Extension of Power Method to Para-Hermitian Matrices: Polynomial Power Method

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Abstract—This document extends the idea of the power method to polynomial para-Hermitian matrices for the extraction of the principal analytic eigenpair. The proposed extension repeatedly multiplies a polynomial vector with a para-Hermitian matrix followed by an appropriate normalization in each iteration. To limit the order growth of the product vector, truncation is performed post-normalization in each iteration. The method is validated through simulation results over an ensemble of randomised para-Hermitian matrices and is shown to perform significantly better than state-of-the-art algorithms.

I. INTRODUCTION

The solution to many broadband array processing problems can be formulated using the space-time covariance matrix $\mathbf{R}[\tau] = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n - \tau]\}$ [1]–[3], which reflects the information in the second order statistics of a data vector $\mathbf{x}[n]$, with $\mathcal{E}\{\cdot\}$ the expectation operator. Analogous to the narrowband covariance matrix $\mathbf{R}[0]$, which has Hermitian symmetry, the z-transform of $\mathbf{R}[\tau]$ — also known as the cross-spectral density (CSD) matrix — possesses para-Hermitian symmetry i.e. $\boldsymbol{R}^{\mathrm{P}}(z) = \boldsymbol{R}^{\mathrm{H}}(1/z^*) = \boldsymbol{R}(z)$ where $\{.\}^{\mathrm{P}}$ denotes the para-conjugate or para-Hermitian operation [4]. To extend the utility of the eigenvalue decomposition (EVD) which enables optimal solutions to many narrowband array problems, we require a polynomial EVD (PEVD) to diagonalise $\mathbf{R}[\tau] \circ\!\!-\!\!\bullet \boldsymbol{R}(z)$, where $\circ\!\!-\!\!\bullet$ denotes a transform pair, for the broadband case. To date, existing PEVD techniques include the second-order sequential best rotation (SBR2) [4], sequential matrix diagonalisation (SMD) [5], DFT based smooth-decomposition [6] and analytic PEVD [7]–[10] algorithms.

Over recent years, PEVD algorithms have seen use in MIMO communications design [11], [12], subband coding [3], blind signal separation [2], speech enhancement [13], beamforming [14]–[16], broadband angle of arrival estimation [17], and many more. In cases such as coding or compaction, which work by extracting the dominant signal component from multichannel data [18], a complete PEVD may be unnecessary, and it often suffices to extract the largest eigenvalue and its corresponding eigenvector, here termed the principal eigenpair. For the standard EVD, this can be accomplished e.g. by applying the power method [19]. Using polynomial matrix notation, with often modest modifications, narrowband algorithms can be readily extended to their equivalent broadband cases; an

example is the generalisation of the multiple signal classification (MUSIC) algorithm [20] to the polynomial MUSIC approach [17].

This paper extends the power iterations method [19] to para-Hermitian matrices in order to extract the principal analytic eigenpair. The extension multiplies a polynomial vector with a given para-Hermitian matrix and then in each iteration performs a normalization to unity everywhere on the unit circle instead of the normalization of the approximated eigenvector to unit length as in the ordinary power method. The order of the product vector, which grows with each iteration due to repeated multiplication, is limited by truncating it either to the support estimate reported in [21] or by trimming trailing values that fall below some threshold. The proposed approach requires that the dominant eigenvalue spectrally majorises the remaining eigenvalues; for estimated CSD matrix, this assumption is satisfied with probability one [22].

Below, Sec. II reviews the power iteration methods, which is then extended to the polynomial case in Sec. III. We assess the polynomial power iteration method through simulations over an ensemble of CSD matrices of moderately large support in Sec. IV. Sec. V provides a summary and an outlook to potential applications.

