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On the Analysis of Ait-Sahalia-Type Model for Rough Volatility Modelling

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Abstract
Fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$ is used widely, for instance, to describe 'rough' volatility data in finance. In this paper, we examine a generalised Ait-Sahalia-type model driven by a fractional Brownian motion with $H < \frac{1}{2}$ and establish theoretical properties such as an existence-and-uniqueness theorem, regularity in the sense of Malliavin differentiability and higher moments of the strong solutions.

Keywords Rough volatility · Malliavin calculus · Fractional Brownian motion · Strong solution · Higher moments

Mathematics Subject Classification (2020) 60H10 · 60H30

1 Introduction

Over the years, SDEs driven by noise with α^--Hölder continuous random paths for $\alpha \in [\frac{1}{2}, 1)$ have been applied to model the dynamical behaviour of volatility of asset prices in finance. See, for example, [1–3] and the references therein. However, in recent years, empirical evidence (see e.g. [4]) has shown that volatility paths of asset prices are more irregular in the sense of α^--Hölder continuity for $\alpha \in (0, \frac{1}{2})$ in

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many instances. This inadequacy actually showed the need for models based on SDEs driven by a noise of low α^--Hölder regularity with $\alpha \in (0, \frac{1}{2})$ which has been used by researchers and practitioners to describe the volatility dynamics of asset prices. These models are driven by rough signals that can capture well the 'roughness' in the volatility process of asset prices. Such rough signals arise, for example, from paths of the fractional Brownian motion (fBm). The fractional Brownian motion is a generalisation of the ordinary Brownian motion. It is a centred self-similar Gaussian process with stationary increments which depends on the Hurst parameter H. The Hurst parameter lies in (0, 1) and controls the regularity of the sample paths in the sense of a.e. (local) H^--Hölder continuity. The smaller the Hurst parameter, the rougher the sample paths and vice versa. For instance, the authors in [5] employ the fractional Brownian motion with $H < \frac{1}{2}$ to model the 'rough' volatility process of asset prices and derive a representation of the sensitivity parameter delta for option prices. Similarly, the authors in [6] also consider an asset price model in connection with the sensitivity analysis of option prices whose correlated 'rough' volatility dynamics is described by means of an SDE driven by a fractional Brownian motion with $H < \frac{1}{2}$. The reader may consult [7, 8] for the coverage of properties and financial applications of the fractional Brownian motion with $H < \frac{1}{2}$ (see also Appendix).

In the context of interest rate modelling, Ait-Sahalia proposed a new class of highly nonlinear stochastic models in [9] for the evolution of interest rates through time after rejecting existing univariate linear-drift stochastic models based on empirical studies. In this model, (short-term) interest rates x_t have the SDE dynamics

$$\mathrm{d}x_t = \left(\alpha_{-1}x_t^{-1} - \alpha_0 + \alpha_1 x_t - \alpha_2 x_t^2\right)\mathrm{d}t + \sigma x_t^\theta \mathrm{d}B_t \tag{1}$$

on $t \geq 0$ with initial value x_0, where $\alpha_{-1}, \alpha_0, \alpha_1, \alpha_2 > 0, \sigma > 0, \theta > 1$ and B_t is a scalar Brownian motion. SDE (1) has been studied by many authors (see e.g. [10, 11]). Besides interest rate modelling, SDE (1) has also been used extensively among academic researchers and market practitioners to describe stochastic volatility and asset price dynamics. For example, in stochastic volatility modelling, the stock price process S_t, $t \geq 0$, may be modelled by the Black–Scholes SDE

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}B_t, \quad t \geq 0, \tag{2}$$

where $\mu \in \mathbb{R}$ is the mean return and $\sigma_t > 0, t \geq 0$, is the volatility process described by the SDE (1). Generally, there are several classes of SDE (1) with parametric restriction. For example, Black–Scholes, Vasicek, Dothan, CIR and CEV models fall under SDE (1).

In the context of 'rough' stochastic volatility modelling, we note that SDE (1) may not provide a good fit since the driving noise is a Brownian motion $B_\cdot$. In this case, we recognise the need to replace the driving noise $B_\cdot$ with a fractional Brownian motion $B_\cdot^H$ and consider a 'rough' volatility model based on the SDE

$$\mathrm{d}x_t = \left(\alpha_{-1}x_t^{-1} - \alpha_0 + \alpha_1 x_t - \alpha_2 x_t^\rho\right)\mathrm{d}t + \sigma x_t^\theta d^\circ B_t^H \tag{3}$$

for $t \geq 0$ and $H \in (0, \frac{1}{2})$, where $\sigma x_t^\theta d^\circ B_t^H$ stands for a stochastic integral in the sense of Russo and Vallois (see Sect. 5). However, since the expected value of $\sigma x_t^\theta d^\circ B_t^H$ (if it exists) is not zero, in general, we observe that the SDE (3) does not necessarily yield the Ornstein–Uhlenbeck dynamics as a special case. In other words, SDE (3) may not be used to capture the mean reversion property, which plays an important role in finance. In order to account for the mean reversion property of SDE (3), we may consider instead the following SDE

$$\mathrm{d}x_t = \left(\alpha_{-1} x_t^{-1} - \alpha_0 + \alpha_1 x_t - \alpha_2 t^{2H-1} x_t^\rho\right) \mathrm{d}t + \sigma x_t^\theta \mathrm{d}B_t^H \tag{4}$$

for $t \geq 0$ with initial value x_0, $t \in (0, 1]$, $H \in (0, \frac{1}{2})$ and $\rho > 1$. The stochastic integral for the fractional Brownian motion in (4) is defined via an integral concept in [7] and related to a Wick–Itô–Skorohod type of integral (see also Sect. 5). We mention that the mean of the stochastic integral in (4) is zero (provided that the mean exists). Therefore, one obtains from SDE (4) the Ornstein–Uhlenbeck dynamics as a special case if one formally chooses α_{-1}, α_2 and θ to be zero.

As mentioned before, the original Ait-Sahalia model has been applied to interest rate modelling. However, the Ait-Sahalia model in our setting, (3) and (4) cannot be employed in interest rate modelling since empirical evidence shows that interest rate paths rather exhibit Hölder continuity with an index bigger than $\frac{1}{2}$ (see [12]). This is also the reason why we in this paper apply the extended Ait-Sahalia model to 'rough' volatility modelling.

Although we also prove an existence and uniqueness result for solutions to SDE (3) (see Theorem 5.5), we mainly focus in this paper on the study of SDE (4). We emphasise that our mathematical methods employed in this paper differ significantly from those used in [13]. For example, in the case of $H < \frac{1}{2}$, we cannot apply the Itô Lemma as in [13] for $H > \frac{1}{2}$, to prove the existence of higher moments of solutions to SDE (3) or (4) (see Sect. 4) but have to resort to other techniques based on, for example, the Clark–Ocone formula and the concept of rough path integrals in the sense of Russo–Vallois.

Finally, we mention some other works related to our article: Let us point out here that SDEs with explosive drifts driven by Hölder continuous noises in the case of fBm with $H > \frac{1}{2}$ were initially analysed in Hu et al. [3], where the authors address the properties of positivity, existence of moments and the Malliavin differentiability of strong solutions. Other interesting and more recent results related to the SDE (8) can be found in Di Nunno et al. [14], who establish for a large class of unbounded and explosive drift vector fields existence and uniqueness of local and global solutions to SDEs with additive noise, which is merely Hölder continuous and not necessarily Gaussian. Further, the work of Di Nunno et al. [15], which appeared after the completion of our article, also deals with the Malliavin differentiability of solutions with general Gaussian Volterra drivers. In addition, we refer to Kubilius [16] and Kubilius and Medziunas [17], where the Ait-Sahalia model for a parameter θ is less than 1 and greater than 1 is investigated as an example, when $H > \frac{1}{2}$. We remark here that SDEs of the type (3) or (4), which involve 'rough path' integrals in the sense of Russo and Vallois, were not studied in the above-mentioned works.

The remainder of the paper is organised as follows: In Sect. 2, we introduce the fractional Ait-Sahalia-type model for rough volatility modelling. We establish an existence and uniqueness result for solutions to SDE (4) in Sect. 5 by studying the properties of solutions to an associated SDE driven by an additive fractional noise (see Sects. 3 and 4). In addition, we also discuss the alternative model (3) in Sect. 5.

