Set-Theoretic and Type-Theoretic Ordinals Coincide

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***Abstract*—In constructive set theory, an ordinal is a hereditarily transitive set. In homotopy type theory (HoTT), an ordinal is a type with a transitive, wellfounded, and extensional binary relation. We show that the two definitions are equivalent if we use (the HoTT refinement of) Aczel's interpretation of constructive set theory into type theory. Following this, we generalize the notion of a type-theoretic ordinal to capture *all* sets in Aczel's interpretation rather than only the ordinals. This leads to a natural class of ordered structures which contains the type-theoretic ordinals and realizes the higher inductive interpretation of set theory. All our results are formalized in Agda.**

I. Introduction

Set theory and dependent type theory are two very different settings in which constructive mathematics can be developed, but not always in comparable ways. Lively discussions on what foundation is "better" are not uncommon. While we do not dare to offer a judgment on this question, we can at least report that the choice of foundation is in a certain sense insignificant for the development of constructive ordinal theory. We consider this an interesting finding since ordinals are fundamental in the foundations of set theory and are used in theoretical computer science in termination arguments [1] and semantics of inductive definitions [2], [3].

In constructive set theory, following Powell's seminal work [4], a standard definition[1] of an ordinal is that of a *transitive* set whose elements are again transitive sets (also cf. Aczel and Rathjen [8]). A set x is transitive if for every $y \in x$ and $z \in y$, we have $z \in x$, i.e., if $y \in x$ implies $y \subseteq x$. Note how this definition makes essential use of how the membership predicate $\in$ in set theory is global, by simultaneously referring to $z \in y$ and $z \in x$. In type theory, on the other hand, the statement "if $y : x$ and $z : y$ then $z : x$" is ill-formed, and so ordinals need to be defined differently. In homotopy type theory, an ordinal is defined to be a type equipped with an order relation that is transitive, extensional, and wellfounded [9, §10.3].

[1]In a classical setting, Cantor's ordinals [5] can be presented in multiple equivalent ways. In a constructive setting, these presentations are not equivalent and are often not as well-behaved as one would wish. Therefore, various reasonable definitions of ordinals are known and have been studied; for example, Taylor [6] gave a constructively better-behaved formulation, and in previous work, we compared several approaches [7].

A priori, the set-theoretic and the type-theoretic approaches to ordinals are thus quite different. One way to compare them is to interpret one foundation into the other. Aczel [10] gave an interpretation of Constructive ZF set theory into type theory using so-called setoids, which was later refined using a higher inductive type $\mathbb{V}$ in the HoTT book [9, §10.5], referred to as the *cumulative hierarchy*. Through this construction, homotopy type theory hosts a model of set theory, and we make use of this to study the set-theoretic approach to ordinals within it.

To be specific, the cumulative hierarchy $\mathbb{V}$ allows us to define a set membership relation $\in$, which makes it possible to consider the type $\mathbb{V}_{\mathrm{ord}}$ of elements of $\mathbb{V}$ that are set-theoretic ordinals. Similarly, we write Ord for the type of all type-theoretic ordinals, i.e., for the type of transitive, extensional, and wellfounded order relations. We show that $\mathbb{V}_{\mathrm{ord}}$ and Ord are equivalent (isomorphic), meaning that we can translate between type-theoretic and set-theoretic ordinals.

This translation by itself would not be satisfactory if it were not well-behaved; what makes it valuable is that it preserves the respective order. A fundamental result of type-theoretic ordinals is that the type Ord of (small) ordinals is itself a type-theoretic ordinal when ordered by inclusion of strictly smaller initial segments (also referred to as *bounded simulations*). To complement this, we show that the type $\mathbb{V}_{\mathrm{ord}}$ of set-theoretic ordinals also canonically carries the structure of a type-theoretic ordinal. The isomorphisms that we construct respect these orderings, and our first main result (Theorem 33) is that Ord and $\mathbb{V}_{\mathrm{ord}}$ are isomorphic as ordinals (and, consequently, equal, by a standard application of univalence). Thus, the set-theoretic and type-theoretic approaches to ordinals coincide in homotopy type theory.

Going further, we dive deeper into the study of the isomorphism $\mathbb{V}_{\mathrm{ord}} \to \mathsf{Ord}$. The analogue to this function in set theory computes the *rank* [2], [8], [11] of sets recursively. While our definition is recursive as well, we show that it is possible to give a conceptually simpler, non-recursive description of the rank of transitive sets, although this requires paying close attention to size issues. Specifically, we show that the rank of a set-theoretic ordinal α is isomorphic to — but not equal to for size reasons — the type of all members of α (Corollary 46).

In the second part of the paper, we generalize the isomorphism between set- and type-theoretic ordinals. Given that the subtype $\mathbb{V}_{\mathrm{ord}}$ of $\mathbb{V}$ is isomorphic to Ord, a type of ordered structures, it is natural to ask what type of ordered structures captures *all* of $\mathbb{V}$. That is, we look for a natural type T of ordered structures such that the diagram on the right commutes. Since $\mathbb{V}$ is $\mathbb{V}_{\mathrm{ord}}$ with transitivity dropped, it is tempting to try to choose T to be Ord without transitivity, i.e., the type of extensional and wellfounded relations. However, such an attempt is too naive to work: consider the type-theoretic ordinal α with two elements $0 < 1$, whose corresponding set in $\mathbb{V}_{\mathrm{ord}}$ is the set $2 = \{\emptyset, \{\emptyset\}\}$. The latter is the set-theoretic transitive closure of the non-transitive set $\{\{\emptyset\}\} \subseteq 2$, but the only extensional, wellfounded order whose order-theoretic transitive closure is α is α itself. In other words, there cannot be an order-preserving isomorphism between $\mathbb{V}$ and the type of extensional, wellfounded order relations, since there is no corresponding order for the set $\{\{\emptyset\}\}$ — we need additional structure to fully capture this set.

$$\begin{tikzcd} \mathbb{V}_{\mathrm{ord}} \arrow[r, "\simeq"] \arrow[d, hook] & \mathsf{Ord} \arrow[d, hook] \\ \mathbb{V} \arrow[r, "\simeq"] & T \end{tikzcd} \tag{1}$$

To this end, we introduce the theory of *(covered) marked extensional wellfounded orders (mewos)*, i.e., extensional, wellfounded relations with additional structure in the form of a *marking*. The idea is that the carrier of the order also contains elements representing elements of elements of the set, with the marking designating the "top-level" elements: the set $\{\{\emptyset\}\}$ is again represented by the order α with two elements $0 < 1$, but with only element 1 marked. Such a marking is *covering* if any element can be reached from a marked top-level element, i.e., if the order contains no "junk". Since every ordinal can be equipped with the trivial covering by marking all elements, the type Ord of ordinals is a subtype of the type $\mathsf{MEWO}_{\mathsf{cov}}$ of covered mewos, as requested by Diagram 1.

The idea of encoding sets as wellfounded structures is not new; see, e.g., [12, §7], [6, §3], [13, §4.7] and Aczel's [14] "canonical picture" [13, Ex 4.22]. Instead, the point is to have a notion that allows for a smooth type-theoretic generalization of the theory of ordinals. Additionally, covered mewos are shown to work predicatively (i.e., without the need to assume resizing axioms), which is not obvious for the previously mentioned approaches.

Aiming for an isomorphism $\mathbb{V} \simeq \mathsf{MEWO}_{\mathsf{cov}}$, we develop the theory of the covered mewos: the type of covered mewos is itself a covered mewo, and it has both a successor operation, and least upper bounds of arbitrary (small) families of covered mewos. Compared to the theory of ordinals, some additional care is required as the orders involved are not assumed to be transitive. Using successors and least upper bounds of mewos, we construct a map $\mathbb{V} \to \mathsf{MEWO}_{\mathsf{cov}}$ by the recursive formula for the rank of a set, and show that it has an inverse.

A. Summary of contributions

- We show that set-theoretic and type-theoretic ordinals coincide (Theorem 33).
- We show that the rank of an ordinal can be defined in a non-recursive way (Corollary 46).
- We show that the model of set theory $\mathbb{V}$ is equivalently represented by the structure of covered marked extensional wellfounded order relations (Theorem 76).

B. Related work

Constructive treatments of ordinals can be found in Joyal and Moerdijk [15] and Taylor [6]. In the context of homotopy type theory, what we call type-theoretic ordinals were developed in the HoTT book [9, §10.3], and their theory significantly expanded by Escardó and collaborators [16]. In previous work [7], [17], we developed a framework for different notions of constructive ordinals, and showed that all ordinals we considered embed into the type-theoretic ordinals in an order-preserving way. In fact, the current paper grew out of an attempt to locate the set-theoretic ordinals somewhere between the countable Brouwer tree ordinals (as considered by, e.g., Brouwer [18], Church [19], Kleene [20], Martin-Löf [21], and Coquand, Lombardi and Neuwirth [22]) and the type-theoretic ordinals in this framework, before we realised that they actually coincide with the latter!

To the best of our knowledge, the first interpretation of Constructive ZF set theory into type theory was given by Aczel [10]. This original interpretation uses so-called setoids and has a form of choice built-in. It was later refined using a higher inductive type $\mathbb{V}$ in the HoTT book [9, §10.5], referred to as the *cumulative hierarchy*, and this is the version we use in the current paper. Gylterud [23] showed that $\mathbb{V}$ can be constructed using only an ordinary inductive type without higher constructors. Although it is not our main motivation, the current paper demonstrates that $\mathbb{V}$ can be realized not as an inductive type at all, but as the collection of all covered marked wellfounded extensional relations (however, the notion of wellfoundedness is defined as an inductive type, and the notion of coveredness uses higher constructors in the form of propositional truncations). Taylor [6] also considers wellfounded extensional relations (which he calls *ensembles*) as "codes" for sets in an elementary topos, but does not consider markings on them. Coverings and markings are what allow us to achieve completeness, i.e., to represent *all* sets in $\mathbb{V}$.

C. Setting, assumptions, and notation

We work in and assume basic familiarity with homotopy type theory as introduced in the HoTT book [9], i.e., Martin-Löf type theory extended with higher inductive types and the univalence axiom. We also follow this book closely regarding notation and denote the Martin-Löf identity type by $a = b$, while $a \equiv b$ is reserved for definitional (also referred to as judgmental) equality. Universe levels are kept implicit, and we write $\mathcal{U}^+$ for the next universe containing the universe $\mathcal{U}$. For an implicitly fixed universe $\mathcal{U}$, we write Prop or $\mathsf{Prop}_{\mathcal{U}}$ for the subtype of propositions, $\mathsf{Prop} :\equiv \Sigma(P : \mathcal{U}).\mathsf{is\text{-}prop}(X)$, where a *proposition* is a type with at most one element ("proof-irrelevant"). Following standard terminology, a *set* is a type whose identity types are propositions.

We write the type of dependent functions $\Pi(x : A).B(x)$ as $\forall(x : A).B(x)$ when $B(x)$ is known to be a family of propositions. Moreover, we denote by $\exists(x : A).B(x)$ the propositional truncation of the type of dependent pairs $\|\Sigma(x : A).B(x)\|$.

