Improving classical optimal age-replacement policies for degrading items

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Abstract. We consider items with observable at inspections degradation. Following an inspection, a decision is made whether to replace an item or to continue operation. When degradation is relatively small, it is cost-beneficial to continue operation and postpone the preventive maintenance. The innovative, probabilistically justified replacement policy defines the postponement in this case. It is based on comparison of the observed degradation with the specially defined reference values. Degradation is modeled by the increasing stochastic process, specifically by the Poisson counting process and by the homogeneous gamma process. Detailed illustrative numerical examples describing the main steps of the developed original methodology are provided. They also show that the proposed policy can result in a significant cost reduction and increase in the replacement times.

Keywords: preventive maintenance; inspection; Poison process; gamma process; failure threshold.

Notations

L	lifetime of an item
$F(t)$	distribution function (cdf)
$\bar{F}(t)$	Survival function
$r(t)$	failure rate
C_r	cost of replacement
C_f	cost of failure
$c(T)$	long-run cost rate
$c_1(T)$	improved long-run cost rate
μ_T	mean duration of a cycle
T^*	optimal black-box replacement time

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T_1^*	improved optimal black-box replacement time
λ	rate of the Poisson process
α, β	parameters of the gamma process
W_t	stochastic process of degradation
$F_w(t)$	distribution function for threshold w
$g(T^*, w_i)$	$\frac{\partial}{\partial w_i} \bar{F}_{w_i}(T^*)$
$\bar{F}_w(t \mid T^*)$	survival function of the remaining lifetime at T*
w_i	observed degradation at T*
w_f	maximal degradation at T* for replacement postponement
$T_{w_f}^*$	new replacement time (postponed)
$c_{w_f}(T_{w_f}^*)$	long-run cost rate when all cycles are conditional
$c_{w_f}(T^*, T_{w_f}^*)$	long-run cost rate when not all cycles are conditional

1. Introduction

Defining optimal maintenance schedule and replacement policy is one of the most important practical and theoretical problems in modern reliability engineering. First studies of preventive maintenance go back to the ground-breaking paper by Barlow and Hunter (1960). Since that time, there were thousands of various published papers on the subject and a number of surveys and books entirely devoted to this topic (Nakagawa (2005), Gertsbakh (2005), Wang and Pham (2006), Alaswad and Xiang (2017) and Wang (2002) to name a few). In our paper, for convenience, we will call the approach developed in Barlow and Hunter (1960) and subsequent papers, where the lifetimes of items are described by 'sudden failures' and degradation is unobserved, the *classical approach* (see also, e.g., Finkelstein et al (2016), Hamidi et al (2016), Badia et al (2002), Finkelstein and Gertsbakh (2016)). Some other worth mentioning in our opinion approaches in maintenance modeling can be found in *recent papers* by Guo and Liang (2022), Pedersen and Vatn (2022), Zhang et al (2021), Wang and Miao (2021), Tambi and Kulkarni (2022) among others.

It is well-known that along with sudden failures there are gradual failures of items often caused by internal degradation. These failures are usually modeled as the first passage times for the appropriate stochastic processes. When deterioration is observable, additional information on the items' states is available and can be used in optimal preventive maintenance (PM) decisions (thus, condition-based). See, e.g., some relevant references on condition-based maintenance: Jardine et al (2006), Castanier et al (2003) Grall et al (2003), Liao et al (2006), Cha et al (2017, 2018), Liao et al (2006), Castro et al (2015), Wang et al. (2021a, 2021b) among others. However, the approach developed in our study is different from those reported in the literature on condition-based maintenance, as we use information on degradation obtained only at inspections and then develop original methodology for obtaining the optimal time for the replacement.

