Positivity Preserving Truncated Scheme for the Stochastic Lotka-Volterra Model with Small Moment Convergence

Yongmei Cai[1], Qian Guo[2]*, Xuerong Mao[3]

[1]School of Mathematical Sciences, University of Nottingham Ningbo China, Ningbo, 315100, China

[2]Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China

[3]Department of Mathematics and Statistics, University of Strathclyde, Glasgow G1 1XH, UK

Abstract

This work concerns with the numerical approximation for the stochastic Lotka-Volterra model originally studied by Mao et al. (2002). The natures of the model including multi-dimension, super-linearity of both the drift and diffusion coefficients and the positivity of the solution make most of the existing numerical methods fail. In particular, the super-linearity of the diffusion coefficient results in the explosion of the 1st moment of the analytical solution at a finite time. This becomes one of our main technical challenges. As a result, the convergence framework is to be set up under the θth moment with $0 < \theta < 1$. The idea developed in this paper will not only be able to cope with the stochastic Lotka-Volterra model but also work for a large class of multi-dimensional super-linear SDE models.

Keywords: Stochastic differential equation, positivity preserving numerical method, multi-dimensional super-linear Lotka-Volterra model, strong convergence

1 Introduction

In 2002, Mao et al. [17] worked on an n-dimensional Lotka-Volterra model

$$dx(t) = \mathrm{diag}(x_1(t), \cdots, x_n(t))[b + Ax(t)dt + \sigma x(t)dB(t)], \tag{1.1}$$

where $x(t) = (x_1(t), \cdots, x_n(t))^T$ is the n population sizes at time t, and the parameters $b = (b_1, \cdots, b_n)^T \in \mathbb{R}^n, A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}, \sigma = (\sigma_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ and $B(t)$ is a scalar Brownian motion. They revealed that the system explosion can even be controlled by some small external variability. Since then, the stochastic population analysis has been widely explored.

The non-linearity of model (1.1) makes it hard to express explicitly the analytical solution. In order to well understand the asymptotic properties of such population model, in addition to the analytical study as is done in [17], it is a good attempt to develop an efficient and reliable numerical algorithm. However, owing to model (1.1)'s natures including multi-dimension, super-linearity and positivity of solution, to the best of our knowledge, an explicit numerical method applicable to (1.1) is hardly found.

*Corresponding author and e-mail: qguo@shnu.edu.cn

This is a peer-reviewed, accepted author manuscript of the following article: Cai, Y., Guo, Q., & Mao, X. (2023). Positivity preserving truncated scheme for the stochastic Lotka–Volterra model with small moment convergence. Calcolo, 60, [24]. https://doi.org/10.1007/s10092-023-00521-9

Recently, numerical methods for stochastic differential equations (SDEs) have been extensively discussed. Among these, the Euler-Maruyama (EM) is easily implementable and has low computational cost. However, Hutzenthaler et al. [8] discovered that the EM method is divergent in finite time for super-linearly growing SDEs. Hence many implicit methods (e.g. [3, 19, 21]) were introduced to cope with the nonlinear scenario.

However, the classical explicit EM method is of the simplest structure, making it more efficient in practice. To take advantages of it, many authors have developed various modified EM methods to handle the SDEs with nonlinear terms, e.g. the tamed EM [9], the truncated EM (TEM) [16], the tamed Milstein EM [20], the multilevel EM [2] and the adaptive EM [10].

Nevertheless, to the best of our knowledge, the above modified EM methods are still unable to cope with a large class of well-known SDE systems with super-linear terms, for instance, the stochastic theta model [7, 12]

$$dS(t) = \phi S(t)dt + \sigma S^{3/2}(t)dB(t),$$

the more general mean-reverting-theta stochastic volatility model [4, 12]

$$\begin{aligned} dS(t) &= \phi_1(\alpha_1 - S(t))dt + \sigma_1 V^{1/2}(t)S^{3/2}(t)dB_1(t) \\ dV(t) &= \phi_2(\alpha_2 - V(t))dt + \sigma_2 V^{3/2}(t)dB_2(t), \end{aligned}$$

and the stochastic interest rate model [1, 7]

$$dR(t) = [\alpha_0 - \alpha_1 R^2(t)]dt + \sigma R^3(t)dB(t).$$

On the other hand, positivity/non-negativity of the exact solutions has been detected in a large class of SDE models in mathematical finance and bio-mathematics. To gurantee the reliability of Monte Carlo simulations for these dynamical systems, another aspect we need to concern about is how to retain this typical property in the numerical algorithm. Such positivity preserving methods have been considered by some authors, but so far most are implicit (e.g. [3, 13]). To save time and cost in practice, an explicit method is in need.

Recently, Mao et al. [18] have proposed a modified TEM approach to cope with the super-linear terms as well as to ensure the positivity of the numerical solution. However, the method does not work for our more general type of the Lotka-Volterra model (1.1) where the diffusion term is also of super-linear growth. Therefore, the main technical challenge is to build up the strong convergence framework in the super-linear sense while preserving positivity. According to Mao et al. [15, 17], the quadratic diffusion term makes system (1.1) so sensitive that even the 1st moment of $x(t)$ diverges to infinity in an finite time interval. This is also revealed and generalised by Li et al. [14]. Therefore, the convergence theory in this paper is to be established under the θth moment with $0 < \theta < 1$.

It is worth emphasizing that the scheme developed in this paper is also applicable to a large branch of popular multi-dimensional super-linear SDE models in various fields including finance, biology, epidemiology and engineering, for example, the above mentioned stochastic theta process [7, 12], the mean-reverting-theta stochastic volatility model [4, 12] and the stochastic interest rate model [1, 7] that are widely used to describe the dynamics of asset price, volatility and other financial quantities. We will give some details later.

2 Notations and Preliminaries

Let $(\Omega, \{\mathcal{F}\}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t)$ be a scalar Brownian motion defined on the probability space. Let $\mathbb{I}_{\bar{\Omega}}$ be the indicator function of a subset $\bar{\Omega}$ of Ω. We denote by $\mathbb{R}^n_+$ the positive cone in $\mathbb{R}^n$, that is, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0 \text{ for } 1 \leqslant i \leqslant n\}$. If E is a vector or matrix, its transpose is denoted by E^T. If $E \in \mathbb{R}^n$, $|E|$ is the Euclidean norm. If E is a matrix, we let $|E| := \sqrt{\text{trace}(E^T E)}$ be its trace norm and $\|E\| := \sup\{|Ex| : |x| = 1\}$ its operator norm.

Throughout this paper, we let the initial value $x(0) = x_0 \in \mathbb{R}^n_+$ be arbitrary and fix two positive real numbers T and $\theta \in (0,1)$. Also, C represents generic positive constants dependent on x_0, T and θ but independent of the real number $\tau > 0$ and the stepsize Δ which will be introduced later on.

Before carrying on the numerical analysis, it would be useful to sort out some known results on the exact solution to model (1.1).

Assumption 2.1. *The diffusion parameters satisfy*

$$\begin{cases} \sigma_{ii} > 0, & \text{for } i = 1, \cdots, n \\ \sigma_{ij} \geqslant 0, & \text{for } i \neq j. \end{cases}$$

Lemma 2.2. *([17]) Under Assumption 2.1, for any initial value $x_0 \in \mathbb{R}^n_+$, there is a unique global solution $x(t) \in \mathbb{R}^n_+$ for all $t \geqslant 0$ almost surely.*

For each real number $\tau > |x_0|$, define a stopping time

$$\lambda = \inf\{t \geqslant 0 : |x(t)| \geqslant \tau \text{ or } x_i(t) \leqslant 1/\tau \text{ for some } i\}. \tag{2.1}$$

Lemma 2.3. *Under Assumption 2.1, for any $T > 0$,*

$$\mathbb{P}(\lambda \leqslant T) \leqslant C\Big[\Big(n^{-1/4}\tau^{1/2} - \frac{1}{2}n\log\tau + \frac{5}{4}n\log n\Big) \wedge \Big(\tau^{-1/2} + \frac{1}{2}\log\tau + \frac{5}{4}\log n\Big)\Big]^{-1}.$$

This is a direct result from [17].

Lemma 2.4. *Let Assumption 2.1 hold. We have*

$$\sup_{0\leqslant t<\infty} \mathbb{E}|x(t)|^\theta < \infty.$$

This is a direct result from [15]. Please note that the 1st moment of $x(t)$ might explode to infinity at a finite time as A could be a positive-definite matrix (see, e.g., [17]). Therefore we will establish our convergence theory under the θth moment with $0 < \theta < 1$.