D. Formalization

All our results have been formalized in the Agda proof assistant, and type checks using Agda 2.6.3. Our formalization of Section II is building on Escardó's TypeTypology library [24], whereas our formalization of Section III is building on the agda/cubical library [25]. The formalization has been archived with the DOI 10.5281/zenodo.7857275, and an HTML rendering of our Agda code is also available at https://tdejong.com/agda-html/st-tt-ordinals/. Throughout (the arXiv version of) our paper, the symbol ⚙ is a clickable link to the corresponding machine-checked statement.

II. Ordinals in type theory and set theory

We start by reviewing both the set-theoretic and type-theoretic approaches to ordinals. We then recall the higher inductive construction of a model of constructive set theory in homotopy type theory [9, §10.5], allowing us to consider the set-theoretic ordinals inside homotopy type theory, and to prove that they coincide with the type-theoretic ordinals. Finally, we revisit a recursive aspect of our proof and provide alternative non-recursive constructions, which require paying close attention to type universe levels.

A. Ordinals in homotopy type theory

The theory of ordinals in homotopy type theory was introduced in the HoTT book [9, §10.3] and significantly expanded on by Escardó and collaborators [16]. One of the core concepts is wellfoundedness which, constructively, is conveniently phrased in terms of accessibility:

Definition 1 (⚙ Accessibility)**.** For a type X equipped with a binary relation $<$, the type family $\mathsf{is\text{-}accessible}_<$ on X is inductively defined by saying that $\mathsf{is\text{-}accessible}_<(x)$ holds if $\mathsf{is\text{-}accessible}_<(y)$ holds for every $y < x$.

The point of accessibility is that it captures the principle of transfinite induction by a *single* inductive definition.

Lemma 2 (⚙ Transfinite induction)**.** *For a type X equipped with a binary relation $<$, every element of X is accessible if and only if for every type family P on X, we have $P(x)$ for all $x : X$ as soon as for every $x : X$, the statement $\forall(y : X).y < x \to P(y)$ implies $P(x)$.* □

Cantor's original definition of ordinal numbers was that of isomorphism classes of well-ordered sets, but using univalence, all representatives of a given isomorphism class in homotopy type theory are identical. Hence, we can use the well-ordered sets directly to represent ordinals. The classical definition of well-order states that every non-empty subset has a minimal element. Constructively, the following (classically equivalent) formulation is better behaved.

Definition 3 (⚙ Type-theoretic ordinal)**.** A binary relation $<$ on a type X is said to be

(i) *prop-valued* if $x < y$ is a proposition for every $x, y : X$;
(ii) *wellfounded* if every element of X is accessible with respect to $<$, i.e., $\forall(x : X).\, \mathsf{is\text{-}accessible}_<(x)$;
(iii) *extensional* if $\forall(z : X).(z < x \leftrightarrow z < y)$ implies $x = y$ for every $x, y : X$; and
(iv) *transitive* if $x < y$ and $y < z$ together imply $x < z$ for every $x, y, z : X$.

A *(type-theoretic) ordinal* is a type X with a binary relation $<$ on X that is prop-valued, wellfounded, extensional, and transitive.

Remark 4. While [9] requires the carrier of an ordinal to be a set (in the sense of HoTT), Escardó [16] observed that this follows from prop-valuedness and extensionality.

We now recall the notion of an initial segment and bounded simulation, which will play fundamental roles in our constructions and proofs.

Definition 5 (⚙ Initial segment, $\alpha \downarrow a$; bounded simulation, $<$)**.** An element a of an ordinal α determines an *initial segment* of α defined as

$$\alpha \downarrow a \;:\equiv\; \Sigma(x : \alpha).x < a,$$

which is again an ordinal with the order induced by α. A *bounded simulation* between ordinals, $p : \alpha < \beta$, is a proof that α is an initial segment of β,

$$\alpha < \beta \;:\equiv\; (\Sigma(b : \beta).\alpha \simeq \beta \downarrow b).$$

Note that the definition of $\alpha < \beta$ above is equivalent to the definition given in [9, Def 10.3.19]; in particular, it is a proposition.

Remark 6. In Definition 5 above, we could have defined a bounded simulation using an identification $\alpha = \beta \downarrow b$, but opted for an equivalence $\alpha \simeq \beta \downarrow b$ instead. These two expressions are equivalent by univalence. However, the latter has the advantage of begin *small*, i.e., living in the same universe as α and β, while the former lives in the next universe.

Theorem 7 (⚙)**.** *The type* Ord *of ordinals in a univalent universe, together with the relation $<$ of bounded simulations, is itself a (large) type-theoretic ordinal.* □

A bounded simulation is a special case of the following more general definition that serves as a notion of morphism between ordinals:

Definition 8 (⚙ Simulation, $\leq$)**.** A *simulation* between two ordinals α and β is a function f between the underlying types satisfying:

(i) *monotonicity:* $x <_\alpha y$ implies $f\,x <_\beta f\,y$ for every two elements $x, y : \alpha$, and
(ii) *the initial segment property:* for every $x : \alpha$ and $y : \beta$, if $y <_\beta f\,x$, then there is a $x' <_\alpha x$ with $f\,x' = y$.

If we have a simulation between α and β, then, motivated by Proposition 9 below, we denote this by $\alpha \leq \beta$.

We stress that $\leq$ is a primitive relation, and *not* given as the disjunction of $<$ and equality — in fact, we have $\alpha \leq \beta \leftrightarrow$

$((\alpha < \beta) + (\alpha = \beta))$ for all ordinals α and β if and only if the law of excluded middle holds [7, Thm 64].

Proposition 9 (⚙). *Simulations make* Ord *into a poset. Moreover, for ordinals α and β, the following are equivalent:*

(i) $\alpha \leq \beta$,
(ii) for every $\gamma : \mathsf{Ord}$, *if* $\gamma < \alpha$, *then* $\gamma < \beta$, *and*
(iii) for every $a : A$, *we have a (necessarily unique)* $b : \beta$ *with* $\alpha \downarrow a = \beta \downarrow b$.

Given $f : \alpha \leq \beta$, *the element* b *in (iii) is given by* $f(a)$. □

The relation $\leq$ on Ord is antisymmetric, which can be used to prove that two ordinals are isomorphic; however, it is often convenient to work with the following alternative description.

Lemma 10 (⚙). *A map between ordinals is an isomorphism if and only if it is bijective and preserves and reflects the order.* □

Univalence implies that isomorphic ordinals in the same universe are equal (as ordinals), which allows us to prove the equalities in the upcoming lemmas.

Lemma 11 (⚙). *For* $p : a < b$ *in an ordinal* α, *iterations of initial segments simplify as follows:* $(\alpha \downarrow a) \downarrow (b, p) = \alpha \downarrow b$. □

Besides initial segments, we will need two additional constructions of ordinals, sums and suprema, as well as a few lemmas expressing how these interact with initial segments.

Definition 12 (⚙ Sum of ordinals, $\alpha + \beta$)**.** Given two ordinals α and β, we construct another ordinal, the *sum* $\alpha + \beta$, by ordering the coproduct of the underlying types of α and β as

$$\begin{aligned} \mathsf{inl}\, a < \mathsf{inr}\, b &:\equiv \mathbf{1}, & \mathsf{inl}\, a < \mathsf{inl}\, a' &:\equiv a <_\alpha a', \\ \mathsf{inr}\, b < \mathsf{inl}\, a &:\equiv \mathbf{0}, & \mathsf{inr}\, b < \mathsf{inr}\, b' &:\equiv b <_\beta b'. \end{aligned}$$

Initial segments of sums obey the following laws:

Lemma 13 (⚙). *For ordinals α and β, and $a : \alpha$, we have:*

(i) $(\alpha + \beta) \downarrow \mathsf{inl}\, a = \alpha \downarrow a$, *and*
(ii) $(\alpha + \mathbf{1}) \downarrow \mathsf{inr} \star = \alpha$. □

Definition 14 (⚙ Supremum of ordinals, $\bigvee_{i:I} \alpha_i$)**.** Given a type $I : \mathcal{U}$ and a family of $\alpha : I \to \mathsf{Ord}$ of ordinals in $\mathcal{U}$, we construct another ordinal, the *supremum* $\bigvee_{i:I} \alpha_i$, as the set quotient of $\Sigma(i : I).\alpha_i$ by the relation

$$(i, x) \approx (j, y) \;:\equiv\; (\alpha_i \downarrow x \simeq \alpha_j \downarrow y)$$

and ordered by

$$[i, x] < [j, y] \;:\equiv\; (\alpha_i \downarrow x < \alpha_j \downarrow y).$$

Note that the distinction between $\simeq$ and $=$ discussed in Remark 6 is important in the definition above. It ensures that the supremum $\bigvee_{i:I} \alpha_i$ lives in the "correct" universe, i.e., is an element of Ord.

The name "supremum" comes from the fact that $\bigvee_{i:I} \alpha_i$ indeed is the supremum (least upper bound) of the family $\alpha : I \to \mathsf{Ord}$ in the poset Ord, as shown in [26, Thm 5.8] which extends [9, Lem 10.3.22]. In particular, we have simulations $\alpha_j \leq \bigvee_{i:I} \alpha_i$ for every $j : I$ given by $x \mapsto [j, x]$.

Lemma 15 (⚙). *Initial segments of suprema obey the following laws for all families* $\alpha : I \to \mathsf{Ord}$ *of ordinals:*

(i) $\bigvee_{i:I} \alpha_i \downarrow [j, x] = \alpha_j \downarrow x$ *for all* $j : I$ *and* $x : \alpha_j$, *and*
(ii) for every $y : \bigvee_{i:I} \alpha_i$, *there exist* $j : J$ *and* $x : \alpha_i$ *for which* $\bigvee_{i:I} \alpha_i \downarrow y = \alpha_j \downarrow x$.

Thus, an initial segment of a supremum is given by an initial segment of a component.

Proof. The first property follows from Proposition 9 and the fact that for every $j : I$, the map $x \mapsto [j, x]$ from α_j to $\bigvee_{i:I} \alpha_i$ is a simulation. The second follows from the first and the surjectivity of the map $[-] : (\Sigma(j : J).\alpha_j) \to \bigvee_{j:J} \alpha_j$. □

B. Ordinals in set theory

In constructive set theory, following Powell [4], the standard definition [8, Def 9.4.1] of an ordinal is simple to state: it is a transitive set whose elements are again transitive sets.

Definition 16 (⚙ Transitive set)**.** A set x is *transitive* if for every z and y with $z \in y$ and $y \in x$, we have $z \in x$.

Note how this definition makes essential use of how the membership predicate $\in$ in set theory is global, by simultaneously referring to $z \in y$ and $z \in x$.

Example 17. The sets $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ are all transitive, but $\{\{\emptyset\}\}$ is not, because $\emptyset$ is not a member.

Definition 18 (⚙ Set-theoretic ordinal)**.** A *set-theoretic ordinal* is a hereditarily transitive set, i.e., a transitive set whose elements are all transitive sets.