II. NARROWBAND POWER ITERATIONS METHOD

Assume we have an instantaneous covariance matrix $\mathbf{R} = \mathbf{R}[0] \in \mathbb{C}^{M \times M}$ for a narrowband sensors array system with its eigenvalues $\lambda_1, \ldots, \lambda_m$ such that $|\lambda_1| > |\lambda_m|$ for $m = 2, \ldots, M$. If we repeatedly multiply a non-zero vector $\mathbf{v}^{(0)} \in \mathbb{C}^M$ against $\mathbf{R}$, we obtain a sequence of vectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(k)}$, which can be written as

$$\mathbf{v}^{(k)} = \mathbf{R}\mathbf{v}^{(k-1)} = \mathbf{R}^k\mathbf{v}^{(0)}, \quad k = 1, 2, \ldots . \tag{1}$$

As $\mathbf{R}$ is Hermitian, its M eigenvectors $\mathbf{q}_1, \ldots, \mathbf{q}_M$ can be selected to form an orthonormal basis for $\mathbb{C}^M$. Thus we can represent $\mathbf{v}^{(0)}$ as a linear combination of eigenvectors,

$$\mathbf{v}^{(0)} = c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + \ldots + c_M\mathbf{q}_M . \tag{2}$$

Substituting $\mathbf{v}^{(0)}$ from (2) into (1), we get

$$\mathbf{v}^k = \mathbf{R}^k \sum_{m=1}^{M} c_m\mathbf{q}_m = \sum_{m=1}^{M} c_m\lambda_m^k\mathbf{q}_m$$

$$= \lambda_1^k(c_1\mathbf{q}_1 + \sum_{m=2}^{M} c_m(\lambda_m/\lambda_1)^k\mathbf{q}_m) . \tag{3}$$

Over a sufficient number of iterations k, the direction $\mathbf{v}^k$ converges to that of $\mathbf{q}_1$ for $c_1 \neq 0$. It means that a random vector repeatedly multiplied with $\mathbf{R}$ converges towards a scaled version of its principal eigenvector provided that the initial vector has a non-zero component in that direction [19]. This constitutes the power method. In order to avoid overflow or underflow, $\mathbf{v}^k$ can be normalised in every iteration step, which makes its converge towards $e^{j\phi}\mathbf{q}_1$, where $\phi = \angle c_1$, reflecting the phase ambiguity of eigenvectors. The rate of convergence depends upon the ratio $|\lambda_2/\lambda_1|$.

III. POLYNOMIAL POWER METHOD (PPM)

In this section, we present the extension of power method to polynomial para-Hermitian matrices.

A. Para-Hermitian Matrix EVD

For an analytic para-Hermitian $\mathbf{R}(z) : \mathbb{C} \to \mathbb{C}^{M \times M}$, an analytic PEVD exist in most cases [23]–[25] as

$$\mathbf{R}(z) = \mathbf{Q}(z)\mathbf{\Lambda}(z)\mathbf{Q}^{\mathrm{P}}(z), \tag{4}$$

where $\mathbf{Q}(z)$ is para-unitary such that $\mathbf{Q}^{\mathrm{P}}(z)\mathbf{Q}(z) = \mathbf{Q}(z)\mathbf{Q}^{\mathrm{P}}(z) = \mathbf{I}$ contains in its columns the analytic eigenvectors, and $\mathbf{\Lambda}(z) = \mathrm{diag}\{\lambda_1(z), \ldots, \lambda_M(z)\}$ is a diagonal and para-Hermitian matrix that holds the analytic eigenvalues. If, on the unit circle, the latter satisfy the inequality

$$|\lambda_m(e^{j\Omega})| \geq |\lambda_{m+1}(e^{j\Omega})| \ \forall \ \Omega \text{ for } m = 1, \ldots, M-1,$$

then we call the eigenvalues of $\mathbf{R}(z)$ spectrally majorised. If $\mathbf{R}(z)$ is estimated from finite data, then its eigenvalues will be spectrally majorised with probability one [22]. In addition, the estimation process leads with probability one to no non-trivial algberaic mulitplicies such that the eigenvalues satisfy

$$|\lambda_1(e^{j\Omega})| > |\lambda_2(e^{j\Omega})| > \ldots > |\lambda_M(e^{j\Omega})|, \ \forall \ \Omega \in \mathbb{R}. \tag{5}$$

While the eigenvalues of a para-Hermitian matrix are unique, the corresponding eigenvectors can be modified by arbitrary analytic allpass functions $\varphi_m(z)$, $m = 1, \ldots, M$ [23]. Therefore, if $\mathbf{q}_m(z)$ is a valid analytic eigenvector of $\mathbf{R}(z)$, then so is $\varphi_m(z)\mathbf{q}_m(z)$.