3 The Positivity Preserving Numerical Framework

In this section, we will elaborate our scheme. Firstly, we rewrite model (1.1) as

$$dx(t) = f(x(t))dt + g(x(t))dB(t) = [f_1(x(t)) + f_2(x(t))]dt + g(x(t))dB(t), \tag{3.1}$$

where

$$f_1(x) := \text{diag}(x_1, x_2, \ldots, x_n)b, \quad f_2(x) := \text{diag}(x_1, x_2, \ldots, x_n)Ax \quad \text{and} \quad g(x) := \text{diag}(x_1, x_2, \ldots, x_n)\sigma x$$

in $\mathbb{R}^n_+$, and otherwise, $f_1(x) = f_2(x) = g(x) = 0$. Notice that $f_1(\cdot)$ is of linear growth, that is,

$$|f_1(x)| \leqslant |b||x| \quad \text{for } x \in \mathbb{R}^n,$$

while $f_2(\cdot)$ and $g(\cdot)$ obey

$$|f_2(x)| \leqslant \|A\||x|^2 \quad \text{and} \quad |g(x)| \leqslant \|\sigma\||x|^2 \quad \text{for } x \in \mathbb{R}^n.$$

Therefore the classical EM method is not applicable. One may consider the modified EM methods, e.g., the truncated EM method (see, e.g., [16]), yet the positivity of the numerical solution is not guaranteed. Consequently, in this paper the truncated method will be further revised to cope with SDE models with positive solutions.

We first find a monotonically increasing continuous function $\eta : [1,\infty) \to \mathbb{R}_+$ so that

$$\sup_{|x|\leqslant a} |f_2(x)| \vee |g(x)| \leqslant \eta(a).$$

Without loss of generality, we choose

$$\eta(a) = (\|A\| \vee \|\sigma\|)a^2$$

for the convenience of further analysis, though not necessary. So the inverse function

$$\eta^{-1}(a) = \sqrt{\frac{a}{\eta(1)}} \quad \text{for } a \geqslant \eta(1).$$

Find a constant $K \geqslant 1 \vee \eta(1)$ and a strictly decreasing function $h : (0, 1] \to [\eta(1), \infty)$ such that

$$\lim_{\Delta \to 0} h(\Delta) = \infty \quad \text{and} \quad h^3(\Delta)\Delta \leqslant K \quad \text{for } \Delta \in (0, 1]. \tag{3.2}$$

Given a stepsize $\Delta \in (0, 1]$, define a mapping $\chi_\Delta : \mathbb{R}^n \to \{x \in \mathbb{R}^n : |x| \leqslant \sqrt{\frac{h(\Delta)}{\eta(1)}}\}$ by

$$\chi_\Delta(x) = \big(|x| \wedge \sqrt{\frac{h(\Delta)}{\eta(1)}}\big)\frac{x}{|x|}$$

with $x/|x| = 0$ for $x = 0$. For any $\Delta \in (0, 1]$ and $\theta \in (0, 1)$, define another mapping $\zeta_\Delta : \mathbb{R}^n \to \mathbb{R}^n_+$ by

$$\zeta_\Delta(x) = (x_1 \vee \Delta^{1/\theta}, \cdots, x_n \vee \Delta^{1/\theta})^T.$$

Note that

$$|\zeta_\Delta(x)| \leqslant \sqrt{(|x_1| \vee \Delta^{1/\theta})^2 + \cdots + (|x_n| \vee \Delta^{1/\theta})^2} \leqslant \sqrt{|x|^2 + n\Delta^{2/\theta}}$$

and hence

$$|\zeta_\Delta(\chi_\Delta(x))| \leqslant \sqrt{\frac{h(\Delta)}{\eta(1)} + n\Delta^{2/\theta}}. \tag{3.3}$$

Given any stepsize $\Delta \in (0, 1]$, we now compute the discrete-time positivity preserving truncated EM (PPTEM) solutions $X_k \approx x(t_k)$ for $t_k = k\Delta$ with $\bar{X}_0 = X_0 = x_0$ and

$$\begin{cases} \bar{X}_{k+1} = X_k + \big(f_1(X_k) + f_2(X_k)\big)\Delta + g(X_k)\Delta B_k, \\ X_{k+1} = \zeta_\Delta(\chi_\Delta(\bar{X}_{k+1})) \end{cases} \tag{3.4}$$

for $k = 0, 1, \cdots$, where $\Delta B_k = B(t_{k+1}) - B(t_k)$. Let $\bar{X}_{k,j}$ and $X_{k,j}$ be the jth elements of $\bar{X}_k$ and X_k respectively. Notice that (3.3) implies

$$|X_k| = |\zeta_\Delta(\chi_\Delta(\bar{X}_k))| \leqslant \sqrt{\frac{h(\Delta)}{\eta(1)} + n\Delta^{2/\theta}},$$

and hence

$$|f_2(X_k)| \leqslant h(\Delta) + n\Delta^{2/\theta}\eta(1) \quad \text{and} \quad |g(X_k)| \leqslant h(\Delta) + n\Delta^{2/\theta}\eta(1).$$

We finally extend $\bar{X}_k$ and X_k to the whole $t \geqslant 0$, namely,

$$\bar{X}(t) := \bar{X}_k \quad \text{and} \quad X(t) := X_k \quad \text{for } t \in [t_k, t_{k+1}). \tag{3.5}$$

Similarly, $\bar{X}_j(t)$ and $X_j(t)$ represent the jth elements of $\bar{X}(t)$ and $X(t)$ respectively. Clearly, $X(t) = \zeta_\Delta\big(\chi_\Delta(\bar{X}(t))\big)$.

4 Strong Convergence

In this part, the strong convergence of our scheme is to be discussed under the same parametric condition as for the exact solution.

Lemma 4.1. *Under Assumption 2.1,*

$$\sup_{0<\Delta\leqslant 1}\sup_{0\leqslant t\leqslant T}\mathbb{E}|\bar{X}(t)|^\theta \leqslant C$$

and

$$\sup_{0<\Delta\leqslant 1}\sup_{0\leqslant t\leqslant T}\mathbb{E}|X(t)|^\theta \leqslant C.$$

Proof. For any integer $k \geqslant 0$ and $j = 1, \cdots, n$,

$$\begin{aligned}\bar{X}_{k+1,j}^2 &= (X_{k,j} + \big(f_{1,j}(X_k) + f_{2,j}(X_k)\big)\Delta + g_j(X_k)\Delta B_k)^2\\ &= X_{k,j}^2 + (f_{1,j}(X_k) + f_{2,j}(X_k))^2\Delta^2 + g_j^2(X_k)(\Delta B_k)^2 + 2X_{k,j}\big(f_{1,j}(X_k) + f_{2,j}(X_k)\big)\Delta\\ &+ 2X_{k,j}g_j(X_k)\Delta B_k + 2\big(f_{1,j}(X_{k,j}) + f_{2,j}(X_{k,j})\big)g_j(X_k)\Delta B_k\Delta.\end{aligned}$$

Then

$$\big(1 + \bar{X}_{k+1,j}^2\big)^{\theta/2} = \big(1 + X_{k,j}^2\big)^{\theta/2}(1 + \gamma_{k,j})^{\theta/2},$$

where

$$\begin{aligned}\gamma_{k,j} = \frac{1}{1 + X_{k,j}^2}\Big[&(f_{1,j}(X_k) + f_{2,j}(X_k))^2\Delta^2 + g_j^2(X_k)(\Delta B_k)^2 + 2X_{k,j}\big(f_{1,j}(X_k) + f_{2,j}(X_k)\big)\Delta\\ &+ 2X_{k,j}g_j(X_k)\Delta B_k + 2\big(f_{1,j}(X_k) + f_{2,j}(X_k)\big)g_j(X_k)\Delta B_k\Delta\Big].\end{aligned}$$

By Taylor's formula,

$$(1 + \gamma_{k,j})^{\theta/2} \leqslant 1 + \frac{\theta}{2}\gamma_{k,j} + \frac{\theta(\theta-2)}{8}\gamma_{k,j}^2 + \frac{\theta(\theta-2)(\theta-4)}{48}\gamma_{k,j}^3.$$