The first three sets of Example 17 are all ordinals, but the fourth is not, because its member $\{\{\emptyset\}\}$ is non-transitive.

The elements of an ordinal are not only transitive sets: they are in fact ordinals again, as shown by the following standard argument.

Lemma 19 (⚙). *Being an ordinal is hereditary: the elements of a set-theoretic ordinal are themselves ordinals.*

Proof. Let x be a set-theoretic ordinal and $y \in x$. Then y is a transitive set by assumption. Moreover, if $z \in y$, then z is again a transitive set, because $z \in x$ by transitivity of x. □

C. Set theory in homotopy type theory

In order to relate the set-theoretic and type-theoretic approaches to ordinals, we recall a higher inductive [9, §6] construction of a model of constructive set theory inside homotopy type theory from [9, §10.5]. The model may be seen as a refinement of Aczel's [10] interpretation of constructive set theory in type theory, and is referred to as the *cumulative hierarchy* in the HoTT book [9] and the *iterative hierarchy* in Gylterud [23].

It is convenient to introduce the following terminology before proceeding.

Definition 20 (⚙ Equal images)**.** Two maps $f : A \to X$ and $g : B \to X$ with the same codomain are said to have *equal images* if for every $a : A$, there exists some $b : B$ such that

$f\,a = g\,b$, and conversely, for every $b : B$, there exists some $a : A$ with $g\,b = f\,a$.

Definition 21 (Cumulative hierarchy $\mathbb{V}$; [9, Def 10.5.1])**.** The *cumulative hierarchy* $\mathbb{V}$ with respect to a type universe $\mathcal{U}$ is the higher inductive type with the following constructors:

(i) for every type $A : \mathcal{U}$ and $f : A \to \mathbb{V}$ we have an element of $\mathbb{V}$, denoted by $\mathbb{V}\text{-set}(A, f)$;

(ii) for every two types $A, B : \mathcal{U}$ and maps $f : A \to \mathbb{V}$ and $g : B \to \mathbb{V}$, if f and g have equal images, then we have an identification $\mathbb{V}\text{-set}(A, f) = \mathbb{V}\text{-set}(B, g)$;

(iii) set-truncation, i.e., for every $x, y : \mathbb{V}$ and $p, q : x = y$, we have an identification $p = q$.

$\mathbb{V}$ is a model of set theory by [9, Thm 10.5.8]. It is instructive to see how to represent the sets $\emptyset$, $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$ from Example 17 in $\mathbb{V}$:

- The empty set $\emptyset$ is represented as $\ulcorner\emptyset\urcorner :\equiv \mathbb{V}\text{-set}(\mathbf{0}, !)$ where $!$ is the unique map from $\mathbf{0}$ to $\mathbb{V}$.
- The singleton set $\{\emptyset\}$ may be represented by setting $\ulcorner\{\emptyset\}\urcorner :\equiv \mathbb{V}\text{-set}(\mathbf{1}, \lambda \star \,.\, \ulcorner\emptyset\urcorner)$.
- Finally, the set $\{\emptyset, \{\emptyset\}\}$ can be encoded as $\mathbb{V}\text{-set}(\mathbf{2}, f)$ where $f(0) :\equiv \ulcorner\emptyset\urcorner$ and $f(1) :\equiv \ulcorner\{\emptyset\}\urcorner$.

The second constructor of $\mathbb{V}$ ensures that the elements have the correct notion of equality. For instance, using the example given directly above, it means that the elements $\mathbb{V}\text{-set}(\mathbf{2}, f)$ and $\mathbb{V}\text{-set}(\mathbb{N}, f \circ \mathsf{isEven})$ are equal.

Observe that $\mathbb{V}$ is a large type, i.e., it lives in the next universe $\mathcal{U}^+$. Following [9, §10.5], we now define the set membership and the subset relation on $\mathbb{V}$, so that we can define set-theoretic ordinals inside $\mathbb{V}$.

Definition 22 (Set membership $\in$ on $\mathbb{V}$)**.** We define the *set membership* relation $\in : \mathbb{V} \to \mathbb{V} \to \mathsf{Prop}_{\mathcal{U}^+}$ inductively as:

$$x \in \mathbb{V}\text{-set}(A, f) \;:\equiv\; \exists(a : A).\, f\,a = x.$$

This is well-defined because $\mathsf{Prop}_{\mathcal{U}^+}$ is a set (in the sense of HoTT), and if f and g have equal images, then $x \in \mathbb{V}\text{-set}(A, f)$ holds exactly when $x \in \mathbb{V}\text{-set}(B, g)$ does.

Definition 23 (Subset relation $\subseteq$)**.** We define the *subset relation* $\subseteq : \mathbb{V} \to \mathbb{V} \to \mathsf{Prop}_{\mathcal{U}^+}$ as

$$x \subseteq y \;:\equiv\; \forall(v : \mathbb{V}).v \in x \to v \in y.$$

The type $\mathbb{V}$ models Myhill's Constructive Set Theory [23], and in fact all of Zermelo-Fraenkel set theory with Choice, if we assume the axiom of choice in type theory [27]. In the following, we will in particular need the following two set-theoretic axioms:

Lemma 24 (Items (i) and (vii) of [9, Thm 10.5.8])**.** *The following two set-theoretic axioms are satisfied by $\mathbb{V}$:*

(i) extensionality: *two elements x and y of $\mathbb{V}$ are equal if and only if $x \subseteq y$ and $y \subseteq x$, and*

(ii) $\in$-induction: *for any prop-valued family $P : \mathbb{V} \to \mathsf{Prop}$, if, for every $x : \mathbb{V}$, we have $P(x)$ whenever $P(y)$ holds for all $y \in x$, then P holds at every element of $\mathbb{V}$.* □

The set membership relation allows us to formulate the set-theoretic notions of Section II-B for $\mathbb{V}$, and hence, to define the type of set-theoretic ordinals in $\mathbb{V}$.

Definition 25 (Type of set-theoretic ordinals)**.** The *type* $\mathbb{V}_{\text{ord}}$ *of set-theoretic ordinals* is the Σ-type of those $x : \mathbb{V}$ such that x is a set-theoretic ordinal in the sense of Definition 18.

The subtype of set-theoretic ordinals is then an example of a type-theoretic ordinal, which we show to be equal to the type Ord of type-theoretic ordinals in the next subsection.

Theorem 26 ()**.** *Set membership makes $\mathbb{V}_{\text{ord}}$ into a type-theoretic ordinal.*

Proof. Wellfoundedness follows from $\in$-induction, and set membership is a transitive relation on $\mathbb{V}_{\text{ord}}$: if we have set-theoretic ordinals $x, y, z : \mathbb{V}_{\text{ord}}$ such that $x \in y$ and $y \in z$, then $x \in z$, because z is a transitive set. For extensionality, assume that we have $x, y : \mathbb{V}_{\text{ord}}$ such that $u \in x \leftrightarrow u \in y$ for every $u : \mathbb{V}_{\text{ord}}$. We need to show that $x = y$. By extensionality in the sense of Lemma 24, it suffices to show that $v \in x \leftrightarrow v \in y$ for all $v : \mathbb{V}$. But if $v \in x$, then $v : \mathbb{V}_{\text{ord}}$, because being a set-theoretic ordinal is hereditary (Lemma 19). Hence, $v \in y$ by assumption. Similarly, $v \in y$ implies $v \in x$, so that $x = y$, as desired. □

D. *Set-theoretic and type-theoretic ordinals coincide*

Having reviewed the necessary preliminaries, we prove in this subsection that the set-theoretic and type-theoretic ordinals coincide. More precisely, we construct an isomorphism of type-theoretic ordinals between $\mathbb{V}_{\text{ord}}$ and Ord by constructing maps in both directions.

Definition 27 (Φ)**.** The map $\Phi : \mathsf{Ord} \to \mathbb{V}$ is defined by transfinite recursion on Ord as

$$\Phi(\alpha) \;:\equiv\; \mathbb{V}\text{-set}(\alpha, \lambda\, a \,.\, \Phi(\alpha \downarrow a))$$

The function Φ is well-defined, because for every $a : \alpha$, the initial segment $\alpha \downarrow a$ is strictly smaller than α, as ordinals.

Lemma 28 ()**.** *The map Φ is injective and preserves and reflects the strict and weak orders, i.e., for every two type-theoretic ordinals $\alpha, \beta : \mathsf{Ord}$, we have*

(i) $\alpha = \beta \;\leftrightarrow\; \Phi\,\alpha = \Phi\,\beta$,

(ii) $\alpha < \beta \;\leftrightarrow\; \Phi\,\alpha \in \Phi\,\beta$, *and*

(iii) $\alpha \leq \beta \;\leftrightarrow\; \Phi\,\alpha \subseteq \Phi\,\beta$.

Proof. That Φ preserves equality is automatic.

a) $\alpha < \beta \Rightarrow \Phi\,\alpha \in \Phi\,\beta$: If $\alpha < \beta$, then we have $b : \beta$ such that $\alpha = \beta{\downarrow}b$. Hence, in this case, we have $\Phi\,\alpha = \Phi(\beta{\downarrow}b)$, viz. $\Phi\,\alpha \in \Phi\,\beta$ by the definitions of $\in$ and Φ.

b) $\alpha \leq \beta \Rightarrow \Phi\,\alpha \subseteq \Phi\,\beta$: For $x \in \Phi\,\alpha$, we get an a with $x = \Phi(\alpha \downarrow a)$ by definition. Proposition 9 gives us b such that $\alpha \downarrow a = \beta \downarrow b$, and hence $x \in \Phi\,\beta$, as desired.

c) Injectivity – $\Phi\,\alpha = \Phi\,\beta \Rightarrow \alpha = \beta$: We do transfinite induction on Ord. Assume $\alpha : \mathsf{Ord}$ and the induction hypothesis: for every element $a : \alpha$ and ordinal $\beta : \mathsf{Ord}$, if $\Phi(\alpha \downarrow a) = \Phi\,\beta$, then $\alpha \downarrow a = \beta$. We must prove that

$\Phi\,\alpha = \Phi\,\beta$ implies $\alpha = \beta$ for all ordinals $\beta : \mathsf{Ord}$. So assume that $\beta : \mathsf{Ord}$ is such that $\Phi\,\alpha = \Phi\,\beta$. We show that $\alpha \leq \beta$; the reverse inequality is proved similarly. By Proposition 9, it suffices to prove that $\alpha \downarrow a < \beta$ for every $a : \alpha$. For such $a : \alpha$ we have $\Phi(\alpha \downarrow a) \in \Phi\,\alpha = \Phi\,\beta$, and hence, there exists some $b : \beta$ with $\Phi(\alpha \downarrow a) = \Phi(\beta \downarrow b)$. Our induction hypothesis then yields $\alpha \downarrow a = \beta \downarrow b$, and hence the desired $\alpha \downarrow a < \beta$.

d) $\Phi\,\alpha \in \Phi\,\beta \Rightarrow \alpha < \beta$: If $\Phi\,\alpha \in \Phi\,\beta$, then there exists some $b : \beta$ with $\Phi\,\alpha = \Phi(\beta \downarrow b)$, and hence $\alpha < \beta$ by injectivity of Φ.

e) $\Phi\,\alpha \subseteq \Phi\,\beta \Rightarrow \alpha \leq \beta$: Suppose $\Phi\,\alpha \subseteq \Phi\,\beta$. Then for every $a : \alpha$, there exists some $b : \beta$ with $\Phi(\alpha \downarrow a) = \Phi(\beta \downarrow b)$. Injectivity of Φ and Proposition 9 imply $\alpha \leq \beta$. ☐

Lemma 29 (⚙). *The map $\Phi : \mathsf{Ord} \to \mathbb{V}$ factors through the inclusion $\mathbb{V}_{\mathrm{ord}} \hookrightarrow \mathbb{V}$.*

Proof. We first show directly that $\Phi\,\alpha$ is a transitive set for every $\alpha : \mathsf{Ord}$: if we have $x, y : \mathbb{V}$ with $x \in y \in \Phi\,\alpha$, then there exists $a : \alpha$ with $x = \Phi(\alpha \downarrow a)$ and hence $b : \alpha \downarrow a$ with $y = \Phi((\alpha \downarrow a) \downarrow b)$. But $(\alpha \downarrow a) \downarrow b$ and $\alpha \downarrow b$ are equal ordinals by Lemma 11, so $y = \Phi(\alpha \downarrow b)$ and thus $y \in \Phi\,\alpha$, as desired.