B. Power Method Extension to Para-Hermitian Matrices

Let us assume some vector of analytic functions $\mathbf{v}^{(0)}[n]$, $\mathbf{v}^{(0)}(z) \in \mathbb{C}^M$. Then we can write $\mathbf{c}(z) = \mathbf{Q}^{\mathrm{P}}(z)\mathbf{v}^{(0)}(z)$, where $\mathbf{Q}(z)$ is the matrix of eigenvectors from (4). Due to the analyticity of both $\mathbf{Q}(z)$ and $\mathbf{v}^{(0)}(z)$, $\mathbf{c}(z)$ is also analytic. Further, parauinitarity of $\mathbf{Q}(z)$ i.e. $\mathbf{Q}(z)\mathbf{Q}^{\mathrm{P}}(z) = \mathbf{I}$ permits

$$\mathbf{v}^{(0)}(z) = \mathbf{c}(z)\mathbf{Q}(z) \tag{6}$$

$$= c_1(z)\mathbf{q}_1(z) + \cdots + c_M(z)\mathbf{q}_M(z), \tag{7}$$

i.e. the vector $\mathbf{v}^{(0)}(z)$ can be expressed as superposition of analytic eigenvectors, weighted by some analytic functions $c_m(z)$, $m = 1, \ldots, M$.

We now want to utilise $\mathbf{v}^{(0)}(z)$ as the initialisation for PPM. Due to analyticity, we work on the unit circle with $z = e^{j\Omega}$, $\Omega \in \mathbb{R}$, where we have

$$\mathbf{v}^{(0)}(e^{j\Omega}) = c_1(e^{j\Omega})\mathbf{q}_1(e^{j\Omega}) + \ldots + c_M(e^{j\Omega})\mathbf{q}_M(e^{j\Omega}). \tag{8}$$

Note that $\mathbf{q}_m(e^{j\Omega})$ can be seen as either the mth analytic eigenvector evaluated at $z = e^{j\Omega}$ or the mth eigenvector of a Hermitian matrix viz. $\mathbf{R}(z)|_{z=e^{j\Omega}}$. This permits us to replicate the procedure described in Sec. II, where by repeated multiplications, we create a sequence of vectors such that after k iterations, we have

$$\mathbf{v}^{(k)}(e^{j\Omega}) = \mathbf{R}(e^{j\Omega})\mathbf{v}^{(k-1)}(e^{j\Omega}) = \mathbf{R}^k(e^{j\Omega})\mathbf{v}^{(0)}(e^{j\Omega}). \tag{9}$$

Combining (8) and (9), we obtain the frequency-dependent version of (3) as

$$\mathbf{v}^{(k)}(e^{j\Omega}) = (\lambda_1(e^{j\Omega}))^k \Big[c_1(e^{j\Omega})\mathbf{q}_1(e^{j\Omega})$$
$$+ \sum_{m=2}^{M} c_m(e^{j\Omega})\left(\frac{\lambda_m(e^{j\Omega})}{\lambda_1(e^{j\Omega})}\right)^k \mathbf{q}_m(e^{j\Omega}) \Big]. \tag{10}$$

From the fact that the estimated $\mathbf{R}(z)$ is spectrally majorised with its eigenvalues satisfying (5), the summation term in (10) will decay to zero for all values of Ω as $k \to \infty$.

What is required that the first term in (10) converges to an unnormalised version of $\mathbf{q}_1(z)$ evaluated at $z = e^{j\Omega}$? Similar to the discussion in Sec. II, we need to ensure that the initialisation $\mathbf{v}^{(0)}(z)$ contains a portion of the principal eigenvector $\mathbf{q}_1(z)$, i.e. that $c_1(z)$ does not vanish. Since $c_1(z)$ must be analytic, unless $c_1(z) = 0 \ \forall z$, on the unit circle $c_1(e^{j\Omega})$ can only possess isolated zero crossings as a result of the uniqueness theorem of analytic functions [26]. Hence $c_1(e^{j\Omega})$ must in general be non-zero except for a finite number of zeros-crossings. Such zero-crossings will generate challenges for PPM; one solution is normalised regularisation which we will introduce further below.