Therefore

$$\begin{aligned}\mathbb{E}\big((1 + \bar{X}_{k+1,j}^2)^{\theta/2}|\mathcal{F}_{t_k}\big) \leqslant (1 + X_{k,j}^2)^{\theta/2}\Big(&1 + \frac{\theta}{2}\mathbb{E}\big(\gamma_{k,j}|\mathcal{F}_{t_k}\big) + \frac{\theta(\theta-2)}{8}\mathbb{E}(\gamma_{k,j}^2|\mathcal{F}_{t_k})\\ &+ \frac{\theta(\theta-2)(\theta-4)}{48}\mathbb{E}(\gamma_{k,j}^3|\mathcal{F}_{t_k})\Big).\end{aligned} \tag{4.1}$$

Noting that for any integer $j \geqslant 1$,

$$\mathbb{E}\big((\Delta B_k)^{2j-1}|\mathcal{F}_{t_k}\big) = 0 \quad \text{and} \quad \mathbb{E}\big((\Delta B_k)^{2j}|\mathcal{F}_{t_k}\big) = (2j-1)!!\Delta^j$$

hold and all components of X_k are positive due to (3.4), and bearing (3.2) in mind, compute

$$\begin{aligned}\mathbb{E}[\gamma_{k,j}|\mathcal{F}_{t_k}] &= \frac{1}{1 + X_{k,j}^2}\Big[\Big(2X_{k,j}\big(f_{1,j}(X_k) + f_{2,j}(X_k)\big) + g_j^2(X_k)\Big)\Delta + (f_{1,j}(X_k) + f_{2,j}(X_k))^2\Delta^2\Big]\\ &\leqslant \frac{1}{1 + X_{k,j}^2}\Big[\Big(2b_jX_{k,j}^2 + 2X_{k,j}^2\sum_{i=1}^n a_{ji}X_{k,i} + X_{k,j}^2\big(\sum_{i=1}^n \sigma_{ji}X_{k,i}\big)^2\Big)\Delta + 2b_j^2X_{k,j}^2\Delta^2\Big]\\ &+ 2f_{2,j}^2(X_k)\Delta^2\\ &\leqslant \frac{1}{1 + X_{k,j}^2}\Big(2X_{k,j}^2\sum_{i=1}^n a_{ji}X_{k,i} + X_{k,j}^2\big(\sum_{i=1}^n \sigma_{ji}X_{k,i}\big)^2\Big)\Delta + 2b_j\Delta + 2b_j^2\Delta^2 +\\ &+ 2\big(h(\Delta) + n\Delta^{2/\theta}\eta(1)\big)^2\Delta^2\\ &\leqslant \frac{1}{1 + X_{k,j}^2}\Big(2X_{k,j}^2\sum_{i=1}^n a_{ji}X_{k,i} + X_{k,j}^2\big(\sum_{i=1}^n \sigma_{ji}X_{k,i}\big)^2\Big)\Delta + C\Delta,\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\gamma_{k,j}^2|\mathcal{F}_{t_k}] &= \frac{1}{(1+X_{k,j}^2)^2}\mathbb{E}\Big[\Big((f_{1,j}(X_k)+f_{2,j}(X_k))^2\Delta^2 + g_j^2(X_k)(\Delta B_k)^2 + 2X_{k,j}\big(f_{1,j}(X_k)+f_{2,j}(X_k)\big)\Delta \\
&\quad + 2X_{k,j}g_j(X_k)\Delta B_k + 2\big(f_{1,j}(X_k)+f_{2,j}(X_k)\big)g_j(X_k)\Delta B_k\Delta\Big)^2|\mathcal{F}_{t_k}\Big] \\
&\geqslant \frac{1}{(1+X_{k,j}^2)^2}\mathbb{E}\Big[4X_{k,j}^2 g_j^2(X_k)(\Delta B_k)^2 + 4X_{k,j}g_j(X_k)\Delta B_k\Big((f_{1,j}(X_k)+f_{2,j}(X_k))^2\Delta^2 \\
&\quad + g_j^2(X_k)(\Delta B_k)^2 + 2X_{k,j}\big(f_{1,j}(X_k)+f_{2,j}(X_k)\big)\Delta + 2\big(f_{1,j}(X_k)+f_{2,j}(X_k)\big)g_j(X_k)\Delta B_k\Delta\Big)|\mathcal{F}_{t_k}\Big] \\
&\geqslant \frac{4}{(1+X_{k,j}^2)^2}X_{k,j}^2\big(X_{k,j}\sum_{i=1}^n \sigma_{ji}X_{k,i}\big)^2\Delta - \frac{8}{(1+X_{k,j}^2)^2}X_{k,j}\big(|f_{1,j}(X_k)|+|f_{2,j}(X_k)|\big)|g_j(X_k)|^2\Delta^2 \\
&\geqslant \frac{4}{(1+X_{k,j}^2)^2}X_{k,j}^4\big(\sum_{i=1}^n \sigma_{ji}X_{k,i}\big)^2\Delta - \frac{8}{(1+X_{k,j}^2)^2}\Big(|b_j|X_{k,j}^2\big(h(\Delta)+n\Delta^{2/\theta}\eta(1)\big)^2 \\
&\quad + X_{k,j}\big(h(\Delta)+n\Delta^{2/\theta}\eta(1)\big)^3\Big)\Delta^2 \\
&\geqslant \frac{4}{(1+X_{k,j}^2)^2}X_{k,j}^4\big(\sum_{i=1}^n \sigma_{ji}X_{k,i}\big)^2\Delta - C\Delta
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[\gamma_{k,j}^3|\mathcal{F}_{t_k}] &= \frac{1}{(1+X_{k,j}^2)^3}\mathbb{E}\Big[\Big((f_{1,j}(X_k)+f_{2,j}(X_k))^2\Delta^2 + g_j^2(X_k)(\Delta B_k)^2 + 2X_{k,j}\big(f_{1,j}(X_k)+f_{2,j}(X_k)\big)\Delta \\
&\quad + 2X_{k,j}g_j(X_k)\Delta B_k + 2\big(f_{1,j}(X_k)+f_{2,j}(X_k)\big)g_j(X_k)\Delta B_k\Delta\Big)^3|\mathcal{F}_{t_k}\Big] \\
&= \frac{1}{(1+X_{k,j}^2)^3}\mathbb{E}\Big[\Big(2X_{k,j}\big(f_{1,j}(X_k)+f_{2,j}(X_k)\big)\Delta + g_j^2(X_k)(\Delta B_k)^2 + (f_{1,j}(X_k)+f_{2,j}(X_k))^2\Delta^2\Big)^3 \\
&\quad + 3\Big(2X_{k,j}\big(f_{1,j}(X_k)+f_{2,j}(X_k)\big)\Delta + g_j^2(X_k)(\Delta B_k)^2 + (f_{1,j}(X_k)+f_{2,j}(X_k))^2\Delta^2\Big) \\
&\quad \cdot\Big(2X_{k,j}g_j(X_k)\Delta B_k + 2\big(f_{1,j}(X_k)+f_{2,j}(X_k)\big)g_j(X_k)\Delta B_k\Delta\Big)^2|\mathcal{F}_{t_k}\Big] \\
&\leqslant \frac{1}{(1+X_{k,j}^2)^3}\mathbb{E}\Big[9\cdot 2^3 X_{k,j}^3|f_{1,j}(X_k)+f_{2,j}(X_k)|^3\Delta^3 + 9|g_j(X_k)|^6(\Delta B_k)^6 \\
&\quad + 9|f_{1,j}(X_k)+f_{2,j}(X_k)|^6\Delta^6 + 48X_{k,j}^3|f_{1,j}(X_k)+f_{2,j}(X_k)||g_j(X_k)|^2\Delta(\Delta B_k)^2 \\
&\quad + 48X_{k,j}|f_{1,j}(X_k)+f_{2,j}(X_k)|^3|g_j(X_k)|^2\Delta^3(\Delta B_k)^2 + 24|g_j(X_k)|^4X_{k,j}^2(\Delta B_k)^4 \\
&\quad + 24|g_j(X_k)|^4|f_{1,j}(X_k)+f_{2,j}(X_k)|^2(\Delta B_k)^4\Delta^2 + 24|f_{1,j}(X_k)+f_{2,j}(X_k)|^2X_{k,j}^2|g_j(X_k)|^2\Delta^2(\Delta B_k)^2 \\
&\quad + 24|f_{1,j}(X_k)+f_{2,j}(X_k)|^4|g_j(X_k)|^2\Delta^4(\Delta B_k)^2|\mathcal{F}_{t_k}\Big] \\
&\leqslant \frac{C}{(1+X_{k,j}^2)^3}\Big[X_{k,j}^3|f_{1,j}(X_k)|^3\Delta^3 + X_{k,j}^3|f_{2,j}(X_k)|^3\Delta^3 + |g_j(X_k)|^6\Delta^3 + |f_{1,j}(X_k)|^6\Delta^6 \\
&\quad + |f_{2,j}(X_k)|^6\Delta^6 + X_{k,j}^3|f_{1,j}(X_k)||g_j(X_k)|^2\Delta^2 + X_{k,j}^3|f_{2,j}(X_k)||g_j(X_k)|^2\Delta^2 \\
&\quad + X_{k,j}|f_{1,j}(X_k)|^3|g_j(X_k)|^2\Delta^4 + X_{k,j}|f_{2,j}(X_k)|^3|g_j(X_k)|^2\Delta^4 + X_{k,j}^2|g_j(X_k)|^4\Delta^2 \\
&\quad + |f_{1,j}(X_k)|^2|g_j(X_k)|^4\Delta^4 + |f_{2,j}(X_k)|^2|g_j(X_k)|^4\Delta^4 + X_{k,j}^2|f_{1,j}(X_k)|^2|g_j(X_k)|^2\Delta^3 \\
&\quad + X_{k,j}^2|f_{2,j}(X_k)|^2|g_j(X_k)|^2\Delta^3 + |f_{1,j}(X_k)|^4|g_j(X_k)|^2\Delta^5 + |f_{2,j}(X_k)|^4|g_j(X_k)|^2\Delta^5\Big] \\
&\leqslant \frac{C}{(1+X_{k,j}^2)^3}\Big[|b_j|^3X_{k,j}^6\Delta^3 + h^3(\Delta)X_{k,j}^3\Delta^3 + h^6(\Delta)\Delta^3 + |b_j|^6X_{k,j}^6\Delta^6 + h^6(\Delta)\Delta^6 \\
&\quad + |b_j|h^2(\Delta)X_{k,j}^4\Delta^2 + h^3(\Delta)X_{k,j}^3\Delta^2 + |b_j|^3h^2(\Delta)X_{k,j}^4\Delta^4 + X_{k,j}h^5(\Delta)\Delta^4 + h^4(\Delta)X_{k,j}^2\Delta^2 \\
&\quad + |b_j|^2h^4(\Delta)X_{k,j}^2\Delta^4 + h^6(\Delta)\Delta^4 + |b_j|^2h^2(\Delta)X_{k,j}^4\Delta^3 + h^4(\Delta)X_{k,j}^2\Delta^3 + |b_j|^4h^2(\Delta)X_{k,j}^4\Delta^5 \\
&\quad + h^6(\Delta)\Delta^5 + \Delta\Big] \\
&\leqslant C\Delta.
\end{aligned}$$