Now we prove that $\Phi\,\alpha$ is a set-theoretic ordinal for every $\alpha : \mathsf{Ord}$ by transfinite induction on Ord. We just established that $\Phi\,\alpha$ is a transitive set and if $x \in \Phi\,\alpha$, then $x = \Phi(\alpha \downarrow a)$ for some $a : \alpha$, so that x must be a transitive set by the induction hypothesis. ☐

Thus, one half of the desired isomorphism is given by $\Phi : \mathsf{Ord} \to \mathbb{V}_{\mathrm{ord}}$. We define a map in the other direction now.

Definition 30 (⚙ Ψ). We define $\Psi : \mathbb{V} \to \mathsf{Ord}$ recursively by

$$\Psi(\mathbb{V}\text{-}\mathsf{set}(A, f)) \;:\equiv\; \bigvee_{a:A} (\Psi(f\,a) + \mathbf{1}).$$

This map is well-defined because Ord is a set, and if f and g have equal images then the suprema $\Psi(\mathbb{V}\text{-}\mathsf{set}(A, f))$ and $\Psi(\mathbb{V}\text{-}\mathsf{set}(B, g))$ are seen to coincide.

Remark 31. This function above assigns the *rank* to a set and is well-known in set theory, see for example [2, p. 743] and [8, Def 9.3.4].

Proposition 32 (⚙). *When restricted to $\mathbb{V}_{\mathrm{ord}}$, the map Ψ is a section of Φ, i.e., for $x : \mathbb{V}_{\mathrm{ord}}$, we have $\Phi(\Psi\,x) = x$.*

Proof. Since we are proving a proposition, the induction principle of $\mathbb{V}$ implies that it suffices to prove that for every $A : \mathcal{U}$ and $f : A \to \mathbb{V}$ such that $\mathbb{V}\text{-}\mathsf{set}(A, f)$ is a set-theoretic ordinal, the equality $\Phi(\Psi(\mathbb{V}\text{-}\mathsf{set}(A, f))) = \mathbb{V}\text{-}\mathsf{set}(A, f)$ holds, assuming the *induction hypothesis*: $\Phi(\Psi(f\,a)) = f\,a$ holds for all $a : A$. (Note that every $f\,a$ is a set-theoretic ordinal if $\mathbb{V}\text{-}\mathsf{set}(A, f)$ is.) We compute that

$$\Phi(\Psi(\mathbb{V}\text{-}\mathsf{set}(A, f))) = \mathbb{V}\text{-}\mathsf{set}(s, \lambda\, y \,.\, \Phi(s \downarrow y)),$$

where $s :\equiv \bigvee_{a:A}(\Psi(f\,a) + \mathbf{1})$. We now use the second constructor of $\mathbb{V}$ to prove that $\mathbb{V}\text{-}\mathsf{set}(s, \lambda\, y \,.\, \Phi(s \downarrow y))$ is equal to $\mathbb{V}\text{-}\mathsf{set}(A, f)$, i.e., we show that $\lambda\, y \,.\, \Phi(s \downarrow y)$ and f have the same image. It is convenient to set up some notation: we write c_a for $\Psi(f\,a) + \mathbf{1}$.

In one direction, suppose that $a : A$, then

$$f\,a = \Phi(\Psi(f\,a)) = \Phi(c_a \downarrow \mathsf{inr}\,\star) = \Phi(s \downarrow [a, \mathsf{inr}\,\star]),$$

where the first equality holds by induction hypothesis and the second and third by Lemmas 13 and 15, respectively.

Conversely, if we have $y : s$, then by Lemma 15 there exist some $a : A$ and $w : c_a$ such that $s \downarrow y = c_a \downarrow w$. There are now two cases: either $w = \mathsf{inr}\,\star$ or $w = \mathsf{inl}\,x$ with $x : \Psi(f\,a)$. If $w = \mathsf{inr}\,\star$, then, as before,

$$\Phi(s \downarrow y) = \Phi(c_a \downarrow \mathsf{inr}\,\star) = \Phi(\Psi(f\,a)) = f\,a.$$

So suppose that $w = \mathsf{inl}\,x$ with $x : \Psi(f\,a)$. It is here that we use our assumption that $\mathbb{V}\text{-}\mathsf{set}(A, f)$ is a set-theoretic ordinal. Indeed, since $\Psi(f\,a) \downarrow x$ is an initial segment of $\Psi(f\,a)$, we have $\Phi(\Psi(f\,a) \downarrow x) \in \Phi(\Psi(f\,a)) = f\,a$ by Lemma 28 and the induction hypothesis. But $f\,a \in \mathbb{V}\text{-}\mathsf{set}(A, f)$ and the latter is a transitive set, so $\Phi(\Psi(f\,a) \downarrow x) \in \mathbb{V}\text{-}\mathsf{set}(A, f)$. By definition of set membership, this means that there exists some $a' : A$ with $\Phi(\Psi(f\,a) \downarrow x) = f(a')$. Finally,

$$\Phi(s \downarrow y) = \Phi(c_a \downarrow \mathsf{inl}\,x) = \Phi(\Psi(f\,a) \downarrow x) = f(a'),$$

where the second equality holds by Lemma 13. Hence, f and $\Phi(s \downarrow -)$ have the same image, completing the proof. ☐

We are now ready to prove the main theorem of Section II: the type-theoretic and set-theoretic ordinals coincide.

Theorem 33 (⚙). *The ordinals Ord and $\mathbb{V}_{\mathrm{ord}}$ are isomorphic (as type-theoretic ordinals). Hence, by univalence, they are equal.*

Proof. By Lemma 29 we have a map $\Phi : \mathsf{Ord} \to \mathbb{V}_{\mathrm{ord}}$. Moreover, it is an injection by Lemma 28 and a (split) surjection by Proposition 32. Hence, Φ is a bijection. But Lemma 28 tells us that it also preserves and reflects the strict orders, so it is an isomorphism of ordinals by Lemma 10. ☐

E. Revisiting the rank of a set

The recursive nature of the map Ψ from Definition 30 that computes the rank of a set in $\mathbb{V}$ is convenient for proving properties by induction. It is possible, however, to give a conceptually simpler and non-recursive description, although this requires paying close attention to size issues.

Definition 34 (⚙ Type of elements, $\mathbb{T}\,x$). Given an element $x : \mathbb{V}$, we write $\mathbb{T}\,x$ for its type of elements, i.e.,

$$\mathbb{T}\,x \;:\equiv\; \Sigma(y : \mathbb{V}).y \in x.$$

Proposition 35 (⚙). *If $x : \mathbb{V}$ is a set-theoretic ordinal, then $\mathbb{T}\,x$ ordered by $\in$ is a type-theoretic ordinal.*

Proof. Since being a set-theoretic ordinal is hereditary, we have $\mathbb{T}\,x = \Sigma(y : \mathbb{V}_{\mathrm{ord}}).y \in x$, so that the former inherits the ordinal structure from $\mathbb{V}_{\mathrm{ord}}$. ☐

It now becomes important to pay close attention to type universe parameters, so we will annotate them with subscripts. Notice that the $\mathbb{T}$-operation does not define a map $\mathbb{V}_{\mathcal{U}} \to \mathsf{Ord}_{\mathcal{U}}$ like Ψ does, but rather a map $\mathbb{V}_{\mathcal{U}} \to \mathsf{Ord}_{\mathcal{U}^+}$, because the cumulative hierarchy $\mathbb{V}_{\mathcal{U}}$ with respect to the universe $\mathcal{U}$ is itself a type in the next universe $\mathcal{U}^+$.

Still, we will prove that $\Psi(x)$ and $\mathbb{T}\,x$ are *isomorphic* ordinals for every set-theoretic ordinal $x : \mathbb{V}_{\mathcal{U}}$, even though they cannot be equal due to their different sizes. However, we can do a bit better by observing, as in the HoTT book [9, Lem 10.5.5], that the cumulative hierarchy is locally small (in the sense of Rijke [28]), meaning its identity types are $\mathcal{U}$-valued up to equivalence. Then we observe that $\mathbb{T}(\mathbb{V}\text{-}\mathsf{set}(A, f))$ is equal to the image of f, which is equivalent to a type in $\mathcal{U}$ thanks to the fact that $\mathbb{V}$ is a locally small set. This general fact on small images of maps into locally small sets is a "set replacement principle", discussed by Rijke [28] and de Jong and Escardó [26]. Specifically, the image of f is equivalent to the set quotient $A/{\sim}$, where A is the domain of f and $\sim$ relates two elements if f identifies them. We then make the quotient $A/{\sim}$ into an ordinal by defining $[a] < [b]$ as $f\,a \in f\,b$. Finally, we can resize $A/{\sim}$ to an ordinal in $\mathcal{U}$ by using that $\mathbb{V}$ is locally small and by employing a $\mathcal{U}$-valued membership relation, as explained below.

We stress that none of the above constructions rely on propositional resizing principles.

1) The cumulative hierarchy is locally small: We again follow [9, §10.5] in defining a recursive bisimulation relation that makes $\mathbb{V}$ a locally small type.

