Improving the age composition of dynamic populations of manufactured items

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Abstract. Populations of manufactured degrading items that are incepted into operation with a given rate are considered. To improve the age composition of these dynamic populations, the truncation of items' lifetimes is proposed and the corresponding optimization problem is studied. The impact of shocks on the population age structures is also described. The effect of these 'interventions' for dynamic and static populations has not been studied in the literature.

Keywords: Age composition; truncation; failure rate; burn-in; shocks

1. Introduction

Dynamic populations of statistically identical items continuously manufactured/incepted into operation are often characterized by the distribution of a random age at each instant of chronological time that is called the age composition. Age composition, that is widely used in demography and population biology, is an important characteristic describing the current properties of a population (see, e.g., Keyfitz and Casewell (2005), Finkelstein and Vaupel (2015)). Much less attention was paid to this characteristic in reliability and operation research studies. However, it also contains a valuable information for studies in these areas. For instance, if the mean age in a population of operating items at some chronological instant of time is relatively large and the items are degrading (i.e., the corresponding failure rates are increasing), the decision on some kind of preventive maintenance (PM) can be considered. However, this decision should take into account the whole population (specifically, as will be shown, its size) and therefore, differs from the conventional PM strategies for one item/system that are widely explored in the literature (Nakagawa (2005), Gertsbakh (2005), Badia et al (2020) to name a few).

The *population-based approach to 'PM'* of items is innovative and, therefore, was not addressed in the literature so far. Our recent two-page discussion note (Cha and Finkelstein, 2022) can be considered as an extended abstract for the current paper, outlining its main objectives.

Populations of items are operating usually in random environments that can be effectively modeled by the process of external shocks. The effect of shocks on operating items and systems in conventional settings has been extensively studied in the literature (see, e.g., Finkelstein and Cha (2013), Eryilmaz and Tekin (2020), to name a few). However, this effect on dynamic or static

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populations that are characterized by the age composition of items was also not addressed in the literature so far. Obviously, shocks mostly affect the more aged items, leaving items with smaller ages unharmed with larger probability. By 'static populations' we understand populations with the fixed size (without inception of the new items), as e.g., a population of used items (see Section 4).

Note that, the effect of shocks on populations of 'new' items that are characterized by the same zero initial age is widely used in practice in the framework of burn-in procedures (Block and Savits (1997), Cha and Finkelstein (2011)). It also worthwhile noting that although we are considering sufficiently large populations, for which relevant statistical reasoning can be applied, at some instances, it is important to keep in mind that they are finite in the sense, e.g., that the total output generated by the population of items (e,g., road machines or wind turbines in a large wind farm) can be defined and relevant in practice.

In applications, the user does not want to have an 'old' fleet of degrading items as 'old' items are more prone to failures that can be costly. In this paper, as discussed above, we consider settings that essentially decrease the age of a fleet of items. However, this improvement comes at a cost of a smaller size of a population. Therefore, the corresponding optimization problems can be considered.

We believe that the prospective practical value of our study is three-fold. First, as just stated above and reflected in the title, the age of a population is stochastically smaller after applying truncation or external shocks. Secondly, truncation for stationary dynamic populations can be considered as the practically important preventive maintenance operation, which is different from that for a single item as the size of a population is now taken into account. Finally, shocks can be considered as the type of burn-in (widely used in conventional practice) for populations described by the age structure. All these applications were not considered in the literature before.

2. Preliminaries

Let $N(x,t)$ be the age-specific population size at time *t*, i.e., the number of items of age *x* at time *t*. Let X_t denote a random age at time *t* of an item that is picked out at random from a population of size $\int_0^t N(u,t)du$ and define the corresponding pdf to be called the "*age composition*" (see, e.g., Keiding (1990), Arthur and Vaupel (1984), Cha and Finkelstein (2018)):

$$\pi_t(x) = \frac{N(x,t)}{\int_0^t N(u,t)du}, \; 0 \le x \le t \tag{1}$$

Assuming that populations are sufficiently large, (1) can be used for description of X_t. Note that, $1/\int_0^t N(u,t)du$ can be interpreted as a probability of choosing any item (equal probabilities) from the population of size $\int_0^t N(u,t)du$.