Thus, for a sufficiently large value of k, $\mathbf{v}^{(k)}(e^{j\Omega})$ becomes

$$\mathbf{v}^{(k)}(e^{j\Omega}) = (\lambda_1(e^{j\Omega}))^k c_1(e^{j\Omega})\mathbf{q}_1(e^{j\Omega}). \tag{11}$$

If we normalise (11) to unit norm $\forall \ \Omega \in \mathbb{R}$, which we will discuss in Sec. III-C, we get

$$\lim_{k \to \infty} \mathbf{v}_{\mathrm{norm}}^{(k)}(e^{j\Omega}) = \varphi_1(e^{j\Omega})\mathbf{q}_1(e^{j\Omega}), \tag{12}$$

where $\varphi_1(e^{j\Omega})$ is an arbitrary allpass filter noting the ambiguity of analytic eigenvectors discussed in Sec. III-A. With an analytic $\varphi(z)$, $\mathbf{v}^{(k)}(z)$ will generally be an absolutely convergent but infinite series. Since we cannot control either $c_1(e^{j\Omega})$ or $\varphi_1(z)$, we are bound to perform truncation after normalization. Through repeated truncation, we are effectively performing phase smoothing that helps to minimise the order [6], [9].

C. Normalisation and Principal Eigenvalue

For $\mathbf{v}^{(k)}(z)$ to formally take on properties of an eigenvector, it requires normalisation, such that for the normalised vector we have $\mathbf{v}_{\mathrm{norm}}^{(k),\mathrm{P}}(z)\mathbf{v}_{\mathrm{norm}}^{(k)}(z) = 1$. This can be accomplished as

$$\mathbf{v}_{\mathrm{norm}}^{(k)}(e^{j\Omega}) = \frac{\mathbf{v}^{(k)}(e^{j\Omega})}{\sqrt{\mathbf{v}^{(k),\mathrm{H}}(e^{j\Omega})\mathbf{v}^{(k)}(e^{j\Omega}) + \varepsilon}}, \tag{13}$$

where $0 < \varepsilon \ll 1$ is the regularisation parameter. This regularisation serves two purposes: (1) it prevents a division

by zero, in case the singularity of $c_1(e^{j\Omega})$ causes problems; (2) at frequencies near a singularity of $c_1(e^{j\Omega})$, it will create a bias term, that may deflect a poor initialisation $\boldsymbol{v}^{(0)}(z)$. However, if $\boldsymbol{c}(e^{j\Omega})$ is missing the components of $\boldsymbol{q}_1(e^{j\Omega})$, PPM will not reinsert them.

To evaluate (13), we employ a DFT domain approach. For a sufficiently large DFT length K, we compute (13) for a discrete set of frequencies $\Omega_i = 2\pi i/K$, $i = 0, \ldots, (K - 1)$. Since the overall result is potentially infinite but absolutely convergent due to analyticity, we can then obtain $\mathbf{v}_{\mathrm{norm}}^{(k)}[n] \circ\!\!-\!\!\bullet\; \boldsymbol{v}_{\mathrm{norm}}^{(k)}(z)$ via a K-point inverse DFT in good approximation. The solution may be non-causal and exceed the order necessary for a good approximation. Therefore, after performing normalization, a shift-corrected trimming of $\mathbf{v}_{\mathrm{norm}}^{(k)}[n]$ may be applied to restore causality and to limit the order of the normalised product in (13). This will be discussed in Sec. III-D.

