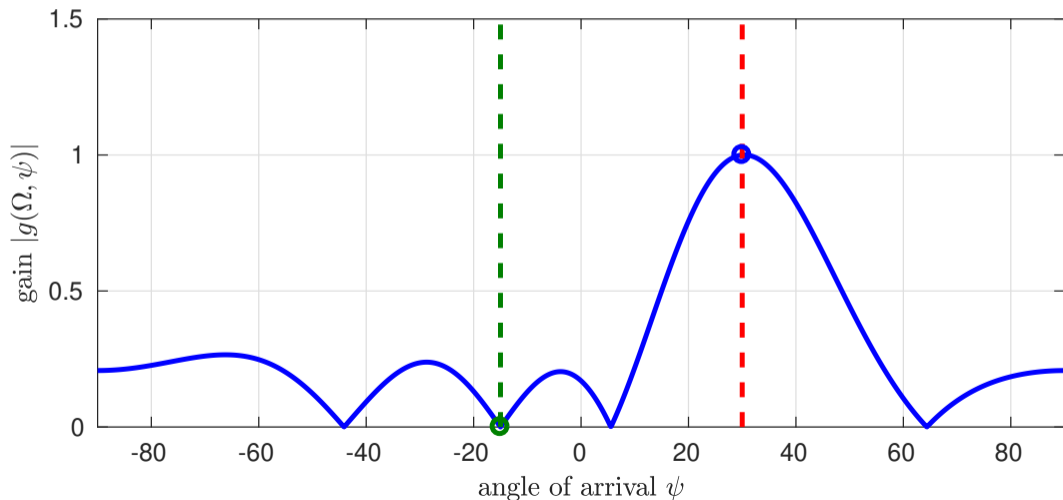- $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:
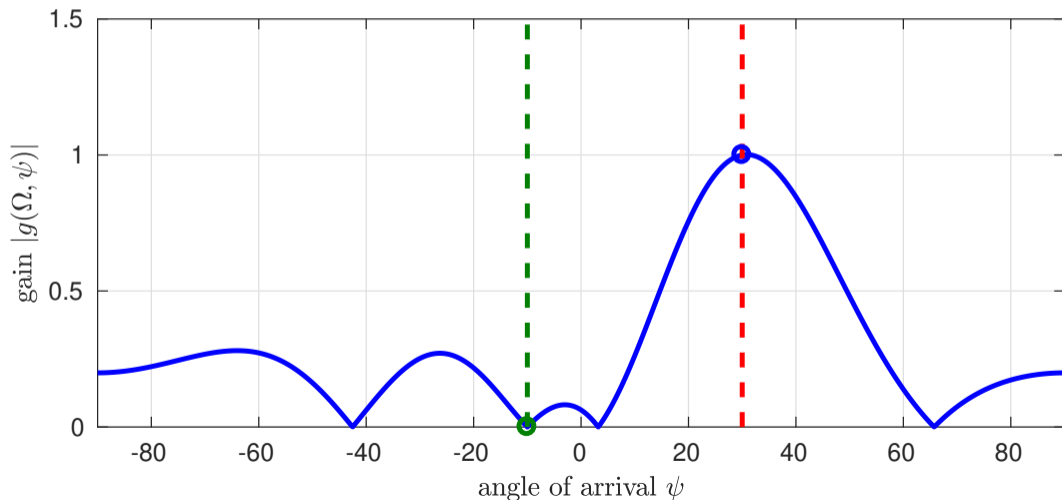
Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

University of Strathclyde Engineering

Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:
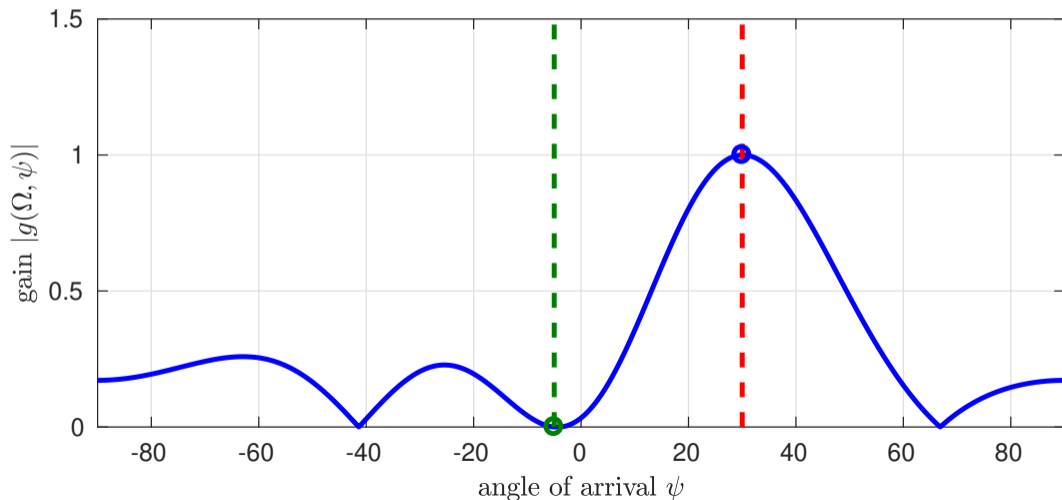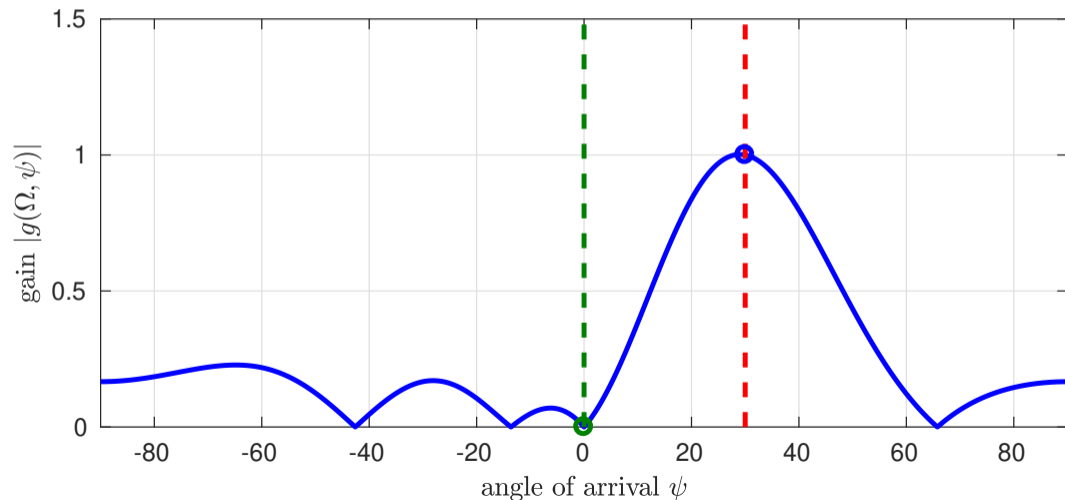
Beamforming Example — Variable Interferer I

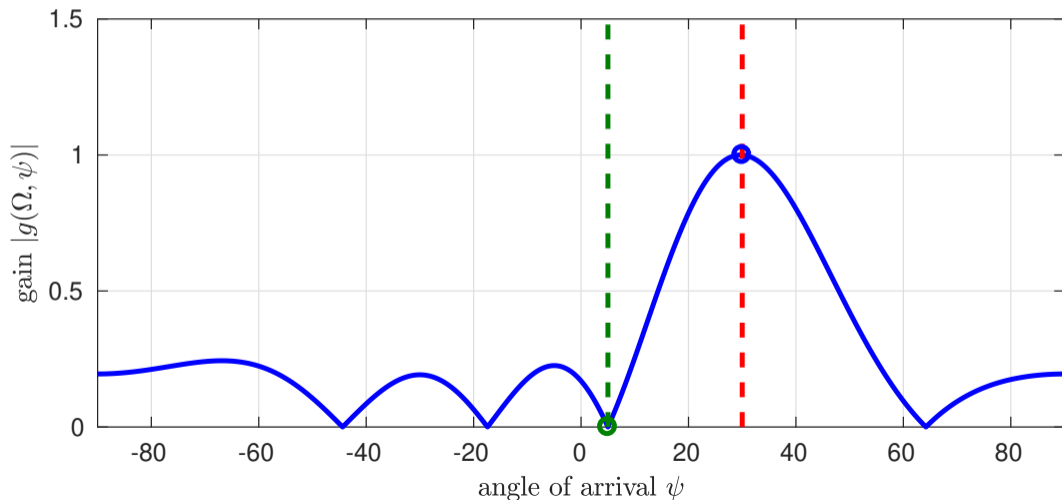▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

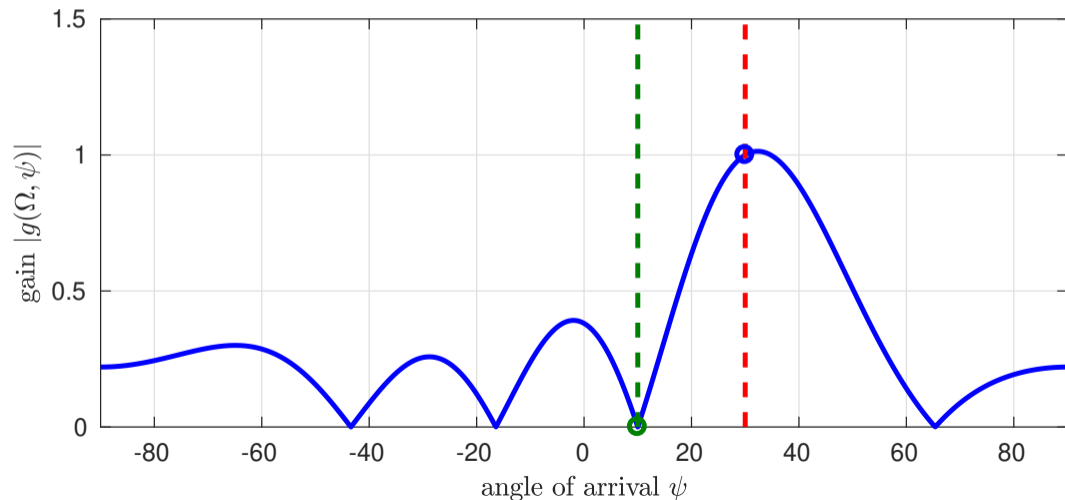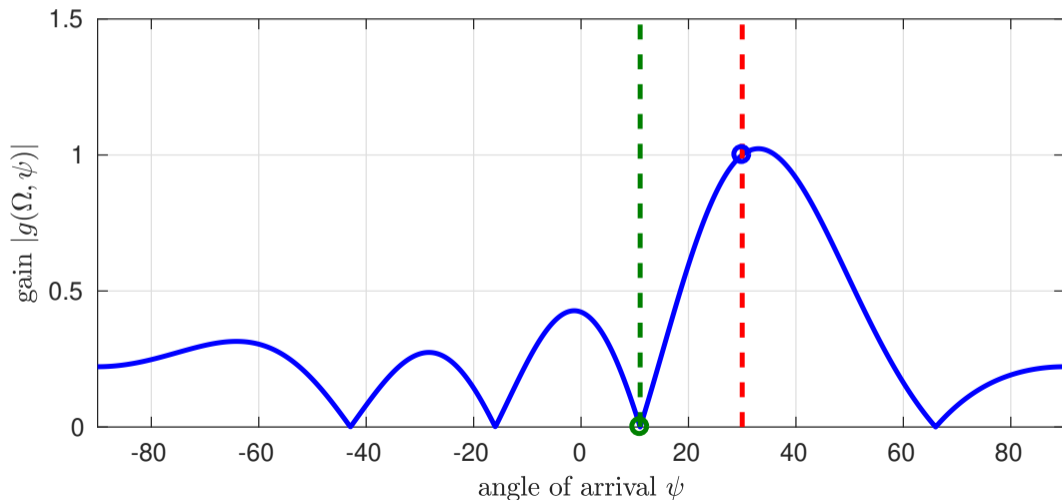▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:
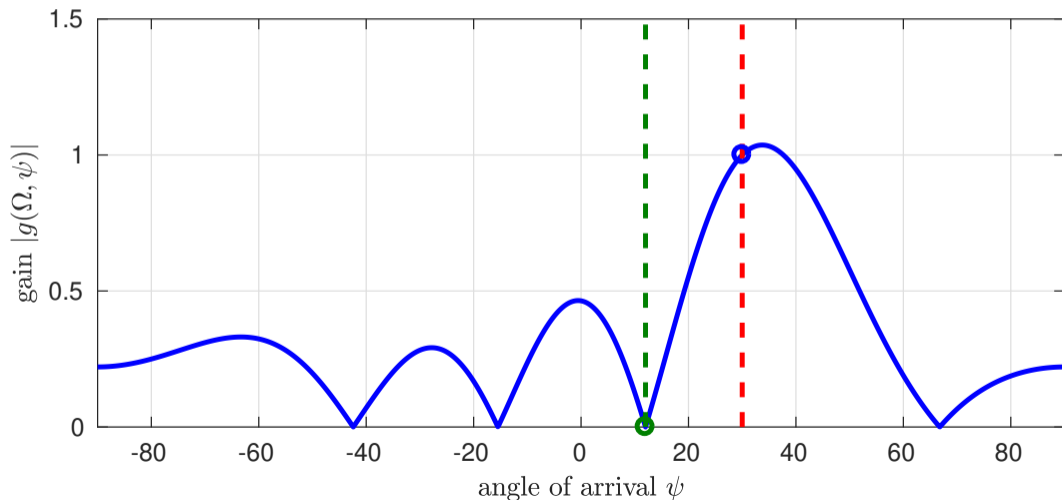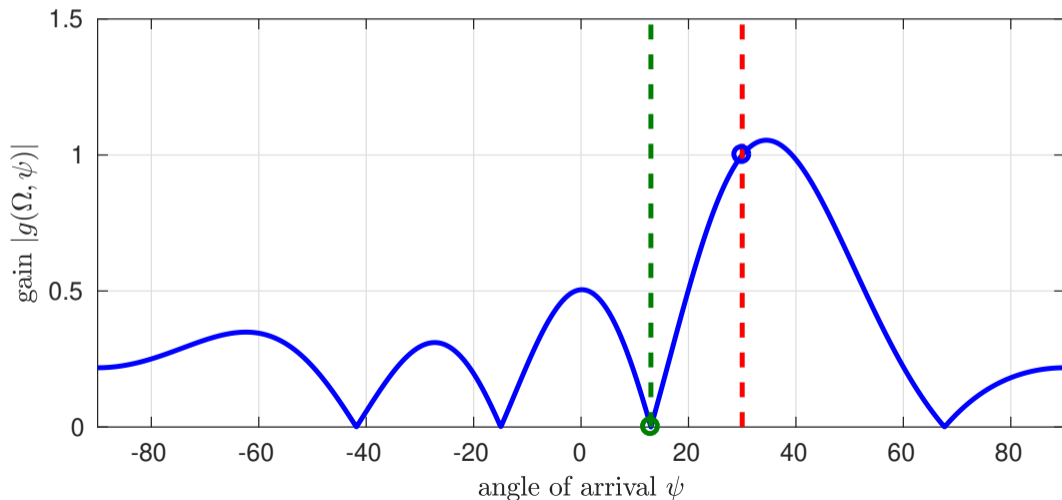
Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