Assume now that new items are incepted into operation with rate $B(t)$ at time *t*. Thus, $B(t)$ is the number of new items incepted into operation in the small unit interval of time. In what follows, for convenience and having in mind the forthcoming integrations, we will (loosely) use notation

$B(t)dt$ for the number of items manufactured in $[t, t+dt)$. Assuming that the lifetimes of items manufactured at any $t \geq 0$ are i.i.d. with the CDF $F(x)(\bar{F}(x) \equiv 1 - F(x))$, and following the standard approaches used in demographic research, the age composition of the population at time t can be *defined* as (see also Cha and Finkelstein (2018), Hazra et al (2018))

$$\pi_t(x) = \frac{B(t-x)\bar{F}(x)}{\int_0^t B(t-u)\bar{F}(u)du} \cdot I(0 \leq x \leq t), \qquad (2)$$

where $I(0 \leq x \leq t)$ is the corresponding indicator; the numerator, multiplied by dx, is the expected number of survived items having age x (i.e., in the interval $[x, x+dx)$), whereas the denominator *defines* the expected size of the population at time *t.* For convenience, we will drop "expected" from what follows, where appropriate. Thus, we see that in (1) the age composition is defined via random quantities, whereas in (2) it is defined by the expectations of these quantities. However, as in (2) we have the defined deterministic pattern of inception $B(t)$, $\pi_t(x)$ has a 'proper' density meaning in this case as well. To see this, first note that $\int_0^t B(t-u)du$ is the total number of items incepted into operation in (0,t). Furthermore:

-Denote by $P_1(t, dx)$ the probability that any item incepted into operation in $[0, t)$ is alive at *t* and has age in [*x, x*+*dx*]

- Denote by $P(t)$ the probability that any item incepted into operation in $[0, t)$ is alive at *t.* Then

$$\frac{B(t-x)\bar{F}(x)dx}{\int_0^t B(t-u)\bar{F}(u)du} = \frac{\int_0^t B(t-u)du\ P_1(t, dx)}{\int_0^t B(t-u)du\ P(t)} = \frac{P_1(t, dx)}{P(t)},$$

which also defines the conditional probability that an item has an age in [*x, x*+*dx*] given that it is alive at *t*. The latter perfectly complies with the 'density meaning' in $\pi_t(x)dx$.

Setting $B(t) = B, t \to \infty,$ we arrive at the well-known density of the equilibrium distribution:

$$\pi_\infty(x) = \frac{\bar{F}(x)}{\int_0^\infty \bar{F}(u)du} \equiv \frac{\bar{F}(x)}{\mu}, \qquad (3)$$

where μ denotes the mean that corresponds to the CDF $F(x)$. Relationship (3) also characterizes the pdf of the remaining lifetime in this set up (see, e.g., Finkelstein and Vaupel (2015)).

As it was discussed in the Introduction, we want to decrease the age X_t, e.g., in the sense of the usual stochastic ordering (Shaked and Shanthikumar (2007)), i.e.,

$$\tilde{X}_t \leq_{st} X_t \text{ when } \tilde{\Pi}_t(x) \geq \Pi_t(x), x \geq 0, \qquad (4)$$

where $\tilde{X}_t$ and $\tilde{\Pi}_t(x)$ denote the decreased age and its CDF accordingly. In what follows, the stronger ordering in the sense of the likelihood ratio will be also used

$$\tilde{X}_t \le_{lr} X_t \text{ when } \tilde{\pi}_t(x)/\pi_t(x) \text{ is decreasing in } x \ge 0. \quad (5)$$

3. Truncation for dynamic populations

In order to decrease the consequences of sudden failures of aged items (e.g., with increasing failure rate $\lambda(x)$), it can be reasonable to discard them on reaching certain age. This, in a way, resembles the conventional preventive maintenance (PM), however, the situation here is different, as discarding items comes at a price of reducing the population size of operating items. The latter can be critical in applications. The trade-off between the increasing probabilities of failures (manifested by the increasing failure rates of items) and the decreasing population size can define the corresponding optimization problem for obtaining the optimal truncation value. In what follows, we describe the main steps of our innovative approach.