The corresponding principal eigenvalue can be determined via the Rayleigh quotient

$$R_{Q,k}(z) = \frac{\boldsymbol{v}^{(k),\mathrm{P}}(z)\boldsymbol{R}(z)\boldsymbol{v}^{(k)}(z)}{\boldsymbol{v}^{(k),\mathrm{P}}(z)\boldsymbol{v}^{(k)}(z)} \tag{14}$$

$$= \boldsymbol{v}_{\mathrm{norm}}^{(k),\mathrm{P}}(z)\boldsymbol{R}(z)\boldsymbol{v}_{\mathrm{norm}}^{(k)}(z) \;, \tag{15}$$

i.e. by a weighted inner product of the eigenvector estimate. We thus have $\lambda_1(z) = \lim_{k\to\infty} R_{Q,k}(z)$.

D. Order Limitation by Shift-Corrected Truncation

Below, we define two approaches to limit the order of $\boldsymbol{v}_{\mathrm{norm}}^{(k)}(z)$.

1) Order Limitation to Eigenvector Support: It is possible to accurately estimate the support of $\boldsymbol{Q}(z)$ before extracting any of its columns constituting the eigenvectors [21]. This estimated support N can be exploited by truncating $\mathbf{v}_{\mathrm{norm}}^{(k)}[n]$ subject to an optimal shift-correction akin to [27], such that

$$\tilde{\mathbf{v}}_{\mathrm{norm}}^{(k)}[n] = \mathbf{v}_{\mathrm{norm}}^{(k)}[n - \Delta_{\mathrm{opt}}]p_N[n] \;, \tag{16}$$

where $p_N[n]$ is a rectangular window of size N, and the shift Δ_{opt} is calculated via

$$\Delta_{\mathrm{opt}} = \arg\max_{\Delta} \sum_{n=0}^{N-1} \|\mathbf{v}[n - \Delta]\|_2^2 \;. \tag{17}$$

The truncation in (16) has to be performed in each iteration to ease the computational burden of the multiplication and normalization steps. This repeated truncation of the time domain vector is analogous to phase smoothing in the frequency domain. Its purpose is to find the eigenvector with minimum time-domain support given its ambiguity w.r.t. an arbitrary allpass filter $\phi_1(z)$, as discussed in Sec. III-A.

2) Direct Truncation of Coefficients: This method truncates the outer lags of $\mathbf{v}_{\mathrm{norm}}^{(k)}[n]$ on either end if its norm $|\mathbf{v}_{\mathrm{norm}}^{(k)}[n]|_2$ falls below a low threshold. Since for a low threshold the resulting order will be high, this method may produce higher-order approximations compared to the previous method, particularly if the truncation length N is inappropriate for intermediate solution after insufficient iteration steps. While the

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Algorithm 1: PPM Algorithm

Input: $\boldsymbol{R}(z), \epsilon, k_{max}$
Output: $\hat{\mathbf{q}}_1(z), \hat{\lambda}_1(z)$
$\boldsymbol{v}^{(0)}(z) \in \mathbb{C}^M, \quad k \leftarrow 0, \; \gamma = \infty$;
$\tilde{\boldsymbol{v}}_{\mathrm{norm}}^{(0)}(z) \leftarrow$ normalise & order limit $\boldsymbol{v}^{(0)}(z)$;
while $\gamma > \epsilon$ & $k < k_{max}$ **do**
    $k \leftarrow k + 1$;
    $\boldsymbol{v}^{(k)}(z) \leftarrow \boldsymbol{R}(z)\tilde{\boldsymbol{v}}_{\mathrm{norm}}^{(k-1)}(z)$;
    $\boldsymbol{v}_{\mathrm{norm}}^{(k)}(z) \leftarrow$ normalisation $\boldsymbol{v}^{(k)}(z)$;
    $\tilde{\boldsymbol{v}}_{\mathrm{norm}}^{(k)}(z) \leftarrow$ order limitation of $\boldsymbol{v}_{\mathrm{norm}}^{(k)}(z)$;
    update γ
end
$\hat{\boldsymbol{q}}_1(z) = \tilde{\boldsymbol{v}}_{\mathrm{norm}}^{(k)}(z)$;
$\hat{\lambda}_1(z) = \tilde{\boldsymbol{v}}_{\mathrm{norm}}^{(k),\mathrm{P}}(z)\boldsymbol{R}(z)\tilde{\boldsymbol{v}}_{\mathrm{norm}}^{(k)}(z)$;
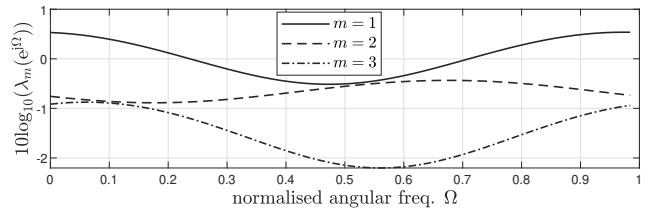