2 The Fractional Ait-Sahalia Model

Throughout this paper unless specified otherwise, we employ the following notation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$ null sets). Denote $\mathbb{E}$ as the expectation corresponding to $\mathbb{P}$. Suppose that B_t^H, $0 \leq t \leq 1$, is a scalar fractional Brownian motion (fBm) with Hurst parameter $H \in (0, \frac{1}{2})$ and B_t, $0 \leq t \leq 1$, is a scalar Brownian motion defined on this probability space.
In what follows, we are interested to study the SDE

$$x_t = x_0 + \int_0^t \left(\alpha_{-1} x_s^{-1} - \alpha_0 + \alpha_1 x_s - \alpha_2 s^{2H-1} x_s^{\rho}\right) \mathrm{d}s + \int_0^t \sigma x_s^{\theta} \mathrm{d}B_s^H, \tag{5}$$

$x_0 \in (0, \infty)$, $0 \leq t \leq 1$, where $H \in (\frac{1}{3}, \frac{1}{2})$, $\tilde{\theta} > 0$, $\rho > 1 + \frac{1}{H\tilde{\theta}}$, $\theta := \frac{\tilde{\theta}+1}{\tilde{\theta}}$, $\sigma > 0$ and $\alpha_i > 0$, $i = -1, \ldots, 2$. Here, the stochastic integral term with respect to $B_\cdot^H$ in (5) is defined by means of an integral concept introduced by Russo and Vallois [18]. See Sect. 5.

As already mentioned in introduction, solutions to the SDE (5) can be used as a model (fractional Ait-Sahalia model) for the description of the dynamics of (rough) volatility in finance. In fact, in this paper, we aim at establishing the existence and uniqueness of strong solutions $x_t > 0$ to SDE (5). In doing so, we show that such solutions can be obtained as transformations of solutions to the SDE

$$y_t = x + \int_0^t \tilde{f}(s, y_s) \mathrm{d}s - \tilde{\sigma} B_t^H, \quad 0 \leq t \leq 1, \quad H \in \left(0, \frac{1}{2}\right), \tag{6}$$

where

$$\tilde{f}(s, y) = \alpha_{-1}\left(-\tilde{\theta} y^{2\tilde{\theta}+1}\right) + \alpha_0 y^{\tilde{\theta}+1} - \alpha_1 \frac{y}{\tilde{\theta}} + \alpha_2 s^{2H-1} \frac{1}{\tilde{\theta}^{\rho}} y^{-\tilde{\theta}\rho+\tilde{\theta}+1} - \tilde{\sigma} H s^{2H-1} y^{-1}(\tilde{\theta}+1), \tag{7}$$

where $\tilde{\sigma} > 0$, $0 < s \leq 1$, $0 < y < \infty$. However, after having applied the transformation, we have to restrict $H \in (1/3, 1/2)$ to make sense of the stochastic integral in SDE (5). See Sect. 5 for further details.

In the sequel, we want to prove the following new properties for solutions to SDE (6):

- Existence and uniqueness of positive strong solutions (Corollary 3.1),

- Regularity of solutions in the sense of Malliavin differentiability (Theorem 4.2),
- Existence of higher moments (Theorem 4.1).

3 Existence and Uniqueness of Solutions to Singular SDEs with Additive Fractional Noise for $H < \frac{1}{2}$

In this section, we wish to analyse the following generalisation of the SDE (6) given by

$$x_t = x_0 + \int_0^t b(s, x_s)ds + \sigma B_t^H, \quad 0 \le t \le 1, \quad H \in \left(0, \frac{1}{2}\right), \quad \sigma > 0. \tag{8}$$

We require the following conditions

(A1) $b \in C\big((0, 1) \times (0, \infty)\big)$ and has a continuous spatial derivative $b' := \frac{\partial}{\partial x} b$ such that

$$b'(t, x) \le K_t, \quad 0 < t < 1, \quad x \in (0, \infty),$$

where $K_t := t^{2H-1}K$ for some $K \ge 0$.

(A2) There exist $\kappa_1 > 0$, $\alpha > \frac{1}{H} - 1$ and $h_1 > 0$ such that $b(t, x) \ge h_1 t^{2H-1} x^{-\alpha}$, $t \in (0, 1]$, $x \le \kappa_1$.

(A3) There are $\kappa_2 > 0$ and $h_2 > 0$ such that $b(t, x) \le h_2 t^{2H-1}(x + 1)$, $t \in (0, 1]$, $x \ge \kappa_2$.

Theorem 3.1 *Suppose that* (A1–A3) *hold. Then, for all* $x_0 > 0$ *the* SDE (8) *has a unique strong positive solution* x_t, $0 \le t \le 1$.

Proof Without loss of generality, let $\sigma = 1$. We are required to establish the following analytical properties.

(i) Uniqueness: Suppose $x_.$ and $y_.$ are two solutions to (8). Then,

$$x_t - y_t = \int_0^t \big(b(s, x_s) - b(s, y_s)\big)\mathrm{d}s.$$

So, using the product rule, the mean value theorem and (A1), we get

$$\begin{aligned}(x_t - y_t)^2 &= 2\int_0^t \big(b(s, x_s) - b(s, y_s)\big)(x_s - y_s)\mathrm{d}s \\ &\le 2\int_0^t K_s(x_s - y_s)^2\mathrm{d}s.\end{aligned}$$

Hence, Gronwall's lemma implies that

$$x_t - y_t = 0, \quad 0 \le t \le 1.$$

(ii) Existence: Let $x_0 > 0$. Because of the regularity assumptions imposed on b, we know that Eq. (8) has (path-by-path) local solutions. Define the stopping times

$$\tau_0 := \inf\{t \in [0,1] : x_t = 0\} \quad \text{and} \quad \tau_n := \inf\{t \in [0,1] : x_t \geq n\},$$

where $\inf \emptyset := 1^+$. Just as in [13], we want to prove that $\tau_0 = 1^+$ and $\lim_{n\to\infty} \tau_n = 1^+$. Here, 1^+ stands for an artificially added element larger than 1. Suppose that $\tau_0 \leq 1$. Then, there is a $\hat{\tau}_0 \in (0, \tau_0]$ such that $x_t \leq \kappa_1$ for all $(\hat{\tau}_0, \tau_0]$. By (A2), we know that $b(t,x) > 0$ for $x \in (0, \kappa_1)$ and $t > 0$. Hence,

$$0 = x_{\tau_0} = x_t + \int_t^{\tau_0} b(s, x_s)\mathrm{d}s + B^H_{\tau_0} - B^H_t, \quad t \in (\hat{\tau}_0, \tau_0]. \tag{9}$$

This implies

$$x_t \leq \left| B^H_{\tau_0} - B^H_t \right| \leq ||B^H_\cdot||_\beta (\tau_0 - t)^\beta, \; t \in (\hat{\tau}_0, \tau_0] \text{ for } \beta \in (0, H). \tag{10}$$

Here, $|| \cdot ||_\beta$ denotes the Hölder-seminorm given by

$$||f||_\beta = \sup_{0 \leq s < t \leq 1} \frac{|f(s) - f(t)|}{(t-s)^\beta}$$

for β-Hölder continuous functions f. So, we also obtain that

$$\begin{aligned} ||B^H_\cdot||_\beta (\tau_0 - t)^\beta &\geq \left| B^H_{\tau_0} - B^H_t \right| \geq \int_t^{\tau_0} b(s, x_s)\mathrm{d}s \\ &\geq h_1 \int_t^{\tau_0} s^{2H-1} x_s^{-\alpha} \mathrm{d}s \geq \frac{h_1}{||B^H_\cdot||^\alpha_\beta} \int_t^{\tau_0} s^{2H-1} \frac{1}{(\tau_0 - s)^{\alpha\beta}} \mathrm{d}s \\ &\geq \frac{h_1}{||B^H_\cdot||^\alpha_\beta} \tau_0^{2H-1} \int_t^{\tau_0} \frac{1}{(\tau_0 - s)^{\alpha\beta}} \mathrm{d}s. \end{aligned}$$