Let Δ denote the truncation age at which items should be discarded. Then the initial distribution $F(x)$ is substituted by

$$F_\Delta(x) = \begin{cases} F(x), 0 \le x < \Delta, \\ 1, \quad x \ge \Delta. \end{cases} \quad (6)$$

It can be easily seen that $F_\Delta(x) \ge F(x)$ and therefore, the lifetime after truncation is, obviously, smaller than that before it in the sense of the usual stochastic order.

Relationship (2) in this case turns to

$$\pi_{t,\Delta}(x) = \frac{B(t-x)\bar{F}(x)I(0 \le x < \Delta)}{\int_0^t B(t-u)\bar{F}(u)I(0 \le u < \Delta)du} \cdot I(0 \le x \le t). \quad (7)$$

Setting $B(t) = B, t \to \infty,$ similar to (3), we arrive at the new equilibrium density (in what follows the stationary case will be considered for simplicity and general illustration of our approach)

$$\pi_{\infty,\Delta}(x) = \begin{cases} \dfrac{\bar{F}(x)}{\int_0^\Delta \bar{F}(u)du}, \quad 0 \le x < \Delta \\ 0, \quad x \ge \Delta \end{cases} \quad (8)$$

Note that,

$$B\int_0^\infty \bar{F}(u)du = B\mu; \quad B\int_0^\Delta \bar{F}(u)du = B\mu_\Delta \quad (9)$$

define, as follows from (2) and (7), the sizes of populations at t for $B(t) = B, t \to \infty,$ in the non-truncated and truncated settings, accordingly.

Intuitively, it is clear that truncation should result in a smaller random age $X_{t,\Delta}$. But in what stochastic sense? Dividing (8) by (3)

$$\frac{\pi_{\infty,\Delta}(x)}{\pi_\infty(x)} = \begin{cases} \int_0^\infty \bar{F}(u)du \,/ \int_0^\Delta \bar{F}(u)du, 0 \le x \le \Delta, \\ 0, x \ge \Delta, \end{cases}$$

which is, obviously, a decreasing function of *x,* and therefore, in accordance with (5), the random age with truncation is smaller than that without it in the sense of the likelihood ratio ordering, i.e., $X_{\infty,\Delta} \leq_{lr} X_{\infty}$. It is well known that in this case the weaker usual stochastic ordering (4) holds for these random ages (Shaked and Shanthikumar, 2007).

Let us choose an item at random from a stationary population with the truncation level Δ. The following 'ordinary' mixture defines the population failure rate (PFR) for the described stationary population. Note that, in this case we do not need the conditional mixing distributions as the structure of the stationary population is such that the principle "the weaker subpopulations are dying out first" is not relevant (Finkelstein and Cha, 2013). Thus, $\Lambda(\Delta)dt$ defined by (10) can be interpreted as the probability of failure in $[t, t+dt)$ of an item chosen at random from the population defined by the age composition (7).

$$\Lambda(\Delta) = \int_0^{\Delta} \lambda(x)\pi_{\infty,\Delta}(x)dx, \tag{10}$$

where $\lambda(x)$ is the baseline failure rate. It can be seen that, when $\lambda(x)$ is increasing, this function is monotonically increasing with Δ to the 'nontruncated' limit

$$\Lambda(\infty) = \int_0^{\infty} \lambda(x)\pi_{\infty}(x)dx.$$

Indeed, $\Lambda(\Delta)$ can be interpreted as an expectation of $\lambda(X_{\infty,\Delta})$ with respect to the distribution of $X_{\infty,\Delta}$, $\pi_{\infty,\Delta}(x)$. For $\Delta_1 < \Delta_2$, from (8)

$$\frac{\pi_{\infty,\Delta_1}(x)}{\pi_{\infty,\Delta_2}(x)} = \begin{cases} \dfrac{\int_0^{\Delta_2} \bar{F}(u)du}{\int_0^{\Delta_1} \bar{F}(u)du}, & 0 \leq x < \Delta_1, \\ 0, & \Delta_1 \leq x < \Delta_2 \end{cases} \tag{11}$$

which is decreasing function of x, meaning that $X_{\infty,\Delta_1} \leq_{lr} X_{\infty,\Delta_2}$. Furthermore, $\lambda(x)$ is increasing function of x. Therefore, we have $\Lambda(\Delta_1) \leq \Lambda(\Delta_2)$ (Shaked and Shanthikumar, 2007).