Thus, on observing degradation at inspection at the predetermined time, a decision is made whether to replace an item immediately or to continue operation. Obviously, when degradation is relatively small, it can be cost-beneficial to postpone replacement. Motivated by this general reasoning, we propose *a new replacement policy* that suggests this postponement in the probabilistically justified way. There are three main steps that describe our methodology:

1. In accordance with the classical replacement model (Barlow and Hunter (1960), Nakagawa (2005)), we obtain the optimal (black box scenario) replacement time for an item that does not take into account the values of degradation that could be observed at inspections.
2. For the inspections executed at the black-box optimal replacement times, we obtain in the innovative way the maximal level of degradation that justifies possible postponement of an actual replacement when the observed degradation is smaller than this value.
3. If the deterioration observed at the black-box optimal replacement time is smaller than the defined level, then replacement is postponed and is executed at the optimally obtained new time. Otherwise, the replacement is executed immediately after inspection

The developed approach for optimal replacement postponement has some connection with that, discussed in our recent paper (Cha and Finkelstein (2020)) on a different topic of life extension (LE). In the current study, the settting, methodology and analytical justifications are different, whereas observing degradation and making decisions based on that is common. However, the latter is a fairly general concept. For some qualitative methodology used in the LE applications see, e.g., Ochella et al (2022).

As follows from the foregoing description, the starting point in our model is obtaining the black-box optimal time of replacement in accordance with the classical approach. However, at some instances, due to logistics, failure of inspection equipment and other reasons, the necessary inspection with observation of degradation cannot be executed or not relied on at that time. For instance, this can happen when the qualified personnel and the sophisticated diagnostic equipment that are not always available are needed. In this case, a replacement is carried out at the black-box optimal time, which is a convenient and well-justified in practice ‘back-up’ option.

On the other hand, when inspection can be executed, the decision to postpone the replacement or not is taken in accordance with the policy proposed in this paper. This combination is one of the advantages of our model compared with the trivariate optimization in Finkelstein et al (2020), where the policy ‘fails’ when inspection cannot be executed, as there is no reasonable alternatives in this case. In the aforementioned paper, the ‘joint’ optimization of the time of inspection, time of replacement and of the degradation threshold (the latter will be also a part of our model) is performed. The other important distinction from existing approaches is that our model can work also ‘cycle by cycle’ with decisions on postponement based on the specific realizations of degradation at inspections. Analytical justifications of the possible superiority of this policy over that described in Finkelstein et al (2020) should be explored in the future research, as the latter also employs the interval-based and not the realization-based approach for the observed degradation.

As a specific stochastic process for modeling deterioration, we consider the gamma process, which is widely used in reliability modeling of degrading items due to its clear meaning and mathematical tractability (see, e.g., Cinlar (1980), Pan and Balakrishnan (2011), Liao et al (2006), Cha and Finkelstein (2018)). An excellent survey on applications of gamma processes in maintenance modeling can be found in van Noortwijk (2009). For mostly methodological, illustrative purposes, we also use the homogeneous Poisson counting process that describes our approach in a simpler, more tractable and speaking for itself way.

Operation and maintenance of numerous engineering systems comply with assumptions of the proposed modeling. As a real-world example we can think about the gyro-based marine navigational equipment and assume that the gyro unit replacements are scheduled in a black-box scenario optimal way. The inspection measures degradation (which is an increasing in time error in longitude or/and latitude) using the reference device that can work only during very short relatively rare intervals of time providing accurate values of navigational parameters (inspection in our terms). Degradation of a gyro is monotone and can be modeled by the gamma process. If this degradation is relatively small (this is defined in what follows), then the next replacement can be postponed. This, as it will be shown, decreases the overall maintenance costs as the gyro unit can be quite expensive and also can decrease the number of spare gyro units that are used for replacements. Optimal replacements of some offshore equipment can be also modeled in the proposed way when the inspections are fairly cheap and simple and can be performed on cite, whereas the replacements are performed by the maintenance crew ‘ordered’ as a result of this inspection (this, of course, implies relatively short time that is needed for the maintenance crew to prepare and reach the cite.