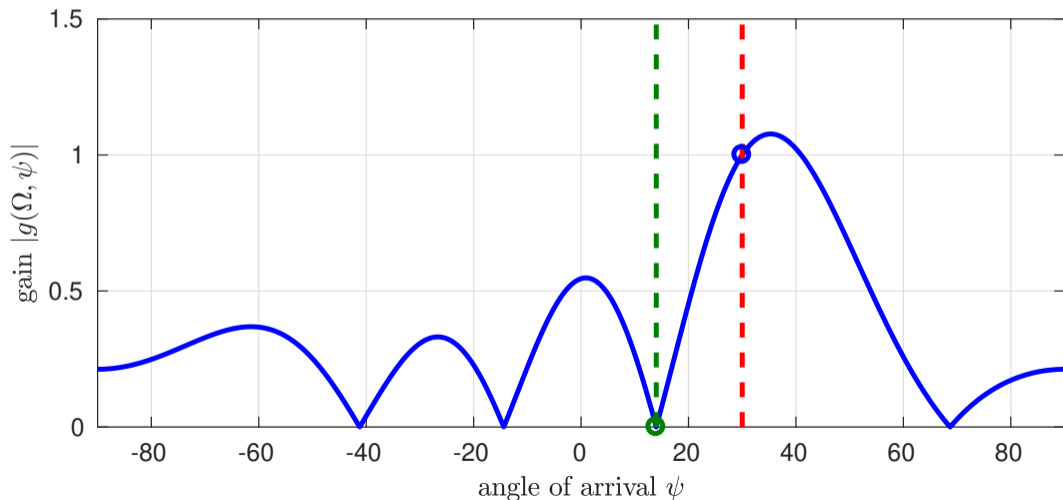▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

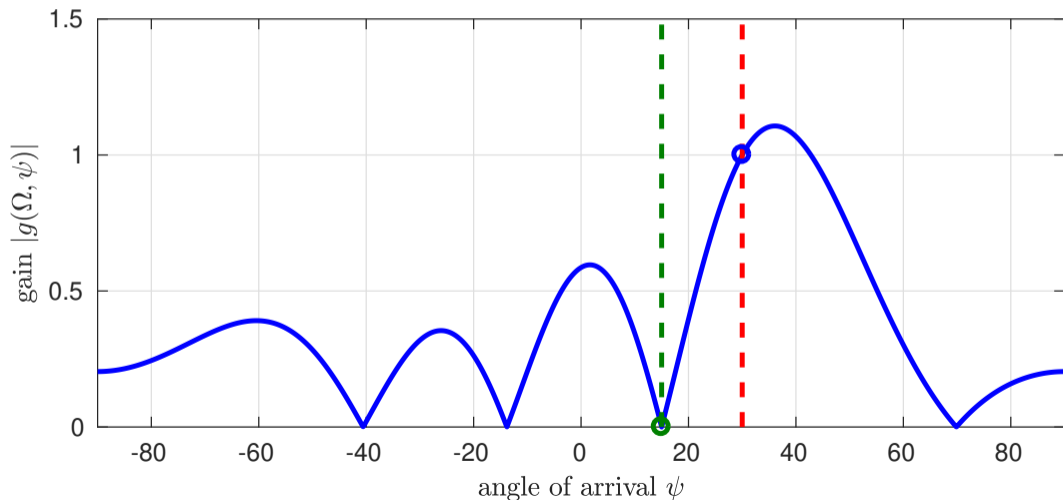▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

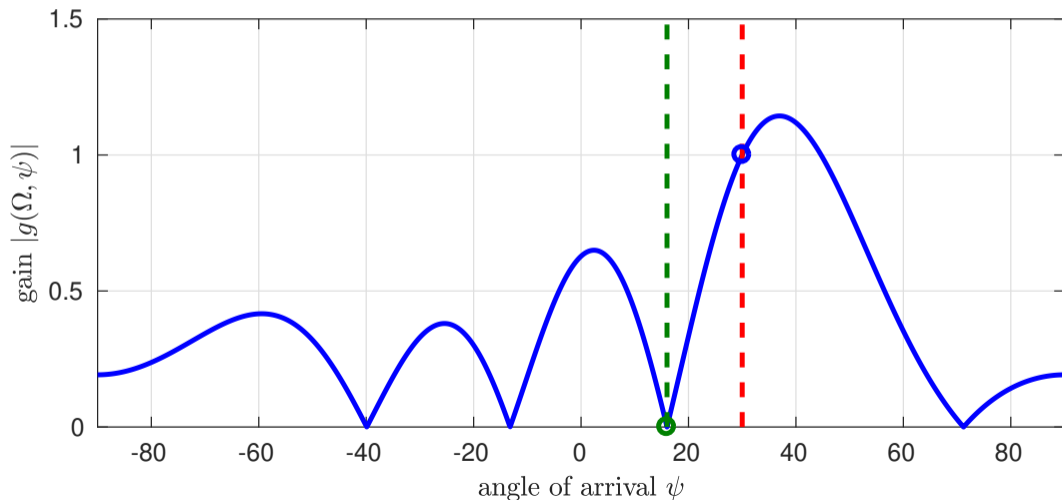# Beamforming Example — Variable Interferer I

▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer I

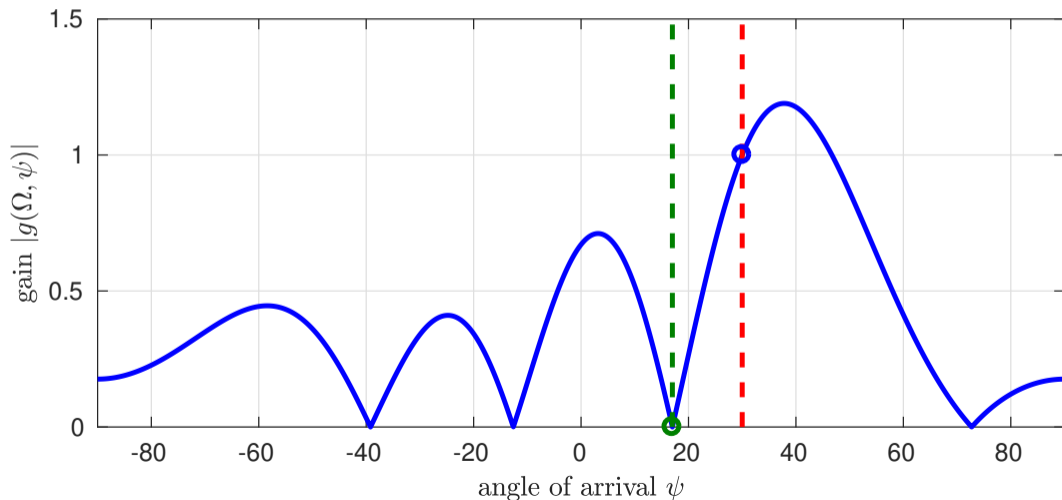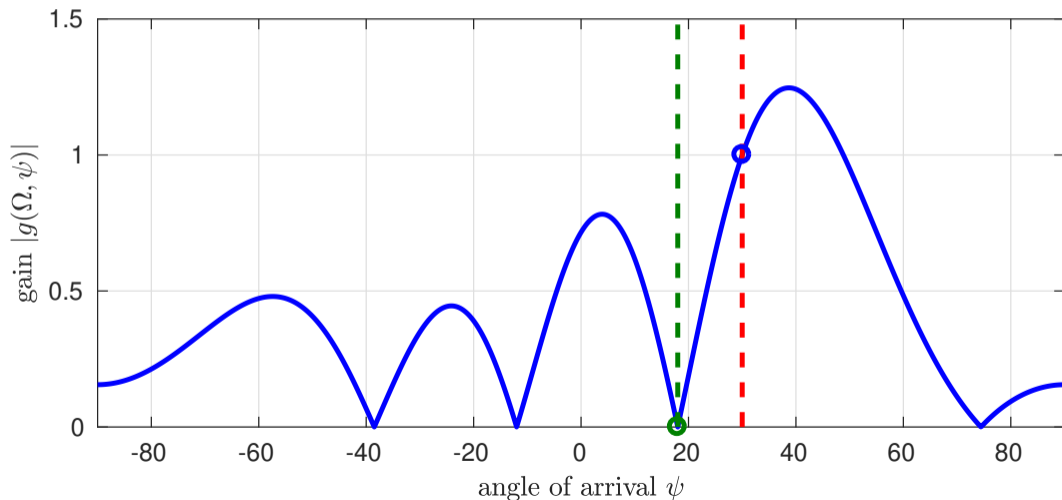▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:
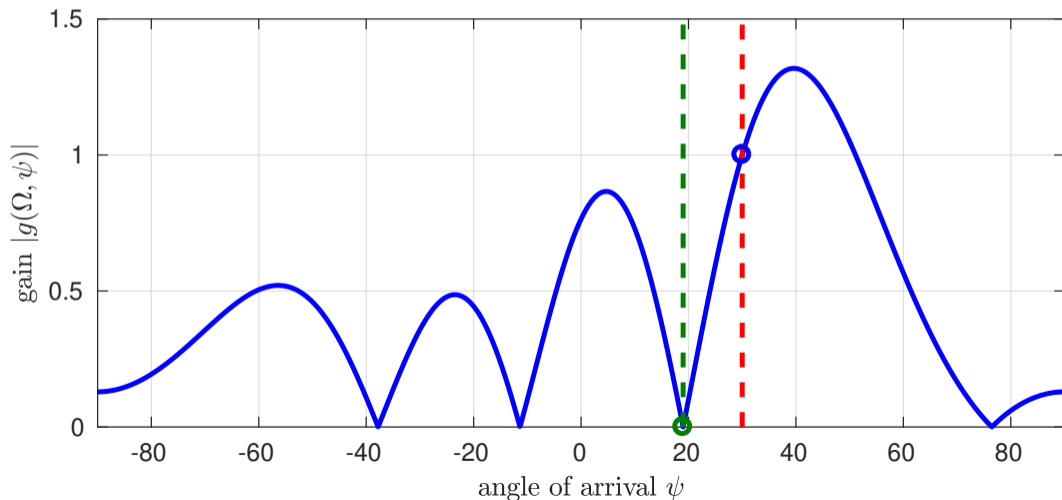
Beamforming Example — Variable Interferer I
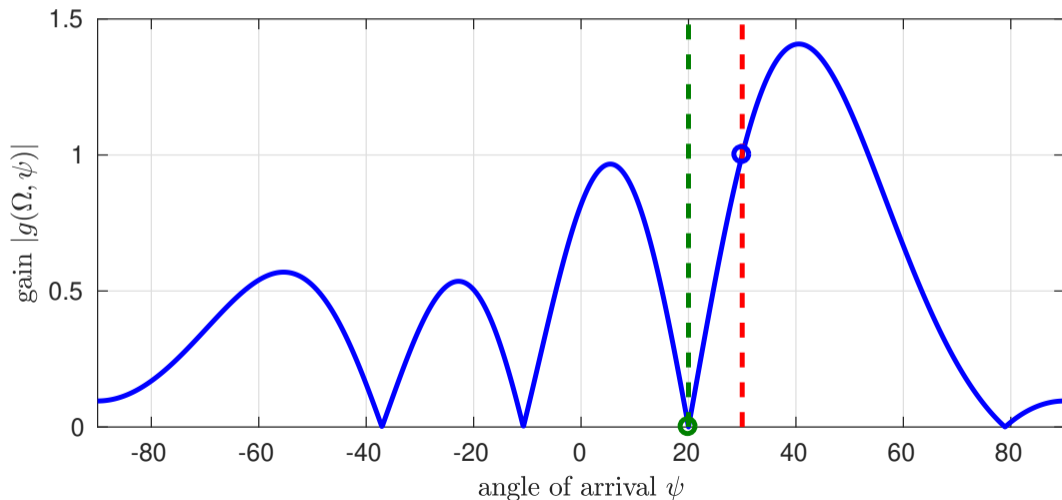
▶ $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:

Beamforming Example — Variable Interferer II

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

Beamforming Example — Variable Interferer II

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

Beamforming Example — Variable Interferer II

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

Beamforming Example — Variable Interferer II

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

Beamforming Example — Variable Interferer II

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

Beamforming Example — Variable Interferer II

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

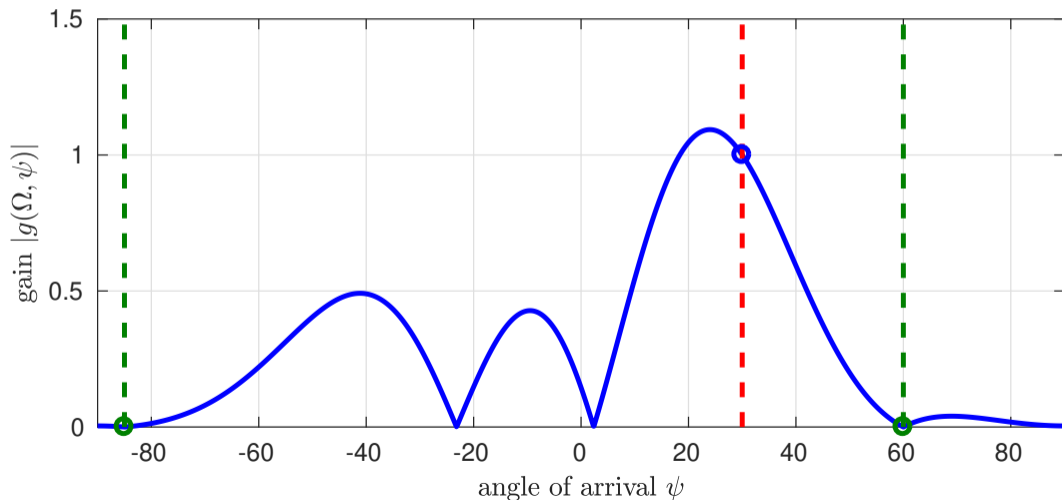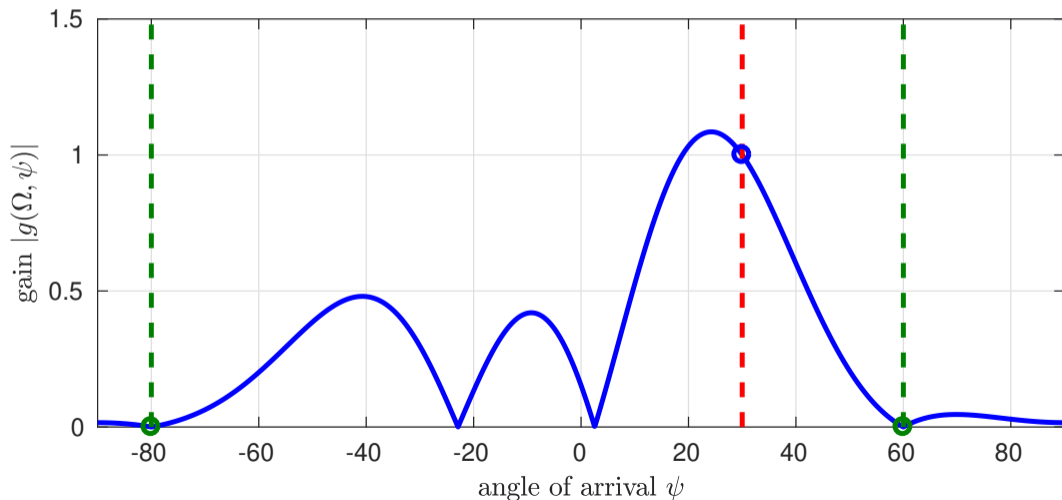▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