Definition 36 (⚙ Bisimulation [9, Def 10.5.4])**.** The *bisimulation* relation $\approx\, : \mathbb{V}_{\mathcal{U}} \to \mathbb{V}_{\mathcal{U}} \to \mathsf{Prop}_{\mathcal{U}}$ is inductively defined by

$$\begin{aligned}\mathbb{V}\text{-}\mathsf{set}(A, f) \approx \mathbb{V}\text{-}\mathsf{set}(B, g) :\equiv{}& (\forall(a : A).\exists(b : B).f\,a \approx g\,b)\\ &\times (\forall(b : B).\exists(a : A).g\,b \approx f\,a).\end{aligned}$$

Lemma 37 (⚙ Lemma 10.5.5 of [9])**.** *For every $x, y : \mathbb{V}_{\mathcal{U}}$, we have an equivalence of propositions $(x = y) \simeq (x \approx y)$.* □

Hence, the bisimulation relation captures equality on $\mathbb{V}$, but has the advantage that it has values in $\mathcal{U}$ rather than $\mathcal{U}^+$. This also allows us to define a $\mathcal{U}$-valued membership relation.

Definition 38 (⚙ $\in_{\mathcal{U}}$)**.** Define $\in_{\mathcal{U}}\, : \mathbb{V}_{\mathcal{U}} \to \mathbb{V}_{\mathcal{U}} \to \mathsf{Prop}_{\mathcal{U}}$ inductively by $x \in_{\mathcal{U}} \mathbb{V}\text{-}\mathsf{set}(A, f) :\equiv \exists(a : A).f\,a \approx x$.

Lemma 39 (⚙)**.** *For every $x, y : \mathbb{V}_{\mathcal{U}}$, we have an equivalence of propositions $(x \in y) \simeq (x \in_{\mathcal{U}} y)$.*

Proof. By $\mathbb{V}$-induction and Lemma 37. □

2) The set quotients: Throughout this subsection, assume that we are given $A : \mathcal{U}$ and $f : A \to \mathbb{V}$ such that $\mathbb{V}\text{-}\mathsf{set}(A, f)$ is a set-theoretic ordinal. We show that the type of elements of $\mathbb{V}\text{-}\mathsf{set}(A, f)$ is given by a suitable quotient of A. This simple quotient can capture all the elements of $\mathbb{V}\text{-}\mathsf{set}(A, f)$ precisely because $\mathbb{V}\text{-}\mathsf{set}(A, f)$ is hereditarily transitive.

Definition 40 (⚙)**.** We write $A/{\sim}$ for the set quotient of A by the $\mathcal{U}^+$-valued equivalence relation $a \sim b :\equiv (f\,a = f\,b)$. Similarly, we write $A/{\sim_{\mathcal{U}}}$ for the set quotient of A by the $\mathcal{U}$-valued equivalence relation given by $a \sim_{\mathcal{U}} b :\equiv (f\,a \approx f\,b)$.

The important thing to note in the above definition is that $A/{\sim} : \mathcal{U}^+$, while $A/{\sim_{\mathcal{U}}} : \mathcal{U}$. It is easy to prove that the latter is a *small replacement* of the former:

Lemma 41 (⚙)**.** *Writing* $\mathrm{im}\, f :\equiv \Sigma(v : \mathbb{V}).\exists(a : A).f\,a = v$ *for the image of f, we have $(A/{\sim_{\mathcal{U}}}) \simeq (A/{\sim}) = (\mathrm{im}\, f)$.* □

We define relations on the quotients that make them into large and small type-theoretic ordinals, respectively.

Definition 42 (⚙)**.** We define a $\mathcal{U}^+$-valued binary relation $\prec$ on $A/{\sim}$ by $[a] \prec [b] :\equiv (f\,a \in f\,b)$. Similarly, we define a $\mathcal{U}$-valued relation $\prec_{\mathcal{U}}$ on $A/{\sim_{\mathcal{U}}}$ by $[a] \prec_{\mathcal{U}} [b] :\equiv (f\,a \in_{\mathcal{U}} f\,b)$.

Proposition 43 (⚙)**.** *The relation $\prec$ makes $A/{\sim}$ into an ordinal in $\mathcal{U}^+$, and $\prec_{\mathcal{U}}$ makes $A/{\sim_{\mathcal{U}}}$ into an ordinal in $\mathcal{U}$.*

Proof. For transitivity, it suffices to prove that $[a] \prec [b]$ and $[b] \prec [c]$ together imply $[a] \prec [c]$ for all $a, b, c : A$. But this follows from the fact that $f(c)$ is a transitive set which holds because it is an element of the set-theoretic ordinal $\mathbb{V}\text{-}\mathsf{set}(A, f)$. For extensionality, assume that $x \prec [a] \leftrightarrow x \prec [b]$ for every $x : A/{\sim}$. We have to prove that $[a] = [b]$, i.e., that $f\,a = f\,b$. We show that $f\,a \subseteq f\,b$ and note that the reverse inclusion is proved similarly. Suppose that we have $x : \mathbb{V}$ with $x \in f\,a$. Then because $f\,a$ is a member of the transitive set $\mathbb{V}\text{-}\mathsf{set}(A, f)$, we get $x \in \mathbb{V}\text{-}\mathsf{set}(A, f)$. Hence, there exists some $c : A$ with $f(c) = x$. But then $f(c) = x \in f\,a$, and so $[c] \prec [a]$. Hence, $[c] \prec [b]$ by assumption, and therefore, $x = f(c) \in f\,b$, as desired. Further, to see that every element of $A/{\sim}$ is accessible, we prove the following statement by transfinite induction in the ordinal $(\mathbb{V}, \in)$: for every $x : \mathbb{V}$ and every $a : A$, if $f\,a = x$, then $[a]$ is accessible. So let $x : \mathbb{V}$ and $a : A$ be such that $f\,a = x$ and assume the induction hypothesis that for every $y \in x$ and $b : A$, if $f\,b = y$, then $[b]$ is accessible. For accessibility of $[a]$, it suffices to prove that every $[b]$ is accessible whenever we have $b : A$ with $[b] \prec [a]$. But given such a $b : A$ we have $f\,b \in f\,a = x$, and hence accessibility of $[b]$ by induction hypothesis. The claim about $(A/{\sim_{\mathcal{U}}}, \prec_{\mathcal{U}})$ is proved analogously. □

Finally, the quotient is equal to the type of elements:

Lemma 44 (⚙)**.** *For every $A : \mathcal{U}$ and $f : A \to \mathbb{V}$, the ordinals $(A/{\sim}, \prec)$ and $(\mathbb{T}(\mathbb{V}\text{-}\mathsf{set}(A, f)), \in)$ are equal.*

Proof. By Lemma 10, we only need to verify that the bijection from Lemma 41 preserves and reflects the order, but this is clear because $[a] \prec [b]$ holds exactly when $f\,a \in f\,b$. □

3) Alternative descriptions of the rank: We are now ready to prove the main result of this subsection: we show that the rank of $\mathbb{V}\text{-}\mathsf{set}(A, f)$, as recursively computed by Ψ, is equal to the quotient $A/{\sim_{\mathcal{U}}}$, thus providing a simpler non-recursive description of its rank.

Theorem 45 (⚙)**.** *The ordinals $\Psi(\mathbb{V}\text{-}\mathsf{set}(A, f))$ and $A/{\sim_{\mathcal{U}}}$ are equal.*

Proof. Because Φ is injective with inverse Ψ (Lemma 28 and Proposition 32), it suffices to show that

$$\Phi(A/{\sim_{\mathcal{U}}}) = \mathbb{V}\text{-set}(A, f).$$

By definition of Φ and equality on $\mathbb{V}$, it is enough to prove

$$\Phi(A/{\sim_{\mathcal{U}}} \downarrow [a]) = f\, a$$

for every $a : A$. We slightly generalize this statement so that it becomes amenable to a proof by transfinite induction on $A/{\sim_{\mathcal{U}}}$. Namely, we show that for every $a' : A/{\sim_{\mathcal{U}}}$ and every $a : A$, if $a' = [a]$, then $\Phi(A/{\sim_{\mathcal{U}}} \downarrow [a]) = f\, a$ holds. So suppose that we have $a : A$. We first show that $f\, a \subseteq \Phi(A/{\sim_{\mathcal{U}}} \downarrow [a])$. Now if $x \in f\, a$, then there exists some $b : A$ with $x = f\, b$, because $f\, a$ is a member of the transitive set $\mathbb{V}\text{-set}(A, f)$. But then $f\, b = x \in f\, a$, so $[b] \prec [a]$ and hence $\Phi(A/{\sim_{\mathcal{U}}} \downarrow [b]) = f\, b = x$ by the induction hypothesis. Further, $\Phi(A/{\sim_{\mathcal{U}}} \downarrow [b])$ is an element of $\Phi(A/{\sim_{\mathcal{U}}} \downarrow [a])$, because $[b] \prec [a]$ and $(A/{\sim_{\mathcal{U}}} \downarrow [a]) \downarrow [b] = A/{\sim_{\mathcal{U}}} \downarrow [b]$ by Lemma 11. Hence, $x = \Phi(A/{\sim_{\mathcal{U}}} \downarrow [b]) \in \Phi(A/{\sim_{\mathcal{U}}} \downarrow [a])$, as desired. For the other inclusion, suppose that $x \in \Phi(A/{\sim_{\mathcal{U}}} \downarrow [a])$. By another application of Lemma 11, we see that there exists some $b : A$ such that $[b] \prec [a]$ and $x = \Phi(A/{\sim_{\mathcal{U}}} \downarrow [b])$. Then $x = \Phi(A/{\sim_{\mathcal{U}}} \downarrow [b]) = f\, b$ by the induction hypothesis, but also $[b] \prec [a]$, so that $x = f\, b \in f\, a$, as we wished to show. □

Corollary 46 **().** *For every $x : \mathbb{V}_{\mathcal{U}}$, the ordinals $\Psi(x)$ and $\mathbb{T}\, x$ are isomorphic, but not equal, because the latter lives in a larger universe.*

Proof. Since we are proving a proposition, $\mathbb{V}$-induction implies that it is enough to prove that $\Psi(\mathbb{V}\text{-set}(A, f))$ and $\mathbb{T}(\mathbb{V}\text{-set}(A, f))$ are isomorphic ordinals, for every $A : \mathcal{U}$ and $f : A \to \mathbb{V}$. But this holds by the following chain of isomorphisms of ordinals:

$$\begin{aligned}
\Psi(\mathbb{V}\text{-set}(A, f)) &\simeq A/{\sim_{\mathcal{U}}} && \text{(by Theorem 45)}\\
&\simeq A/{\sim} && \text{(by Lemma 41)}\\
&\simeq \mathbb{T}(\mathbb{V}\text{-set}(A, f)) && \text{(by Lemma 44).} \quad \square
\end{aligned}$$

III. Generalizing from ordinals to sets

Since we now understand the subtype of $\mathbb{V}$ that consists of exactly the hereditarily transitive sets, it is a natural goal to characterize the full type $\mathbb{V}$ to complete the square on the left, by generalizing the notion of type-theoretic ordinals. Since arbitrary $\mathbb{V}$-sets are not necessarily transitive, we certainly need to give up transitivity. However, as discussed in the introduction, doing so and simply considering *extensional wellfounded relations* is insufficient to complete the square. Our solution is to further equip them with *covering markings*. We then develop a generalization of the theory of type-theoretic ordinals that matches $\mathbb{V}$.

$$\begin{tikzcd}
\mathbb{V}_{\mathrm{ord}} \arrow[r, "\simeq"] \arrow[d, hook] & \mathsf{Ord} \arrow[d, hook] \\
\mathbb{V} \arrow[r, "\simeq"] & ?
\end{tikzcd}$$

Giving up transitivity as an assumption, we at times need to consider the transitive and reflexive-transitive closure of a given relation $<$, i.e., the smallest proposition-valued such relations that include $<$. We denote them by $<^+$ and $<^*$ respectively. In type theory, it is standard to implement $<^+$ and $<^*$ using inductive families describing sequences of steps, which then can be propositionally truncated to ensure proof-irrelevance.