Substituting the above three inequalities in (4.1) infers

$$\mathbb{E}\Big[\sum_{j=1}^{n}\big(1+\bar{X}_{k+1,j}^{2}\big)^{\theta/2}|\mathcal{F}_{t_k}\Big]\leqslant\sum_{j=1}^{n}\big(1+X_{k,j}^{2}\big)^{\theta/2}+\Delta\theta\sum_{j=1}^{n}\big(1+X_{k,j}^{2}\big)^{\theta/2-2}H_j(X_k),$$

where

$$\begin{aligned}H_j(X_k)=&\,C(1+X_{k,j}^2)^2+(1+X_{k,j}^2)X_{k,j}^2\sum_{i=1}^{n}a_{ji}X_{k,i}+\frac{1}{2}(1+X_{k,j}^2)X_{k,j}^2\Big(\sum_{i=1}^{n}\sigma_{ji}X_{k,i}\Big)^2\\&-\frac{1}{2}(2-\theta)X_{k,j}^4\Big(\sum_{i=1}^{n}\sigma_{ji}X_{k,i}\Big)^2\\=&\,C(1+X_{k,j}^2)^2+(1+X_{k,j}^2)X_{k,j}^2\sum_{i=1}^{n}a_{ji}X_{k,i}+\frac{1}{2}X_{k,j}^2\Big(\sum_{i=1}^{n}\sigma_{ji}X_{k,i}\Big)^2\\&-\frac{1}{2}(1-\theta)X_{k,j}^4\Big(\sum_{i=1}^{n}\sigma_{ji}X_{k,i}\Big)^2.\end{aligned}$$

Notice that

$$\begin{aligned}&\sum_{j=1}^{n}\big(1+X_{k,j}^2\big)^{\theta/2-2}H_j(X_k)\\&\leqslant C\sum_{j=1}^{n}(1+X_{k,j}^2)^2+\sum_{j=1}^{n}\sum_{i=1}^{n}(1+X_{k,j}^2)X_{k,j}^2a_{ji}X_{k,i}+\frac{1}{2}\sum_{j=1}^{n}X_{k,j}^2\Big(\sum_{i=1}^{n}\sigma_{ji}X_{k,i}\Big)^2\\&-\frac{1}{2}(1-\theta)\sum_{j=1}^{n}X_{k,j}^4\Big(\sum_{i=1}^{n}\sigma_{ji}X_{k,i}\Big)^2\\&\leqslant C\sum_{j=1}^{n}(1+X_{k,j}^2)^2+\frac{4}{5}n\sum_{j=1}^{n}(1+X_{k,j}^2)^{5/4}X_{k,j}^{5/2}+\frac{1}{5}\sum_{j=1}^{n}\sum_{i=1}^{n}|a_{ji}|^5X_{k,i}^5\\&+\frac{1}{2}n|\sigma|^2\sum_{j=1}^{n}X_{k,j}^4-\frac{1}{2}(1-\theta)\sum_{j=1}^{n}\sigma_{jj}^2X_{k,j}^6\\&\leqslant C\end{aligned}$$

under Assumption 2.1. Here, note that the last term is dominant for large $X_{k,j}$ and it is negative. We thus conclude

$$\mathbb{E}\Big[\sum_{j=1}^{n}\big(1+\bar{X}_{k+1,j}^{2}\big)^{\theta/2}|\mathcal{F}_{t_k}\Big]\leqslant\sum_{j=1}^{n}\big(1+X_{k,j}^{2}\big)^{\theta/2}+C\Delta.$$

It then follows by iteration that

$$\begin{aligned}\mathbb{E}\Big[\sum_{j=1}^{n}\big(1+X_{k,j}^2\big)^{\theta/2}\Big] &\leqslant \mathbb{E}\Big[\sum_{j=1}^{n}\big(1+\bar{X}_{k,j}^2+\Delta^{2/\theta}\big)^{\theta/2}\Big]\\ &\leqslant \mathbb{E}\Big[\sum_{j=1}^{n}\big(1+\bar{X}_{k,j}^2\big)^{\theta/2}\Big]+n\Delta\\ &=\mathbb{E}\Big[\mathbb{E}\Big(\sum_{j=1}^{n}\big(1+\bar{X}_{k,j}^2\big)^{\theta/2}|\mathcal{F}_{t_{k-1}}\Big)\Big]+n\Delta\\ &\leqslant \mathbb{E}\Big[\sum_{j=1}^{n}(1+X_{k-1,j}^2)^{\theta/2}\Big]+C\Delta\\ &\leqslant \sum_{j=1}^{n}(1+x_{0,j}^2)^{\theta/2}+Ck\Delta\\ &\leqslant C+CT.\end{aligned}$$

Finally, we have

$$\begin{aligned}\sup_{0<\Delta\leqslant 1}\sup_{0\leqslant t\leqslant T}\mathbb{E}|\bar{X}(t)|^\theta &= \sup_{0<\Delta\leqslant 1}\sup_{0\leqslant k\Delta\leqslant T}\mathbb{E}|\bar{X}_k|^\theta\\ \leqslant \sup_{0<\Delta\leqslant 1}\sup_{0\leqslant k\Delta\leqslant T}\sum_{j=1}^{n}\mathbb{E}\big(1+\bar{X}_{k,j}^2\big)^{\theta/2}&\leqslant C\end{aligned}$$

and similarly, the other assertion also follows. □

To explore the strong convergence of our method, let us first consider the SDE

$$d\psi(t)=f_\tau(\psi(t))dt+g_\tau(\psi(t))dB(t) \tag{4.2}$$

on $t\geqslant 0$ with initial value $\psi(0)=x_0$, where

$$f_\tau(\psi)=f(\bar{\zeta}_\tau(\bar{\chi}_\tau(\psi))) \quad \text{and} \quad g_\tau(\psi)=g(\bar{\zeta}_\tau(\bar{\chi}_\tau(\psi))),$$

with

$$\bar{\chi}_\tau(\psi)=\Big(|\psi|\wedge\big(\tau+\frac{1}{2\tau}\big)\Big)\frac{\psi}{|\psi|}$$

and

$$\bar{\zeta}_\tau(\psi)=\Big(\psi_1\vee\frac{1}{2\tau},\cdots,\psi_n\vee\frac{1}{2\tau}\Big)^T$$

are analogues of $\chi_\Delta(\cdot)$ and $\zeta_\Delta(\cdot)$ defined in Section 3 respectively.