Beamforming Example — Variable Interferer II

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

Beamforming Example — Variable Interferer II

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

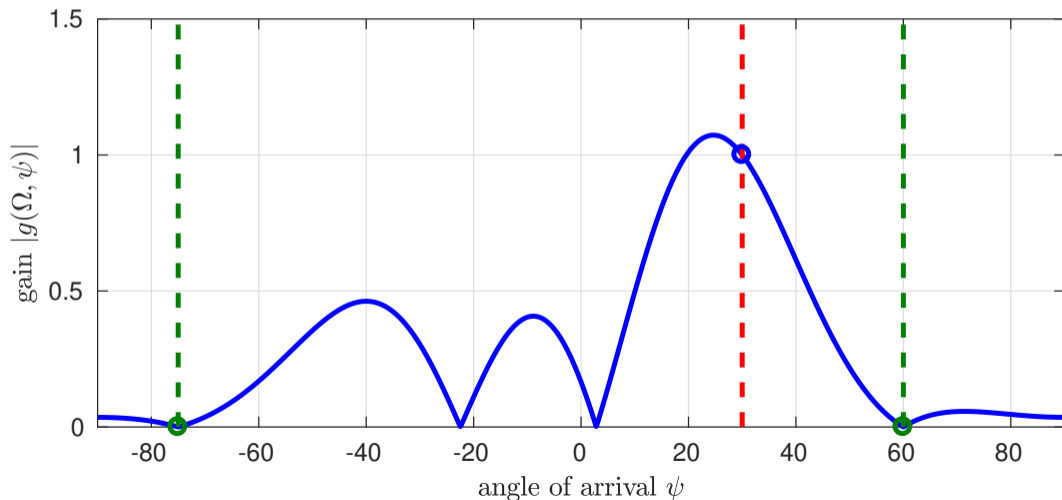▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

Beamforming Example — Variable Interferer II

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

Beamforming Example — Variable Interferer II

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

Beamforming Example — Variable Interferer II

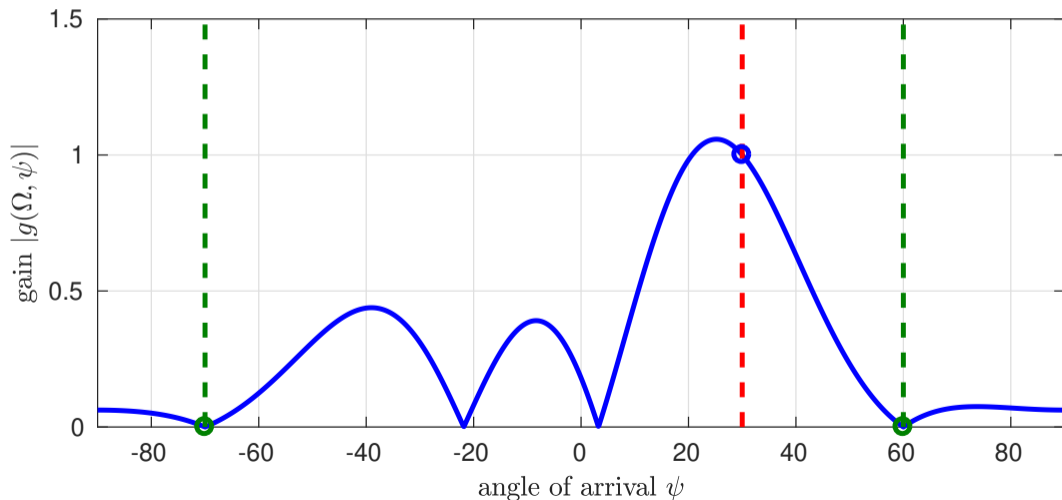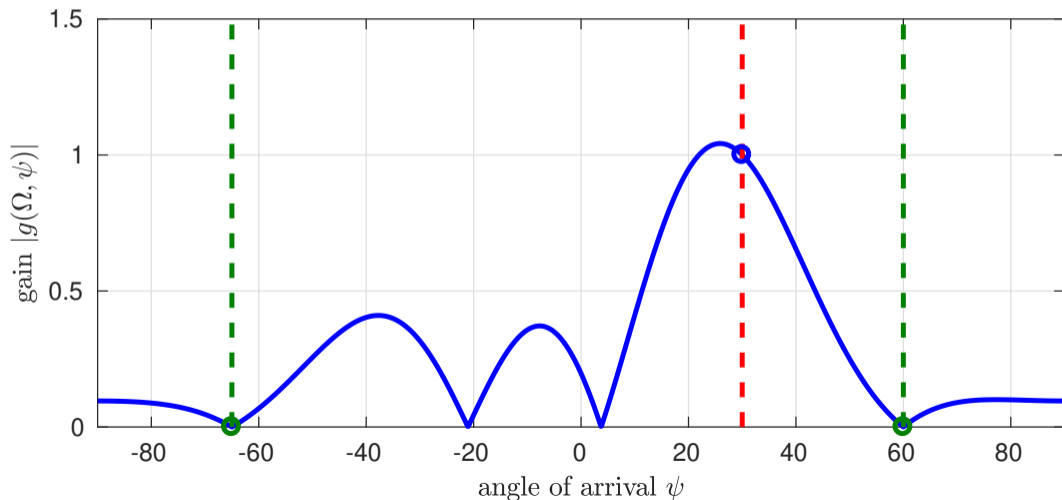▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

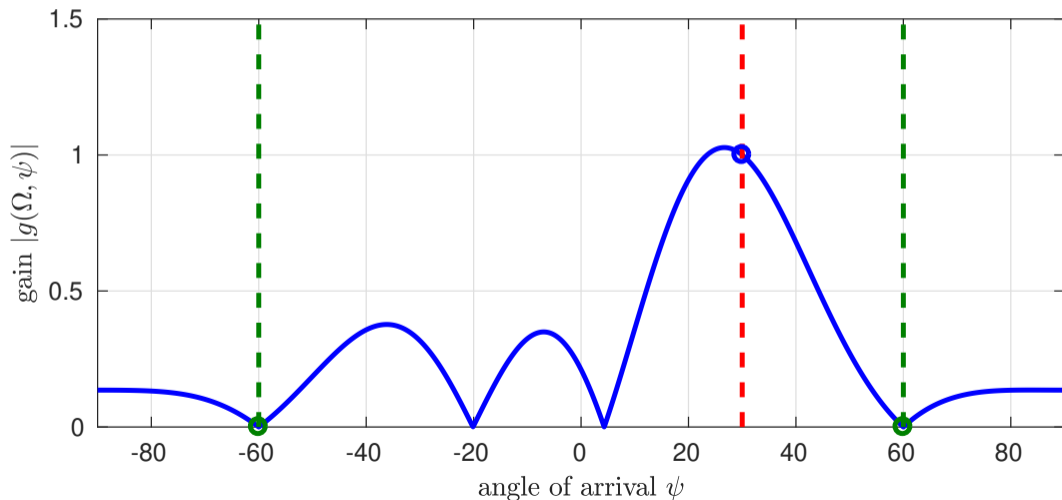▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

Beamforming Example — Variable Interferer II

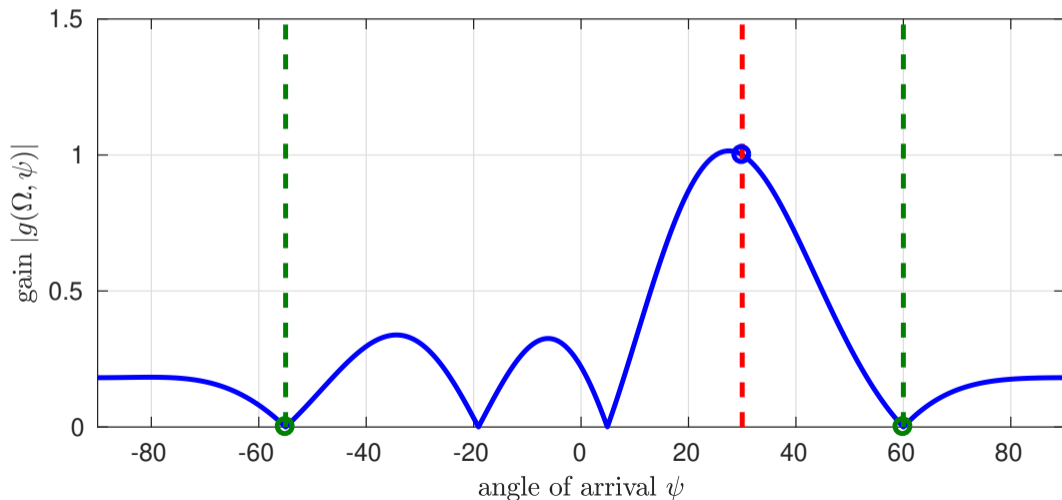- $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

Beamforming Example — Variable Interferer II

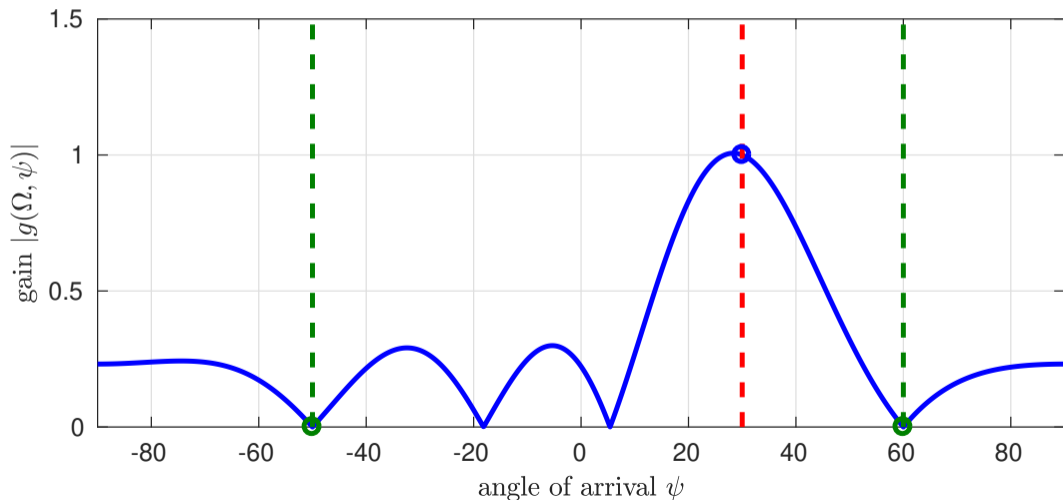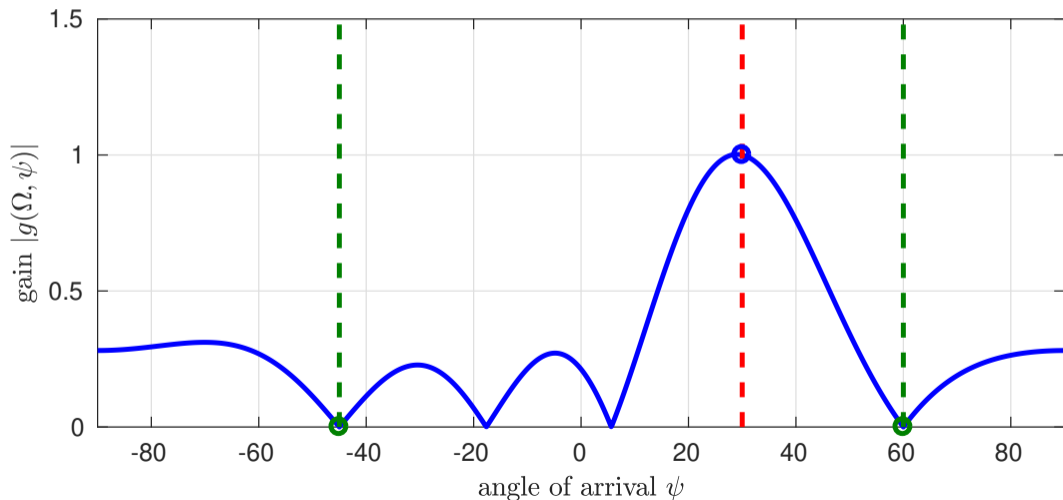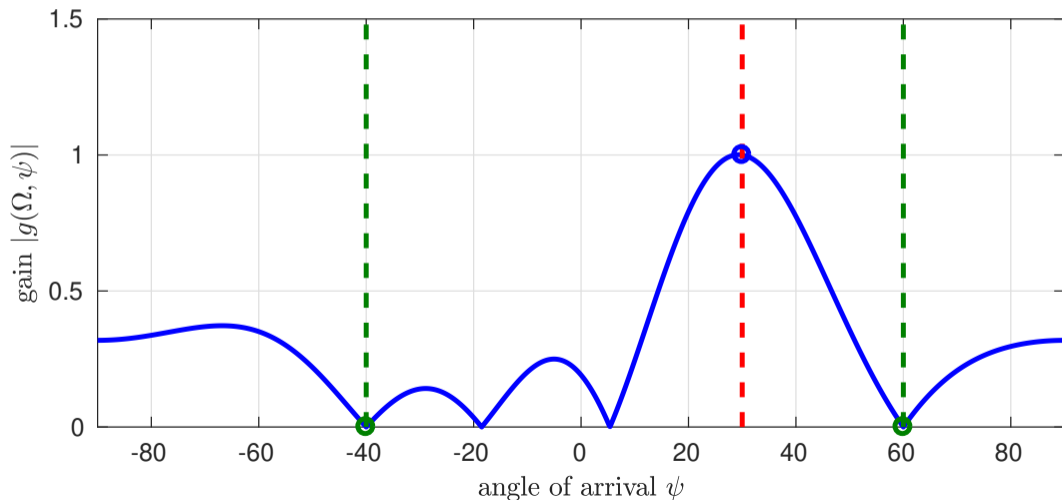▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:

Beamforming Example — Variable Interferer III

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:
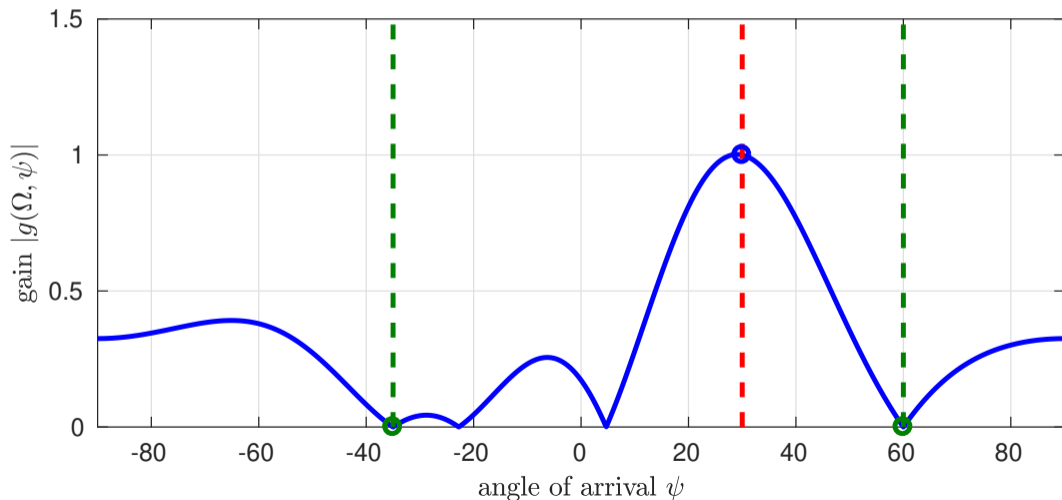
Beamforming Example — Variable Interferer III