A. Mewos: marked extensional wellfounded order relations

We start by defining the generalization of type-theoretic ordinals that we need to complete the above square.

Definition 47 **(Mewo).** A *marked extensional wellfounded order* (*mewo*) is a triple $(X, <, \mathrm{m})$, where X is a type, $<$ is a binary relation on X that is extensional, wellfounded, and valued in propositions, i.e., $(X, <)$ is an ensemble in the sense of Taylor [6], and $\mathrm{m} : X \to \mathsf{Prop}$ is a prop-valued predicate on X (called a *marking*).

We say that $x : X$ is *marked* if $\mathrm{m}(x)$, and *covered* if there exists a marked x_0 such that $x <^* x_0$. A *covered mewo* is a mewo where every element is covered.

We write MEWO for the type of mewos, and $\mathsf{MEWO}_{\mathsf{cov}}$ for its subtype of covered mewos. From now on, we keep the order and the marking implicit, overloading the symbols $<$ and m whenever required, and denote a mewo only by its carrier X. The *subtype of marked elements* of X is the total space of m,

$$\mathrm{M}_X :\equiv \Sigma(x : X).\, \mathrm{m}(x)$$

and we implicitly apply the first projection to treat elements of M_X as elements of X. With this convention, a mewo is covered if we can show

$$\forall(x : X).\exists(x_0 : \mathrm{M}_X).x <^* x_0.$$

Remark 48. As for ordinals, the extensionality of the relation $<$ implies that X is necessarily a set. Further note that, by univalence, an equality $X = Y$ between mewos is an equivalence $e : X \simeq Y$ that preserves and reflects both order and marking, i.e., satisfies $(x_1 < x_2) \leftrightarrow (e\, x_1 < e\, x_2)$ and $\mathrm{m}(x) \leftrightarrow \mathrm{m}(e\, x)$. The identical characterization holds if the mewos in consideration are covered since coveredness is a propositional property.

Our second main result is that $\mathsf{MEWO}_{\mathsf{cov}}$ is the missing corner in the discussed square as shown on the right, cf. Theorem 76. This means that a covered mewo simultaneously behaves like a generalized type-theoretic ordinal *and* a set in $\mathbb{V}$. The first connection is easy to make precise:

$$\begin{tikzcd}
\mathbb{V}_{\mathrm{ord}} \arrow[r, "\simeq"] \arrow[d, hook] & \mathsf{Ord} \arrow[d, hook] \\
\mathbb{V} \arrow[r, "\simeq"] & \mathsf{MEWO}_{\mathsf{cov}}
\end{tikzcd} \tag{2}$$

Example 49 (Ordinals as covered mewos)**.** Given a type-theoretic ordinal, we get a covered mewo by forgetting the transitivity of the order and marking everything.

We will later see that, if we view a mewo as a $\mathbb{V}$-set, it is exactly the marked elements that become elements of the set (while the unmarked ones become elements of elements of …). Therefore, for type-theoretic ordinals, everything will be an element of the corresponding $\mathbb{V}$-set. This is already determined by the top horizontal map in (2), i.e., the map Φ from Definition 27. Based on this observation, Example 49 guides and motivates much of our theory of mewos.

B. Order relations between mewos

The main concepts that we need to generalize from type-theoretic ordinals are the relations $\leq$ and $<$ between mewos. The above square (2) means that these relations necessarily need to correspond to the relations $\subseteq$ and $\in$ between $\mathbb{V}$-sets. To begin, the concept of a *simulation* between type-theoretic ordinals is straightforward to generalize to mewos:

Definition 50 (Simulation, $\leq$)**.** Given mewos X and Y, a function $f : X \to Y$ is a *simulation* if it fulfills the following properties:

(i) it preserves the markings: $\forall x.\, \mathrm{m}(x) \to \mathrm{m}(f\,x)$;
(ii) it is monotone: $x_1 < x_2 \to f\,x_1 < f\,x_2$;
(iii) it has the initial segment property, i.e., its image is downwards closed in a strong sense:

$$\forall x_2.\forall(y < f\,x_2).\exists(x_1 < x_2).f\,x_1 = y. \qquad (3)$$

We write $X \leq Y$ for the type of simulations.

An example of a function that fails to be a simulation precisely because it does not preserve markings is the identity function on the order $0 < 1$, if we mark both 0 and 1 in the domain, but only 1 in the codomain. In set theory, this corresponds to the fact that $\{\emptyset, \{\emptyset\}\}$ is not a subset of $\{\{\emptyset\}\}$.

Lemma 51 ()**.** *For mewos X, Y, and Z, we have the following properties of simulations:*

(i) The underlying function f of a simulation $X \leq Y$ is injective: $f\,x_1 = f\,x_2$ implies $x_1 = x_2$.
(ii) There is at most one simulation between any two mewos, i.e., $X \leq Y$ is a proposition.
(iii) Simulations are antisymmetric, i.e.

$$X \leq Y \to Y \leq X \to X = Y.$$

(iv) We have the trivial simulation $X \leq X$ and simulations can be composed, i.e.

$$X \leq Y \to Y \leq Z \to X \leq Z.$$

(v) $X = Y$ is a proposition, i.e., MEWO *is a set.*
(vi) In the property (3) of the definition of a simulation, the symbol $\exists$ can equivalently be replaced by Σ.

Proof. The arguments are copies of the proofs for type-theoretic ordinals (cf. [9, Lem 10.3.12, Cor 10.3.13&15, Lem 10.3.16]). □

Definition 52 (Initial segment, $X \downarrow^+ x$)**.** If X is a mewo and $x : X$, then the *initial segment* $X \downarrow^+ x$ is the mewo of elements transitively below x, with the canonical inherited order. The marked elements are the immediate predecessors of x.

That is, in detail, the carrier of $X \downarrow^+ x$ is given by the type $\Sigma(x' : X).(x' <^+ x)$, the order by $(x_1, s) < (x_2, t) :\equiv (x_1 < x_2)$, and the marking by $\mathrm{m}(x_1, s) :\equiv (x_1 < x)$.

Lemma 53 ()**.** *The mewo $(X \downarrow^+ x)$ is covered for every x.*

Proof. Given (x_1, s) with $s : (x_1 <^+ x)$, we wish to show that there exists a marked p such that $x_1 <^* p$. Since we are proving a proposition, we may assume that s is a sequence $x_1 < \ldots < x_n < x$, and x_n is marked by definition. □

For type-theoretic ordinals, a bounded simulation is a simulation whose domain is equivalent to an initial segment under a certain element of its codomain. For mewos, we ensure that the latter property is true by definition. Some caveats and subtleties are discussed in Section III-C below.

Definition 54 (Bounded simulation, $<$)**.** A *bounded simulation* between mewos X and Y is a pair (y, e), where $y : M_Y$ is a marked element in Y and $e : X \simeq (Y \downarrow^+ y)$ an equivalence of mewos. We write $X < Y$ for the type of such pairs.

It is important that the above definition specifies that y is marked, in line with our earlier explanation that exactly the marked elements of a mewo correspond to elements of a $\mathbb{V}$-set. We will later (Corollary 59) see that the type $X < Y$ is a proposition. For now, let us observe the following:

Lemma 55 ()**.** *The relation $<$ is wellfounded on* MEWO *and* $\mathsf{MEWO_{cov}}$.

Proof. Since $\mathsf{MEWO_{cov}} \hookrightarrow \mathsf{MEWO}$ is order-preserving, it suffices to check that MEWO is wellfounded. Thus, we need to show that every mewo X is accessible, i.e., that all its predecessors are accessible. By definition, every predecessor is of the form $X \downarrow^+ x_0$ for some marked $x_0 : X$.

Exploiting that the order on X itself is wellfounded, we show by transfinite induction on x the more general statement that every $X \downarrow^+ x$ is accessible, no matter whether x is marked. Thus, assume that, for all $z < x$, we have that $X \downarrow^+ z$ is accessible. We need to prove that all predecessors of $X \downarrow^+ x$, i.e., all $(X \downarrow^+ x) \downarrow^+ (x_1, p)$, are accessible. An adaption of Lemma 11 for mewos shows that this mewo is equal to $X \downarrow^+ x_1$, which is accessible by the induction hypothesis. □

C. Subtleties caused by markings

Observing how bounded simulations interact with other (possibly bounded) simulations reveals the complete change of view we are forced to make when generalizing from ordinals to mewos. This is indeed intended since we claim (and prove) that $\leq$ and $<$ correspond to $\subseteq$ and $\in$, and for arbitrary sets, the latter relations fail to have many properties that one might associate with the former relations.

The first point is that $X < Y$ generally does not imply $X \leq Y$. A bounded simulation $X < Y$ gives rise to a function $f : X \to Y$ via the composition of the function underlying $e : X = (Y \downarrow^+ y)$ and the first projection $(Y \downarrow^+ y) \to Y$. However, the first projection is in general not a simulation as it may not preserve markings. A counter-example is the covered mewo $\circ \leftarrow \bullet$, i.e., the mewo with two comparable elements, the larger of which is marked (denoted by $\bullet$), while the smaller is not (denoted by $\circ$). Since $(\circ \leftarrow \bullet) \downarrow^+ \bullet$ is, by definition, simply $\bullet$, there is a bounded simulation from $\bullet$ to

$\circ \leftarrow \bullet$. However, there is no simulation as the marking is not preserved. The crux here is that the operation $\downarrow^+$ changes the marking. The translation to the language of sets is that $\{\emptyset\}$ is an element, but not a subset, of $\{\{\emptyset\}\}$.

Secondly, the $<$ relation on mewos is not transitive. The principle that "an initial segment of an initial segment is an initial segment" does not hold. A simple counter-example is the empty mewo $\emptyset$ together with $\bullet$ and $\circ \leftarrow \bullet$. We have bounded simulations $\emptyset < \bullet < (\circ \leftarrow \bullet)$, but no bounded simulation $\emptyset < (\circ \leftarrow \bullet)$. In this case, the translation is that $\emptyset$ is an element of $\{\emptyset\}$, which itself is an element of $\{\{\emptyset\}\}$; the latter however does not have $\emptyset$ as an element.

For technical reasons, it is occasionally useful to use mewos that ensure that the discussed properties do hold. This can be achieved by changing the marking to the trivial one:

Definition 56 (⚙ Trivializing the marking, $\overline{X}$)**.** If X is a mewo, we write $\overline{X}$ for the mewo that has the same carrier and order as X, but where every element is marked.