The paper is organized as follows. Some necessary results needed for further discussion are given in Section 2. Section 3 describes the relevant failure models. The optimal policy is discussed in Section 4, whereas illustrative examples are given in Section 5. Finally, the concluding remarks are formulated in Section 6.

2. Preliminaries

Let $F(t)$, $\bar{F}(t)$, and $r(t)$ denote the Cdf, the survival function and the failure rate describing the lifetime L of a degrading item (i.e., with the failure mode due to degradation), respectively. Assume that degradation is manifested by the increasing $r(t)$. Let an item be incepted into operation at t=0. To reduce the consequences of sudden failures, the preventive replacement by as good as new item is performed at time *T*. In the classical age replacement model (Barlow and Hunter (1960), we have the following well-known relationship for the expected long-run cost rate to be minimized

$$c(T) = \frac{C_r \bar{F}(T) + C_f F(T)}{\mu_T} \quad , \tag{1}$$

where, C_r is the cost of preventive replacement and $C_f > C_r$ is the cost of failure that usually includes the cost of replacement and other consequences due to a sudden failure (e.g., operational delays), and $\mu_T = \int_0^T \bar{F}(x)dx$. Note that, for the stochastically ‘better’ system: $\bar{F}_1(t) \geq \bar{F}(t), t \geq 0$, we have:

$$c_1(T) = \frac{C_r \bar{F}_1(T) + C_f F_1(T)}{\mu_{T,1}} \leq c(T) , \tag{2}$$

where $\mu_{T,1}$ corresponds to the Cdf $F_1(t)$. Denote by T^* and T_1^* the optimal PM times minimizing $c(T)$ and $c_1(T)$, respectively. Then, according to Nakagawa (2005), Finkelstein et al (2020),

$$c(T^*) = (C_f - C_r)\lambda(T^*) ,$$

$$c_1(T_1^*) = (C_f - C_r)\lambda_1(T_1^*) ,$$

where, the failure rate $\lambda_1(t)$ is also assumed to be increasing. It can be also easily shown that in this case:

$$c_1(T_1^*) \leq c(T^*) ; \; \lambda_1(T_1^*) \leq \lambda(T^*) . \tag{3}$$

However, (3), in general, does not mean that

$$T_1^* \geq T^* . \tag{4}$$

We will call the ‘classical scenario’ described above the *black-box* scenario, as the degradation processes in the items are not specified, whereas degradation is manifested by the increasing failure rates.

Assume now that the observable monotonically increasing, homogeneous deterioration process $\{W_t, t \geq 0\}, W_0 = 0$, with independent increments describes deterioration of an item. A failure occurs when the process reaches the deterministic level *w*. Then, the lifetime of a system, *L* can be described by the following survival function

$$P(L > t) \equiv \bar{F}_w(t) = P(W_t \leq w) . \tag{5}$$

We assume that $P(L > t)$ is an absolutely continuous distribution with respect to t for each fixed w. Denote the corresponding Cdf by $F_w(t) = 1 - \bar{F}_w(t) = P(W_t > w)$ and the marginal density by $f(t, w) = \frac{\partial}{\partial t} F_w(t)$. The Cdf of the remaining lifetime upon reaching deterioration $0 \leq w_i \leq w$ at inspection is, due to homogeneity, $F_{w-w_i}(t)$. In fact, $F_w(t)$ and $F(t)$ describe the same random variable *L*, whereas the former one employs, according to the assumption, the additional information on the degradational process. In what follows, we will employ only $F_w(t)$, whereas $F(t)$ was introduced for the general setting and the ‘connection’ with existing literature.