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

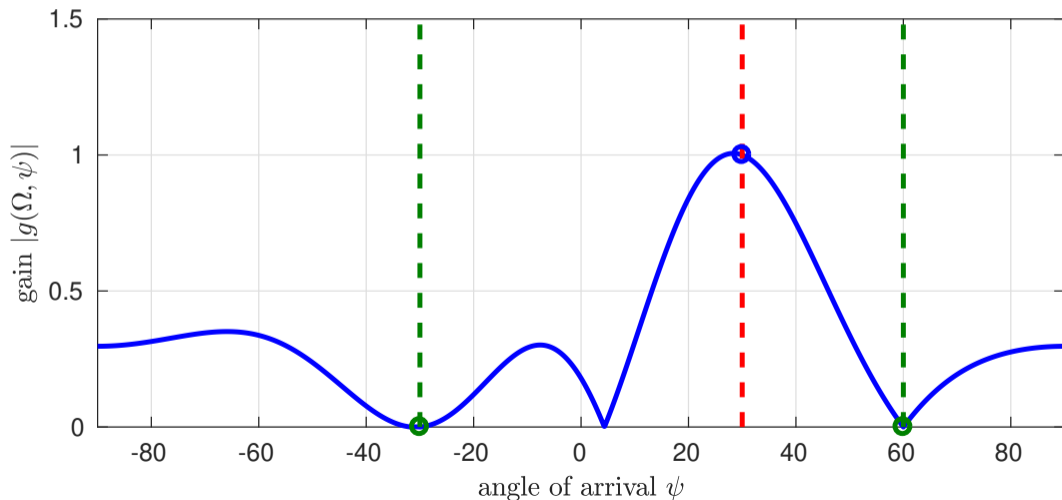▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:
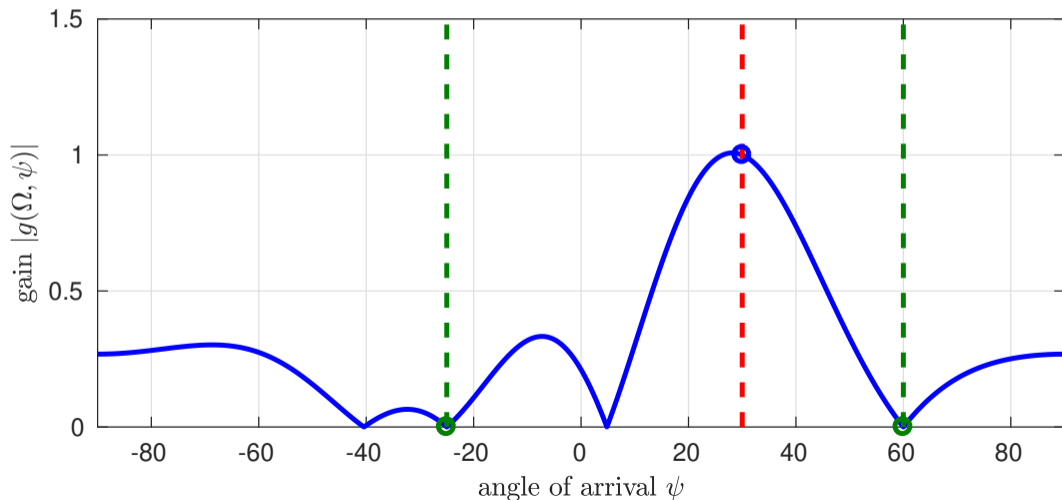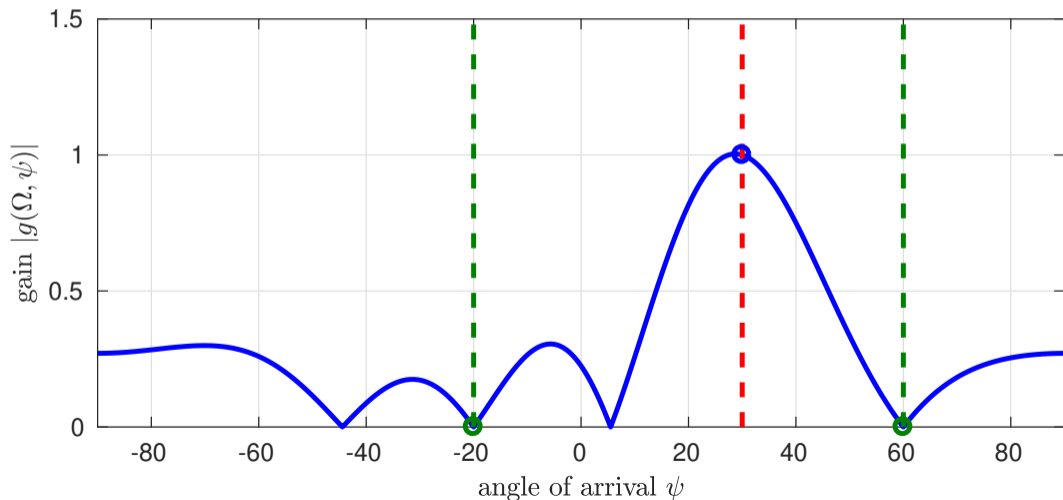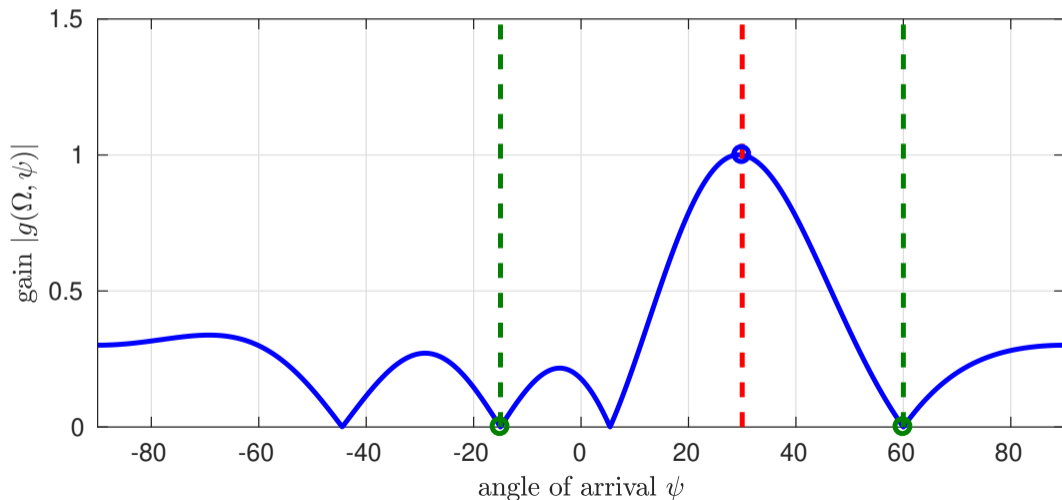
Beamforming Example — Variable Interferer III

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

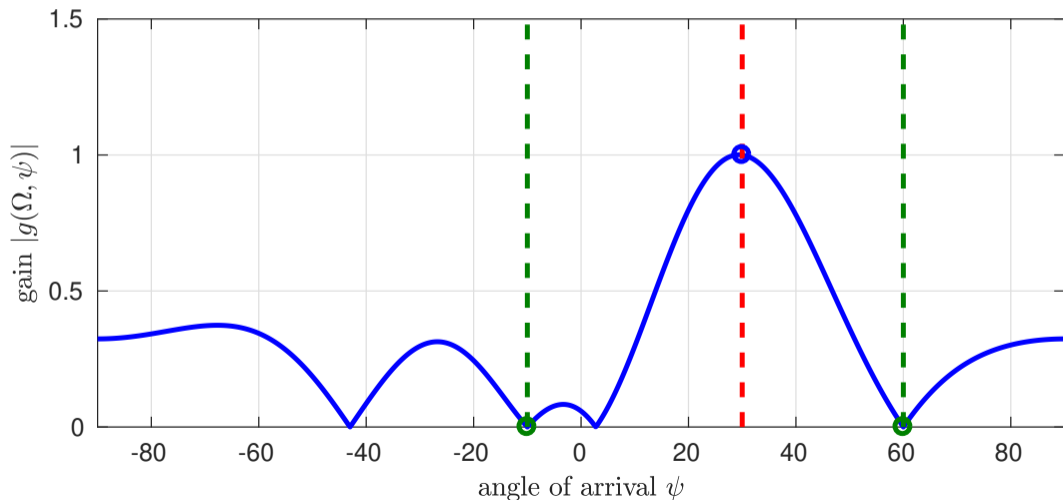▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

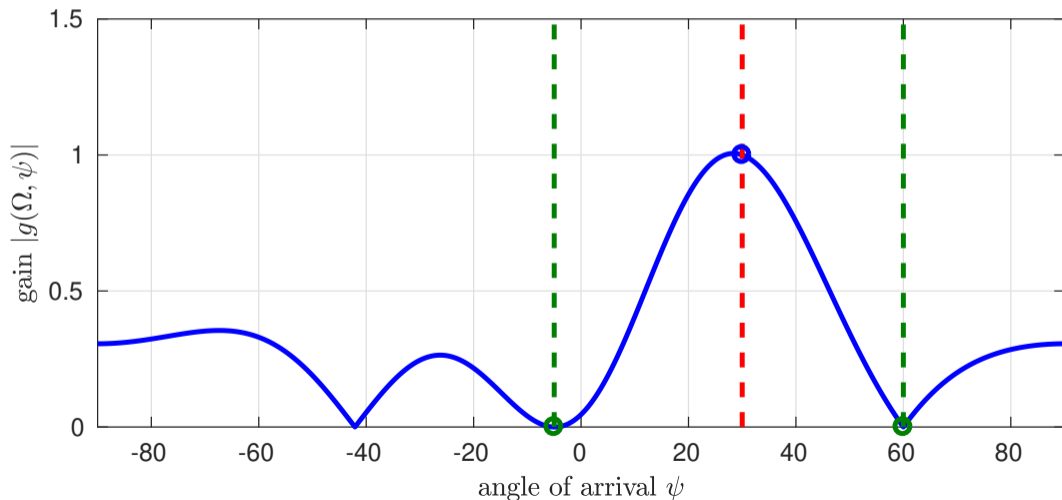▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

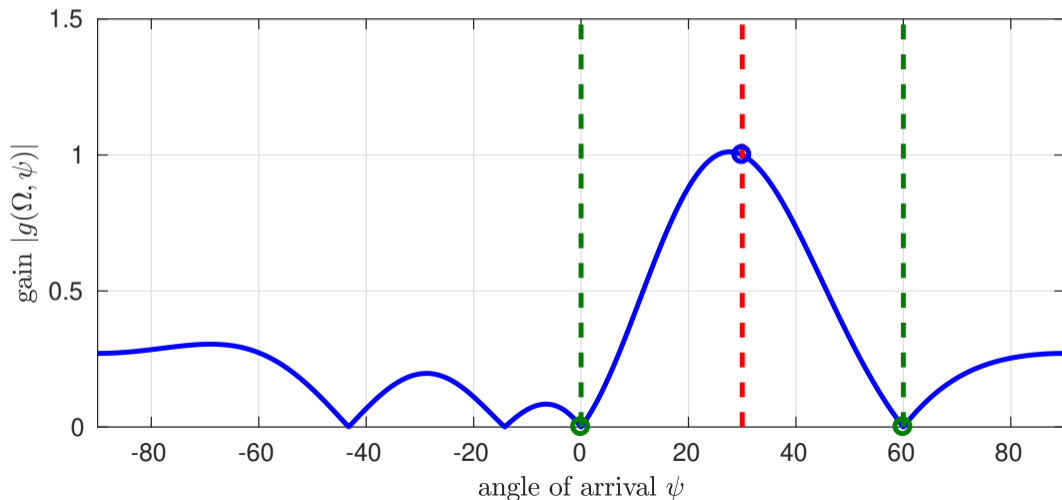▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:
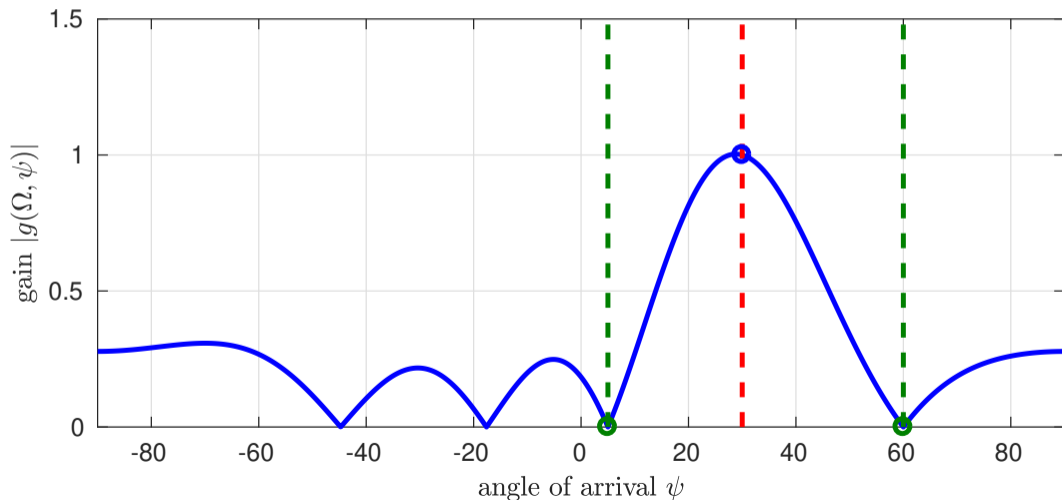
Beamforming Example — Variable Interferer III

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

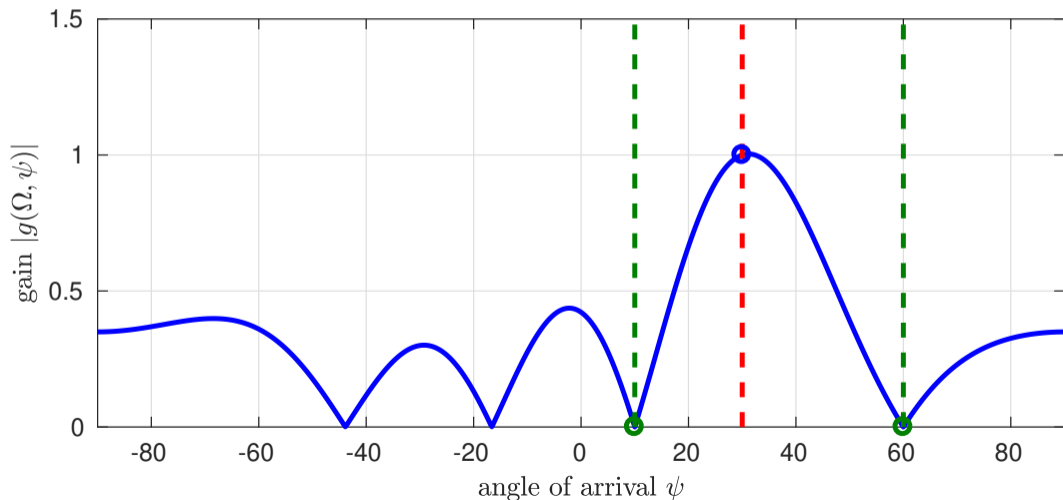▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

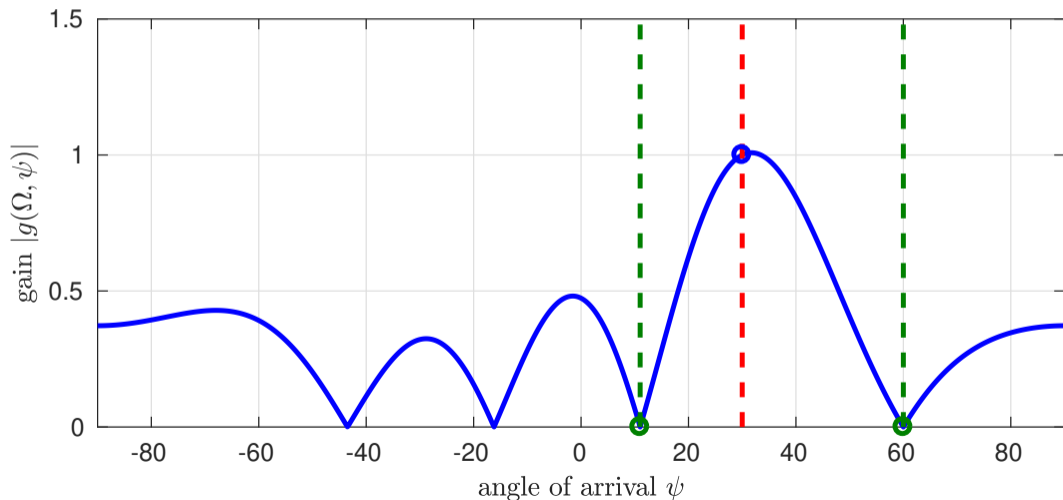▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