If $\alpha\beta \geq 1$, we get a contradiction. For $\alpha\beta < 1$, we find that

$$||B^H_\cdot||_\beta (\tau_0 - t)^\beta \geq \frac{h_1}{||B^H_\cdot||^\alpha_\beta} \tau_0^{2H-1} \frac{(\tau_0 - t)^{1-\alpha\beta}}{1 - \alpha\beta}, \quad t \in (\hat{\tau}_0, \tau_0].$$

Hence,

$$0 = \lim_{t \to \tau_0} ||B^H_\cdot||_\beta (\tau_0 - t)^{\beta + \alpha\beta - 1} \geq \frac{h_1 \tau_0^{2H-1}}{||B^H_\cdot||^\alpha_\beta (1 - \alpha\beta)} > 0.$$

So, $\tau_0 = 1^+$. Assume now that

$$\tau_\infty := \lim_{n \to \infty} \tau_n \leq 1.$$

Then (compare [13]), we can distinguish between the following two cases:

1. Case: There is $\widehat{\tau_1}$ such that $x_{\widehat{\tau_1}} = \kappa_2 + x_0$ and $x_t \geq \kappa_2 + x_0$ for all $t \in (\widehat{\tau_1}, \tau_\infty)$.
2. Case: For all $n \in \mathbb{N}$ with $n > \kappa_2 + x_0$ and $\epsilon > 0$, one can find an interval $(\widehat{\tau_1}, \widehat{\tau_2}) \subset (\tau_\infty - \epsilon, \tau_\infty)$ such that $x_{\widehat{\tau_1}} = \kappa_2 + x_0$ and

$$\kappa_2 + x_0 \leq \inf_{t \in (\widehat{\tau_1}, \widehat{\tau_2})} x_t \leq n \leq \sup_{t \in (\widehat{\tau_1}, \widehat{\tau_2})} x_t.$$

Thus, by using (A3), we obtain that

$$x_t \leq \kappa_2 + x_0 + ||B_\cdot^H||_\beta \tau_\infty^\beta + h_2 \tau_\infty^{2H} (2H)^{-1} + h_2 \int_{\hat{\tau}_1}^t s^{2H-1} x_s \mathrm{d}s.$$

So, by letting

$$\alpha = \kappa_2 + x_0 + ||B_\cdot^H||_\beta \tau_\infty^\beta + h_2 \tau_\infty^{2H} (2H)^{-1},$$

it follows from Gronwall's lemma that

$$\begin{aligned} x_t &\leq \alpha + \int_{\hat{\tau}_1}^t \alpha h_2 s^{2H-1} \exp\left(\int_s^t h_2 u^{2H-1} \mathrm{d}u \right) \mathrm{d}s \\ &\leq \gamma + \int_0^1 \gamma h_2 s^{2H-1} \exp\left(\int_s^1 h_2 u^{2H-1} \mathrm{d}u \right) \mathrm{d}s, \end{aligned}$$

where $\gamma := \kappa_2 + x_0 + ||B_\cdot^H||_\beta + \frac{h_2}{2H}$. The latter estimate leads to a contradiction. □

As a consequence of Theorem 3.1, we obtain the following result:

Corollary 3.2 *Suppose that $x \in (0, \infty)$ and $\rho > \frac{1}{H\tilde{\theta}} + 1$, where ρ and $\tilde{\theta}$ are parameters of $\tilde{f}$ in (7). Then, there exists a unique strong solution $y_t > 0$ to* SDE (6).

Proof Let $\epsilon = \frac{H}{2}$. Then,

$$\tilde{f}(s, y) = \tilde{g}_1(s, y) + \tilde{g}_2(s, y),$$

where

$$\tilde{g}_1(s, y) := \alpha_{-1} \left(-\tilde{\theta} y^{2\tilde{\theta}+1} \right) + \alpha_0 y^{\tilde{\theta}+1} - \alpha_1 \frac{y}{\tilde{\theta}} + \epsilon \tilde{\sigma} s^{2H-1} y^{-1} (\tilde{\theta} + 1),$$

and

$$\tilde{g}_2(s, y) := \alpha_2 s^{2H-1} \frac{1}{\tilde{\theta}^\rho} y^{-\tilde{\theta}\rho + \tilde{\theta} + 1} - (H + \epsilon) \tilde{\sigma} s^{2H-1} y^{-1} (\tilde{\theta} + 1).$$

We see that

$$\tilde{g}_1(s, y) \geq \alpha_{-1}(-\tilde{\theta} y^{2\tilde{\theta}+1}) + \alpha_0 y^{\tilde{\theta}+1} - \alpha_1 \frac{y}{\tilde{\theta}} + \epsilon \tilde{\sigma} y^{-1} (\tilde{\theta} + 1)$$

$$\geq 0$$

for all $s \in (0, 1]$ and $y \in (0, y_0)$ for some $y_0 > 0$. Since

$$-\tilde{\theta}\rho + \tilde{\theta} + 1 < -\frac{1}{H} + 1 < -1,$$

we also find some $y_1 > 0$ such that

$$\begin{aligned}\tilde{g}_2(s, y) &= s^{2H-1} y^{-\tilde{\theta}\rho+\tilde{\theta}+1} \left(\alpha_2 \frac{1}{\tilde{\theta}^\rho} - (H+\epsilon)\tilde{\sigma}(\tilde{\theta}+1) y^{\tilde{\theta}\rho-\tilde{\theta}-2} \right) \\ &\geq h_1 s^{2H-1} y^{-\alpha}\end{aligned}$$

for all $s \in (0, 1]$ and $y \in (0, y_1]$, where $h_1 > 0$ and $\alpha := \tilde{\theta}\rho - \tilde{\theta} - 1$. So,

$$\tilde{f}(s, y) \geq h_1 s^{2H-1} y^{-\alpha}$$

for all $s \in (0, 1]$, $y \in (0, y_1)$ for some $y_1 > 0$, which shows that $\tilde{f}$ satisfies (A2). As for (A3), we see that there exists some $y_2 \geq 1$ such that

$$\tilde{f}(s, y) \leq s^{2H-1} \left(\alpha_2 \frac{1}{\tilde{\theta}^\rho} y^{-\tilde{\theta}\rho+\tilde{\theta}+1} - H\tilde{\sigma} y^{-1}(\tilde{\theta}+1) \right) \leq h_2 s^{2H-1}(1+y)$$

for all $s \in (0, 1]$, $y \in (y_2, \infty)$ and some $h_2 > 0$. We have that

$$\tilde{f}'(s, y) = f_1(s, y) + f_2(s, y),$$

where

$$f_1(s, y) := -\alpha_{-1}\tilde{\theta}(2\tilde{\theta}+1) y^{2\tilde{\theta}} + \alpha_0(\tilde{\theta}+1) y^{\tilde{\theta}} - \frac{\alpha_1}{\tilde{\theta}}$$

and

$$f_2(s, y) := s^{2H-1} \left(\alpha_2 \frac{1}{\tilde{\theta}^\rho} (-\tilde{\theta}\rho + \tilde{\theta} + 1) y^{-\tilde{\theta}\rho+\tilde{\theta}} + H\tilde{\sigma}(\tilde{\theta}+1) y^{-2} \right).$$

So, there exist $y_1, y_2 > 0$ such that

$$\tilde{f}'(s, y) \leq f_1(s, y) \leq K \leq s^{2H-1} K = K_s$$

for all $s \in (0, 1]$, $y \in (0, y_1)$ as well as

$$\tilde{f}'(s, y) \leq f_2(s, y) \leq s^{2H-1} K = K_s$$

for all $s \in (0,1]$, $y \in (y_2, \infty)$ and some $K > 0$. On the other hand, we see that

$$\tilde{f}'(s, y) \le K_1 + s^{2H-1} K_2 \le s^{2H-1} K = K_s$$

for all $s \in (0,1]$, $y_0 \in [y_1, y_2]$ for some $K_1, K_2, K > 0$. Altogether, we see that $\tilde{f}$ also satisfies (A1). Since $-B_\cdot^H$ is a fractional Brownian motion, the proof follows. □

4 Malliavin Differentiability and Existence of Higher Moments of Solutions

In this section, we want to show that the solution x to the SDE

$$x_t = x + \int_0^t \tilde{f}(s, x_s) ds - \tilde{\sigma} B_t^H, \quad 0 \le t \le 1, \quad x > 0, \tag{11}$$

is Malliavin differentiable in the direction of $B_\cdot^H$ for $H \in (0, \frac{1}{2})$ and where $\tilde{\sigma} > 0$ is an arbitrary constant. Furthermore, we verify that solutions x_t to (11) belong to L^q for all $q \ge 1$. For this purpose, let $\tilde{f}_n : (0,1] \times \mathbb{R} \to \mathbb{R}$, $n \ge 1$ be a sequence of bounded, globally Lipschitz continuous (and smooth) functions such that

(i) $\tilde{f} \mid_{[\frac{1}{n}, n]} = \tilde{f} \mid_{(0,1] \times [\frac{1}{n}, n]}$ for all $n \ge 1$,

(ii) $\tilde{f}_n'(s, x) \le K_s$ for all $(s, x) \in (0,1] \times \mathbb{R}$, $n \ge 1$, where K_s is defined in (A1).