For illustration of our findings, we will consider two processes of deterioration. The Poisson process, as the simpler one, is used mostly for methodological purposes, whereas the gamma process is already often used for modeling deterioration in practice.

a. Poisson process. Let degradation in the item be modelled by the homogeneous Poisson process with rate λ. Let N be the integer failure threshold, $N = 1, 2, \ldots$ Then (5) turns into

$$\bar{F}_N(t) = \exp\{-\lambda t\}\sum_{i=0}^{N-1}\frac{(\lambda t)^i}{i!}, \quad (6)$$

whereas the corresponding failure rate, which is monotonically increasing to λ, is given by

$$r(t,N) = \frac{\lambda^N t^{N-1}}{(N-1)!\sum_{k=0}^{N-1}\frac{(\lambda t)^k}{k!}}. \quad (7)$$

Moreover, importantly

$$r(t,m) < r(t,n), m > n, t > 0, \quad (8)$$

whereas, $r(0,N) = 0, N > 1; r(0,1) = \lambda$.

The remaining lifetime on observing $n < N$ events is described by the Cdf

$$F_{N-n}(t) = 1 - \exp\{-\lambda t\}\sum_{i=0}^{N-n-1}\frac{(\lambda t)^i}{i!}. \quad (9)$$

Therefore, the corresponding failure rate is also increasing in n for each $t > 0$.

b. Gamma process of degradation. For the *homogeneous* gamma process with parameters $\alpha > 0, \beta > 0$, the lifetime to reach the threshold w is described by the following survival function (see, e.g., van Noortwijk (2009))

$$\bar{F}_w(t) = 1 - \frac{\Gamma(\alpha t, \beta w)}{\Gamma(\alpha t)}, \quad (10)$$

where $\Gamma(a) = \int_0^\infty z^{a-1}\exp\{-z\}dz; \Gamma(a,x) = \int_x^\infty z^{a-1}\exp\{-z\}dz$,

Then the remaining lifetime is defined by

$$\bar{F}_{w-\tilde{w}}(t) = 1 - \frac{\Gamma(\alpha t, \beta(w-\tilde{w}))}{\Gamma(\alpha t)}. \quad (11)$$

It is shown in Liao et al. (2006) that the failure rate that corresponds to (10), $\lambda(t,w)$ is an increasing function.

Remark 1. In what follows, we assume that degradational failures are self-announced. This is the case in many practical situations. However, it should be noted that when this is not the case, the general classical approach described by Equations (1)-(4) is not applicable and so does our model to be developed.

3. The degradation-based failure models

If an item that was incepted into operation at $t = 0$ is operable at time $t > 0$, there is obviously, some 'resource' (that can be loosely understood as remaining lifetime) left. However, for the described black-box scenario and due to increasing with time risk of failure, it is not cost-wise beneficial to continue operation after the optimal time T^*, as follows from the corresponding minimization of $c(T)$ problem. On the other hand, if the degradation observed at inspection at optimal T^* is relatively small, there can be substantial resource left, or in other words, the risk of failure is not 'critical'. Then it seems natural to postpone replacement to some new optimal value $T_1^* > T^*$ with $c_1(T_1^*) \le c(T^*)$ (see (3)). Note that, in the black-box scenario of comparing optimal PM for two items described in the previous section, as mentioned, (4) was not necessarily true, whereas for the model to be described further, it will be the case. After this general description, we proceed with a more formal reasoning.

Denote the random value of degradation observed at T^* by $W_{T^*} \in [0, w)$. Then the remaining lifetime distribution for realization $W_{T^*} = w_i < w$ is $F_{w-w_i}(t)$, whereas the black-box remaining lifetime is described by the survival function

$$\bar{F}_w(t \mid T^*) = \frac{\bar{F}_w(t+T^*)}{\bar{F}_w(T^*)} \; . \tag{12}$$

On the other hand, for our specific model of deterioration, the same relationship can be written in a different way as

$$\bar{F}_w(t \mid T^*) = \int_0^w \frac{g(T^*, w_i)}{\bar{F}_w(T^*)} \bar{F}_{w-w_i}(t) dw_i \,, \tag{13}$$

where $g(T^*, w_i) = \dfrac{\partial}{\partial w_i} \bar{F}_{w_i}(T^*)$.