In the language of $\mathbb{V}$-sets, $\overline{X}$ is the union of all the sets represented by elements (of elements of elements $\ldots$) of X. Note that $\overline{X}$ is still not transitive and thus not a type-theoretic ordinal. Nevertheless, this operation allows us to recover several important properties of type-theoretic ordinals:

Lemma 57 (⚙)**.** *For given mewos X, Y, and Z, we have:*

(i) for every $x : X$, the first projection $(X \downarrow^+ x) \to \overline{X}$ is a simulation;
(ii) $X < Y \to X \leq \overline{Y}$;
(iii) $X < Y \to Y < Z \to X < \overline{Z}$.

Proof. As the conditions involving markings now are vacuously true, the arguments for type-theoretic ordinals apply. $\square$

As a demonstration of how this is useful, we can show the following technical lemma:

Lemma 58 (⚙)**.** *Given a mewo X, the function $x \mapsto X \downarrow^+ x$ is injective: $(X \downarrow^+ x_1) = (X \downarrow^+ x_2)$ implies $x_1 = x_2$.*

Proof. We show that $f : (X \downarrow^+ x_1) \leq (X \downarrow^+ x_2)$ implies that any predecessor of x_1 is also a predecessor of x_2; extensionality of X then gives the claimed injectivity. To do this, let us consider the following diagram:

$$\begin{tikzcd} X \downarrow^+ x_1 \arrow[rr, "f"] \arrow[dr, "\mathsf{fst}"'] & & X \downarrow^+ x_2 \arrow[dl, "\mathsf{fst}"] \\ & \overline{X} & \end{tikzcd}$$

All maps are simulations and, by uniqueness of simulations (Lemma 51), the diagram necessarily commutes. Given a predecessor $x < x_1$, it is marked in $X \downarrow^+ x_1$ by construction, and since f preserves markings, $f\,x$ is marked as well, i.e., we have $f\,x < x_2$. But since the diagram commutes, we have $f\,x = x$ as elements of X. $\square$

A consequence is that bounded simulations are unique:

Corollary 59 (⚙)**.** *For mewos X and Y, the type $X < Y$ of bounded simulations is a proposition.*

Proof. By definition, $X < Y \equiv \Sigma(y : Y).(X = (Y \downarrow^+ y))$. Assume $(y, p), (y', q) : X < Y$. By the above lemma, we then have $y = y'$ since $(Y \downarrow^+ y) = X = (Y \downarrow^+ y')$, and $p = q$ since MEWO is a set. Hence $(y, p) = (y', q)$, as desired. $\square$

D. Simulations and coverings

As we have seen, bounded simulations and simulations are tricky to compare. The first step towards improving this situation is to characterize a simulation via initial segments:

Lemma 60 (⚙)**.** *Let X and Y be mewos. Further, let $f : X \to Y$ be a function between the carriers that preserves markings, i.e., such that $\mathrm{m}(x) \to \mathrm{m}(f\,x)$. The following are equivalent:*

(i) f is a simulation.
(ii) for all $x : X$, we have $(X \downarrow^+ x) = (Y \downarrow^+ (f\,x))$.

Proof. $(i) \Rightarrow (ii)$: An equality of mewos is a surjective simulation that preserves and reflects the markings. The simulation f is monotone and thus can be restricted to a simulation $\bar{f} : X \downarrow^+ x \leq Y \downarrow^+ (f\,x)$. Monotonicity of f guarantees that markings are preserved, while the initial segment property ensures that markings are reflected. Finally, by induction on the number of steps, the initial segment property for $<$ can be extended to $<^+$; hence every y in $Y \downarrow^+ (f\,x)$ has a preimage.

$(ii) \Rightarrow (i)$: Assume that, for every x, we have an equality $e_x : X \downarrow^+ x = Y \downarrow^+ (f\,x)$ of mewos.

It is a standard result that the transitive closure of a wellfounded relation is wellfounded. Using this we show, by transfinite induction on x, that f is a simulation at point x:

- for $x_1 < x$ we have $f\,x_1 < f\,x$;
- for $y_1 < f\,x$, there is $x_1 < x$ such that $f\,x_1 = y_1$.

The induction hypothesis states that f is a simulation at every point z with $z <^+ x$ or, in other words, that the composition $(X \downarrow^+ x) \xrightarrow{\mathsf{fst}} \overline{X} \xrightarrow{f} \overline{Y}$ is a simulation (cf. Definition 56). Therefore, the diagram

$$\begin{tikzcd} X \downarrow^+ x \arrow[r, "e_x"] \arrow[d] & Y \downarrow^+ (f\,x) \arrow[d] \\ \overline{X} \arrow[r, "f"] & \overline{Y} \end{tikzcd} \tag{4}$$

commutes by uniqueness of simulations (Lemma 51). We can now easily check that f is a simulation at point x. First, $z < x$ means that z is a marked element in $X \downarrow^+ x$, thus $e_x\,z$ is marked in $Y \downarrow^+ (f\,x)$, translating to $\mathsf{fst}(e_x\,z) < f\,x$, and commutativity of (4) implies $\mathsf{fst}(e_x\,z) = f\,z$. Second, let $y_1 < f\,x$ be given. This means that y_1 is marked in $Y \downarrow^+ (f\,x)$ and we get the marked x_1 as the unique preimage of y_1 under the equivalence e_x. $\square$

While we have seen in Section III-C that $<$ is not transitive and does not necessarily imply $\leq$, we now get the following familiar property:

Corollary 61 (⚙)**.** *For mewos X, Y and Z, we have*

$$X < Y \to Y \leq Z \to X < Z.$$

Proof. We have $X = (Y \downarrow^+ y)$ by assumption and $(Y \downarrow^+ y) = (Z \downarrow^+ f\, y)$ by Lemma 60. ☐

One may view Lemma 60 as stating that a function is a simulation if and only if it behaves like a simulation pointwise (or *locally*). We now consider such functions that are only defined on the marked elements:

Definition 62 (⚙ Partial simulation, $\leq_{\mathrm{M}}$)**.** A *partial simulation* between mewos X and Y is a function $f : \mathrm{M}_X \to \mathrm{M}_Y$ that preserves initial segments,

$$\mathsf{psim}(f) \;:\equiv\; \forall (x : \mathrm{M}_X).(X \downarrow^+ x) = (Y \downarrow^+ f\, x),$$

and we write $X \leq_{\mathrm{M}} Y \;:\equiv\; \Sigma(f : \mathrm{M}_X \to \mathrm{M}_Y).\mathsf{psim}(f)$.

A convenient alternate representation is the following:

Lemma 63 (⚙)**.** *The type of partial simulations $X \leq_{\mathrm{M}} Y$ is equivalent to the type*

$$\forall (x : \mathrm{M}_X).\exists (y : \mathrm{M}_Y).X \downarrow^+ x = Y \downarrow^+ y.$$

and hence a proposition.

Proof. We can calculate

$$\begin{aligned} X \leq_{\mathrm{M}} Y \equiv\;& \Sigma(f : \mathrm{M}_X \to \mathrm{M}_Y).\forall x.(X \downarrow^+ x) = (Y \downarrow^+ f\, x) \\ \simeq\;& \Pi(x : \mathrm{M}_X).\Sigma(y : \mathrm{M}_X).(X \downarrow^+ x) = (Y \downarrow^+ y) \\ \simeq\;& \forall (x : \mathrm{M}_X).\exists (y : \mathrm{M}_X).(X \downarrow^+ x) = (Y \downarrow^+ y), \end{aligned}$$

where the first step is the definition of $\leq_{\mathrm{M}}$, the second is the "untruncated axiom of choice" [9, Thm 2.15.7], and the last step uses that $(Y \downarrow^+ y_1) = (Y \downarrow^+ y_2)$ implies $y_1 = y_2$ by Lemma 58, which means that Σ and $\exists$ are equivalent. ☐

A mewo can have the property that its marking alone already fully determines how it maps into other mewos. The notions introduced above allow us to make this precise:

Definition 64 (⚙ Principality)**.** The marking of a mewo X is *principal* if, for all mewos Y, the canonical restriction map $(X \leq Y) \to (X \leq_{\mathrm{M}} Y)$ given by Lemma 60 is an equivalence.

In other words, for any chosen codomain Y, the marking of X is principal if a (necessarily unique) partial simulation out of X already determines a (necessarily unique) simulation out of X. However, being principal is actually simply a "relative" description of the "absolute" property of being covering:

Lemma 65 (⚙)**.** *A marking covers if and only if it is principal.*

Proof. Let m be a marking on a mewo X.

a) ***covers*** $\Rightarrow$ ***principal****:* Assume we have a partial simulation $f : X \leq_{\mathrm{M}} Y$. For a given $x : X$, we need to find a (necessarily unique) $y : Y$ such that $X \downarrow^+ x = Y \downarrow^+ y$. By the covering property, there exists $x_0 : \mathrm{M}_X$ with $p : x <^* x_0$. By analyzing p, we get either $x = x_0$, in which case the goal is given by the partial simulation, or $x <^+ x_0$. In the latter case, we get $e : (X \downarrow^+ x_0) = (Y \downarrow^+ f\, x_0)$ from the partial simulation. Applying the function underlying e on x, we generate an element $y : Y$ that satisfies the required property. If x is marked, then the (unique) y that we find is necessarily equal to the one given by the partial simulation, which is marked by assumption.

b) ***principal*** $\Rightarrow$ ***covers****:* Assume m is principal. Let $\widehat{X}$ be the mewo of all elements covered by M_X, defined as

$$\widehat{X} \;:\equiv\; \Sigma(x : X).\exists (x_0 : \mathrm{M}_X).(x <^* x_0),$$

with order and marking inherited from X. We have $\widehat{X} \leq X$ by projection. We also have $X \leq_{\mathrm{M}} \widehat{X}$ by definition and thus $X \leq \widehat{X}$ by principality, meaning that the two mewos are equal by antisymmetry. In other words, m covers all of X. ☐

We have seen in Lemma 55 that $<$ is wellfounded on MEWO and $\mathsf{MEWO}_{\mathsf{cov}}$. The observation that principality and covering coincide allows us to show that, in the latter case, the order is also extensional:

Theorem 66 (⚙)**.** *The structure* $(\mathsf{MEWO}_{\mathsf{cov}}, <)$ *is an extensional wellfounded order.*

Proof. Wellfoundedness has been established in Lemma 55. Extensionality follows antisymmetry (Lemma 51) as soon as we can show that

$$\forall (Z : \mathsf{MEWO}_{\mathsf{cov}}).(Z < X) \to (Z < Y) \tag{5}$$

implies $X \leq Y$. Thus, let us prove this property.

Let X and Y be covered mewos. By principality and Lemma 63, we need to show that for every $x : \mathrm{M}_X$ there exists some $y : \mathrm{M}_Y$ with $X \downarrow^+ x = Y \downarrow^+ y$. By definition, the predecessors of X are exactly the mewos of the form $X \downarrow^+ x$ for marked x, so that this formula is equivalent to the assumption (5). ☐

In contrast, the relation $<$ is clearly not extensional on MEWO, as there are many different mewos without predecessors, namely exactly those with completely empty markings.