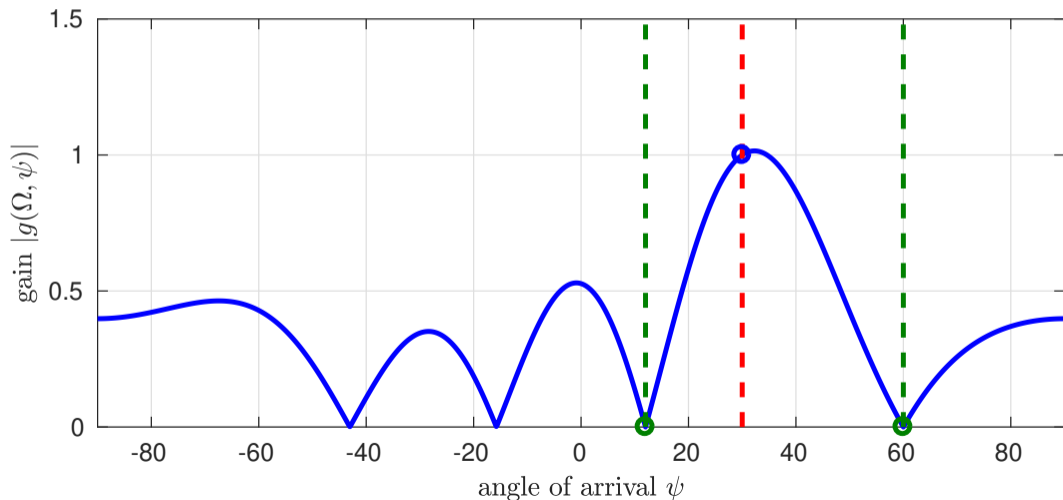▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

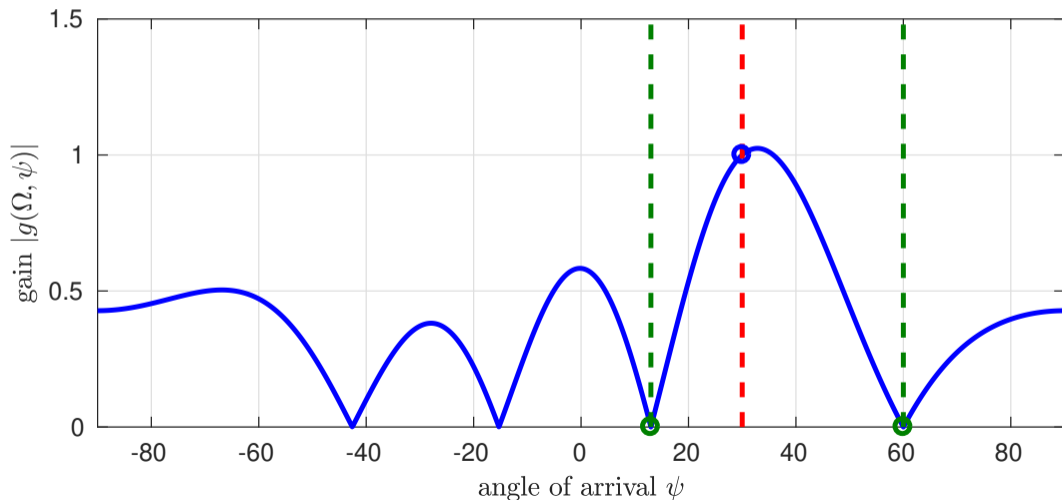▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:
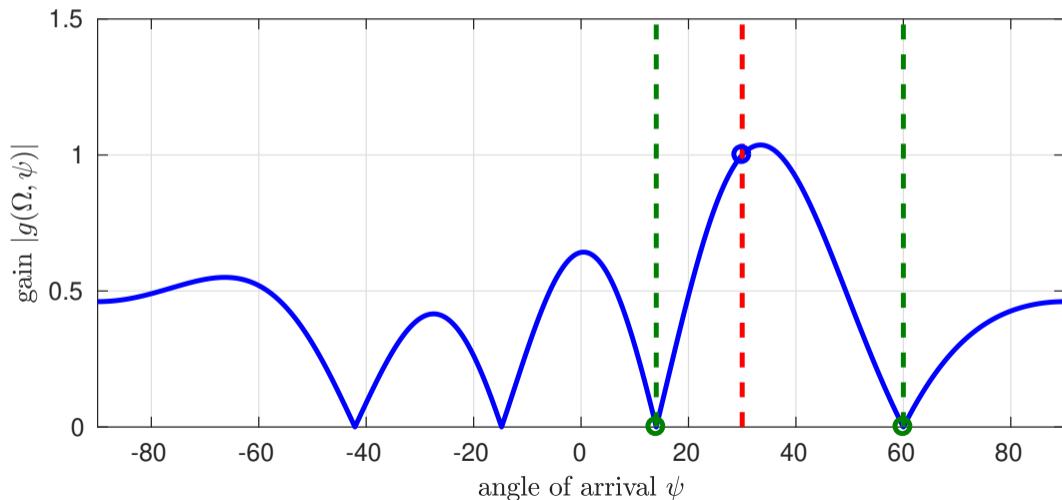
Beamforming Example — Variable Interferer III

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

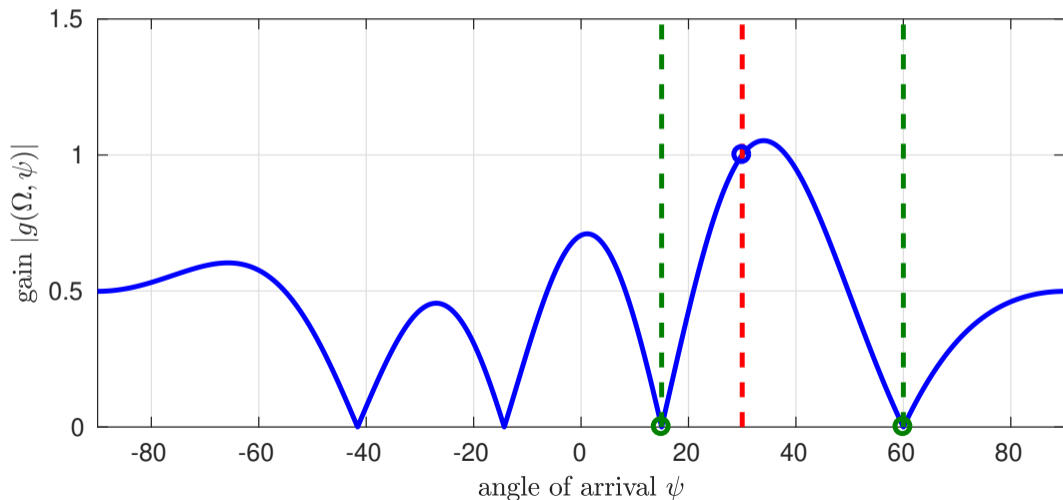▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

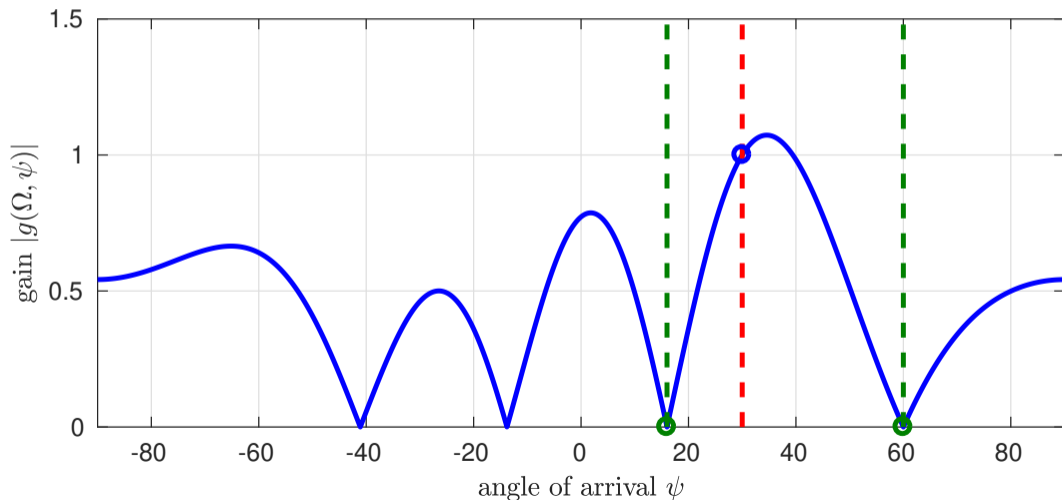▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

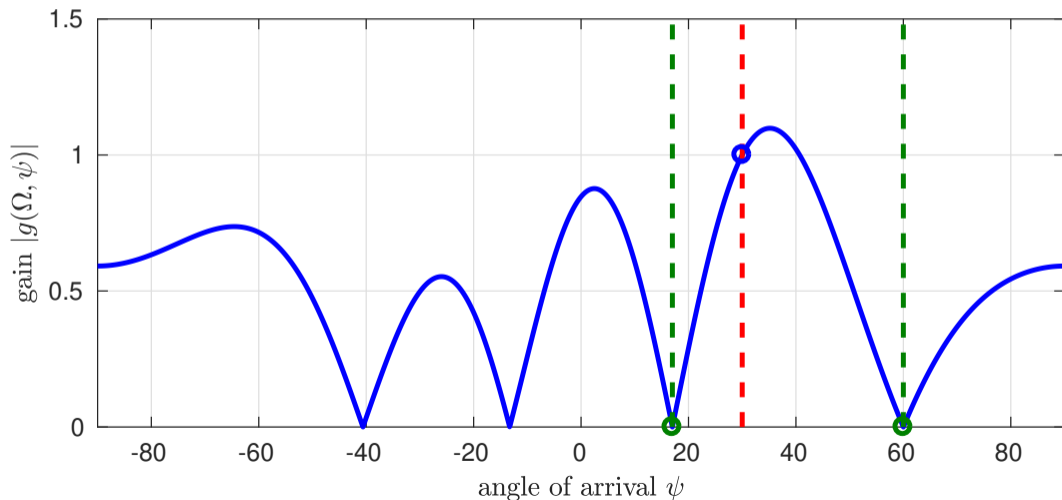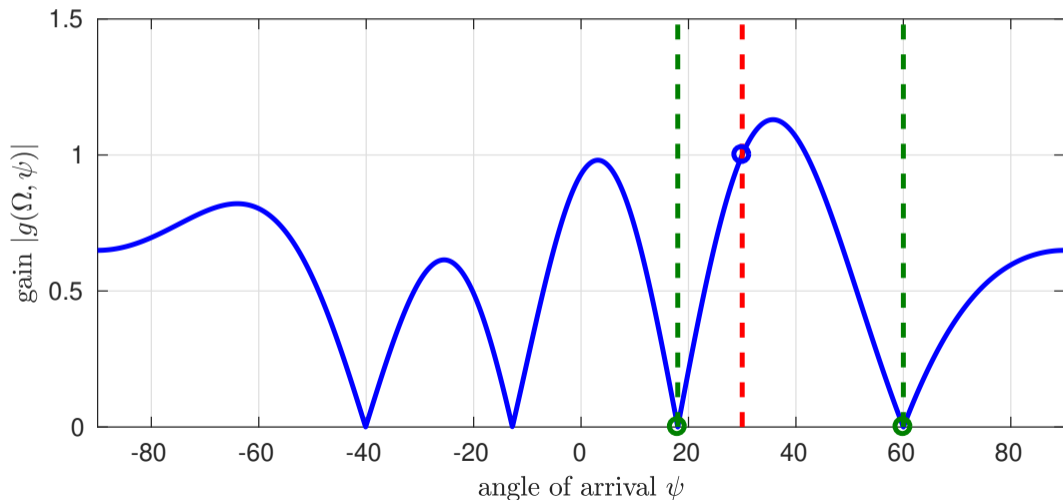- $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:
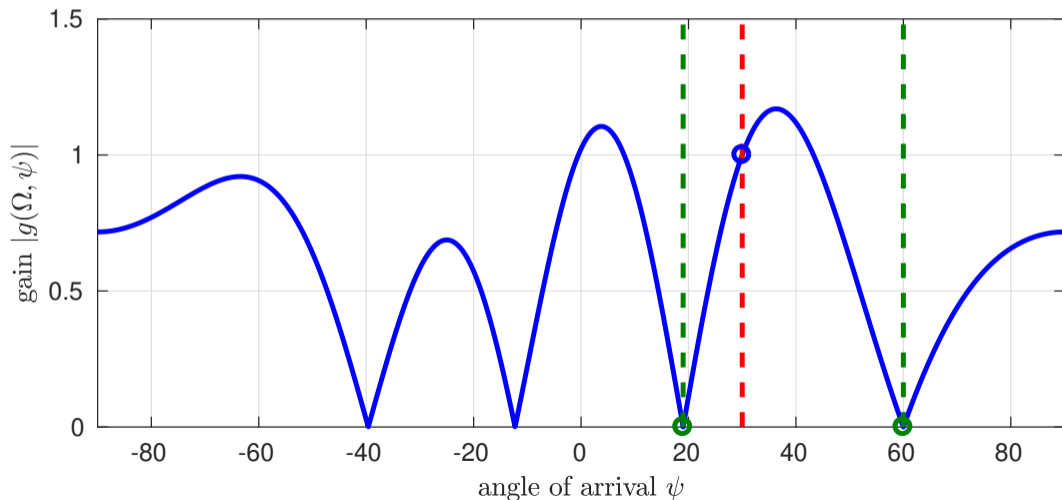
Beamforming Example — Variable Interferer III

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Beamforming Example — Variable Interferer III