So, we see that

$$\tilde{f}_n'(s, x) \underset{n \to \infty}{\longrightarrow} \tilde{f}(s, x)$$

for all $(s, x) \in (0,1] \times (0, \infty)$. Denote by $D_\cdot^H$ and $D_\cdot$, the Malliavin derivative in the direction of $B_\cdot^H$ and $W_\cdot$, respectively. Here, $W_\cdot$ is the Wiener process with respect to the representation

$$B_t^H = \int_0^t K_H(t, s) \mathrm{d}W_s, \quad t \ge 0. \tag{12}$$

See Appendix. Since $-B_\cdot^H$ is a fractional Brownian motion, let us without loss of generality assume in (11) that $\tilde{\sigma} = -1$. Because of the regularity of the functions $\tilde{f}_n$, $n \ge 1$, we find that the solutions $x_\cdot^n$ to

$$x_t^n = x + \int_0^t \tilde{f}_n(s, x_s) \mathrm{d}s + B_t^H, \quad x > 0, \quad 0 \le t \le 1$$

are Malliavin differentiable with Malliavin derivative $D_u^H x_t$ satisfying the equation

$$D_u^H x_t^n = \int_u^t \tilde{f}_n'(s, x_s^n) D_u^H x_s^n \mathrm{d}s + \chi_{[0,t]}(u).$$

Hence,

$$D_u^H x_t^n = \chi_{[0,t]}(u) \exp\left(\int_u^t \tilde{f}_n'(s, x_s^n) \mathrm{d}s\right) \quad \lambda \times \text{P-a.e.}$$

for all $0 \leq t \leq 1$ (λ Lebesgue measure). Further, using the transfer principle between $D_\cdot^H$ and $D_\cdot$ (see [5, Proposition 5.2.1]), we have that

$$K_H^* D_\cdot^H x_t = D_\cdot x_t \tag{13}$$

where $K_H^* : \mathcal{H} \to L^2([0, T])$ is given by

$$(K_H^* y)(s) = K_H(T, s) y(s) + \int_s^T (y(t) - y(s)) \frac{\partial}{\partial t} K_H(t, s) \mathrm{d}t \tag{14}$$

for

$$\frac{\partial}{\partial t} K_H(t, s) = c_H \left(H - \frac{1}{2}\right) \left(\frac{1}{2}\right)^{H - \frac{1}{2}} (t - s)^{H - \frac{3}{2}}. \tag{15}$$

Here $\mathcal{H} = I_{T^-}^{\frac{1}{2} - H}(L^2)$. See Appendix. On the other hand, using (13), we also see that

$$D_u x_t^n = \int_u^t \tilde{f}_n' \left(s, x_s^n\right) D_u x_s^n \mathrm{d}s + K_H(t, u) \tag{16}$$

in $L^2([0, t] \times \Omega)$ for all $0 \leq t \leq 1$. Set

$$Y_t^n(u) = D_u x_t^n - K_H(t, u).$$

Then,

$$Y_t^n(u) = \int_u^t \left\{ \tilde{f}_n' \left(s, x_s^n\right) Y_s^n(u) + \tilde{f}_n \left(s, x_s^n\right) K_H(s, u) \right\} \mathrm{d}s.$$

Using the fundamental solution of the equation

$$\dot{\Phi}(t) = \tilde{f}_n'(t, x_t^n) \cdot \Phi(t), \quad \Phi(u) = 1.$$

We then obtain that

$$Y_t^n(u) = \int_u^t \exp\left(\int_s^t \tilde{f}' \left(r, x_r^n\right) \mathrm{d}r\right) \tilde{f}_n'(s, x_s^n) K_H(s, u) \mathrm{d}s.$$

Hence,

$$D_u x_t^n = \int_u^t \exp\left(\int_s^t \tilde{f}' \left(r, x_r^n\right) \mathrm{d}r\right) \tilde{f}_n'(s, x_s^n) K_H(s, u) \mathrm{d}s + K_H(t, u)$$

$$= J_1^n(t,u) + J_2^n(t,u) + K_H(t,u), \quad u < t, \quad \lambda \times \text{P-a.e.},$$

where

$$J_1^n(t,u) := \int_u^t \exp\left(\int_s^t \tilde{f}'(r,x_r)\mathrm{d}r\right)\left(\tilde{f}_n'(s,x_s^n) - K_s\right) K_H(s,u)ds$$

and

$$J_2^n(t,u) := \int_u^t \exp\left(\int_s^t \tilde{f}'(r,x_r)\mathrm{d}r\right) K_s \cdot K_H(s,u)\mathrm{d}s.$$

Without loss of generality, let $T = t = 1$. Then,

$$\begin{aligned} \int_0^1 (D_u x_1^n)^2 \mathrm{d}u \ &\leq C\left\{\int_0^1 (J_1^n(1,u))^2\mathrm{d}u + \int_0^1 (J_2^n(1,u))^2\mathrm{d}u \right. \\ &\quad \left. + \int_0^1 (K_H(1,u))^2\mathrm{d}u\right\}. \end{aligned} \tag{17}$$

Using Fubini's theorem, we get that

$$\begin{aligned} &\int_0^1 (J_1^n(1,u))^2\mathrm{d}u \\ &= \int_0^1 \left(\int_0^1 \chi_{[u,1]}(s) \exp\left(\int_s^t \tilde{f}'(r,x_r)\mathrm{d}r\right)\left(\tilde{f}_n'(s,x_s^n) - K_s\right) K_H(s,u)\mathrm{d}s\right)^2 \mathrm{d}u \\ &= \int_0^1\int_0^1 \left\{\exp\left(\int_{s_1}^1 \tilde{f}'(r,x_r)\mathrm{d}r\right)\left(\tilde{f}_n'(s_1,x_{s_1}^n) - K_{s_1}\right)\right. \\ &\quad \left. \times \exp\left(\int_{s_2}^1 \tilde{f}'(r,x_r)\mathrm{d}r\right)\left(\tilde{f}_n'(s_2,x_{s_2}^n) - K_{s_2}\right)\int_0^{s_1\wedge s_2} K_H(s_1,u)K_H(s_2,u)\mathrm{d}u\right\}\mathrm{d}s_1\mathrm{d}s_2. \end{aligned}$$

From (12), we see for the covariance function

$$R_H(s_1,s_2) = \mathbb{E}\left[B_{s_1}^H \cdot B_{s_2}^H\right]$$

that

$$R_H(s_1,s_2) = \int_0^{s_1\wedge s_2} K_H(s_1,u)K_H(s_2,u)\mathrm{d}u.$$

Since

$$0 \leq R_H(s_1,s_2) = \frac{1}{2}\left(s_1^{2H} + s_2^{2H} - |s_1 - s_2|^{2H}\right) \leq 1, \quad H < \frac{1}{2}$$

and

$$\left(\tilde{f}_n'(s_1,x_{s_1}^n) - K_{s_1}\right) \cdot \left(\tilde{f}_n'(s_2,x_{s_2}^n) - K_{s_2}\right) \geq 0$$

for $0 < s_1, s_2 \leq 1$, we find that

$$\begin{aligned}\int_0^1 (J_1^n(1,u))^2 \mathrm{d}u &\leq \left(\int_0^1 \left(\exp\left(\int_s^t \tilde{f}'(r,x_r)\mathrm{d}r\right)\left(\tilde{f}_n'(s,x_s^n) - K_s\right)\mathrm{d}s\right)^2 \\ &= \left\{-\exp\left(\int_s^1 \tilde{f}'(r,x_r)\mathrm{d}r\right)\Big|_{s=0}^1 - \int_0^1 K_s \exp\left(\int_s^1 \tilde{f}'(r,x_r)\mathrm{d}r\right)\mathrm{d}s\right\}^2 \\ &\leq \left(\exp\left(\int_0^1 K_r \mathrm{d}r\right) + \int_0^1 K_s \mathrm{d}s \cdot \exp\left(\int_0^1 K_r \mathrm{d}r\right)\right)^2.\end{aligned}$$