Thus, it is better to have smaller Δ, as it decreases the probability of failure of a chosen item. But, as was discussed, this comes at a price of a smaller number of items in a population. This effect can be taken into account by considering the *relative* PFR when $\Lambda(\Delta)$ in (10) is divided by the size of a population $B\mu_{\Delta}$ (see (9). Using (8)-(10),

$$\Lambda_r(\Delta) = \frac{\int_0^{\Delta} \lambda(x)\pi_{\infty,\Delta}(x)dx}{B\mu_{\Delta}} = \frac{\int_0^{\Delta} \lambda(x)\bar{F}(x)dx}{B(\mu_{\Delta})^2} = \frac{F(\Delta)}{B(\mu_{\Delta})^2}. \tag{12}$$

Indeed, when we have a finite population of i.i.d. items (e.g., the fleet of road machines employed in construction of the national road) the failure of one item have different consequences for a population, say of 100 and 200 items. Obviously, these consequences are smaller for the latter population. The measure (12) captures this reasoning and the goal is to find Δ that minimizes (12).

These general considerations can be also made more specific via the cost-wise discussion.

Let the cost of a failure of an item be C_f, whereas the productivity rate of one item is *r*. In practice, it is important to minimize the relative loss that compares the loss of one item with the gain provided by the whole population. Thus, for the arbitrary chosen item from a population, we can decrease the *probability* of a loss in a small interval of time dt, i.e., the expected loss, by truncating, but by this truncation we decrease the size of a population decreasing the possible 'gain'. Indeed, let the time interval be Δt, then the expected relative loss in this interval for an item chosen at random from the population of size $B\mu_\Delta$ is, as can be easily seen, proportional to $\Lambda_r(\Delta)$.

Thus, the following optimization problem should be considered,

$$\Lambda_r(\Delta^*) = \min_{\Delta>0} \Lambda_r(\Delta). \tag{13}$$

Remark 1. We see that our reasoning here differs from the conventional optimal PM framework. This is because we are, in fact, dealing with the whole population and therefore, two factors are in the trade-off, i.e., the increasing baseline failure rate, and the population size.

The sign of the derivative $\Lambda_r'(\Delta)$ in (12) is defined by the sign of the function:

$$G(\Delta) = \lambda(\Delta)\int_0^\Delta \bar{F}(u)du - 2\int_0^\Delta \lambda(u)\bar{F}(u)du = \lambda(\Delta)\int_0^\Delta \bar{F}(u)du - 2F(\Delta), \tag{14}$$

whereas $\Lambda'(\Delta)$ (see (10)) is defined by the following non-negative function (as $\lambda(x)$ is increasing)

$$\lambda(\Delta)\int_0^\Delta \bar{F}(u)du - \int_0^\Delta \lambda(u)\bar{F}(u)du = \lambda(\Delta)\int_0^\Delta \bar{F}(u)du - F(\Delta) \ge 0 \tag{15}$$

and, therefore, $\Lambda(\Delta)$ is increasing. However, due to multiplier 2 in (14), $G(\Delta)$ is not necessarily positive now.

Let $\lambda(0) \ne 0$. Then $\lim_{\Delta\to 0}\Lambda_r(\Delta) = \infty$, whereas $G(\Delta) < 0$ for sufficiently small Δ. On the other hand, $G(0) = 0$. Thus $\Lambda_r(\Delta)$ is decreasing in the vicinity of origin. Let $\lim_{x\to\infty}\lambda(x) = \infty$. Then $G(\infty) = \infty$ and a finite minimum $\Lambda_r(\Delta^*)$ exists. Other cases can be also effectively and elementarily analyzed; however, some additional conditions can be implemented for existence of an optimal solution.