Fig. 1. Eigenvalues for the example matrix $\boldsymbol{R}(z)$.

direct truncation is simple, there currently does not exists any well-defined criterion to determine the appropriate truncation threshold.

E. Stopping Criterion

We want to know how closely aligned two polynomial vectors $\boldsymbol{v}^{(k)}(z)$ and $\boldsymbol{v}^{(k-1)}(z)$ are. Evaluated on the unit circle, we can check the Hermitian angle $\angle\{\boldsymbol{a}(z), \boldsymbol{b}(z)\}$ between any two polynomial vectors $\boldsymbol{a}(z), \boldsymbol{b}(z) : \mathbb{C} \to \mathbb{C}^M$ via

$$\angle\{\boldsymbol{a}(e^{j\Omega}), \boldsymbol{b}(e^{j\Omega})\} = \mathrm{acos}\left(\frac{|\boldsymbol{a}^H(e^{j\Omega})\boldsymbol{b}(e^{j\Omega})|}{\|\boldsymbol{a}(e^{j\Omega})\|_2 \cdot \|\boldsymbol{b}(e^{j\Omega})\|_2}\right) \;. \tag{18}$$

Exploiting this, we now define a metric as $\alpha(\Omega) = \angle\{\boldsymbol{v}^{(k)}(e^{j\Omega}), \boldsymbol{v}^{(k-1)}(e^{j\Omega})\}$; note that we retain the normalisation in (18) in case of errors due to truncation. If two successive estimates are aligned, we have $\alpha(\Omega) = 0 \; \forall\Omega$. To measure an overall deviation,

$$\gamma = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\alpha(\Omega)|^2 \mathrm{d}\Omega \tag{19}$$

is a suitable criterion for convergence, while $\alpha(\Omega)$ permits to check on any frequency-dependent differences in convergence.

IV. SIMULATIONS AND RESULTS

A. Order Limitation and Initialisation Effects

We evaluate the impact of the two types of order limitations, mentioned in Sec-III-D, on convergence and efficiency by

considering a matrix $\boldsymbol{R}(z): \mathbb{C} \to \mathbb{C}^{3\times 3}$ with eigenvalues

$$\lambda_1(z) = \tfrac{6+j}{100}z + 1.01 + \tfrac{6-j}{100}z^{-1}$$
$$\lambda_2(z) = \tfrac{-1+2j}{100}z + 0.86 + \tfrac{-1-2j}{100}z^{-1}$$
$$\lambda_3(z) = \tfrac{5+2j}{100}z + 0.71 + \tfrac{5+2j}{100}z^{-1}$$

as illustrated in Fig. 1. The matrix of eigenvectors $\boldsymbol{Q}(z)$ is defined by a sequence of elementary paraunitary operations [28],

$$\boldsymbol{Q}(z) = [\boldsymbol{q}_1(z), \dots, \boldsymbol{q}_3(z)] = \prod_{i=1}^{4}(\mathbf{I} + (z^{-1}-1)\mathbf{e}_i\mathbf{e}_i^{\mathrm{H}}),$$

with unit norm vectors $\mathbf{e}_i/\sqrt{2}$, $i = 1, \dots 4$,

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{e}_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

With 500 random initializations for $\mathbf{v}^{(0)}[n]$, Algorithm 1 is executed over $k = 300$ iterations, recording both the vector alignment γ of (19) and the normalised error in the principal eigenvalue,

$$\xi_\lambda = \frac{\sum_\tau |\lambda[\tau] - \hat{\lambda}[\tau]|^2}{\sum_\tau |\lambda[\tau]|^2} . \tag{20}$$

Both types of limitation strategies are tested. For the second method, the threshold for the truncation of trailing coefficients is set to 10^{-3}.