Similarly, we also obtain that

$$\int_0^1 \left(J_2^n(1,u)\right)^2 \mathrm{d}u \leq C(K,H)$$

for a constant $C(K,H) < \infty$. We also have that

$$\int_0^1 (K_H(1,u))^2 \mathrm{d}u = \mathbb{E}\left[\left(B_1^H\right)^2\right] = 1.$$

Altogether, we get that

$$\mathbb{E}\left[\int_0^1 \left(D_u x_1^n\right)^2 \mathrm{d}u\right] \leq C(K,H) \tag{18}$$

for all $n \geq 1$ for a constant $C(K,H) < \infty$. Define now the stopping times τ_n by

$$\tau_n = \inf\left\{0 \leq t \leq 1; x_t \notin \left[\frac{1}{n}, n\right]\right\} \quad (\inf \emptyset = \infty)$$

Then, we know from the proof of the existence of solutions in the previous section that $\tau_n \nearrow \infty$ for $n \to \infty$. So,

$$\begin{aligned}x_{t\wedge\tau_n}^n - x_{t\wedge\tau_n} &= \int_0^{t\wedge\tau_n} \left\{\tilde{f}_n\left(s, x_s^n\right) - \tilde{f}(s,x_s)\right\}\mathrm{d}s \\ &= \int_0^t \chi_{[0,\tau_n)}(s)\left\{\tilde{f}_n\left(s, x_{s\wedge\tau_n}^n\right) - \tilde{f}_n(s, x_{s\wedge\tau_n})\right\}\mathrm{d}s.\end{aligned}$$

Hence,

$$\left|x_{t\wedge\tau_n}^n - x_{t\wedge\tau_n}\right| \leq K_n \int_0^t \left|x_{s\wedge\tau_n}^n - x_{s\wedge\tau_n}\right| \mathrm{d}s$$

for a Lipschitz constant K_n. Then, Gronwall's lemma implies that

$$x_{t\wedge\tau_n}^n = x_{t\wedge\tau_n}$$

for all t, n P-a.e. Since $\tau_n \nearrow \infty$ for $n \to \infty$ a.e., we have that

$$x_t^n \underset{n\to\infty}{\to} x_t \tag{19}$$

for all t P-a.e. Using the Clark–Ocone formula (see [7]), we get that

$$x_1^n = \mathbb{E}\left[x_1^n\right] + \int_0^1 \mathbb{E}\left[D_s x_1^n | \mathcal{F}_s\right] \mathrm{d}W_s,$$

where $\{\mathcal{F}\}_{0\le t\le 1}$ is the filtration generated by $W_\cdot$. It follows that

$$\begin{aligned}\mathbb{E}[(x_1^n - \mathbb{E}[x_1^n])^2] &= \mathbb{E}\left[\int_0^1 \left(\mathbb{E}[D_s x_1^n | \mathcal{F}_s]\right)^2 \mathrm{d}s\right] \\ &\le \mathbb{E}\left[\int_0^1 \mathbb{E}[(D_s x_1^n)^2 | \mathcal{F}_s] \mathrm{d}s\right] = \int_0^1 \mathbb{E}\left[\left(D_s x_1^n\right)^2\right] \mathrm{d}s.\end{aligned}$$

So, we see from (18) that

$$\mathbb{E}\left[\left(x_1^n - \mathbb{E}\left[x_1^n\right]\right)^2\right] \le C(K, H) < \infty.$$

for all $n \ge 1$. We also have that

$$\left|\left|x_1^n - \mathbb{E}[x_1^n]\right| - \left|x_1 - \mathbb{E}[x_1^n]\right|\right| \le \left|x_1^n - x_1\right| \underset{n\to\infty}{\longrightarrow} 0$$

because of (19). So,

$$\varliminf_{n\to\infty} \left|x_1^n - \mathbb{E}\left[x_1^n\right]\right| = \varliminf_{n\to\infty} \left|x_1 - \mathbb{E}\left[x_1^n\right]\right|.$$

Suppose that $\mathbb{E}[x_1^n]$, $n \ge 1$ is unbounded. Then, there exists a subsequence $n_k, k \ge 1$ such that

$$\left|\mathbb{E}\left[x_1^{n_k}\right]\right| \underset{n\to\infty}{\longrightarrow} \infty.$$

It follows from the lemma of Fatou and the positivity of x_t that

$$\begin{aligned}\infty &= \mathbb{E}\left[\varliminf_{k\to\infty} \left(\left|x_1 - |\mathbb{E}[x_1^{n_k}]|\right|\right)^2\right] \\ &\le \mathbb{E}\left[\varliminf_{k\to\infty} \left(\left|x_1 - \mathbb{E}[x_1^{n_k}]\right|\right)^2\right] \\ &= \mathbb{E}\left[\varliminf_{k\to\infty} \left(\left|x_1^{n_k} - \mathbb{E}[x_1^{n_k}]\right|\right)^2\right] \\ &\le \varliminf_{k\to\infty} \mathbb{E}\left[\left|x_1^{n_k} - \mathbb{E}[x_1^{n_k}]\right|^2\right] \le C < \infty,\end{aligned}$$

which is a contradiction. Hence,

$$\sup_{n\geq 1} |\mathbb{E}[x_1^n]| < \infty.$$

Further, we also obtain from the Burkholder–Davis–Gundy inequality and (18) that

$$\begin{aligned}
\mathbb{E}[|x_1^n|^{2p}] &< C_p \left(|\mathbb{E}[x_1^n]|^{2p} + \mathbb{E}\left[\left(\int_0^1 \mathbb{E}[D_s x_1^n | \mathcal{F}_s] dW_s \right)^{2p} \right] \right) \\
&\leq C_p \left(|\mathbb{E}[x_1^n]|^{2p} + \mathbb{E}\left[\left(\sup_{0\leq u\leq 1} \left| \int_0^u \mathbb{E}[D_s x_1^n | \mathcal{F}_s] dW_s \right| \right)^{2p} \right] \right) \\
&\leq C_p \left(|\mathbb{E}[x_1^n]|^{2p} + m_p \mathbb{E}\left[\left(\int_0^1 \mathbb{E}[D_s x_1^n | \mathcal{F}_s]^2 ds \right)^{p} \right] \right) \\
&\leq C(p, K, H) \qquad (20)
\end{aligned}$$

for $n \geq 1$. So, it follows from (19) and the lemma of Fatou that

$$\mathbb{E}[|x_1|^{2p}] \leq \varliminf_{n\to\infty} \mathbb{E}[|x_1^n|]^{2p} \leq C(p, K, H) < \infty$$

for all $p \geq 1$. So, we obtain the following result:

Theorem 4.1 *Let x_t, $0 \leq t \leq 1$ be the solution to (11). Then, $x_t \in L^q(\Omega)$ for all $q \geq 1$ and $0 \leq t \leq 1$.*

In addition, we obtain from Lemma 1.2.3 in [7] in connection with estimate (20) that x_1 is Malliavin differentiable in the direction of $W_\cdot$. The latter, in combination with (13), also entails the Malliavin differentiability of x_1 with respect to $B_\cdot^H$. Thus, we have also shown the following result:

Theorem 4.2 *The positive unique strong solution x_t to (11) is Malliavin differentiable in the direction of $B_\cdot^H$ and $W_\cdot$ for all $0 \leq t \leq 1$.*