Clearly, SDE (4.2) is with global Lipschitz coefficients $f_\tau(\cdot)$ and $g_\tau(\cdot)$. So SDE (4.2) has a unique global positive solution on $t\geqslant 0$.

For each stepsize $\Delta\in(0,1]$, apply EM method to (4.2) by computing the EM solution $\Psi_k\approx\psi(t_k)$ for $t_k=k\Delta$ with $\Psi_0=x_0$ and

$$\Psi_{k+1}=\Psi_k+f_\tau(\Psi_k)\Delta+g_\tau(\Psi_k)\Delta B_k \quad \text{for } k=0,1,\cdots. \tag{4.3}$$

Then we define

$$\Psi(t)=\sum_{k=0}^{\infty}\Psi_k\mathbb{I}_{[t_k,t_{k+1})}(t)$$

and the Itô process

$$\hat{\Psi}(t)=x_0+\int_0^t f_\tau(\Psi(s))ds+\int_0^t g_\tau(\Psi(s))dB(s).$$

It is known [11] that

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|\hat{\Psi}(t)-\psi(t)|^{\theta}\Big)\leqslant C_{\tau}\Delta^{\theta/2}, \tag{4.4}$$

where C_τ is a positive constant dependent on τ and T but independent of Δ.

Lemma 4.2. *Let $j>1$ be an integer large enough for*

$$\Big(\frac{2j}{2j-1}\Big)^{\theta}(T+1)^{\theta/2j}\leqslant 2.$$

Then

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|\hat{\Psi}(t)-\Psi(t)|^{\theta}\Big)\leqslant C_{\tau}\Delta^{\theta(j-1)/2j}.$$

Proof. Let d be the integer part of T/Δ. By the Hölder inequality, we see

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|\hat{\Psi}(t)-\Psi(t)|^{\theta}\Big)\\
&\leqslant \mathbb{E}\Big(\max_{0\leqslant i\leqslant d}\sup_{t_i\leqslant t\leqslant t_{i+1}}|f_{\tau}(\Psi_i)(t-t_i)+g_{\tau}(\Psi_i)(B(t)-B(t_i))|^{\theta}\Big)\\
&\leqslant C_{\tau}\Delta^{\theta}+C_{\tau}\mathbb{E}\Big(\max_{0\leqslant i\leqslant d}\sup_{t_i\leqslant t\leqslant t_{i+1}}|B(t)-B(t_i)|^{\theta}\Big)\\
&\leqslant C_{\tau}\Delta^{\theta}+C_{\tau}\Big[\mathbb{E}\Big(\max_{0\leqslant i\leqslant d}\sup_{t_i\leqslant t\leqslant t_{i+1}}|B(t)-B(t_i)|^{2j}\Big)\Big]^{\theta/2j}.
\end{aligned}$$

By the Doob martingale inequality, we further deduce

$$\begin{aligned}
&\mathbb{E}\Big(\max_{0\leqslant i\leqslant d}\sup_{t_i\leqslant t\leqslant t_{i+1}}|B(t)-B(t_i)|^{2j}\Big)\\
&\leqslant \sum_{i=0}^{d}\mathbb{E}\Big(\sup_{t_i\leqslant t\leqslant t_{i+1}}|B(t)-B(t_i)|^{2j}\Big)\leqslant\Big(\frac{2j}{2j-1}\Big)^{2j}\sum_{i=0}^{d}\mathbb{E}|B(t_{i+1})-B(t_i)|^{2j}\\
&\leqslant\Big(\frac{2j}{2j-1}\Big)^{2j}\sum_{i=0}^{d}(2j-1)!!\Delta^{j}\leqslant\Big(\frac{2j}{2j-1}\Big)^{2j}(2j-1)!!(T+1)\Delta^{j-1}.
\end{aligned}$$

Substituting this back yields

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|\Psi(t)-\hat{\Psi}(t)|^{\theta}\Big)\\
&\leqslant C_{\tau}\Delta^{\theta}+C_{\tau}\Big[\Big(\frac{2j}{2j-1}\Big)^{2j}(2j-1)!!(T+1)\Delta^{j-1}\Big]^{\theta/2j}\\
&\leqslant C_{\tau}\Delta^{\theta}+C_{\tau}j^{\theta/2}\Delta^{\theta(j-1)/2j},
\end{aligned}$$

where the inequality

$$[(2j-1)!!]^{1/j}\leqslant\frac{1}{j}\sum_{i=1}^{j}(2i-1)=j$$

is used in the last step. The required assertion hence follows. □

Theorem 4.3. *Let Assumption 2.1 hold. For any $T>0$,*

$$\lim_{\Delta\to 0}\mathbb{E}|X(T)-x(T)|^{\theta}=0.$$

Proof. Let $\epsilon \in (0,1)$. Define

$$\Omega_1 = \{\lambda > T\}.$$

According to Lemma 2.3, one can choose a $\tau = \tau(\epsilon)$ sufficiently large for

$$\mathbb{P}(\Omega_1^c) = \mathbb{P}(\lambda \leqslant T) \leqslant \epsilon/2,$$

i.e.

$$\mathbb{P}(\Omega_1) \geqslant 1 - \epsilon/2.$$

With this τ, notice that for any $\omega \in \Omega_1$,

$$x(t) = \psi(t), \quad t \in [0,T].$$

We hence see from (4.4) that

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T} |\hat{\Psi}(t) - x(t)|^\theta \mathbb{I}_{\Omega_1}(\omega)\Big) \leqslant C_\tau \Delta^{\theta/2}. \tag{4.5}$$

Recall that C_τ is a positive constant dependent on τ and T but independent of Δ. This and the Chebyshev inequality give

$$\mathbb{P}\Big(\sup_{0\leqslant t\leqslant T} |\hat{\Psi}(t) - x(t)|\mathbb{I}_{\Omega_1}(\omega) \geqslant \frac{1}{2\tau}\Big)\leqslant (2\tau)^\theta C_\tau \Delta^{\theta/2}, \quad \forall \Delta \in (0,1],$$

Letting

$$\Omega_2 := \Big\{\sup_{0\leqslant t\leqslant T} |\hat{\Psi}(t) - x(t)| < \frac{1}{2\tau}\Big\} \cap \Omega_1,$$

one can thus find a $\Delta_1 \in (0,1]$ sufficiently small for

$$\mathbb{P}(\Omega_2) \geqslant 1 - \epsilon, \quad \Delta \in (0, \Delta_1].$$

Given any $\omega \in \Omega_2$ and $\Delta \in (0, \Delta_1]$, we observe

$$\sup_{0\leqslant t\leqslant T} |\Psi(t)| \leqslant \sup_{0\leqslant t\leqslant T} |\hat{\Psi}(t)| \leqslant \sup_{0\leqslant t\leqslant T} |x(t)| + \sup_{0\leqslant t\leqslant T} |\hat{\Psi}(t) - x(t)| < \tau + \frac{1}{2\tau} \tag{4.6}$$

and

$$\begin{aligned}\inf_{0\leqslant t\leqslant T} \Psi_i(t) &\geqslant \inf_{0\leqslant t\leqslant T} \hat{\Psi}_i(t) \geqslant \inf_{0\leqslant t\leqslant T} x_i(t) - \sup_{0\leqslant t\leqslant T} |x_i(t) - \hat{\Psi}_i(t)| \\ &> \inf_{0\leqslant t\leqslant T} x_i(t) - \sup_{0\leqslant t\leqslant T} |x(t) - \hat{\Psi}(t)| > \frac{1}{\tau} - \frac{1}{2\tau} = \frac{1}{2\tau}.\end{aligned} \tag{4.7}$$

Now choose $\Delta^* \in (0, \Delta_1]$ so small that

$$\eta^{-1}(h(\Delta^*)) \geqslant \tau + \frac{1}{2\tau} \quad \text{and} \quad \Delta^{*1/\theta} \leqslant \frac{1}{2\tau}.$$