The results, illustrated in Fig. 2, show that the generally less costly Algorithm 1 with truncation method 1 (in black) can terminate within shorter time. The execution time for the truncation method depends on the truncation threshold. If the coefficient truncation threshold is small, the execution time will be higher because of more relaxed order limitation and hence higher cost of each iteration. For the given ensemble experiment, the order of the resulting $\boldsymbol{v}_{\mathrm{norm}}^{(k)}(z)$ is 4 and 49 under the order limitation method 1 and the truncation method 2, respectively. Despite all this, the direct truncation method employed by Algorithm 1 achieves greater accuracy in both the principal eigenvector and eigenvalue as evident from Fig. 2.

To assess the impact of a singularity in $c_1(\mathrm{e}^{j\Omega})$, we use a simple system with $\lambda_1(\mathrm{e}^{j\Omega}) = 1$ and $\lambda_2(\mathrm{e}^{j\Omega}) = \frac{1}{2}$. With eigenvectors $\boldsymbol{q}_{1,2}(z) = [1; \pm z^{-1}]/\sqrt{2}$ and initialisation $\boldsymbol{v}^{(0)}(z) = [(1 - z^{-1})\boldsymbol{q}_1(z) + (1 + z^{-1})\boldsymbol{q}_2(z)]/\sqrt{2}$, we have $c_1(\mathrm{e}^{j\Omega}) = 1$ for $\Omega = 0$. Nonetheless, a regularisation $\varepsilon = 0.1$ leads to the convergence of the Hermitian angle in Fig. 3, showing that the singularity is overcome, even though the bias introduced by ε causes a noise floor of around -50dB to the alignment of the estimated principal eigenvector with the ground truth.

B. Comparison with State-Of-The-Art

We compare the proposed PPM approach to the state-of-the-art algorithms SBR2 [4] and SMD [5], through an ensemble of 10^3 randomised spectrally majorised para-Hermitian matrices $\boldsymbol{R}(z) \in \mathbb{C}^{4\times 4}$ based on the source model in [5]. In this
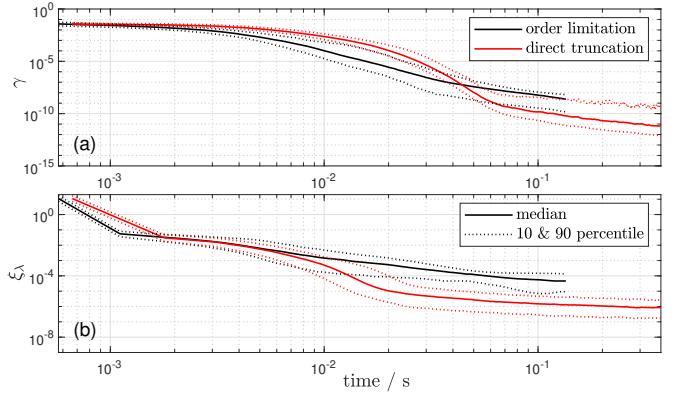

Fig. 2. Effects of types of order limitation on algorithm convergence, measuring (a) angle deviation metric γ, and (b) accuracy of the extracted eigenvalue, ξ_λ.
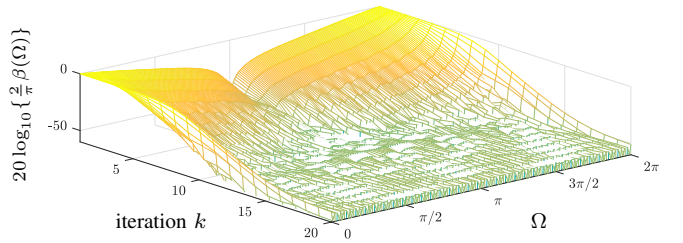