Denote by $w_f < w$ some deterministic level of degradation to be defined. We want to compare (12)-(13) with $\bar{F}_{w-w_i}(t)$. The reason for this comparison is as follows: the remaining resource/lifetime (13), is such that it is *not optimal* to continue operation of an item and the preventive replacement should be executed at T^*. This is due to the fact that T^* was obtained as an optimal time of replacement minimizing the long-run cost rate (1). However, if the remaining resource/lifetime is stochastically larger (see Shaked and Shantikumar (2007)) than defined in (13), it is reasonable to assume that this resource can be utilized and the optimal PM can be postponed to some new optimal time.

As the family of $\bar{F}_{w-w_i}(t)$ is ordered, i.e., for $w_{i,1} \ge w_{i,2}$

$$\bar{F}_{w-w_{i,2}}(t) \ge \bar{F}_{w-w_{i,1}}(t), t \ge 0 \,, \tag{14}$$

w_f could be found as the maximal value of w_i such that if $w_i \le w_f$, then

$$\bar{F}_w(t \mid T^*) \le \bar{F}_{w-w_i}(t) \,, \tag{15}$$

which means that the remaining lifetime after observing degradation $w_i \le w_f$ at T^* is stochastically larger than that in the black-box scenario. It is obvious for our setting with ordered lifetimes that $w_f < w$ exists. This will be also illustrated graphically in our examples.

Let $\bar{F}_{w,w_f}(t\,|\,T^*)$ denote the conditional survival function (for our failure model) for the remaining lifetime on condition that the degradation process is in $[0,w_f)$ at T^*, i.e.,

$$\bar{F}_{w,w_f}(t\,|\,T^*)=\int_0^{w_f}\frac{g(T^*,w_i)}{\bar{F}_{w_f}(T^*)}\bar{F}_{w-w_i}(t)dw_i\ . \tag{16}$$

As follows from our previous considerations while defining w_f

$$\bar{F}_{w,w_f}(t\,|\,T^*)\ge \bar{F}_w(t\,|\,T^*),\ t>0\ .$$

or, equivalently, from (13) and (16):

$$\int_0^{w_f}\frac{g(T^*,w_i)}{\bar{F}_{w_f}(T^*)}\bar{F}_{w-w_i}(t)dw_i\ge\int_0^{w}\frac{g(T^*,w_i)}{\bar{F}_w(T^*)}\bar{F}_{w-w_i}(t)dw_i \tag{17}$$

Denote now by $F_{w,w_f}(t,T^*)$ the Cdf of a random variable that is defined as follows. It is the event (failure) if the process reaches w in [0,T*) or reaches w in $[T^*,\infty)$ conditioned on $0\le w_i\le w_f$ at T^*. Thus, the corresponding piece-wise survival function is

$$\bar{F}_{w,w_f}(t,T^*)=\begin{cases}\bar{F}_w(t),\ 0\le t<T^*\\ \bar{F}_w(T^*)\displaystyle\int_0^{w_f}\frac{g(T^*,w_i)}{\bar{F}_{w_f}(T^*)}\bar{F}_{w-w_i}(t-T^*)dw_i,\ t\ge T^*\end{cases}$$
$$=\begin{cases}\bar{F}_w(t),\ 0\le t<T^*\\ \dfrac{\bar{F}_w(T^*)}{\bar{F}_{w_f}(T^*)}\displaystyle\int_0^{w_f}g(T^*,w_i)\bar{F}_{w-w_i}(t-T^*)dw_i,\ t\ge T^*\end{cases} \tag{18}$$

whereas for the initial 'unconditioned setting' when $0\le w_i\le w$ at T^* and using (13), we have

$$\bar{F}_w(t,T^*)=\begin{cases}\bar{F}_w(t),\ 0\le t<T^*\\ \bar{F}_w(T^*)\displaystyle\int_0^{w}\frac{g(T^*,w_i)}{\bar{F}_w(T^*)}\bar{F}_{w-w_i}(t-T^*)dw_i,\ t\ge T^*\end{cases}$$
$$=\begin{cases}\bar{F}_w(t),\ 0\le t<T^*\\ \displaystyle\int_0^{w}g(T^*,w_i)\bar{F}_{w-w_i}(t-T^*)dw_i,\ t\ge T^*\end{cases} \tag{19}$$