For any $\omega \in \Omega_2$ and $\Delta \in (0, \Delta^*]$, (4.6) and (4.7) imply

$$\Psi(t) = X(t) = \bar{X}(t), \quad \forall t \in [0,T].$$

Making use of this, we derive

$$\begin{aligned}&\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T} |X(t) - x(t)|^\theta \mathbb{I}_{\Omega_2}\Big) = \mathbb{E}\Big(\sup_{0\leqslant t\leqslant T} |\Psi(t) - x(t)|^\theta \mathbb{I}_{\Omega_2}\Big) \\ &\leqslant \mathbb{E}\Big(\sup_{0\leqslant t\leqslant T} |\Psi(t) - \hat{\Psi}(t)|^\theta\Big) + \mathbb{E}\Big(\sup_{0\leqslant t\leqslant T} |\hat{\Psi}(t) - x(t)|^\theta \mathbb{I}_{\Omega_1}\Big).\end{aligned}$$

It then follows from Lemma 4.2 with any $j \geqslant 3$ and (4.5) that

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|X(t)-x(t)|^\theta \mathbb{I}_{\Omega_2}\Big) \leqslant C_\mathcal{T}\Delta^{\theta/3}, \quad \Delta\in(0,\Delta^*]. \tag{4.8}$$

On the other hand, choose a number $\bar\theta\in(\theta,1)$, compute

$$\begin{aligned}&\mathbb{E}\big(|X(T)-x(T)|^\theta \mathbb{I}_{\Omega_2^c}\big)\\ &\leqslant \big(\mathbb{P}(\Omega_2^c)\big)^{1-\theta/\bar\theta}\big(\mathbb{E}|X(T)-x(T)|^{\bar\theta}\big)^{\theta/\bar\theta} \leqslant C\epsilon^{1-\theta/\bar\theta}, \quad \forall\Delta\in(0,\Delta^*]\end{aligned}$$

by Lemmas 2.4 and 4.1. Combining this and (4.8), we derive

$$\mathbb{E}|X(T)-x(T)|^\theta \leqslant C_\mathcal{T}\Delta^{\theta/3}+C\epsilon^{1-\theta/\bar\theta}, \quad \forall\Delta\in(0,\Delta^*].$$

Consequently,

$$\lim_{\Delta\to 0}\mathbb{E}|X(T)-x(T)|^\theta \leqslant C\epsilon^{1-\theta/\bar\theta}.$$

This leads to the required assertion as ϵ is arbitrary.

□

5 Examples

We will highlight in this part the advantages of our method in retaining the true behaviours of SDE models with positive solutions such as system (1.1). This is done by comparing the behaviours of our scheme with the EM and the TEM. Moreover, the reliability of the PPTEM will also be illustrated.

With the following system parameters

$$b=\begin{pmatrix}50\\12\end{pmatrix}, A=\begin{pmatrix}0.6 & 3\\ 2 & 4\end{pmatrix} \text{ and } \sigma=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}, \tag{5.1}$$

a two-dimensional cooperative Lotka-Volterra system

$$dx(t)=[f_1(x(t))+f_2(x(t))]dt+g(x(t))dB(t) \tag{5.2}$$

is formulated, where

$$f_1(x)=\begin{pmatrix}50x_1\\12x_2\end{pmatrix}, \quad f_2(x)=\begin{pmatrix}0.6x_1^2+3x_1x_2\\2x_1x_2+4x_2^2\end{pmatrix}$$

and

$$g(x)=\begin{pmatrix}2x_1^2+x_1x_2\\x_1x_2+2x_2^2\end{pmatrix}$$

in $\mathbb{R}_+^2$, and $f_1(x)=f_2(x)=g(x)=0$ for $x\notin\mathbb{R}_+^2$.

Define $\eta:[1,\infty)\to\mathbb{R}_+$ by

$$\eta(a)=6a^2$$

such that

$$\sup_{x\in\mathbb{R}^2,|x|\leqslant a}|f_2(x)|\vee|g(x)|\leqslant\big(\|A\|\vee\|\sigma\|\big)a\leqslant 6a^2=\eta(a),$$

and hence the inverse function

$$\eta^{-1}(a)=\sqrt{a/6}.$$

Let $h:(0,1]\to[\eta(1),\infty)$ be

$$h(\Delta)=10^5\Delta^{-1/3}.$$

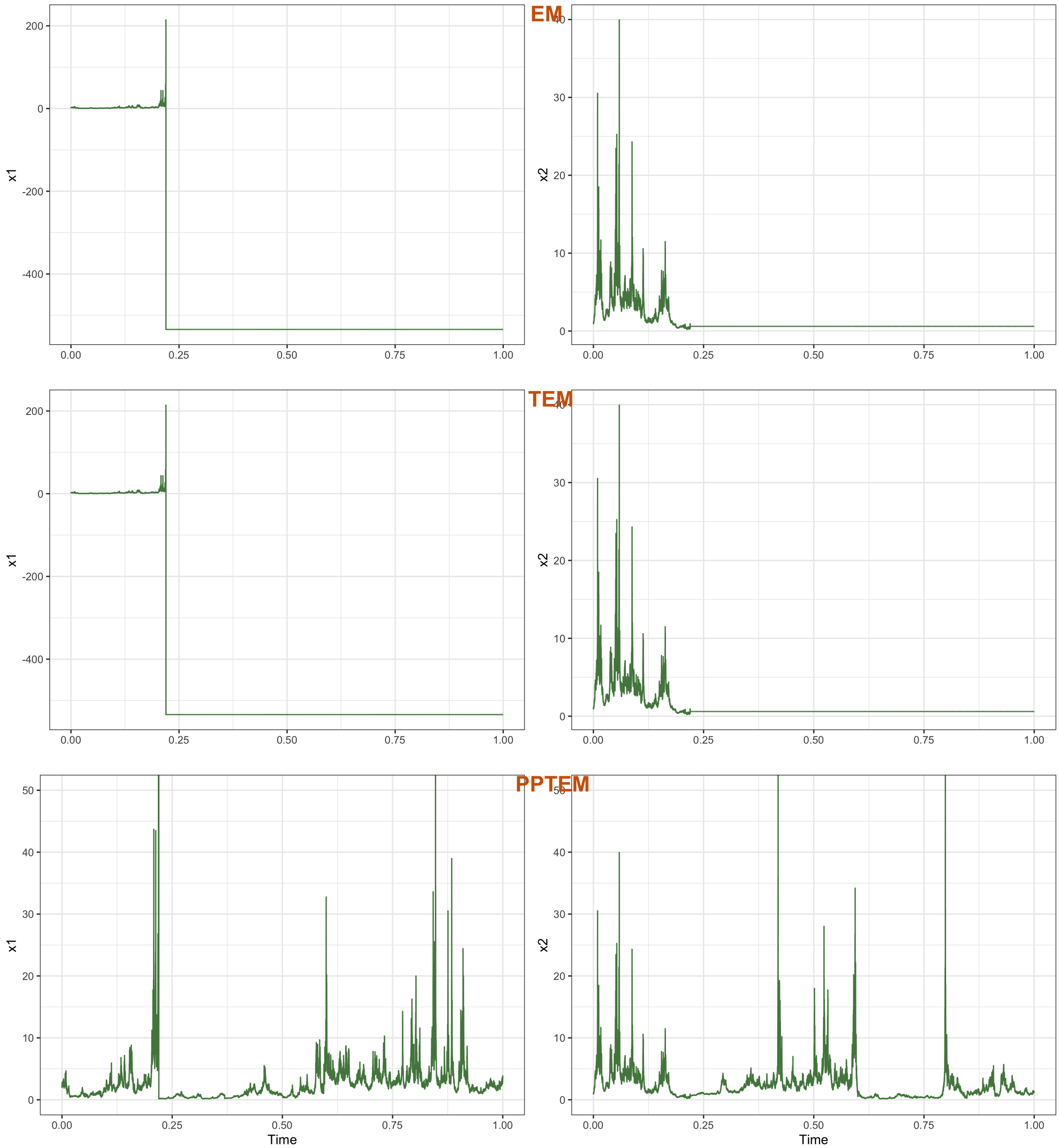

Figure 1: Computer simulations of system (5.2) by the EM, TEM and PPTEM, each with a stepsize $\Delta = 10^{-5}$ and the initial value $x_0 = (2, 1)^T$.