Fig. 3. Convergence of $\beta(\Omega) = \angle\{\mathbf{v}^{(k)}(\mathrm{e}^{j\Omega})\boldsymbol{q}_1(\mathrm{e}^{j\Omega})\}$ in case of a singularity of $c_1(\mathrm{e}^{j\Omega})$ for $\Omega = 0$.

model, we vary the order of the ground truth eigenvectors in $\mathcal{O}\{\boldsymbol{Q}(z)\}$ from 20 to 100 in steps of 20. The proposed method is executed with $k_{\max} = 10^3$ and $\epsilon = 10^{-10}$. The support of $\tilde{\boldsymbol{v}}_{\mathrm{norm}}^{(k)}[n]$ is truncated with a threshold of 10^{-3} via the second method and the regularisation term ε is set to zero to see if we have any issue with random normalization. The SBR2 and SMD algorithms are allowed a maximum of 10^3 iterations with truncation parameter μ, meant to limit the order of any intermediate $\hat{\boldsymbol{q}}_1(z)$ by truncating its outer lags with the same threshold of 10^{-3} as for PPM, while the maximum off-diagonal threshold to is chosen as 10^{-6}.

We evaluate these methods based on the order of extracted eigenvector $\mathcal{O}\{\hat{\boldsymbol{q}}_1(z)\}$, ξ_λ and the execution time. Note that we have not excluded any instance of the ensemble that have possible poor initialisation. The ensemble results are illustrated in Fig. 4 which shows that PPM performance is better than both SBR2 and SMD w.r.t. all three metrics. Especially, the order of the extracted eigenvector is orders of magnitude lower than that of SBR2 and SMD. Although convergence is yet to be proven, the ensemble results suggest that the proposed method converges to an acceptable approximate principal eigenpair as evident from Fig. 4. Furthermore, the ensemble results, which look fairly acceptable at least within 10 to 90 percentile, show that there is low probability to encounter a zero crossings for $c_1(z)$ when randomly initialising $\boldsymbol{v}^{(0)}(z)$.
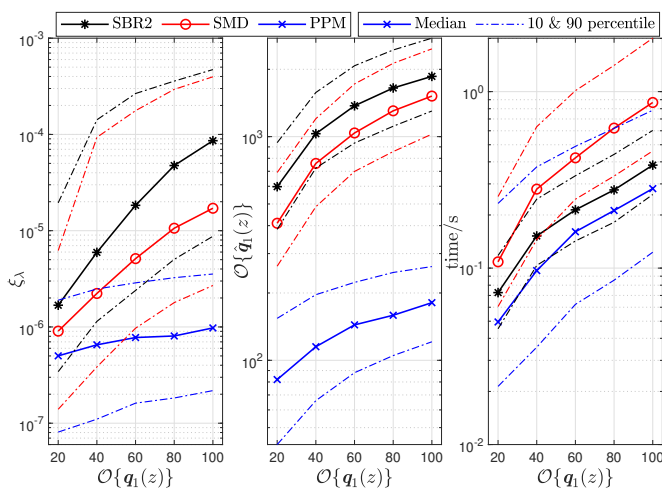
Fig. 4. Performance metrics comparison (a) ξ_λ, (b) $\mathcal{O}\{\hat{q}_1(z)\}$, and (c) execution time of SBR2, SMD and PPM in dependence of the ground truth order of the principal eigenvector.

V. CONCLUSION

We have proposed a polynomial power method for the extraction of the principal eigenpair of a para-Hermitian matrix. The proposed method succeeds as long as the principal eigenvalue does not share non-trivial algebraic multiplicity with the remaining eigenvalues. Simulation results shows that the PPM performs significantly better compared to state-of-the-art algorithms i.e. SBR2 [4] and SMD [5].

The proposed method finds potential applications in areas such as data compaction [18], single broadband source to sensors transfer and source spectral density estimation [29], broadband transient signal detection [30]–[32], and the calculations of an approximate PEVD through deflation via subsequent extractions and eliminations of the principal eigenpairs. The approach can also be generalised to calculating a polynomial singular value decomposition [33].

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