E. Constructions on mewos

Recall the rank function $\Psi : \mathbb{V} \to \mathsf{Ord}$ from Definition 30. Since different $\mathbb{V}$-sets can have the same rank, Ψ is not injective and thus certainly not a simulation. We have seen that we can turn it into a simulation by restricting its domain to $\mathbb{V}_{\mathrm{ord}}$. This is of course not sufficient anymore for our current goal of characterizing *all* of $\mathbb{V}$; instead, we extend the codomain from Ord to MEWO. Doing this requires us to generalize the operations on Ord that we used to construct the rank function. In Definition 12, we recalled the addition of type-theoretic ordinals. While it would be possible to phrase this definition in full generality for mewos, we restrict ourselves for simplicity to the case of interest (the successor), which already contains the crucial ideas.

There is however an important difference. The successor operation for type-theoretic ordinals, if translated to and written in the notation of set theory, maps a set S to $S \cup \{S\}$. This is of course required in order not to leave the realm of *transitive* sets (and orders). For mewos, we need to slightly refine the function so that it corresponds to the (non-transitive) *singleton operation* $S \mapsto \{S\}$.

Definition 67 (⚙ Singleton, $\{X\}$)**.** For a given mewo X, we define the *singleton order* $\{X\}$ to be the marked order with carrier $X + \mathbf{1}$ and the order given as follows:

- $(\mathsf{inl}\, x < \mathsf{inl}\, y)$ if and only if $x < y$;
- $(\mathsf{inl}\, x < \mathsf{inr}\, \star)$ if and only if $\mathrm{m}(x)$;
- $(\mathsf{inr}\, \star < z)$ false for all z.

Finally, we mark the single point $\mathsf{inr}\, \star$.

It is worth pointing out how this *almost* generalizes the successor operation of a type-theoretic ordinal. Since such an ordinal is a completely marked (and transitive) mewo, the second clause above matches exactly the sum operation given in Definition 12 when the second summand is $\mathbf{1}$. However, a faithful generalization of Definition 12 would in the end mark not only $\mathsf{inr}\, \star$, but also all elements that were marked in X.

Another critical point to note is that, for an arbitrary mewo X, the singleton $\{X\}$ need not be a mewo. As an example, consider the mewo $\circ$ with exactly one element, which is unmarked. If we now take its singleton, neither this existing element nor the newly added element has any predecessors. Since they are not equal, extensionality is missing. The obstacle in this example is that the original marking is insufficient. Fortunately, if we start with a *covered* mewo, the successor is not only extensional but also covered again:

Lemma 68 (⚙)**.** *If X is a covered mewo, then so is $\{X\}$.*

Proof. Wellfoundedness is immediate. Regarding extensionality, the interesting case is comparing an element of the form $\mathsf{inl}\, x$ with $\mathsf{inr}\, \star$. It suffices to show that their predecessors are not the same. To do so, observe that x is covered in X, i.e., there exists x_0 with $x <^* x_0$. By construction, x_0 is a predecessor of $\mathsf{inr}\, \star$, while wellfoundedness ensures that it cannot possibly be a predecessor of $\mathsf{inl}\, x$. Coveredness: The element $\mathsf{inr}\, \star$ is marked and thus trivially covered. To see that an arbitrary $\mathsf{inl}\, x$ is covered, note that there exists a marked x_0 with $x <^* x_0$ in X. By construction, we have $\mathsf{inl}\, x_0 < \mathsf{inr}\, \star$, implying that $\mathsf{inl}\, x$ is covered. ☐

The second important construction that we discussed for type-theoretic ordinals is computing suprema (Definition 14). In the case of mewos, the better intuition is to think of *unions*, although the universal property of the supremum is satisfied too, as we will see shortly in Lemma 71.

Definition 69 (⚙ Union of mewos, $\bigcup F$)**.** The *union* $\bigcup F$ of a family of mewos $F : A \to \mathsf{MEWO}$ is defined as follows:

- The carrier is $\Sigma(a : A).F\, a$ quotiented by $\approx$, where we define $(a, x) \approx (b, y)$ to be $(Fa \downarrow^+ x) \simeq (Fb \downarrow^+ y)$ as (covered) mewos;
- and $[a, x] < [b, y]$ is defined as $(F\, a \downarrow^+ x) < (F\, b \downarrow^+ y)$.

We mark $s : \bigcup F$ if and only if there exist $a_0 : A$ and $x_0 : F\, a_0$ with $s = [a_0, x_0]$ such that x_0 is marked in $F\, a_0$.

Remark 70. The explanation given in Remark 6 applies. A priori, the type $(Fa \downarrow^+ x) = (Fb \downarrow^+ y)$ is too large as it lives in a higher universe than the mewos in consideration, which is why we use $\simeq$ in the definition above. The issue is also extensively discussed in Section II-E.

Continuing the observation that mewos act as sets and simultaneously generalize ordinals, we note that the union is also a supremum:

Lemma 71 (⚙)**.** *$\bigcup F$ is the least upper bound of all $F(a)$.*

Proof. $F\, a \leq \bigcup F$ is easy to check. Assume now that we have $F\, a \leq Y$ for every a; we want to prove $\bigcup F \leq Y$. By a calculation analogous to the one in Lemma 63, this goal means we need to show that, for any $z : \bigcup F$, there exists a $y : Y$ such that $(\bigcup F \downarrow^+ z) = (Y \downarrow^+ y)$ and $\mathrm{m}(z) \to \mathrm{m}(y)$. This follows by induction on z, using the uniqueness of y and the assumption for the marking condition. ☐

Lemma 72 (⚙)**.** *If F is a family of covered mewos, then $\bigcup F$ is covered.*

Proof. Let $[a, x]$ be an element of $\bigcup F$; we want to show that $[a, x]$ is covered. By assumption, x is covered in $F\, a$ by some x_0. Since the operation $(F\, a \downarrow^+ \_)$ preserves $<$, it also preserves $<^*$ and we get $F\, a \downarrow^+ x <^* F\, a \downarrow^+ x_0$, giving $[a, x] <^* [a, x_0]$ as required. ☐

Remark 73. Note that, in the situation of Definition 69, we can have $(a, x) \approx (b, y)$ such that x is marked while y is not. The simplest example when this happens is the union of the mewos $\bullet \leftarrow \bullet$ and $\circ \leftarrow \bullet$ (cf. Section III-C for the notation), in set-theoretic notation corresponding to the union of $\{\{\emptyset\}, \emptyset\}$ and $\{\{\emptyset\}\}$. Therefore, it is important to phrase the marking condition in Definition 69 using an *exists* instead of *forall*.

F. $\mathbb{V}$-sets and covered mewos coincide

We are ready to prove our second main theorem, and complete the square (2) by showing that $\mathbb{V}$ and $\mathsf{MEWO}_{\mathsf{cov}}$ coincide. We have seen that the relation $\in$ on $\mathbb{V}$ is wellfounded and extensional. By marking everything, $\mathbb{V}$ is therefore a (large) mewo. Similarly, $\mathsf{MEWO}_{\mathsf{cov}}$ itself is a (large) mewo, using Theorem 66 and total marking. To show that they are equal as such, we construct simulations between them.

Lemma 74 (⚙)**.** *We have a simulation $\mathbb{V} \leq \mathsf{MEWO}_{\mathsf{cov}}$.*

Proof. We define the function $\Psi : \mathbb{V} \to \mathsf{MEWO}_{\mathsf{cov}}$ underlying the simulation by induction on the input by defining

$$\Psi(\mathbb{V}\text{-}\mathsf{set}(A, f)) :\equiv \bigcup_{a:A} (\{\Psi(f\, a)\}) .$$

We need to verify that extensionally equal representatives are mapped to equal mewos, which follows from Lemma 71.

The following observation is helpful to see that Ψ is a simulation: the predecessors (i.e., elements) of $\mathbb{V}\text{-}\mathsf{set}(A, f)$ are exactly the elements of the form $f(a_0)$ for $a_0 : A$, and similarly, via a quick calculation, the predecessors of $\bigcup_{a:A} (\{\Psi(f\, a)\})$ are of the form $\Psi(f\, a_0)$.

Regarding monotonicity, assume we have elements $v_1 \in v_2$ in $\mathbb{V}$. By induction on v_2, we may assume that it is of the form $\mathbb{V}\text{-}\mathsf{set}(A, f)$, and its predecessor v_1 is therefore of the form $f(a_0)$. As we have just seen, we then have the desired $\Psi(f\,a_0) < \Psi(\mathbb{V}\text{-set}(A, f))$. Regarding the second property, we proceed similarly. Given any $y \in \Psi(\mathbb{V}\text{-set}(A, f))$, we know that y is of the form $\Psi(f\,a_0)$, and hence we have $f(a_0) \in \mathbb{V}\text{-set}(A, f)$ as required. ☐

Lemma 75 (⚙). *We have a simulation* $\mathsf{MEWO}_{\mathsf{cov}} \leq \mathbb{V}$.

Proof. We define the function $\Phi : \mathsf{MEWO}_{\mathsf{cov}} \to \mathbb{V}$ by

$$\Phi(X) \;:\equiv\; \mathbb{V}\text{-set}\big(\mathrm{M}_X, \lambda\, x_0\,.\,\Phi(X \downarrow^+ x_0)\big).$$

The predecessors of X are of the form $X \downarrow^+ x_0$ for $x_0 : \mathrm{M}_X$, while the elements of $\Phi(X)$ are $\Phi(X \downarrow^+ x_0)$ for $x_0 : \mathrm{M}_X$. Therefore, the simulation properties for Φ follow analogously to how we derived them in the proof of Lemma 74. ☐

By Lemmas 74 and 75, and antisymmetry, we get:

Theorem 76 (⚙). *The structures* $(\mathbb{V}, \in)$ *and* $(\mathsf{MEWO}_{\mathsf{cov}}, <)$ *are equal as covered mewos.* ☐

IV. Conclusion

Working in homotopy type theory, we have shown that the set-theoretic ordinals in $\mathbb{V}$ coincide with the type-theoretic ordinals. Moreover, by generalizing from type-theoretic ordinals to covered mewos, we have captured *all* sets in $\mathbb{V}$.

A natural question is whether similar results can be obtained by working inside set theory instead. E.g., we expect the type-theoretic ordinals in the cubical sets model [29] of homotopy type theory to coincide with the set-theoretic ordinals, using the Mostowski collapse lemma [30]. Another, orthogonal question is whether the presentation of $\mathbb{V}$ as the type of covered mewos can shed any light on the open problem [9, below Cor 10.5.9] of whether $\mathbb{V}$ satisfies the *strong collection* and *subset collection* axioms of Constructive ZF set theory. Moreover, it would be interesting to study how other, different notions of constructive ordinals, such as Taylor's *plumb ordinals* [6], behave in a type-theoretic setting.

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