Thus, comparing (18) and (19), it follows from (17) that

$$\bar{F}_{w,w_f}(t,T^*)\ge\bar{F}_w(t,T^*),\ t\ge 0\ . \tag{20}$$

In the previous section, under the assumption that one item is stochastically larger than the other, i.e., $\bar{F}_1(T)\ge\bar{F}(T)$ and according to the black-box scenario, relationship (3) was obtained, which did not necessarily lead to relationship (4) (the postponement of optimal replacement). In our current, specific case, we also have the comparison (20), but distinct from the general black-box case, the survival functions are now the same in $0 \le t < T^*$ in (18) and (19). Denote the failure rate that corresponds to $\bar{F}_{w,w_f}(t,T^*)$ by $r_{w,w_f}(t,T^*)$ and the failure rate that corresponds to $\bar{F}_w(t,T^*)$, by $r_w(T)$. It is clear that

$$\begin{aligned} r_{w,w_f}(t,T^*) &= r_w(t), 0 \le t < T^* \\ r_{w,w_f}(t,T^*) &< r_w(t), t \ge T^* \end{aligned} \tag{21}$$

with a jump $r_w(t)\text{-}r_{w,w_f}(t,T^*)$ at T^* .

4. Optimal PM policy

In the previous section, the conditional lifetime distribution (18) was derived. Assume first, that all cycles are conditional, i.e., described by (18) meaning that degradation on all cycles is $0 \le w_i \le w_f$ at T^*. Then the expected long run cost rate can be defined in a traditional way. Thus,

$$c_{w_f}(T_{w_f}) = \frac{C_r \bar{F}_{w,w_f}(T_{w_f},T^*) + C_f F_{w,w_f}(T_{w_f},T^*)}{\mu_{T,w_f}}, \tag{23}$$

where

$$\mu_{T,w_f} = \int_0^{T_{w_f}} \bar{F}_{w,w_f}(t,T^*)dt$$

and T_{w_f} denotes the variable replacement time. We also assume that the inspection cost is negligible compared with other costs involved. From (2)-(3), (5) and (20), it follows that the optimal $T^*_{w_f}$ (that should be obtained as in the classical model by minimizing (23)) exists and that

$$c_{w_f}(T^*_{w_f}) \le c(T^*), \tag{24}$$

whereas for optimal times, distinct from our reasoning while discussing (3)-(4), we have now

$$T^*_{w_f} \ge T^* . \tag{25}$$

This follows from the general reasoning and the specific form of the survival function (18). Indeed, when $T_{w_f} \to 0$, $c_{w_f}(T_{w_f}) \to \infty$ and then decreases exactly as $c(T)$ in (1) for $0 \le T < T^*$.Thus $T^*_{w_f}$ can be only larger than T^*, due to our specific setting.

In accordance with the foregoing discussion, we are ready to provide now the sequence of operations that should be executed while implementing the corresponding replacement *policy.*

1. Obtain the black box T^* by minimizing (1),

2. Obtain w_f as the maximal value of w_i such that if $w_i \le w_f$, (15) holds,

3. If an item fails in $[0,T^*)$, replace it,

4. If the observed at T^* degradation $w_i \in [0,w_f)$, obtain $T^*_{w_f}$ by minimizing (23) and replace at this time.

5. If the observed at T^* degradation $w_i \in [w_f, w)$, replace at T^* .

6. Repeat 1-6 after replacement….

Remark 2. As follows from our reasoning, the derivative $c_{w_f}(T_{w_f})$ is not continuous at T^* as well as the corresponding failure rate. Therefore, minimization of (23) cannot be performed in the same analytical manner as in classic optimal replacement modeling (Barlow and Hunter (1960), Nakagawa (2006)). However, as discussed, the optimal solution of the minimization problem exists and it is larger than *T**, whereas the specific values of interest can be obtained numerically.