Example 1. Suppose that $\lambda(t) = 2t^2 + 1$. Previously, we have compared

$$\pi_\infty(x) = \frac{\bar{F}(x)}{\int_0^\infty \bar{F}(u)du} \quad \text{and} \quad \pi_{\infty,\Delta}(x) = \begin{cases} \dfrac{\bar{F}(x)}{\int_0^\Delta \bar{F}(u)du}, & 0 \le x < \Delta \\ 0, & x \ge \Delta \end{cases},$$

and it has been found that the random age with truncation is stochastically smaller than that without it. To illustrate it, by setting $\Delta = 1$, the Cdfs of $\pi_{\infty}(x)$ and $\pi_{\infty,\Delta}(x)$, which are denoted by $\Pi_{\infty}(x)$ and $\Pi_{\infty,\Delta}(x)$, have been obtained in Fig. 1.

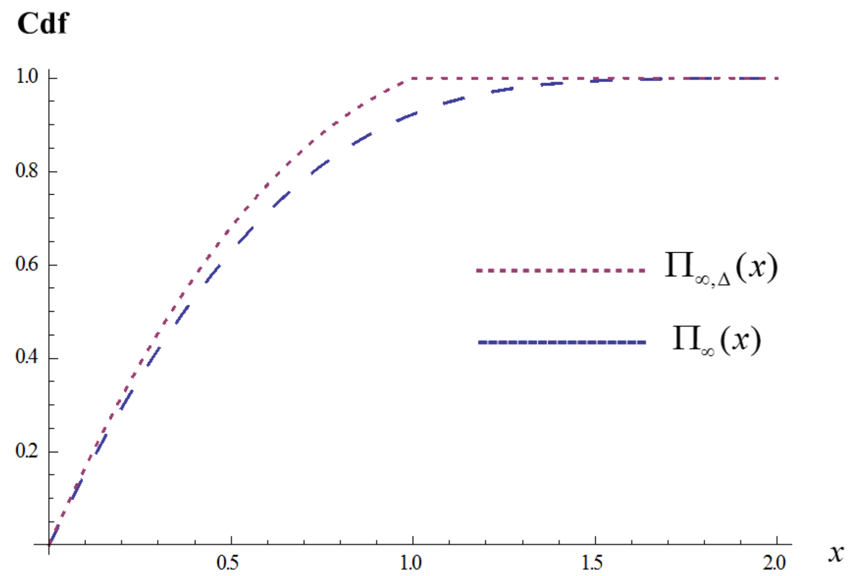

Fig. 1 The graphs for $\Pi_{\infty}(x)$ and $\Pi_{\infty,\Delta}(x)$

Consider now optimization of $\Lambda_r(\Delta)$. The curve for $\Lambda_r(\Delta)$ in this case with $B = 0.01$ is given in Fig. 2. In this case, the optimal truncation level is obtained as $\Delta^* = 0.95$. Suppose now that $\lambda(t) = 3t^2 + 1$ and the curve for $\Lambda_r(\Delta)$ with the same $B = 0.01$ is given in Fig. 3, where the optimal truncation level is obtained as $\Delta^* = 0.80$. Observe that these failure rates are ordered and, therefore $\Lambda_r(\Delta)$ are also ordered as can be seen comparing the curves. From intuitive considerations, we can expect that the truncation level should be smaller for the second case as larger failure rate means larger risk of future failures, which is 'neutralized' by the appropriate truncation. Indeed, numerical results confirm this observation, as $\Delta^* = 0.80$ which is a noticeable decrease compared with $\Delta^* = 0.95$. The figures also show the noticeable decrease in the relative PFR, as compared with the no-truncation case when $\Delta^* = \infty$. It is interesting to perform the relevant sensitivity analysis of parameters of the model. However, this is out of the scope of the current note. Finally, we emphasize once more that the obtained results for populations are different from those for a single item as the trade-off between the increasing probabilities of failures and the decreasing population size is achieved in our approach.