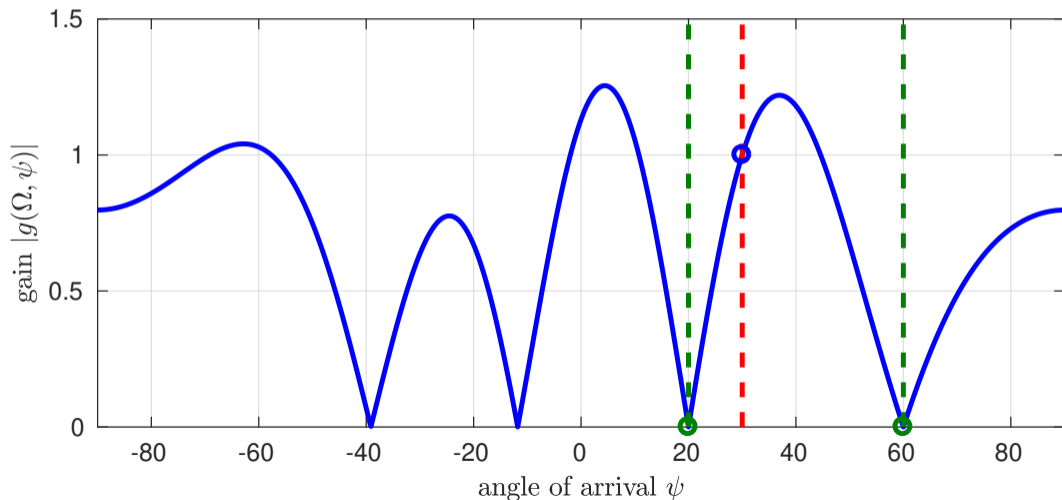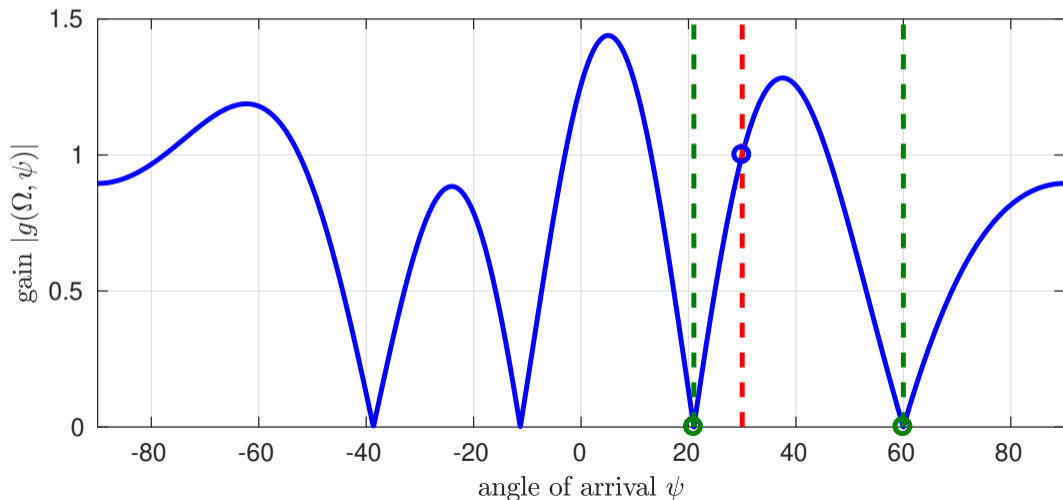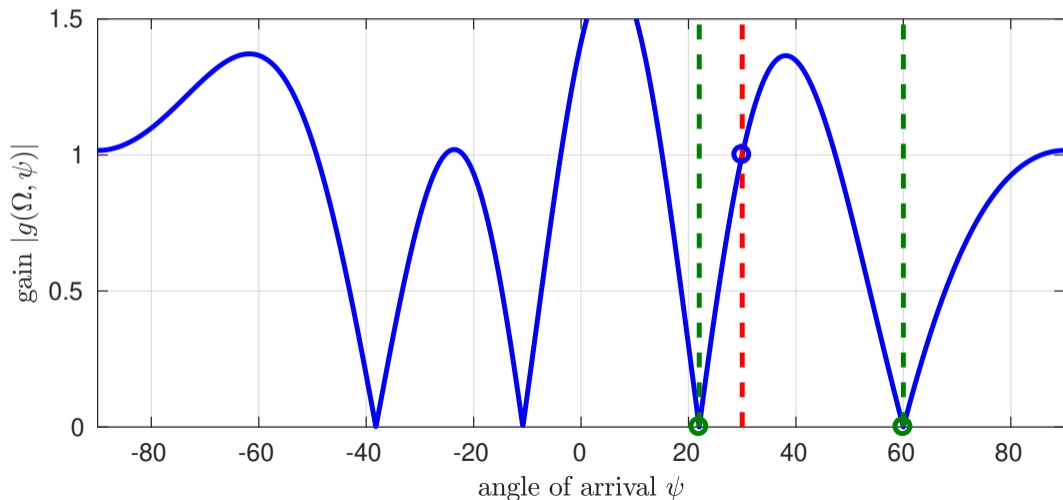▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

▶ $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:

Data Independent Beamforming

- Previous beamformer designs were based on the knowledge of DoA of the signal of interest and of interfering sources;
- remaining degrees of freedom are invested to suppress spatially white noise;
- using the analogy between spatial and temporal processing, classical filter design techniques can be invoked to design arrays with a bandpass-type angular response;
- beamformers based on source parameters (frequency, DoA) rather the actual received waveforms are termed data independent beamformers;
- this is in contrast to statistically optimum beamformers, which take the received signal statistics into account.

1.5 Statistically Optimum Beamforming

▶ Statistically optimum beamformer minimise e.g. the signal power of the beamformer output, $y[n]$;

▶ to avoid the trivial solution $\mathbf{w} = \mathbf{0}$, the signal of interest needs to be protected by constraints;

▶ this results in e.g. the following constrained optimisation problem

$$\min_{\mathbf{w}^*} \mathcal{E}\{|y[n]|^2\} \qquad \text{subject} \quad \text{to} \qquad \mathbf{s}_{\Omega,\vartheta}^{\mathrm{H}} \mathbf{w} = 1 \; ;$$

▶ the solution to this specific statistically optimum beamformer is known as the minimum variance distortionless response (MVDR) [43].

MVDR Beamformer

▶ Solving the MVDR problem: minimise the power of $y[n] = \mathbf{w}^{\mathrm{H}}\mathbf{x}$ subject to the contraint $\mathbf{w}^{\mathrm{H}}\mathbf{s}_{\Omega_0,\vartheta_0} = 1$;

▶ Formulation using a Lagrange multiplier λ:

$$\frac{\partial}{\partial \mathbf{w}^*}\left(\mathbf{w}^{\mathrm{H}}\mathcal{E}\left\{\mathbf{x}\mathbf{x}^{\mathrm{H}}\right\}\mathbf{w} - \lambda(\mathbf{w}^{\mathrm{H}}\mathbf{s}_{\Omega_0,\vartheta_0} - 1)\right) = \mathbf{R}_{xx}\mathbf{w} - \lambda\mathbf{s}_{\Omega_0,\vartheta_0} = \mathbf{0}$$

▶ the solution $\mathbf{w} = \lambda\mathbf{R}_{xx}^{-1}\mathbf{s}_{\Omega_0,\vartheta_0}$ is inserted into the constraint equation to determine λ:

$$\lambda\mathbf{s}_{\Omega_0,\vartheta_0}^{\mathrm{H}}\mathbf{R}_{xx}^{-1}\mathbf{s}_{\Omega_0,\vartheta_0} = 1$$

▶ therefore

$$\mathbf{w}_{\mathrm{MVDR}} = \left(\mathbf{s}_{\Omega_0,\vartheta_0}^{\mathrm{H}}\mathbf{R}_{xx}^{-1}\mathbf{s}_{\Omega_0,\vartheta_0}\right)^{-1}\mathbf{R}_{xx}^{-1}\mathbf{s}_{\Omega_0,\vartheta_0}$$

▶ this statistically optimum beamformer has various other names, e.g. Capon beamformer [8, 42].

MVDR Beamformer — Simple Case

▶ In the case of spatially white noise input, such that

$$\mathbf{R}_{xx} = \sigma_{xx}^2 \mathbf{I} \qquad \longrightarrow \qquad \mathbf{R}_{xx}^{-1} = \sigma_{xx}^{-2} \mathbf{I}$$

the MVDR solution reduces to

$$\mathbf{w}_{\mathrm{MVDR}} = \frac{\mathbf{s}_{\Omega_0, \vartheta_0}}{\|\mathbf{s}_{\Omega_0, \vartheta_0}\|_2^2} = \frac{\mathbf{s}_{\Omega_0, \vartheta_0}}{M} \quad ;$$

▶ this is identical to the data independent beamformer in the absence of interference (i.e. no spatially structured noise);

Generalised Sidelobe Canceller (GSC)

▶ The generalised sidelobe canceller is a specific implementation of the MVDR beamformer; it transforms the constrained MVDR problem into an unconstrained optimisation problem;

▶ a first guess at the solution is performed by the quiescent beamformer $\mathbf{w}_\mathrm{q}$, which is identical to the previously defined data independent beamformer, obtained by inverting the constraint equation

$$\mathbf{C}^\mathrm{H}\mathbf{w}_\mathrm{q} = \mathbf{f} \qquad \longrightarrow \qquad \mathbf{w}_\mathrm{q} = \left(\mathbf{C}^\mathrm{H}\right)^\dagger \mathbf{f}$$
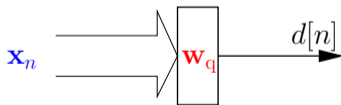
▶ the quiescent beamformer eliminates interferers captured by $\mathbf{C}$ and $\mathbf{f}$, but passes the signal of interest, any interferers unaccounted for in the constraints, and spatially distributed noise.

GSC — Idea

▶ GSE idea: produce array signals that are free of any contribution from the signal of interest, and use the resulting signal vector $\mathbf{u}[n]$ to eliminate remaining interference from the quiescent output:
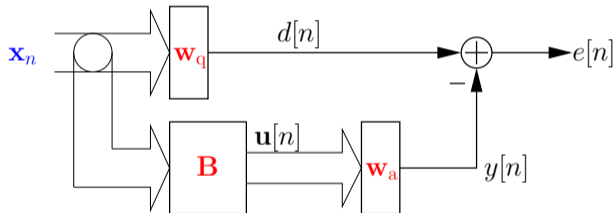
▶ the blocking matrix $\mathbf{B}$ eliminates the signal of interest and any interferers captured by the constraints;

▶ the vector $\mathbf{w}_a$ will be based on the statistics of $\mathbf{u}[n]$ and $d[n]$ to minimise the beamformer output variance $\mathcal{E}\big\{|e[n]|^2\big\}$.

GSC — Blocking Matrix

▶ In order to project away from the constraints,

$$\mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \begin{bmatrix} \mathbf{s}_{\Omega_0,\vartheta_0} & \mathbf{s}_{\Omega_1,\vartheta_1} & \cdots & \mathbf{s}_{\Omega_{r-1},\vartheta_{r-1}} \end{bmatrix} = \mathbf{0}$$

▶ assuming that the r constraints are linearly independent, the singular value decomposition of the constraint matrix yields

$$\mathbf{B} \cdot \begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_0^{\perp} \end{bmatrix} \begin{bmatrix} \sigma_0 & & & \\ & \ddots & & \mathbf{0} \\ & & \sigma_{r-1} & \\ \hline & \mathbf{0} & & \mathbf{0} \end{bmatrix} \cdot \mathbf{V}^{\mathrm{H}} = \mathbf{0}$$

▶ the matrix $\mathbf{U}_0^{\perp} \in \mathbb{C}^{M \times (M-r)}$ spans the nullspace of $\mathbf{C}^{\mathrm{H}}$, and

$$\mathbf{B} = (\mathbf{U}_0^{\perp})^{\mathrm{H}} \in \mathbb{C}^{(M-r) \times M}$$

has the required property, as $(\mathbf{U}_0^{\perp})^{\mathrm{H}} \cdot \begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_0^{\perp} \end{bmatrix} \mathbf{\Sigma} = [\mathbf{0} \ \ \mathbf{I}] \cdot \mathbf{\Sigma} = \mathbf{0}$.

GSC — Unconstrained Optimisation

▶ The beamforming vector $\mathbf{w}_a$ is adjusted to minimise the output power;

▶ the MMSE or Wiener solution is given by

$$\mathbf{w}_a = \mathbf{R}_{uu}^{-1} \cdot \mathbf{p} = \frac{\mathbf{B}\mathbf{R}_{xx}(\mathbf{C}^H)^\dagger \mathbf{f}}{\mathbf{B}\mathbf{R}_{xx}\mathbf{B}^H}$$

with the covariance matrix

$$\mathbf{R}_{uu} = \mathcal{E}\{\mathbf{u}[n] \cdot \mathbf{u}^H[n]\} = \mathbf{B}\,\mathcal{E}\{\mathbf{x}[n] \cdot \mathbf{x}^H[n]\}\,\mathbf{B}^H = \mathbf{B}\mathbf{R}_{xx}\mathbf{B}^H$$

and the cross-correlation vector

$$\mathbf{p} = \mathcal{E}\{\mathbf{u}[n] \cdot d^*[n]\} = \mathbf{B}\mathbf{R}_{xx}\mathbf{w}_q$$

▶ iterative optimisation schemes, such as the least mean squares (LMS) algorithm [16, 64] may be used to approximate the MMSE solution.

1.6 Beamforming and MIMO Processing

University of
Strathclyde
Engineering

- Assume a transmission scenario with an M-element transmit (Tx) antenna array and an N-element receive (Rx) array;
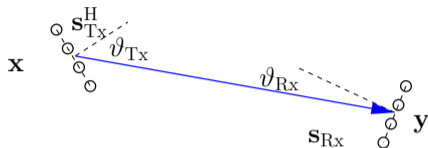
- in the absence of scatterers and any attenuation, the farfield transmission from the transmit antenna is characterised by a steering vector $\mathbf{s}_{\mathrm{Tx}}^{\mathrm{H}}$;
- the incoming waveform at the Rx device is described by another steering vector $\mathbf{s}_{\mathrm{Rx}}$;
- the overall MIMO system between a Tx vector $\mathbf{x} \in \mathbb{C}^M$ and an Rx vector $\mathbf{y} \in \mathbb{C}^N$ is described as

$$\mathbf{y} = \mathbf{s}_{\mathrm{Rx}} \cdot \mathbf{s}_{\mathrm{Tx}}^{\mathrm{H}} \cdot \mathbf{x} \;=\; \mathbf{H}\mathbf{x}$$

- the MIMO system matrix $\mathbf{H} = \mathbf{s}_{\mathrm{Rx}} \cdot \mathbf{s}_{\mathrm{Tx}}^{\mathrm{H}}$ is rank one only.

MIMO Requirements

▶ The farfield assumption is convenient for beamforming, but leads to a rank one MIMO system matrix which is incompatible with the desire to extract multiple independent subchannels or with to achieve diversity;

▶ rich scattering in connection with MIMO usually implies multiple reflections of signals;

▶ together with a sufficiently large antenna spacing means that the farfield assumption is invalid and the MIMO system matrix is not rank deficient;

▶ some suggestions of "sufficiently large spacing" imply an antenna element distance of $d > 10\lambda$;

▶ recall spatial sampling requires $d < \frac{1}{2}\lambda$!

Beamforming with Spatial Aliasing

▶ For a flexible spatial sampling with $d = \alpha\lambda$, $0 < \alpha \in \mathbb{R}$, the steering vector for a waveform with normalised angular frequency Ω and DoA ϑ is

$$\mathbf{y} = e^{j\Omega n} \begin{bmatrix} 1 \\ e^{j2\alpha\Omega\sin(\vartheta)} \\ \vdots \\ e^{j2\alpha(M-1)\Omega\sin(\vartheta)} \end{bmatrix} = \mathbf{s}_{2\alpha\Omega,\vartheta} \cdot e^{j\Omega}$$

▶ inspecting $\mathbf{s}_{2\alpha\Omega,\vartheta}$ the steering vector is aliased to a different frequency $2\alpha\Omega$;

▶ although the correct frequency can be retrieved unambigiously from temporal sampling of any array element, at Ω various different angles could provide the same steering vector $\mathbf{s}_{2\alpha\Omega,\vartheta}$;

▶ the array performs spatial undersampling, resulting in spatial aliasing.