5 Application

In this section, we aim at applying the results of the previous section to obtain a unique strong solution x_t to the SDE

$$x_t = x_0 + \int_0^t \left(\alpha_{-1} x_s^{-1} - \alpha_0 + \alpha_1 x_s - \alpha_2 s^{2H-1} x_s^{\rho}\right) \mathrm{d}s + \int_0^t \sigma x_s^{\theta} \mathrm{d}B_s^H, \qquad (21)$$

$0 \leq t \leq 1$, for $H \in (\frac{1}{3}, \frac{1}{2})$, $\tilde{\theta} > 0$, $\rho > 1 + \frac{1}{H\tilde{\theta}}$, $\sigma > 0$ and $\theta := \frac{\tilde{\theta}+1}{\tilde{\theta}}$. Here, the stochastic integral with respect to $B^H_\cdot$ is defined by

$$\int_0^t g(x_s)\mathrm{d}B^H_s = \int_0^t -Hs^{2H-1}g'(x_s)\mathrm{d}s + \int_0^t g(x_s)d^\circ B^H_s \tag{22}$$

for functions $g \in \mathcal{C}^3((0,\infty);\mathbb{R})$. See also the second Remark 5.3. The stochastic integral on the right-hand side of (22) is the symmetric integral with respect to $B^H_\cdot$ introduced by Russo and Vallois. See, for example, [18] and the references therein. Such an integral denoted by

$$\int_0^t Y_s d^\circ X_s, \quad t \in [0,1] \tag{23}$$

for continuous process $X_\cdot$, $Y_\cdot$ is defined as

$$\lim_{\epsilon \searrow 0} \frac{1}{2\epsilon} \int_0^t Y_s(X_{s+\epsilon} - X_s)ds,$$

provided this limit exists in the ucp-topology. In order to construct a solution to (21), we need a version of the Itô formula for processes $Y_\cdot$, which have a finite cubic variation. A continuous process is said to have a finite strong cubic variation (or 3-variation), denoted by $[Y, Y, Y]$, if

$$[Y, Y, Y] := \lim_{\epsilon \searrow 0} \frac{1}{\epsilon} \int_0^t (Y_{s+\epsilon} - Y_s)^3 ds$$

exists in ucp as well as

$$\sup_{0<\epsilon\leq 1} \frac{1}{\epsilon} \int_0^1 |Y_{s+\epsilon} - Y_s|^3 ds < \infty \quad \text{a.e.}$$

See [18]. Using the concept of finite strong cubic variation, one can show the following Itô formula (see [18]).

Theorem 5.1 *Assume that $Y_\cdot$ is a real valued process with finite strong cubic variation and $g \in \mathcal{C}^3((0,\infty);\mathbb{R})$. Then,*

$$g(Y_t) = g(Y_0) + \int_0^t g'(Y_s)d^\circ Y_s - \frac{1}{12}\int_0^t g'''(Y_s)d[Y, Y, Y]_s, \quad 0 \leq t \leq 1.$$

Remark 5.2 The last term on the right-hand side of the equation is a Lebesgue–Stieltjes integral with respect to the bounded variation process $[Y, Y, Y]$.

Remark 5.3
- We mention that for $Y_\cdot = B^H_\cdot$, $H \in (\frac{1}{3}, \frac{1}{2})$, $[B^H_\cdot, B^H_\cdot, B^H_\cdot]$ is zero a.e.

- If $X_\cdot = B^H_\cdot$ in (22), then it follows from Theorem 6.3.1 in [8] that our stochastic integral in (22) equals the Wick–Itô–Skorohod integral. The latter also gives a justification for the definition of the stochastic integral in (22) in the general case.

Theorem 5.4 *Suppose that* $H \in (\frac{1}{3}, \frac{1}{2})$*,* $\tilde{\theta} > 0$*,* $\sigma > 0$ *and* $\rho > 1 + \frac{1}{H\tilde{\theta}}$*. Let* $\theta = \frac{\tilde{\theta}+1}{\tilde{\theta}}$*. Then, there exists a unique strong and positive solution to the* SDE (21).

Proof Let $y_\cdot$ be the unique strong and positive solution to

$$y_t = x + \int_0^t \tilde{f}(s, y_s)ds - \tilde{\sigma} B^H_t, \quad 0 \le t \le 1, \quad x > 0,$$

where $\tilde{f}$ is defined as in Sect. 2. Define $g \in \mathcal{C}^3((0,\infty);\mathbb{R})$ by $g(y) = \frac{y^{-\tilde{\theta}}}{\tilde{\theta}}$. Then, a modification of Theorem 5.1 (see Lemma 6.1) entails that

$$x_t := g(y_t) = \frac{x^{-\tilde{\theta}}}{\tilde{\theta}} + \int_0^t (-1) y_s^{-(\tilde{\theta}+1)} d^\circ y_s - \frac{1}{12}\int_0^t g'''(y_s) d[y,y,y]_s.$$

Since $[B^H_\cdot, B^H_\cdot, B^H_\cdot]$ is zero a.e. (see Remark 5.3), we observe that $[y,y,y]$ is zero a.e. So,

$$\begin{aligned} x_t &= \frac{x^{-\tilde{\theta}}}{\tilde{\theta}} + \int_0^t (-1) y_s^{-(\tilde{\theta}+1)} d^\circ y_s \\ &= \frac{x^{-\tilde{\theta}}}{\tilde{\theta}} + \int_0^t (-1) y_s^{-(\tilde{\theta}+1)} \tilde{f}(s, y_s) \mathrm{d}s + \int_0^t \tilde{\sigma} y_s^{-(\tilde{\theta}+1)} d^\circ B^H_s \\ &= \frac{x^{-\tilde{\theta}}}{\tilde{\theta}} - \int_0^t \Big\{(-1) y_s^{-(\tilde{\theta}+1)} \tilde{f}(s, y_s) - H\tilde{\sigma} s^{2H-1} y_s^{-(\tilde{\theta}+2)}(\tilde{\theta}+1)\Big\} \mathrm{d}s \\ &\quad + \int_0^t \tilde{\sigma} y_s^{-(\tilde{\theta}+1)} \mathrm{d}B^H_s. \end{aligned}$$

Since we can write $(y_s)^{-(\tilde{\theta}+1)} = \tilde{\theta}^\theta \left(\frac{y_s^{-\tilde{\theta}}}{\tilde{\theta}}\right)^\theta$, we now have

$$\begin{aligned} x_t &= \frac{x^{-\tilde{\theta}}}{\tilde{\theta}} + \int_0^t f\left(s, \frac{y_s^{-\tilde{\theta}}}{\tilde{\theta}}\right) \mathrm{d}s + \int_0^t \tilde{\sigma} y_s^{-(\tilde{\theta}+1)} \mathrm{d}B^H_s \\ &= \frac{x^{-\tilde{\theta}}}{\tilde{\theta}} + \int_0^t f\left(s, \frac{y_s^{-\tilde{\theta}}}{\tilde{\theta}}\right) \mathrm{d}s + \int_0^t \tilde{\sigma}\tilde{\theta}^\theta (x_s)^\theta \mathrm{d}B^H_s, \end{aligned}$$

where $f(s,y) := \alpha_{-1} y^{-1} - \alpha_0 + \alpha_1 y - \alpha_2 s^{2H-1} y^\rho$, $s \in (0,1]$, $y \in (0,\infty)$. So $x_\cdot$ satisfies the SDE (21) if we choose $\tilde{\sigma} = \tilde{\theta}^{-\theta}\sigma$ for $\sigma > 0$. In order to show the uniqueness of solutions to SDE (21), one can apply the Itô formula in Theorem 5.1

to the inverse function g^{-1} given by $g^{-1}(y) = \left(\tilde{\theta}\right)^{-\frac{1}{\tilde{\theta}}} y^{-\frac{1}{\tilde{\theta}}}$ by using the fact that $[B_\cdot^H, B_\cdot^H, B_\cdot^H] = 0$ a.e. for $H \in (\frac{1}{3}, \frac{1}{2})$. □

Finally, using the same arguments as in the proof of Theorem 5.4, we also get the following result for the alternative Ait-Sahalia model (3):

Theorem 5.5 *Retain the conditions of Theorem 5.4 with respect to $H, \tilde{\theta}, \theta$ and ρ. Then, there exists a unique strong solution $x_t > 0$ to* SDE (3).