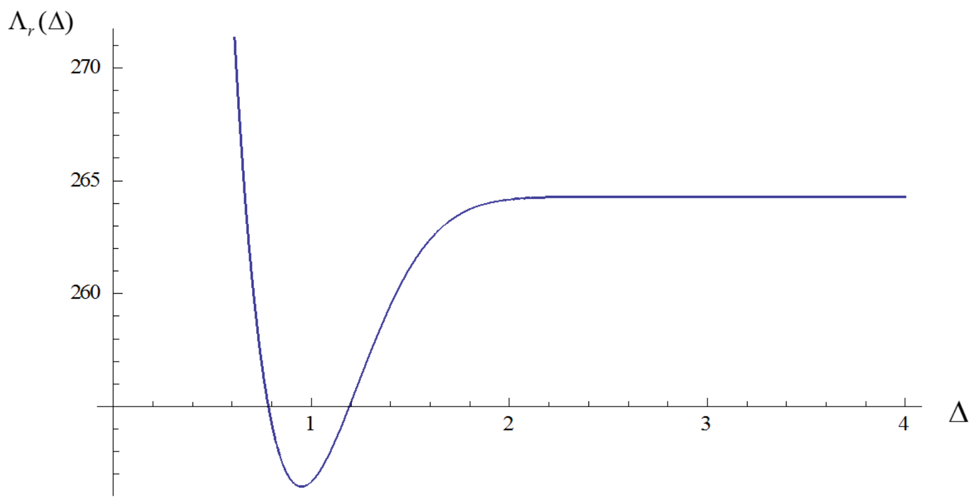

Fig. 2 $\Lambda_r(\Delta)$ as a function of Δ for $\lambda(t) = 2t^2 + 1$, $B = 0.01$.

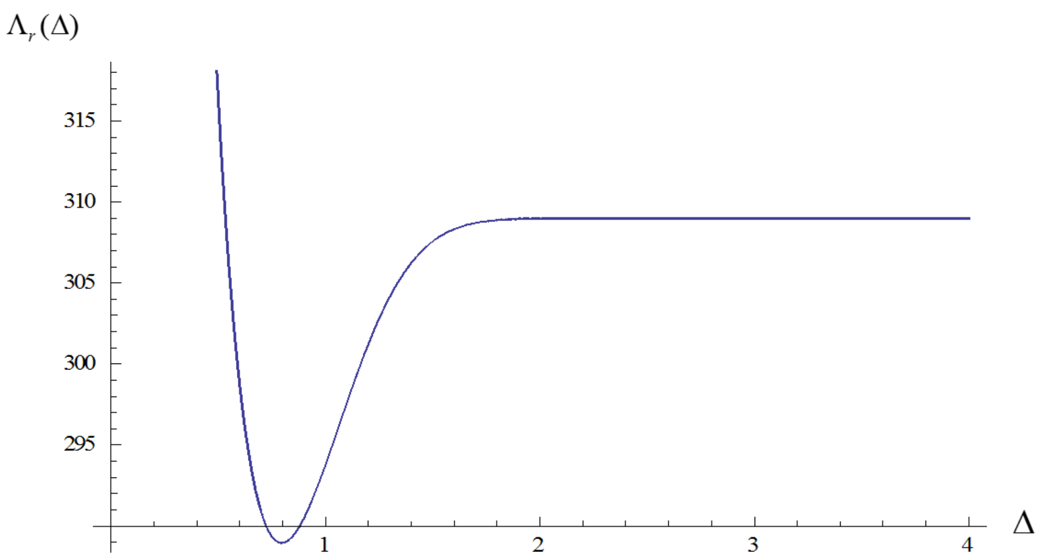

Fig. 3 $\Lambda_r(\Delta)$ as a function of Δ for $\lambda(t) = 3t^2 + 1$, $B = 0.01$.

4. Populations under shocks

In practice, for populations operating in a random environment, it is important to assess an impact of external shocks on the whole dynamic population. That is, what will be the age composition of a population with initial age composition (1) after a shock? Another useful interpretation (although for a 'static' (fixed) population) is as follows: there is a given fleet/population of used items with a random age of usage described by (1). The user decides to improve/decrease this random age, as more aged items are more prone to failure that can result in substantial economic loss. This can be done, e.g., by operation of burn-in, specifically, by the burn-in executed via shocks. Obviously, burn-in can be applied only to specific populations (e.g., electronic items or parts of a larger devices). This problem was approached in the literature on burn-in only for fixed populations of 'new' items that are, obviously' not characterized by the age composition (see, e.g., Block and Savits (1997), Cha and Finkelstein (2011)). In the current note, for the considered set up, we just formulate some initial properties and provide relevant introductory discussion within our general framework, whereas the more detailed and advanced study including relevant optimization will be reported elsewhere.