Spatial Undersampling Example

▶ Beamforming parameters: signal of interest with $\Omega = \frac{\pi}{2}$, direction of arrival $\vartheta = 30°$, $M = 32$ array elements;

▶ data independent beamformer design with correct spatial sampling ($d = \lambda/2$) and incorrect spatial sampling ($d = 10\lambda$):
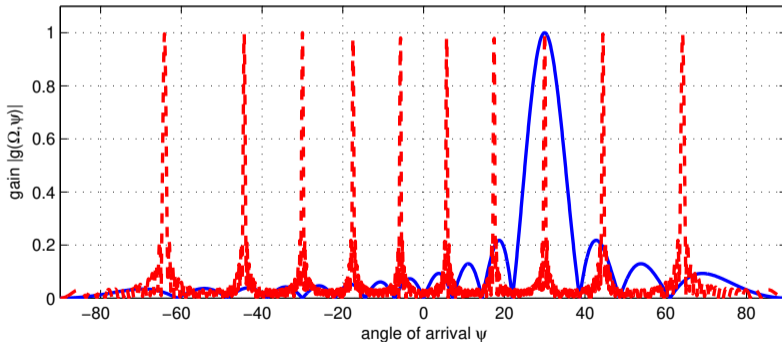
▶ MIMO systems perform beamforming, but may dissipate energy into aliased directions.

1.7 Narrowband Signals

▶ We have previously assumed that a narrowband signal is a complex exponential, $e^{j\Omega n}$;

▶ this permitted to characterise the signal received at the array by means of a steering vector that only depends on Ω and the direction of arrival;

▶ we now relax this restriction: in practice, we deal with bandpass signals of finite bandwidth ω_b and centre frequency ω_c;

▶ for the lth source:

$$u_\ell(t) = \tilde{u}_l(t) \cdot e^{j\omega_c t}, \tag{1}$$

where $\tilde{u}_\ell(t)$ is a baseband signal;
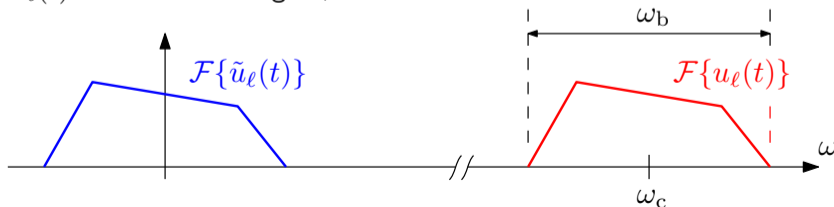
Narrowband Assumption

▶ For a signal to be considered narrowband, the propagation delay across the array must be small w.r.t. any changes in the baseband signal $\tilde{u}_\ell(t)$ (or of the envelope of $u_\ell(t)$);
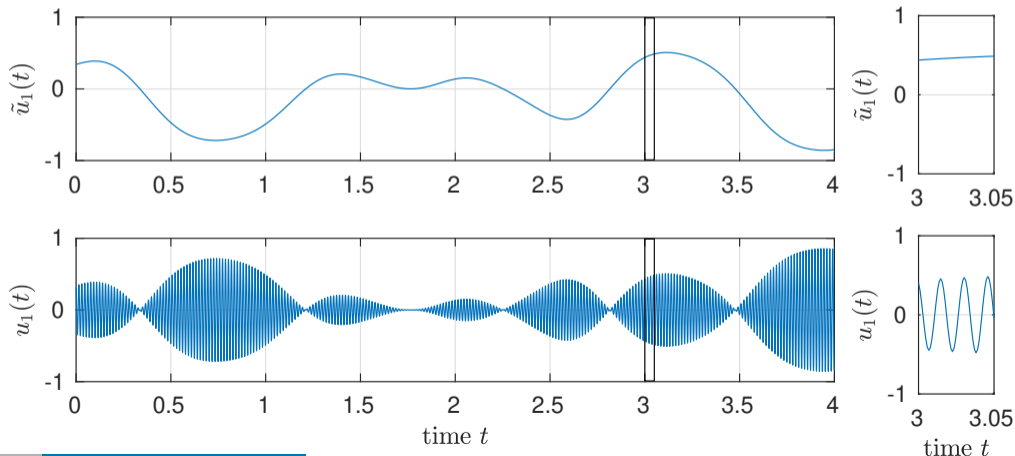
Received Narrowband Array Signal

▶ An array receives a single modulated bandpass signal $u_\ell(t)$:

$$\mathbf{x}(t) = \left[\begin{array}{c} u_\ell(t - \tau_1) \\ \vdots \\ u_\ell(t - \tau_M) \end{array} \right] = \left[\begin{array}{c} \tilde{u}_\ell(t - \tau_1) \\ \vdots \\ \tilde{u}_\ell(t - \tau_M) \end{array} \right] \cdot \left[\begin{array}{c} e^{j\omega_c(t-\tau_1)} \\ \vdots \\ e^{j\omega_{cc}(t-\tau_M)} \end{array} \right] \approx \tilde{u}_\ell(t - \tau_1) e^{j\omega_c - \tau_1} \mathbf{s}_{\vartheta_\ell, \omega_c}$$

(2)

▶ after sampling: $\mathbf{x}[n] = \tilde{u}_\ell[n] \cdot e^{j\omega_c \tau_1} \cdot \mathbf{s}_{\vartheta_\ell, \Omega_c}$;

▶ for the narrowband covariance matrix:

$$\mathbf{R} = \mathcal{E}\big\{\mathbf{x}[n]\mathbf{x}^H[n]\big\} = \mathcal{E}\{\tilde{u}_\ell[n]\tilde{u}_\ell^*[n]\} \, \mathbf{s}_{\vartheta_\ell, \Omega_c} \mathbf{s}_{\vartheta_\ell, \Omega_c}^H = \sigma_\ell^2 \mathbf{s}_{\vartheta_\ell, \Omega_c} \mathbf{s}_{\vartheta_\ell, \Omega_c}^H$$

(3)

▶ for L independent source signals, $\mathcal{E}\{\tilde{u}_\ell[n]\tilde{u}_k^*[n]\} = 0$ for $\ell \neq k$; therefore in the noise-free case:

$$\mathbf{R} = \sum_{\ell=1}^{L} \sigma_\ell^2 \mathbf{s}_{\vartheta_\ell, \Omega_c} \mathbf{s}_{\vartheta_\ell, \Omega_c}^H \ .$$

(4)

Narrowband versus Broadband

How long can a array signal be regarded as narrowband?

- Compton [9]: signals at opposite ends of the array must not be decorelated;
- some rule of thumb: fractional bandwidth $\omega_{\mathrm{b}}/\omega_{\mathrm{c}} \ll 1$ (typically smaller than 5%);
- these rules are somewhat 'fuzzy'; recall that

$$\mathbf{R} = \sum_{\ell=1}^{L} \sigma_\ell^2 \mathbf{s}_{\vartheta_\ell, \Omega_{\mathrm{c}}} \mathbf{s}_{\vartheta_\ell, \Omega_{\mathrm{c}}}^{\mathrm{H}} \; ;$$

- this matrix possesses rank L as long as the steering vectors are linearly independent;
- if the narrowband assumption is no longer satisfied, the approximation in (2) becomes inaccurate, and the rank of $\mathbf{R}$ will increase [65, 66];
- this can also be tied to the array performance [10, 44, 38, 37];
- when must a signal be considered broadband? John McWhirter's "If you need a tap delay line." captures the ambiguity well!

1.8 Broadband MVDR Beamformer

▶ Each sensor is followed by a tap delay line of dimension L, giving a total of ML coefficients in a vector $\mathbf{v} \in \mathbb{C}^{ML}$ [6]
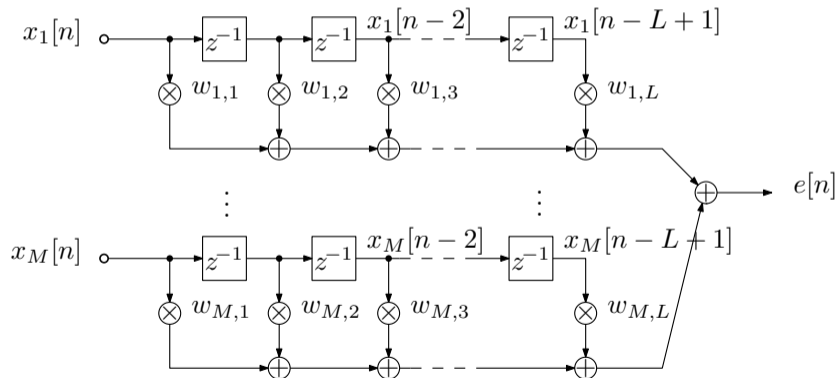
Broadband MVDR Beamformer Constraints

- A larger input vector $\mathbf{x}_n \in \mathbb{C}^{ML}$ is generated, also including lags;

- the general approach is similar to the narrowband system, minimising the power of $e[n] = \mathbf{v}^{\mathrm{H}} \mathbf{x}_n$;

- however, we require several constraint equations to protect the signal of interest, e.g.

$$\mathbf{C} = [\mathbf{s}(\vartheta_{\mathrm{s}}, \Omega_0), \ \mathbf{s}(\vartheta_{\mathrm{s}}, \Omega_1) \ \ldots \ \mathbf{s}(\vartheta_{\mathrm{s}}, \Omega_{L-1})] \tag{5}$$

- these L constraints pin down the response to unit gain at L separate points in frequency:

$$\mathbf{C}^{\mathrm{H}} \mathbf{v} = \mathbf{1} ; \tag{6}$$

- generally $\mathbf{C} \in \mathbb{C}^{ML \times L}$, but simplifications can be applied if the look direction is towards broadside.

Broadband Generalised Sidelobe Canceller

- A quiescent beamformer $\mathbf{v}_q = \left(\mathbf{C}^H\right)^\dagger \mathbf{1} \in \mathbb{C}^{ML}$ picks the signal of interest;

- the quiescent beamformer is optimal for AWGN but generally passes structured interference;

- the output of the blocking matrix $\mathbf{B}$ contains interference only, which requires $[\mathbf{B}\mathbf{C}]$ to be unitary; hence $\mathbf{B} \in \mathbb{C}^{ML \times (M-1)L}$;

- an adaptive noise canceller $\mathbf{v}_a \in \mathbb{C}^{(M-1)L}$ aims to remove the residual interference:
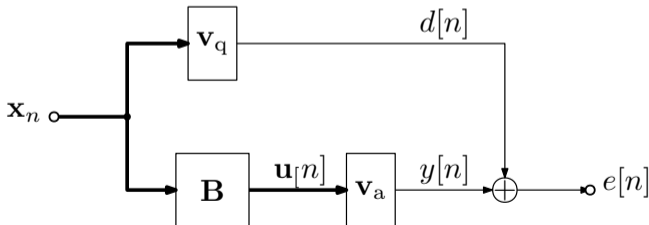
- note: all dimensions are determined by $\{M, L\}$.

Broadband Beamformer Example

▶ We assume a signal of interest from $\vartheta = 30°$;

▶ three interferers with angles $\vartheta_i \in \{-40°, -10°, 80°\}$ active over the frequency range $\Omega = 2\pi \cdot [0.1;\ 0.45]$ at signal to interference ratio of -40 dB;
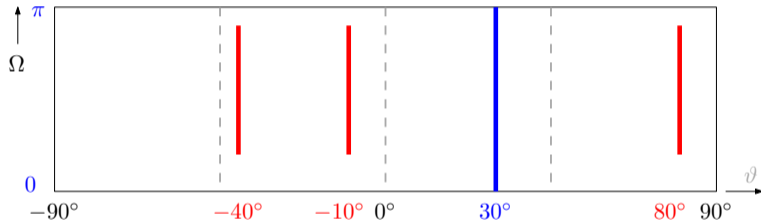
▶ $M = 8$ element linear uniform array is also corrupted by spatially and temporally white additive Gaussian noise at 20 dB SNR;

▶ tap-delay-line length: $L = 150$;

▶ cost per iteration: approx. 2 MMACs (standard), can be reduced to 10 kMACs when efficiently implemented.

Broadband Quiescent Beamformer

▶ Directivity pattern of quiescent standard broadband beamformer:
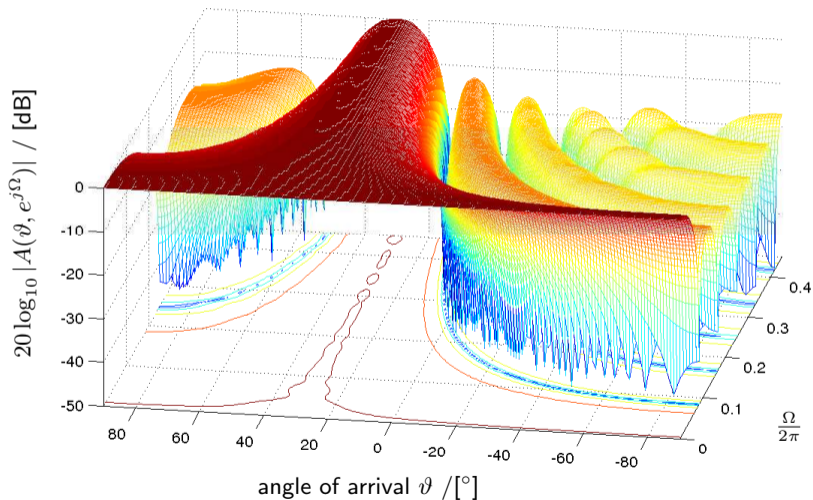
Optimised Broadband Beamformer

▶ Directivity pattern of the broadband beamformer:
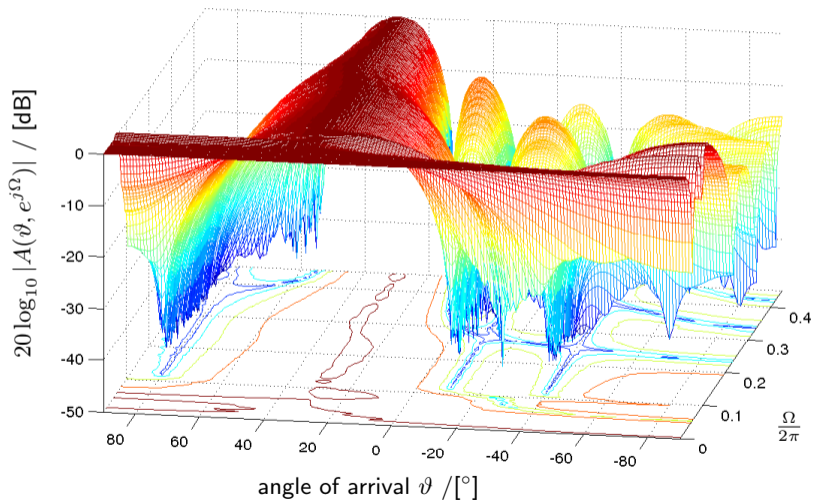
1.9 Summary

▶ Spatial sampling by an array of sensors (e.g. antenna elements) has been explored;

▶ the spatial data window of a narrowband source is characterised by the steering vector;

▶ appropriate data independent beamformers can be designed based on the steering vectors of desired sources and interferers;

▶ statistically optimum beamformers are based on the signal statistics;

▶ a specific statistically optimum beamformer, the generalised sidelobe canceller, has been reviewed — it uses signal statistics to improve the performance of a data independent beamformer derived from the constraint equations;

▶ some similarities and differences between beamforming and MIMO systems have been highlighted;

▶ broadband beamforming requires the inclusion of tap delay lines.

1.10 Related Broadband Beamforming Work

University of
Strathclyde
Engineering

▶ General wideband beamforming: [23];

▶ time domain adaptive broadband beamforming: [6, 7, 15, 18, 27, 35, 43];

▶ discrete Fourier transform domain processing: [21, 36, 11, 55]

▶ subband domain beamforming [25, 45, 60, 61, 62, 63, 59, 55];

▶ frequency-invariant broadband beamforming [22, 26, 27, 49];

▶ polynomial matrix-based beamforming related
work [1, 2, 3, 4, 12, 13, 19, 20, 29, 30, 34, 33, 32, 46, 47, 48, 54, 31] based on polynomial
eigenvalued decomposition theory [51, 52, 5] and
algorithms [28, 39, 41, 40, 58, 53, 57, 56]

References I

[1] M. Alrmah, J. Corr, A. Alzin, K. Thompson, and S. Weiss.
Polynomial subspace decomposition for broadband angle of arrival estimation.
In *Sensor Signal Processing for Defence*, pages 1–5, Edinburgh, Scotland, Sept 2014.

