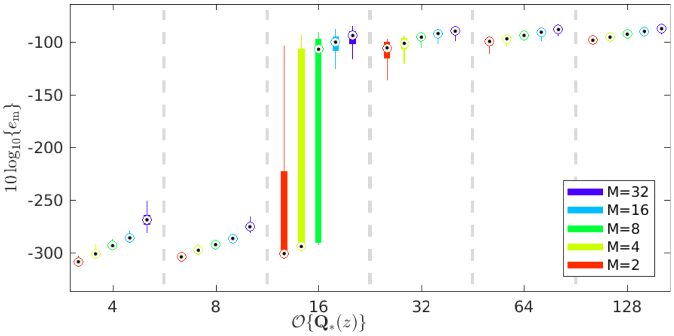

Fig. 4. Ensemble results for mismatch between the obtained solution and the ground truth, dependent on the order $\mathcal{O}\{\boldsymbol{Q}_*(z)\}$ of the ground truth and the spatial dimension M.

$\boldsymbol{Q}_i(z)$, $i = 1, 2$, are set to $\{2, 4, 8, 16, 32, 64\}$, which determines the order $\mathcal{O}\{\boldsymbol{Q}_*(z)\}$ of the ground truth paraunitary matrix. For each value of M and the orders of $\boldsymbol{Q}_i(z)$, $i = 1, 2$, we generate 100 random instances of $\{\boldsymbol{R}(z), \boldsymbol{Q}_*(z)\}$.

To characterise the mismatch e_m between the ideal ground truth paraunitary matrix $\boldsymbol{Q}_*(z)$ and the matrix $\hat{\boldsymbol{Q}}_*(z)$ recovered by the polynomial Procrustes problem, we measure

$$e_\mathrm{m} = \oint_{|z|=1} \|\boldsymbol{Q}_*(z) - \hat{\boldsymbol{Q}}_*(z)\|_\mathrm{F}^2 \frac{\mathrm{d}z}{z} \; . \qquad (25)$$

As with (23), the metric in (25) can be evaluated in the time domain using Parseval's theorem. Since according to Sec. IV-C, small trailing values in $\hat{\boldsymbol{Q}}_*(z)$ falling below a threshold of 10^{-10} are truncated [24], a shift-alignment is necessary. Note however, that after a similar truncation of trailing values below 10^{-10} was applied to $\boldsymbol{Q}_*(z)$, both the truncated ground truth and the Procrustes solution exhibited the same orders across the entire ensemble.

Over the untrimmed order $\mathcal{O}\{\boldsymbol{Q}_*(z)\}$ of the ground truth, Fig. 4 shows the mismatch e_m for different values of M. This metric is close to machine precision for small values of $\mathcal{O}\{\boldsymbol{Q}_*(z)\}$. For larger values of $\mathcal{O}\{\boldsymbol{Q}_*(z)\} \geq 16$, the construction of $\hat{\boldsymbol{Q}}_*(z)$ leads to the truncation of small trailing values, which causes a loss in precision in the results around the truncation value of 10^{-10}. Nonetheless, the size of the mismatch error e_m highlights that the correct paraunitary ground truth matrix is recovered.

VI. Conclusions

We have investigated the challenge of finding the best least-squares fit of a paraunitary matrix to a given matrix of analytic functions. For this purpose, we have extended the narrowband Procrustes problem, which is based on the factorisation afforded by an analytic SVD. The restriction here has been to square matrices with spectrally majorised, distinct singular values. We have shown that a unique solution to the polynomial Procrustes problem exists, for which we have presented an algorithmic implementation.

In simulations, we have demonstrated the approach in the domains of time delay estimation, filter bank completion, and paraunitary matrix recovery. Given a sufficient DFT size, the achievable precision can be close to machine accuracy, or to the level of truncation if trimming is applied to control the polynomial order of the matrices.

For further investigation, there remains the case of multiple identical, in particular zero singular values, the case of general rectangular matrices where singular vectors may be arbitrary within some subspace, and the application to problems where benchmark techniques exist for comparison.

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