Assume that a shock affects the whole population of items described by the age composition (1). Let an impact of a shock be described by the extreme shock model adjusted to the age-specific population, i.e., a shock 'kills' an item of age x with probability $p(x)$ and is survived with the complementary probability $q(x)$. It is natural to assume that $q(x)$ is a decreasing function of age as deteriorating items with larger age are more prone to failure. In accordance with (1), the age composition after a shock, $\pi_{t,q}(x)$, can be written as

$$\pi_{t,q}(x) = \frac{q(x)N(x,t)}{\int_0^\infty q(u)N(u,t)du} \tag{16}$$

On the other hand, (3) turns to

$$\pi_{t,q}(x) = \frac{B(t-x)q(x)\bar{F}(x)}{\int_0^t B(t-u)q(u)\bar{F}(u)du} \cdot I(0 \le x \le t) \tag{17}$$

with a stationary variant (to be considered further for the illustration of our approach, whereas some generalizations to the non-stationary case can be also obtained)

$$\pi_{\infty,q}(x) = \frac{q(x)\bar{F}(x)}{\int_0^\infty q(u)\bar{F}(u)du} \tag{18}$$

Comparing (4) and (18) results in

$$\pi_{\infty,q}(x) = \frac{q(x)\bar{F}(x)}{\int_0^\infty q(u)\bar{F}(u)du} = q(x)\frac{\int_0^\infty \bar{F}(u)du}{\int_0^\infty q(u)\bar{F}(u)du}\frac{\bar{F}(x)}{\int_0^\infty \bar{F}(u)du} = Cq(x)\pi_\infty(x), \tag{19}$$

where the normalizing constant is $C = \int_0^\infty \bar{F}(u)du / \int_0^\infty q(u)\bar{F}(u)du$. Thus

$$\frac{\pi_{\infty,q}(x)}{\pi_\infty(x)} = Cq(x) \tag{20}$$

is decreasing and therefore, in accordance with (5), the random age after a shock is smaller than it was before in the sense of the likelihood ratio ordering, i.e., $\tilde{X}_t \le_{lr} X_t$. It should be also noted that, as follows from (19), this operation with the corresponding age compositions (densities) can be described in terms of the weighted distributions (Behdani et al, 2018).

What happens if we have more than one shock affecting a population? Here we should distinguish between two scenarios. In the first one, we have a 'static' population (no inception of new items) with the given initial age composition (random age). Then, in the course of burn-in, we can apply the second, third, etc shock. After the n-th shock (16) turns to

$$\pi_{t,q,n}(x) = \frac{q^n(x)N(x,t)}{\int_0^\infty q^n(u)N(u,t)du}.$$

In principle, the number of required for burn-in shocks can be obtained, e.g., in the following way. Assume that the user wants the mean age of the population of items to be not larger than A, whereas, this value for the initial population exceeds this value. Thus, the minimal *n* should be obtained, for which $\int_0^\infty x\pi_{\infty,q,n}(x)dx < A$.

For dynamic populations with inception of new items, a population after one shock becomes non-stationary and further reasoning can become very complex. This can be considered elsewhere. However, under certain assumptions, we can proceed in a different way. Assume that the process of shocks affecting the population is a homogeneous Poisson process with rate ρ independent of degradation processes in the items. Then, in accordance with the extreme shock model (every shock results in failure with probability *q(x)* and is harmless with probability 1-*q(x)*), the survival function of an item under the process of shocks can be defined as

$$\bar{F}_q(x) = \exp\left\{-\rho\int_0^x q(u)du\right\}\bar{F}(x). \tag{21}$$

Substituting (21) instead of $\bar{F}(x)$ in (3) results in

$$\pi_{\infty,q}(x) = \frac{\overline{F_q}(x)}{\int_0^\infty \overline{F_q}(u)du}$$

The corresponding likelihood ratio

$$\frac{\pi_{\infty,\Delta}(x)}{\pi_\infty(x)} = \frac{\overline{F_q}(x)\int_0^\infty \bar{F}(u)du}{\int_0^\infty \overline{F_q}(u)du\bar{F}(x)} \sim \exp\left\{-\rho\int_0^x q(u)du\right\}$$

is decreasing and therefore, as previously, $\tilde{X}_t \leq_{lr} X_t$. Thus, the effect of the process of external shocks in the described setting is similar (in a way) to that with truncated lifetimes, i.e., the random age that corresponds to the age composition of a population of shocks is smaller than that without shocks in the sense of the likelihood ratio ordering. This is, of course, counterintuitive from the practical perspective, but the paradox can hopefully be resolved by using an appropriately modified measure (12). This topic needs further investigation.