[2] M. Alrmah and S. Weiss.
Filter bank based fractional delay filter implementation for widely accurate broadband steering vectors.
In *5th IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, Saint Martin, December 2013.

[3] M. Alrmah, S. Weiss, and S. Lambotharan.
An extension of the MUSIC algorithm to broadband scenarios using polynomial eigenvalue decomposition.
In *19th European Signal Processing Conference*, pages 629–633, Barcelona, Spain, August 2011.

[4] A. Alzin, F. Coutts, J. Corr, S. Weiss, I. K. Proudler, and J. A. Chambers.
Adaptive broadband beamforming with arbitrary array geometry.
In *IET/EURASIP Intelligent Signal Processing*, London, UK, December 2015.

[5] G. Barbarino and V. Noferini.
On the Rellich eigendecomposition of para-Hermitian matrices and the sign characteristics of ∗-palindromic matrix polynomials.
Linear Algebra and its Applications, 672:1–27, Sept. 2023.

[6] K. M. Buckley.
Spatial/Spectral Filtering with Linearly Constrained Minimum Variance Beamformers.
IEEE Transactions on Acoustics, Speech, and Signal Processing, ASSP-35(3):249–266, March 1987.

[7] K. M. Buckley and L. J. Griffith.
An Adaptive Generalized Sidelobe Canceller with Derivative Constraints.
IEEE Transactions on Antennas and Propagation, 34(3):311–319, March 1986.

[8] J. Capon.
High-resolution frequency-wavenumber spectrum analysis.
Proceedings of the IEEE, 57(8):1408–1418, Aug 1969.

References II

[9] R. T. Compton.
     Adaptive Antennas.
     Prentice Hall, 1988.

[10] R. T. Compton.
     "The Bandwidth Performance of a Two-Element Adaptive Array with Tapped Delay-Line Processing".
     IEEE Transactions on Antennas and Propagation, Vol.36(No.1):pp.4–14, January 1988.

[11] R. T. Compton.
     "The Relationship Between Tapped Delay-Line and FFT Processing in Adaptive Arrays".
     IEEE Transactions on Antennas and Propagation, Vol.36(No.1):pp.15–26, January 1988.

[12] F. Coutts, K. Thompson, S. Weiss, and I. Proudler.
     Impact of fast-converging PEVD algorithms on broadband AoA estimation.
     In *Sensor Signal Processing for Defence Conference*, pages 1–5, London, UK, December 2017.

[13] F. K. Coutts, I. K. Proudler, and S. Weiss.
     Efficient implementation of iterative polynomial matrix evd algorithms exploiting structural redundancy and parallelisation.
     IEEE Transactions on Circuits and Systems I: Regular Papers, 66(12):4753–4766, Dec. 2019.

[14] G. H. Golub and C. F. Van Loan.
     Matrix Computations.
     John Hopkins University Press, Baltimore, Maryland, 3rd edition, 1996.

[15] K. M. Griffith, L. J.and Buckley.
     Quiescent Pattern Control in Linearly Constrained Adaptive Arrays.
     IEEE Transactions on Acoustics, Speech, and Signal Processing, 35(7):917–926, July 1987.

[16] S. Haykin.
Adaptive Filter Theory.
Prentice Hall, Englewood Cliffs, 2nd edition, 1991.

[17] S. Haykin and K. R. Liu.
Handbook on Array Processing and Sensor Networks.
Wiley-IEEE Press, 2010.

[18] M. W. Hoffman and K. M. Buckley.
Robust Time-Domain Processing of Broadband Microphone Array Data.
IEEE Transactions on Speech and Audio Processing, 3(3):193–203, May 1995.

[19] A. Hogg, V. Neo, S. Weiss, C. Evers, and P. Naylor.
A polynomial eigenvalue decomposition MUSIC approach for broadband sound source localization.
In *IEEE Workshop on Applications of Signal Processing to Audio and Acoustics*, New Paltz, NY, Oct. 2021.

[20] C. L. Koh, S. Redif, and S. Weiss.
Broadband GSC beamformer with spatial and temporal decorrelation.
In *17th European Signal Processing Conference*, pages 889–893, Glasgow, Scotland, August 2009.

[21] C. L. Koh and S. Weiss.
Overlap-Save Broadband GSC Beamforming Algorithms Using Alternative Constraints.
In *International ITG/IEEE Workshop on Smart Antennas*, Duisburg, Germany, April 4-5 2005.

[22] W. Liu and S. Weiss.
New Class of Broadband Arrays with Frequency-Invariant Beam Patterns.
In *IEEE International Conference on Acoustics, Speech, and Signal Processing*, volume II, pages 185–188, Montreal, Canada, May 17–21 2004.

References IV

[23] W. Liu and S. Weiss.
Wideband Beamforming — Concepts and Techniques.
Wiley, 2010.

[24] W. Liu, S. Weiss, and L. Hanzo.
A Novel Method for Partially Adaptive Broadband Beamforming.
In *IEEE Workshop on Signal Processing Systems*, pages 361–372, Antwerp, September 2001.

[25] W. Liu, S. Weiss, and L. Hanzo.
Subband Adaptive Generalized Sidelobe Canceller for Broadband Beamforming.
In *11th IEEE Signal Processing Workshop on Statistical Signal Processing*, volume 1, pages 591–594, Singapore, August 2001.

[26] W. Liu, S. Weiss, and L. Hanzo.
A generalized sidelobe canceller employing two-dimensional frequency Invariant filters.
IEEE Transactions on Antennas and Propagation, 53(7):2339–2343, July 2005.

[27] W. Liu, S. Weiss, and C.-L. Koh.
Constrained Adaptive Broadband Beamforming Algorithm in the Frequency Domain.
In *Sensor Array and Multichannel Signal Processing Workshop*, volume 1, pages 94–98, Barcelona,Spain, July 18–21 2004.

[28] J. G. McWhirter, P. D. Baxter, T. Cooper, S. Redif, and J. Foster.
An EVD Algorithm for Para-Hermitian Polynomial Matrices.
IEEE Transactions on Signal Processing, 55(5):2158–2169, May 2007.

[29] N. Moret, A. Tonello, and S. Weiss.
Mimo precoding for filter bank modulation systems based on psvd.
In *IEEE 73rd Vehicular Technology Conference*, pages 1–5, May 2011.
(best paper award).

References V

[30] V. Neo, C. Evers, S. Weiss, and P. Naylor.
Signal compaction using polynomial EVD for spherical array processing with applications.
IEEE Transactions on Audio, Speech, and Language Processing, submitted December 2022.

[31] V. Neo, S. Redif, J. McWhirter, J. Pestana, I. Proudler, S. Weiss, and N. P.A.
Polynomial eigenvalue decomposition for multichannel broadband signal processing.
IEEE Signal Processing Magazine, submitted November 2023.

[32] V. W. Neo, E. d'Olne, A. H. Moore, and P. A. Naylor.
Fixed beamformer design using polynomial eigenvalue decomposition.
In *2022 International Workshop on Acoustic Signal Enhancement (IWAENC)*, pages 1–5, Sep. 2022.

[33] V. W. Neo, S. Weiss, S. W. McKnight, A. O. T. Hogg, and P. A. Naylor.
Polynomial eigenvalue decomposition-based target speaker voice activity detection in the presence of competing talkers.
In *17th International Workshop on Acoustic Signal Enhancement*, Bamberg, Germany, Sept. 2022.

[34] V. W. Neo, S. Weiss, and P. A. Naylor.
A polynomial subspace projection approach for the detection of weak voice activity.
In *Sensor Signal Processing for Defence Conference*, pages 1–5, London, UK, Sept. 2022.

[35] D. Nunn.
Performance Assessments of a Time-Domain Adaptive Processor in a Broad-Band Environment.
IEE Proc. F: Radar and Signal Processing, 130(1):139–146, January 1983.

[36] D. Nunn.
Suboptimal Frequency-Domain Adaptive Antenna Processing Algorithm for Broad-Band Environments.
IEE Proc. F: Radar and Signal Processing, 134(4):341–351, July 1987.

References VI

[37] M. Oudin and J. P. Delmas.
Robustness of adaptive narrowband beamforming with respect to bandwidth.
IEEE Transactions on Signal Processing, 56(4):1532–1538, April 2008.

[38] T. Qin, H. Zhang, and X. Zhang.
Criterion for narrowband beamforming.
Electronics Letters, 40:846–847(1), July 2004.

[39] S. Redif, J. McWhirter, and S. Weiss.
Design of FIR paraunitary filter banks for subband coding using a polynomial eigenvalue decomposition.
IEEE Transactions on Signal Processing, 59(11):5253–5264, November 2011.

[40] S. Redif, S. Weiss, and J. McWhirter.
Sequential matrix diagonalization algorithms for polynomial EVD of parahermitian matrices.
IEEE Transactions on Signal Processing, 63(1):81–89, January 2015.

[41] S. Redif, S. Weiss, and J. G. McWhirter.
An approximate polynomial matrix eigenvalue decomposition algorithm for para-hermitian matrices.
In *11th IEEE International Symposium on Signal Processing and Information Technology*, pages 421–425, Bilbao, Spain, December 2011.

[42] H. L. Van Trees.
Detection, Estimation and Modulation Theory: Optimum Array Processing.
Wiley, New York, 2002.

[43] B. D. Van Veen and K. M. Buckley.
Beamforming: A Versatile Approach to Spatial Filtering.
IEEE Acoustics, Speech, and Signal Processing Magazine, 5(2):4–24, April 1988.

[44] E. W. Vook and R. T. Compton, Jr.
Bandwidth Performance of Linear Adaptive Arrays with Tapped Delay-Line Processing.
IEEE Transactions on Aerospace and Electronic Systems, Vol.28(No.3):pp.901–908, July 1992.

[45] S. Weiss.
Analysis and Fast Implementation of Oversampled Modulated Filter Banks.
In J. G. McWhirter and I. K. Proudler, editors, *Mathematics in Signal Processing V*, chapter 23, pages 263–274. Oxford University Press, March 2002.

[46] S. Weiss, M. Alrmah, S. Lambotharan, J. McWhirter, and M. Kaveh.
Broadband angle of arrival estimation methods in a polynomial matrix decomposition framework.
In *IEEE 5th International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, pages 109–112, Dec 2013.

[47] S. Weiss, S. Bendoukha, A. Alzin, F. Coutts, I. Proudler, and J. Chambers.
MVDR broadband beamforming using polynomial matrix techniques.
In *23rd European Signal Processing Conference*, pages 839–843, Nice, France, September 2015.

[48] S. Weiss, C. Delaosa, J. Matthews, I. Proudler, and B. Jackson.
Detection of weak transient signals using a broadband subspace approach.
In *International Conference on Sensor Signal Processing for Defence*, pages 65–69, Edinburgh, Scotland, Sept. 2021.

[49] S. Weiss, M. Hadley, and J. Wilcox.
Implementation of a flexible frequency-invariant broadband beamformer based on fourier properties.
In *Sensor Signal Processing for Defence Conference*, pages 1–5, London, UK, December 2017.

[50] S. Weiss, W. Liu, R. W. Stewart, and I. K. Proudler.
A Generalised Sidelobe Canceller Architecture based on Oversampled Subband Decompositions.
In *5th International Conference on Mathematics in Signal Processing*, Warwick, December 2000.

[51] S. Weiss, J. Pestana, and I. K. Proudler.
On the existence and uniqueness of the eigenvalue decomposition of a parahermitian matrix.
IEEE Transactions on Signal Processing, 66(10):2659–2672, May 2018.

[52] S. Weiss, J. Pestana, I. K. Proudler, and F. K. Coutts.
Corrections to "on the existence and uniqueness of the eigenvalue decomposition of a parahermitian matrix".
IEEE Transactions on Signal Processing, 66(23):6325–6327, Dec 2018.

[53] S. Weiss, I. Proudler, F. Coutts, and J. Deeks.
Extraction of analytic eigenvectors from a parahermitian matrix.
In *International Conference on Sensor Signal Processing or Defence*, Edinburgh, UK, 2020.

[54] S. Weiss, I. Proudler, F. Coutts, and F. Khattak.
Eigenvalue decomposition of a parahermitian matrix: extraction of analytic eigenvectors.
IEEE Transactions on Signal Processing, 71:1642–1656, Apr. 2023.

[55] S. Weiss and I. K. Proudler.
Comparing Efficient Broadband Beamforming Architectures and Their Performance Trade-Offs.
In *14th International Conference on Digital Signal Processing*, volume I, pages 417–422, Santorini, Greece, July 1–3 2002.
(invited paper).

[56] S. Weiss, I. K. Proudler, G. Barbarino, J. Pestana, and J. G. McWhirter.
On properties and structure of the analytic singular value decomposition.
IEEE Transactions on Signal Processing, 2023.
to be submitted.

[57] S. Weiss, I. K. Proudler, and F. K. Coutts.
Eigenvalue decomposition of a parahermitian matrix: extraction of analytic eigenvalues.
IEEE Transactions on Signal Processing, 69:722–737, 2021.

References IX

[58] S. Weiss, I. K. Proudler, F. K. Coutts, and J. Pestana.
Iterative approximation of analytic eigenvalues of a parahermitian matrix EVD.
In *IEEE International Conference on Acoustics, Speech and Signal Processing*, Brighton, UK, May 2019.

[59] S. Weiss, A. Stenger, R. Rabenstein, and R. Stewart.
Lower Error Bound for Oversampled Subband Adaptive Filters.
IEE Electronics Letters, 34(16):1555–1557, August 1998.

[60] S. Weiss, A. Stenger, R. Stewart, and R. Rabenstein.
Steady-State Performance Limitations of Subband Adaptive Filters.
IEEE Transactions on Signal Processing, 49(9):1982–1991, September 2001.

[61] S. Weiss, R. W. Stewart, and W. Liu.
A Broadband Adaptive Beamforming Structure with Scaled Aperture.
In *Thirty-Sixth Asilomar Conference on Signals, Systems and Computers*, volume 2, pages 1298–1302, Pacific Grove, CA, November 3–6 2002.

[62] S. Weiss, R. W. Stewart, M. Schabert, I. K. Proudler, and M. W. Hoffman.
An Efficient Scheme for Broadband Adaptive Beamforming.
In *33rd Asilomar Conference on Signals, Systems, and Computers*, volume I, pages 496–500, Pacific Grove, CA, November 1999.

[63] S. Weiss, R. W. Stewart, A. Stenger, and R. Rabenstein.
Performance Limitations of Subband Adaptive Filters.
In *9th European Signal Processing Conference*, volume III, pages 1245–1248, Rodos, Greece, September 1998.

[64] B. Widrow and S. D. Stearns.
Adaptive Signal Processing.
Prentice Hall, Englewood Cliffs, New York, 1985.

[65] M. Zatman.
How narrow is narrowband? [adaptive array signal processing].
In *Conference Record of the Thirty-First Asilomar Conference on Signals, Systems and Computers (Cat. No.97CB36136)*, volume 2, pages 1341–1345 vol.2, Nov 1997.

[66] M. Zatman.
How narrow is narrowband?
IEE Proceedings - Radar, Sonar and Navigation, 145:85–91(6), April 1998.