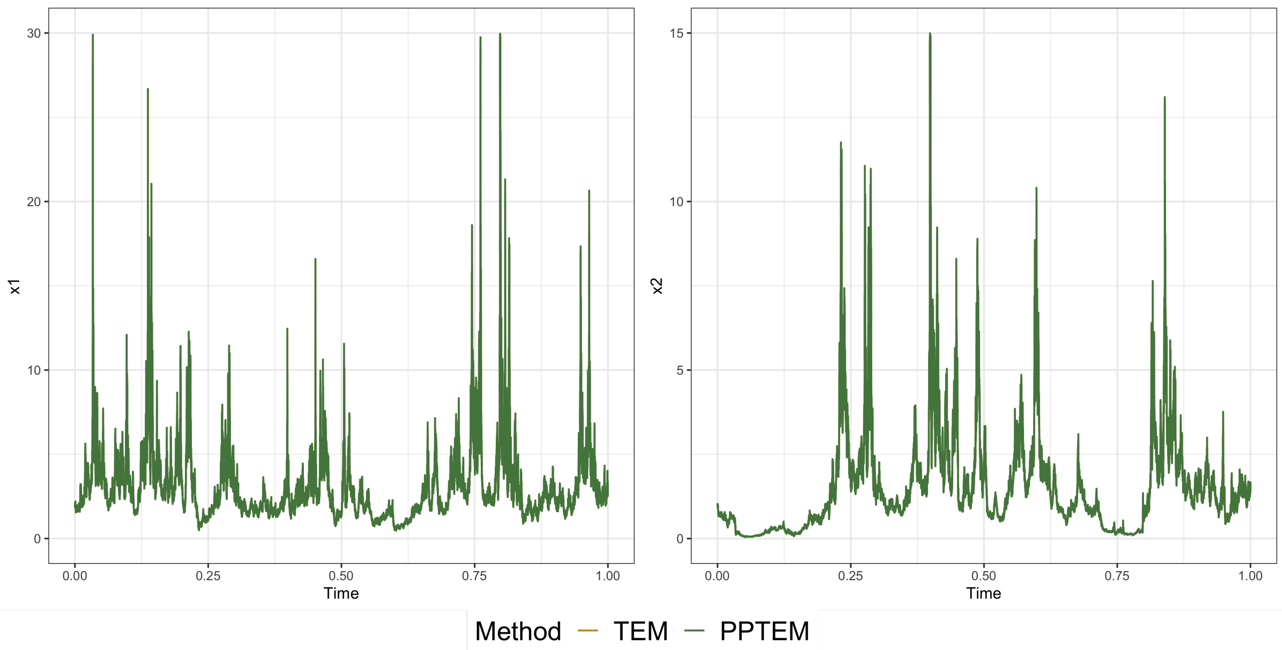

Figure 2: Computer simulations of system (5.2) by the TEM and the PPTEM, both with a stepsize $\Delta = 10^{-6}$ and the initial value $x_0 = (2,1)^T$.

The mapping $\zeta_\Delta(\chi_\Delta(x)) : \mathbb{R}^2 \to \left\{x \in \mathbb{R}^2_+ : |x| \leqslant \sqrt{\frac{10^5}{6}\Delta^{-1/3} + 2\Delta^{2/\theta}}\right\}$ is of the form

$$\zeta_\Delta(\chi_\Delta(x)) = \left(\left(x_1 \wedge \sqrt{\frac{10^5}{6}}\Delta^{-1/6}\frac{x_1}{|x|}\right) \vee \Delta^{1/\theta}, \left(x_2 \wedge \sqrt{\frac{10^5}{6}}\Delta^{-1/6}\frac{x_2}{|x|}\right) \vee \Delta^{1/\theta}\right) \quad \text{for } x \in \mathbb{R}^2.$$

We then formulate the PPTEM solution $X(t)$ via (3.4) and then (3.5). According to Theorem 4.3,

$$\lim_{\Delta \to 0} \mathbb{E}|X(T) - x(T)|^\theta = 0 \tag{5.3}$$

for any $T > 0$ and $\theta \in (0,1)$ under Assumption 2.1. Let us set $\theta = 1/2$ and $T = 1$ from now on.

We are now ready to compare our PPTEM with the EM [15] and the TEM method [16]. The EM scheme is well-known so is omitted here. The TEM applied to SDE (5.2) is to compute the discrete-time TEM solution $W_k \approx x(t_k)$ with $W_0 = x_0$ and

$$W_{k+1} = W_k + [f_1(W_k) + f_2(\chi_\Delta(W_k))]\Delta + g(\chi_\Delta(W_k))\Delta B_k.$$

Also let $W_{k,j}$ be the jth element of W_k. We then extend W_k to the whole $t \geqslant 0$ that is denoted by $W(t)$. Mao [16] shows

$$\lim_{\Delta \to 0} \mathbb{E}|W(T) - x(T)|^\theta = 0 \quad \text{for any } T > 0. \tag{5.4}$$

Figure 1 performs the computer simulations using the EM, TEM and the PPTEM, each with the initial value $x_0 = (2,1)^T$ and the stepsize $\Delta = 10^{-5}$. Although Mao et al. [17] have proved the existence and uniqueness of a global positive solution to this system under Assumption 2.1, it is not surprising that both the EM and TEM paths drop below 0 at around $t = 0.22$. In contrast, we see the PPTEM prevents the paths from hitting 0 during time interval $[0,1]$. This reflects that the typical feature - the positivity of the true solution has been retained by the PPTEM.

Furthermore, to demonstrate the convergence of the PPTEM, a TEM simulation is generated with the stepsize $\Delta = 10^{-6}$ for time interval $[0,1]$. We regard this as a reliable sample of the analytical solution. With the same stepsize, the PPTEM solution is also sketched. From Figure 2, the two paths are so close to each other that no difference can be detected by eyes.

As a conjecture of the convergence rate, we then do the following investigation. Firstly, a simulated path $\{W_k, k = 1, \cdots, 10^8\}$ by TEM is performed with stepsize $\Delta = 10^{-8}$, representing the true solution

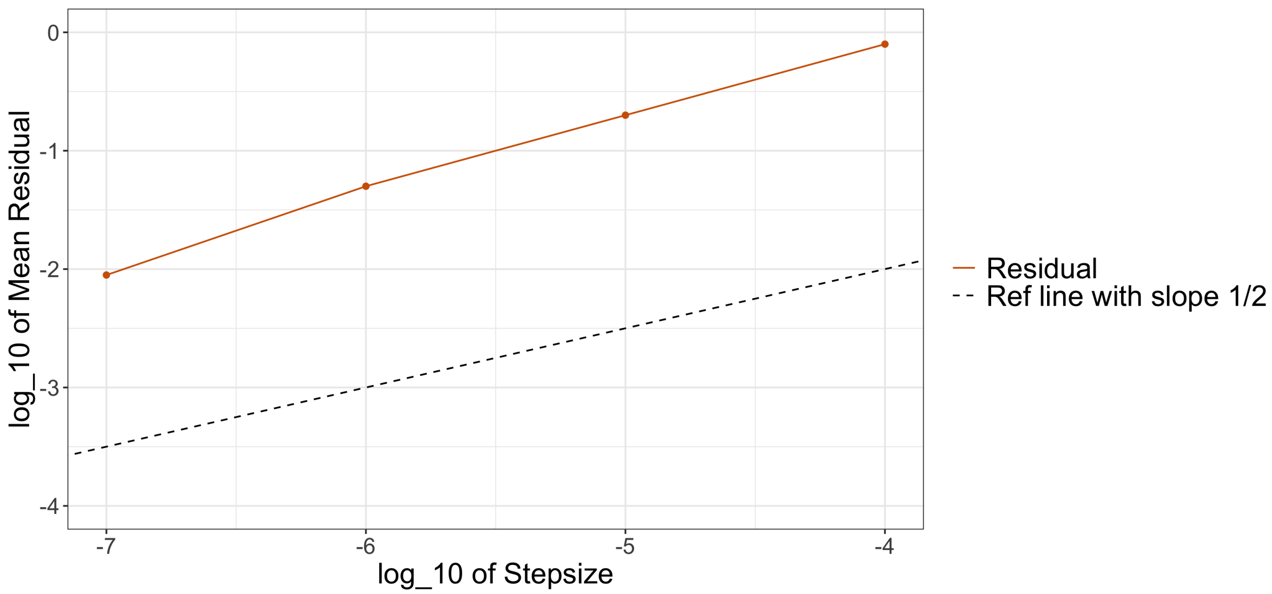

Figure 3: The mean residuals by the PPTEM with stepsize $\Delta = 10^{-7}, 10^{-6}, 10^{-5}$ and 10^{-4} respectively.

of SDE (5.2). Next, four simulations by our PPTEM are produced with stepsize $\Delta = 10^{-7}, 10^{-6}, 10^{-5}$ and 10^{-4} respectively for a duration of 1. We do the above for 100 times and compute the mean residuals by

$$\Big[\sum_{m=1}^{100}\Big((X^m_{10^7,1} - W^m_{10^8,1})^2 + (X^m_{10^7,2} - W^m_{10^8,2})^2\Big)^{1/4}\Big/100\Big]^2$$

for $\Delta = 10^{-7}$ and

$$\Big[\sum_{m=1}^{100}\Big((X^m_{10^6,1} - W^m_{10^8,1})^2 + (X^m_{10^6,2} - W^m_{10^8,2})^2\Big)^{1/4}\Big/100\Big]^2$$

for $\Delta = 10^{-6}$, etc. Figure 3 suggests a convergence rate of order close to 1/2, though it has not been proved theoretically. We leave it as a future work.