Proof Just as in the proof of Theorem 5.4, we can consider the SDE (6), where the vector field $\tilde{f}$ now is given by

$$\tilde{f}(s, y) = \alpha_{-1}(-\tilde{\theta} y^{2\tilde{\theta}+1}) + \alpha_0 y^{\tilde{\theta}+1} - \alpha_1 \frac{y}{\tilde{\theta}} + \alpha_2 \frac{1}{\tilde{\theta}^\rho} y^{-\tilde{\theta}\rho+\tilde{\theta}+1} \tag{24}$$

for $0 < y < \infty$. Then, as in the proof of Corollary (3.2) one immediately verifies that $\tilde{f}$ satisfies the assumptions of Theorem 3.1, which yields a unique strong solution $y_t > 0$ to (6) in this case. In exactly the same way, we also obtain the results of Theorem 4.1 and Theorem 4.2 with respect to $\tilde{f}$ in (24). Finally, we can apply the Itô formula as in the proof of Theorem 5.4 and construct a unique strong solution $x_t > 0$ to (3) based on $y_\cdot$. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

Lemma 6.1 *Let y_t, $0 \le t \le 1$, be the positive strong solution of the* SDE *in the proof of Theorem 5.4 and let $f : (0, \infty) \to (0, \infty)$; $x \to x^{-\alpha}$, where $\alpha > 0$. Then,*

$$f(y_t) = f(y_0) + \int_0^t f'(y_s) d^\circ y_s, \quad 0 \le t \le 1, \; a.e.$$

Proof The proof is based on the same arguments for the proof of the Itô formula as in [18] and the fact that $\min_{t \in [0,2]} y_t > 0$ a.e. (see the proof of Theorem 3.1) for a solution y. on $[0, 2] \supset [0, 1]$. Consider the following type of Taylor formula

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2} f''(a)(b-a)^2 + \frac{1}{6} f^{(3)}(a)(b-a)^3 + R(a, b)(b-a)^3$$

for $a, b \in (0, \infty)$, where

$$R(a, b) := \int_0^1 \frac{\phi^2}{2} (f^{(3)}(\phi a + (1-\phi)b) - f^{(3)}(a)) d\phi.$$

So for $\epsilon \in (0, 1)$ and $s \in [0, 1]$, we obtain

$$f(y_{s+\epsilon}) = f(y_s) + f'(y_s)(y_{s+\epsilon} - y_s) + \frac{1}{2} f''(y_s)(y_{s+\epsilon} - y_s)^2 - \frac{1}{6} f^{(3)}(y_s)(y_{s+\epsilon} - y_s)^3 + R(y_s, y_{s+\epsilon})(y_{s+\epsilon} - y_s)^3$$

and

$$f(y_s) = f(y_{s+\epsilon}) - f'(y_{s+\epsilon})(y_{s+\epsilon} - y_s) + \frac{1}{2} f''(y_{s+\epsilon})(y_{s+\epsilon} - y_s)^2 - \frac{1}{6} f^{(3)}(y_{s+\epsilon})(y_{s+\epsilon} - y_s)^3 - R(y_{s+\epsilon}, y_s)(y_{s+\epsilon} - y_s)^3.$$

The latter entails that

$$\frac{1}{\epsilon} \int_0^t (f(y_{s+\epsilon}) - f(y_s)) \mathrm{d}s = \sum_{i=1}^4 J_{i,\epsilon}(t),$$

where

$$J_{1,\epsilon}(t) := \frac{1}{2\epsilon} \int_0^t (f'(y_{s+\epsilon}) + f'(y_s))(y_{s+\epsilon} - y_s) \mathrm{d}s,$$

$$J_{2,\epsilon}(t) := -\frac{1}{4\epsilon} \int_0^t (f''(y_{s+\epsilon}) - f''(y_s))(y_{s+\epsilon} - y_s)^2 \mathrm{d}s,$$

$$J_{3,\epsilon}(t) := \frac{1}{12\epsilon}\int_0^t (f^{(3)}(y_{s+\epsilon}) + f^{(3)}(y_s))(y_{s+\epsilon} - y_s)^3 \mathrm{d}s,$$

and

$$J_{4,\epsilon}(t) := -\frac{1}{2\epsilon}\int_0^t (R(y_s, y_{s+\epsilon}) + R(y_{s+\epsilon}, y_s))(y_{s+\epsilon} - y_s)^3 \mathrm{d}s.$$

Since the process $f(y_t)$, $0 \le t \le 1$, is continuous, we see that

$$\frac{1}{\epsilon}\int_0^\cdot (f(y_{s+\epsilon}) - f(y_s))\mathrm{d}s \xrightarrow[\epsilon \searrow 0]{\text{ucp}} f(y_\cdot) - f(y_0).$$

On the other hand, using the mean value theorem, we find that

$$\begin{aligned} J_{2,\epsilon}(t) &= -\frac{1}{4\epsilon}\int_0^t \psi_\epsilon(s)(y_{s+\epsilon} - y_s)^3 \mathrm{d}s \\ &= -\frac{1}{4\epsilon}\int_0^t (\psi_\epsilon(s) - \psi_0(s))(y_{s+\epsilon} - y_s)^3 \mathrm{d}s + \frac{1}{4\epsilon}\int_0^t \psi_0(s)(y_{s+\epsilon} - y_s)^3 \mathrm{d}s, \end{aligned}$$

where

$$\psi_\epsilon(s) := \int_0^1 f^{(3)}(\phi y_s + (1-\phi)y_{s+\epsilon})d\phi$$

and

$$\psi_0(s) := f^{(3)}(y_s).$$

We have for $s \in [0, 1]$ that

$$\begin{aligned} |\psi_\epsilon(s) - \psi_0(s)| &= \left|\int_0^1 (f^{(3)}(\phi y_s + (1-\phi)y_{s+\epsilon}) - f^{(3)}(y_s))d\phi\right| \\ &\le \alpha(\alpha+1)(\alpha+3)\int_0^1 \frac{|(y_s)^{\alpha+3} - (\phi y_s + (1-\phi)y_{s+\epsilon})^{\alpha+3}|}{(\phi y_s + (1-\phi)y_{s+\epsilon})^{\alpha+3}(y_s)^{\alpha+3}} d\phi \\ &\le \alpha(\alpha+1)(\alpha+3)\int_0^1 \frac{|(y_s)^{\alpha+3} - (\phi y_s + (1-\phi)y_{s+\epsilon})^{\alpha+3}|}{\left(\phi \min_{s\in[0,2]} y_s + (1-\phi)\min_{s\in[0,2]} y_s\right)^{\alpha+3}(\min_{s\in[0,2]} y_s)^{\alpha+3}} d\phi \\ &\le \alpha(\alpha+1)(\alpha+2)\frac{1}{(\min_{s\in[0,2]} y_s)} 2(\alpha+3)\sup_{\phi\in[0,1]}\sup_{s\in[0,1]} |(y_s)^{\alpha+3} - (\phi y_s + (1-\phi)y_{s+\epsilon})^{\alpha+3}| \\ &\xrightarrow[\epsilon\searrow 0]{} 0 \quad \text{a.e.}, \end{aligned}$$

because of

$$\sup_{\phi\in[0,1]}\sup_{s\in[0,1]} |y_s - \phi y_s - (1-\phi)y_{s+\epsilon}| \le \sup_{s\in[0,1]} |y_s - y_{s+\epsilon}| \xrightarrow[\epsilon\searrow 0]{} 0 \quad \text{a.e.}$$

and uniform continuity. So,

$$\sup_{s\in[0,1]} |\psi_\epsilon(s) - \psi_0(s)| \underset{\epsilon\searrow 0}{\longrightarrow} 0 \quad \text{a.e.}$$

Hence,

$$\begin{aligned}&\left|\frac{1}{4\epsilon}\int_0^t (\psi_\epsilon(s) - \psi_0(s))(y_{s+\epsilon} - y_s)^3 ds\right| \\ &\quad \le \sup_{s\in[0,1]} |\psi_\epsilon(s) - \psi_0(s)| \cdot \sup_{\epsilon>0}\frac{1}{4\epsilon}\int_0^t |y_{s+\epsilon} - y_s|^3 ds \underset{\epsilon\searrow 0}{\longrightarrow} 0 \quad \text{a.e.},\end{aligned}$$

since $y_{\cdot}$ is of strong 3-variation.