Example 2. Suppose that the failure rate of $\bar{F}(x)$ is given, as in Example 1, by $\lambda(t) = 2t^2 + 1$ and $q(t) = q$, $\forall t \ge 0$. Then

$$\bar{F}_q(x) = \exp\{-\rho q x\}\bar{F}(x).$$

We are interested now in the age distribution (Cdf) in a stationary population, which is, as follows from (3),

$$\Pi_{\infty,q}(x) = \frac{\int_0^x \overline{F_q}(u)du}{\int_0^\infty \overline{F_q}(u)du}$$

For $\rho = 3.5$, the corresponding Cdfs have been plotted for $q = 0; 0.10; 0.20; 0.40$ (see Fig. 4), where $q = 0$ corresponds to the case with no shocks.

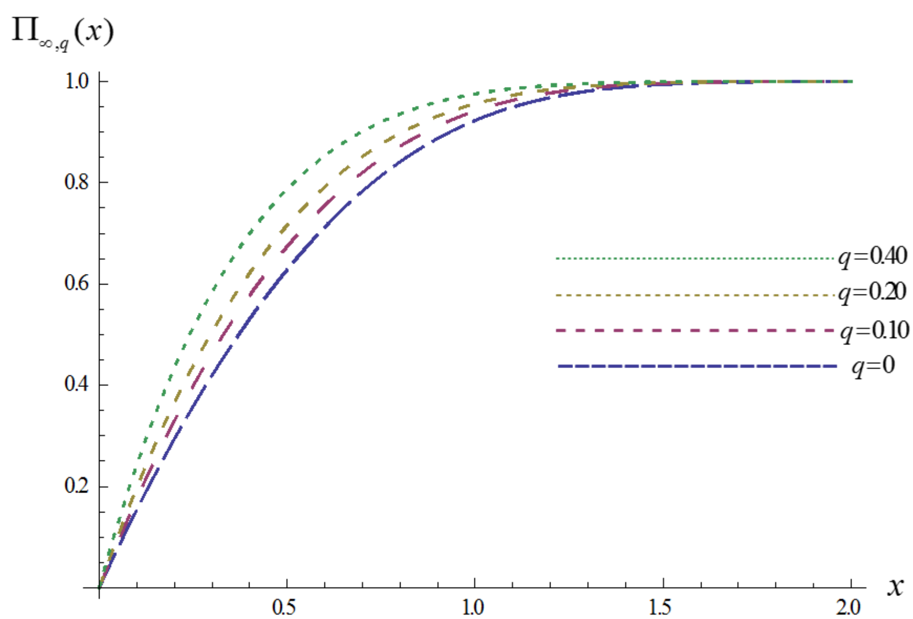

Fig. 4. The graph for $\Pi_{\infty,q}(x)$ for $q = 0, 0.01, 0.20, 0.40$

Note that, when $q_1 < q_2$,

$$\frac{\pi_{\infty,q_2}(x)}{\pi_{\infty,q_1}(x)} = \frac{\int_0^\infty \overline{F_{q_1}}(u)du}{\int_0^\infty \overline{F_{q_2}}(u)du} \frac{\overline{F_{q_2}}(x)}{\overline{F_{q_1}}(x)},$$

is decreasing in x, which implies the likelihood ratio ordering between the age distributions $\pi_{\infty,q_1}(x)$ and $\pi_{\infty,q_2}(x)$. Accordingly, in Figure 4, the corresponding Cdfs are ordered, as likelihood ratio ordering is stronger than the usual stochastic ordering (Shaked and Shanthikumar (2007)).

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