The three studies suggest that, when the stepsize is not small enough that both the EM and TEM get into trouble approximating the exact solution, the PPTEM produces a plausible simulation that retains the positivity. On the other hand, when the stepsize is small enough, our PPTEM is close enough to the exact solution, reflecting the convergence of the PPTEM scheme. In conclusion, compared to most of the existing well-known numerical methods, the PPTEM is able to handle the super-linear SDE systems with positive solutions, in the sense of not only the convergence performance, but meanwhile the positivity preserving.

6 Discussion

In this paper, we mainly interpret our positivity preserving numerical method using the n-dimensional Lotka-Volterra system (1.1). However, not limited to (1.1), the principle of the PPTEM is also applicable to various multi-dimensional super-linear SDE models. For instance, consider the stochastic theta process [7, 12]

$$dS(t) = \phi S(t)dt + \sigma S^{\kappa}(t)dB(t), \tag{6.1}$$

and more generally, the two-dimensional mean-reverting-theta stochastic volatility model [4, 12]

$$\begin{aligned} dS(t) &= \phi_1(\alpha_1 - S(t))dt + \sigma_1\sqrt{V(t)}S^{\kappa_1}(t)dB_1(t) \\ dV(t) &= \phi_2(\alpha_2 - V(t))dt + \sigma_2 V^{\kappa_2}(t)dB_2(t). \end{aligned} \tag{6.2}$$

It has been proved that (6.1) has a unique global non-negative/positive solution for any initial value $S(0) > 0$ for $\kappa \geqslant 0.5$ [5, 6, 15], and so does (6.2) for $\kappa_1, \kappa_2 \geqslant 0.5$ [4]. Following the theory proposed in this paper, one can justify that our PPTEM is a strong convergence to SDE (6.1) when $\kappa \geqslant 1$, and to SDE (6.2) when $\kappa_1, \kappa_2 \geqslant 1$, while one may refer to [6] for the case when $\kappa \in [0.5, 1]$. Note that Baduraliya and Mao [4] have investigated the convergence in probability of EM to model (6.2). While with the PPTEM, we have not only achieved strong convergence but also ensured positivity. Next consider the stochastic interest rate model [1, 7]

$$dR(t) = [\alpha_0 - \alpha_1 R^{\kappa_3}(t)]dt + \sigma R^{\kappa_4}(t)dB(t), \tag{6.3}$$

where the presence of a positive global solution has been verified for $\kappa_3, \kappa_4 > 1$. One can show in the same way that the PPTEM works for (6.3) for $\kappa_3, \kappa_4 > 1$. Consequently, this work is not trivial and essentially of full value.

7 Conclusion

This paper concerns about a positivity preserving numerical method for SDEs with positive solutions. A significant advantage of this method is that it works for the multi-dimensional scenario and allows the SDEs of super-linearly growing coefficients. Therefore our theory has covered a wide range of SDEs. The strong convergence is established in the super-linear sense, while the positivity is preserved. The advantages of this method were fully demonstrated by the numerical examples.

Acknowledgements

The authors would like to thank the editors and reviewers for their professional comments. Y. Cai acknowledges the financial support from Zhejiang Natural Science Foundation (LQ22A010009). Q. Guo would like to thank the National Natural Science Foundation of China (11871343) for their financial support. X. Mao would like to thank the Royal Society (WM160014, Royal Society Wolfson Research Merit Award), the Royal Society of Edinburgh (RSE1832), Shanghai Administration of Foreign Experts Affairs (21WZ2503700, the Foreign Expert Program), for their financial support.

Declarations

Confict of interest The authors declare that they have no confict of interest.

References

[1] Y. Ait-Sahalia. Testing continuous-time models of the spot interest rate. *The review of financial studies*, 9(2):385–426, 1996.

[2] D. F. Anderson, D. J. Higham, and Y. Sun. Multilevel Monte Carlo for stochastic differential equations with small noise. *SIAM Journal on Numerical Analysis*, 54(2):505–529, 2016.

[3] J. A. D. Appleby, M. Guzowska, C. Kelly, and A. Rodkina. Preserving positivity in solutions of discretised stochastic differential equations. *Applied Mathematics and Computation*, 217(2):763–774, 2010.

[4] C. H. Baduraliya and X. Mao. The Euler–Maruyama approximation for the asset price in the mean-reverting-theta stochastic volatility model. *Computers & Mathematics with Applications*, 64(7):2209–2223, 2012.

[5] S. Cai, Y. Cai, and X. Mao. A stochastic differential equation SIS epidemic model with two independent Brownian motions. *Journal of Mathematical Analysis and Applications*, 474(2):1536 – 1550, 2019.

[6] Y. Cai, J. Hu, and X. Mao. Positivity and boundedness preserving numerical scheme for the stochastic epidemic model with square-root diffusion term. *Applied Numerical Mathematics*, 182:100–116, 2022.

[7] K. C. Chan, G. A. Karolyi, F. A. Longstaff, and A. B. Sanders. An empirical comparison of alternative models of the short-term interest rate. *The Journal of Finance*, 47(3):1209–1227, 1992.

[8] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden. Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 467(2130):1563–1576, 2011.

[9] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden. Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients. *Annals of Applied Probability*, 22(4):1611–1641, 2012.

[10] C. Kelly and G. J. Lord. Adaptive time-stepping strategies for nonlinear stochastic systems. *IMA Journal of Numerical Analysis*, 38(3):1523–1549, 2018.

[11] P. E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer, 1992.

[12] A. L. Lewis. Option valuation under stochastic volatility. *Finance Press*, 2009.

[13] L. Li and D. Taguchi. On a positivity preserving numerical scheme for jump-extended CIR process: the alpha-stable case. *BIT Numerical Mathematics*, 59(3):747–774, 2019.

[14] X. Li, X. Mao, and G. Yin. Explicit numerical approximations for stochastic differential equations in finite and infinite horizons: truncation methods, convergence in pth moment and stability. *IMA Journal of Numerical Analysis*, 39(2):847–892, 2019.

[15] X. Mao. *Stochastic differential equations and applications*. Elsevier, Horwood, Chichester, 2007.

[16] X. Mao. The truncated Euler–Maruyama method for stochastic differential equations. *Journal of Computational and Applied Mathematics*, 290:370–384, 2015.

[17] X. Mao, G. Marion, and E. Renshaw. Environmental Brownian noise suppresses explosions in population dynamics. *Stochastic Processes and their Applications*, 97(1):95 – 110, 2002.

[18] X. Mao, F. Wei, and T. Wiriyakraikul. Positivity preserving truncated Euler–Maruyama Method for stochastic Lotka–Volterra competition model. *Journal of Computational and Applied Mathematics*, 394:113566, 2021.

[19] M. V. Tretyakov and Z. Zhang. A fundamental mean-square convergence theorem for SDEs with locally Lipschitz coefficients and its applications. *SIAM Journal on Numerical Analysis*, 51(6):3135–3162, 2013.

[20] X. Wang and S. Gan. The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients. *Journal of Difference Equations and Applications*, 19(3):466–490, 2013.

[21] X. Wang, J. Wu, and B. Dong. Mean-square convergence rates of stochastic theta methods for sdes under a coupled monotonicity condition. *BIT Numerical Mathematics*, pages 1–32, 2020.