Further, since $\psi_0(t)$, $0 \le t \le 1$, is a continuous process, it follows from Remark 2.6, (6) in [18] that

$$\frac{1}{4\epsilon}\int_0^{\cdot} \psi_0(s)(y_{s+\epsilon} - y_s)^3 ds \underset{\epsilon\searrow 0}{\overset{\text{ucp}}{\longrightarrow}} \frac{1}{4}\int_0^{\cdot} \psi_0(s) d[y, y, y]_s = 0,$$

due to $[y, y, y]_s = 0$. So,

$$J_{2,\epsilon}(\cdot) = \underset{\epsilon\searrow 0}{\overset{\text{ucp}}{\longrightarrow}} 0.$$

We also see that

$$\begin{aligned}J_{3,\epsilon}(t) &= \frac{1}{12\epsilon}\int_0^t f^{(3)}(y_s)(y_{s+\epsilon} - y_s)^3 ds + \frac{1}{12\epsilon}\int_0^t f^{(3)}(y_{s+\epsilon})(y_{s+\epsilon} - y_s)^3 ds \\ &= J_{3,\epsilon}^{(1)}(t) + J_{3,\epsilon}^{(2)}(t).\end{aligned}$$

Because of Remark 2.6, (6) in [18], we have again

$$J_{3,\epsilon}^{(1)}(\cdot) \underset{\epsilon\searrow 0}{\overset{\text{ucp}}{\longrightarrow}} 0.$$

Further, Remark 2.6, (5) in [18] implies that

$$J_{3,\epsilon}^{(2)}(\cdot) \underset{\epsilon\searrow 0}{\overset{\text{ucp}}{\longrightarrow}} 0.$$

As for the process $J_{4,\epsilon}(\cdot)$, we can use the same arguments as in the case of $J_{2,\epsilon}(\cdot)$ based on uniform continuity and the strong 3-variation of $y_{\cdot}$ and obtain that

$$J_{4,\epsilon}(\cdot) \underset{\epsilon\searrow 0}{\overset{\text{ucp}}{\longrightarrow}} 0.$$

Altogether, we get in connection with Remark 3.2, (1) in [18] that

$$\lim_{\epsilon \searrow 0} J_{1,\epsilon}(\cdot)$$

exists in the ucp-topology and must be equal to

$$\int_0^{\cdot} f'(y_s) d^\circ y_s.$$

□

For some of the proofs in this article, we need to recall some basic concepts from fractional calculus (see [19, 20]).

Let $a, b \in \mathbb{R}$ with $a < b$. Let $f \in L^p([a, b])$ with $p \geq 1$ and $\alpha > 0$. Then, the *left-* and *right-sided Riemann–Liouville fractional integrals* are defined as

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - y)^{\alpha - 1} f(y) dy$$

and

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y - x)^{\alpha - 1} f(y) dy$$

for almost all $x \in [a, b]$. Here, Γ is the gamma function.

Let $p \geq 1$ and let $I_{a+}^{\alpha}(L^p)$ (resp. $I_{b-}^{\alpha}(L^p)$) be the image of $L^p([a, b])$ of the operator I_{a+}^{α} (resp. I_{b-}^{α}). If $f \in I_{a+}^{\alpha}(L^p)$ (resp. $f \in I_{b-}^{\alpha}(L^p)$) and $0 < \alpha < 1$, then we can define the *left-* and *right-sided Riemann–Liouville fractional derivatives* by

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x - y)^{\alpha}} dy$$

and

$$D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_x^b \frac{f(y)}{(y - x)^{\alpha}} dy.$$

The left- and right-sided derivatives of f can be represented as

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1 - \alpha)} \left(\frac{f(x)}{(x - a)^{\alpha}} + \alpha \int_a^x \frac{f(x) - f(y)}{(x - y)^{\alpha + 1}} dy \right)$$

and

$$D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(1 - \alpha)} \left(\frac{f(x)}{(b - x)^{\alpha}} + \alpha \int_x^b \frac{f(x) - f(y)}{(y - x)^{\alpha + 1}} dy \right).$$

The above definitions imply that

$$I_{a+}^{\alpha}\left(D_{a+}^{\alpha}f\right)=f$$

for all $f \in I_{a+}^{\alpha}(L^p)$ and

$$D_{a+}^{\alpha}\left(I_{a+}^{\alpha}f\right)=f$$

for all $f \in L^p([a,b])$ and similarly for I_{b-}^{α} and D_{b-}^{α}.

Denote by $B^H=\{B_t^H, t\in[0,T]\}$ a d-dimensional *fractional Brownian motion* with Hurst parameter $H \in (0,\frac{1}{2})$. The latter means that $B_\cdot^H$ is a centred Gaussian process with a covariance function given by

$$\begin{aligned}(R_H(t,s))_{i,j} &:= E\left[B_t^{H,(i)}B_s^{H,(j)}\right]\\ &= \delta_{ij}\frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right), \quad i,j=1,\ldots,d,\end{aligned}$$

where δ_{ij} is one, if $i=j$, or zero else.

In the sequel, we also shortly recall the construction of the fractional Brownian motion, which can be found in [7]. For convenience, we restrict ourselves to the case $d=1$.

Denote by $\mathcal{E}$ the class of step functions on $[0,T]$, and let $\mathcal{H}$ be the Hilbert space which one gets through the completion of $\mathcal{E}$ with respect to the inner product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}} = R_H(t,s).$$

The latter provides an extension of the mapping $1_{[0,t]} \mapsto B_t$ to an isometry between $\mathcal{H}$ and a Gaussian subspace of $L^2(\Omega)$ with respect to B^H. Let $\varphi \mapsto B^H(\varphi)$ be this isometry.

If $H<\frac{1}{2}$, one finds that the covariance function $R_H(t,s)$ can be represented as

$$R_H(t,s)=\int_0^{t\wedge s} K_H(t,u)K_H(s,u)du, \tag{25}$$

where

$$\begin{aligned}K_H(t,s) = c_H\Bigg[\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} + \left(\frac{1}{2}-H\right)s^{\frac{1}{2}-H}\\ \int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du\Bigg].\end{aligned} \tag{26}$$

Here, $c_H=\sqrt{\frac{2H}{(1-2H)\beta(1-2H,H+\frac{1}{2})}}$ and β is the beta function. See [7, Proposition 5.1.3].

Using the kernel K_H, one can obtain via (25) an isometry K_H^* between $\mathcal{E}$ and $L^2([0,T])$ such that $(K_H^* 1_{[0,t]})(s) = K_H(t,s)1_{[0,t]}(s)$. This isometry allows for an extension to the Hilbert space $\mathcal{H}$, which has the following representations in terms of fractional derivatives

$$(K_H^*\varphi)(s) = c_H\Gamma\left(H+\frac{1}{2}\right)s^{\frac{1}{2}-H}\left(D_{T^-}^{\frac{1}{2}-H}u^{H-\frac{1}{2}}\varphi(u)\right)(s)$$

and

$$\begin{aligned}(K_H^*\varphi)(s) =& c_H\Gamma\left(H+\frac{1}{2}\right)\left(D_{T^-}^{\frac{1}{2}-H}\varphi(s)\right)(s)\\ &+ c_H\left(\frac{1}{2}-H\right)\int_s^T \varphi(t)(t-s)^{H-\frac{3}{2}}\left(1-\left(\frac{t}{s}\right)^{H-\frac{1}{2}}\right)dt.\end{aligned}$$

for $\varphi \in \mathcal{H}$. One can also prove that $\mathcal{H} = I_{T^-}^{\frac{1}{2}-H}(L^2)$. See [21] and [22, Proposition 6].

We know that K_H^* is an isometry from $\mathcal{H}$ into $L^2([0,T])$. Thus, the d-dimensional process $W = \{W_t, t \in [0,T]\}$ defined by

$$W_t := B^H((K_H^*)^{-1}(1_{[0,t]})) \tag{27}$$

is a Wiener process and the process B^H has the representation

$$B_t^H = \int_0^t K_H(t,s)dW_s. \tag{28}$$

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