Remark 3. It may happen that the degradation at the optimally obtained $T^*_{w_f}$ is still relatively small and, consequently, an item still has some sufficient remaining lifetime/resource. In this case, the proposed procedure can be adjusted, for the second postponement however, it is rather cumbersome (e.g., the new value of w_f should be obtained). Note that, in practice, due to gradual deterioration, one step is usually sufficient for obtaining the well pronounced postponement, whereas the next steps result on average with a non-significant postponements. See also similar reasoning for lifetime extension in Cha and Finkelstein (2020).

Thus, for all *realizations* of cycles (recall that the cycle is a duration between two consecutive replacements) with $w_i \in [0,w_f)$ at *T**, we can decrease the cost rate by postponing *T** to $T^*_{w_f}$. However, there are other realizations of cycles with $w_i \in [w_f, w)$ at *T** with the corresponding lifetimes that are stochastically smaller (see Shaked and Shantikumar (2007)) than those defined by (19). Therefore, the optimal time of replacement for these realizations should be 'theoretically' smaller than *T**, but distinct from the previous case, when the replacement can be postponed, we cannot decrease it now, as *T** was predetermined as the black box-one and the inspection is performed accordingly. Thus, by postponing replacement on relevant realizations of cycles we improve the expected long-run cost rate that corresponds to $T^*_{w_f}, T^*$, which have been already obtained in the defined optimal way. Thus, the minimum of the expected long-run cost-rate function *for the defined above replacement policy* is obviously achieved at these values.

In the following, we use notation defined for relationships (5) and (13). We will derive now the expected long-run cost rate for the described policy. For this some relevant probabilities should be defined. We have four events that describe the cycle in our model:

Event 1. An item fails before T^* and is replaced at the failure time. The corresponding probability of this event is

$$P(W_{T*} > w) = P(L \le T^*) = \int_w^\infty g(T^*, w_i) dw_i \,.$$

The conditional Cdf and the pdf of the failure time of an item for $0 \le t \le T^*$ in this case are

$$P(L \le t \mid L \le T^*) = \frac{F_w(t)}{F_w(T^*)}; \qquad \frac{f(t,w)}{F_w(T^*)} = \frac{f(t,w)}{P(W_{T^*} > w)}, \quad 0 \le t \le T^*$$

accordingly.

Event 2. An item continues to operate after T^* (thus, its degradation at T^* is smaller than w_f), but fails before $T^*_{w_f}$.

Probability of this event is

$$P(W_{T*} \le w_f, W_{T^*_{w_f}} > w) = \int_0^{w_f} P(W_{T^*_{w_f}} - W_{T*} > w - w_i) g(T^*, w_i) dw_i$$

$$= \int_0^{w_f} \int_{w-w_i}^{\infty} g(T^*_{w_f} - T^*, u) g(T^*, w_i) du dw_i \,,$$

where homogeneity and the independent increments property of the degradation process were used. The conditional Cdf of the failure time in this case is defined as

$$P(L < t \mid W_{T*} \le w_f, W_{T^*_{w_f}} > w)$$

$$= P(W_t > w \mid W_{T*} \le w_f, W_{T^*_{w_f}} > w) = \frac{P(W_{T*} \le w_f, W_t > w)}{P(W_{T*} \le w_f, W_{T^*_{w_f}} > w)}$$

$$= \frac{\int_0^{w_f} P(W_t - W_{T^*} > w - w_i) g(T^*, w_i) dw_i}{P(W_{T*} \le w_f, W_{T^*_{w_f}} > w)} = \frac{\int_0^{w_f} F_{w-w_i}(t - T^*) g(T^*, w_i) dw_i}{P(W_{T*} \le w_f, W_{T^*_{w_f}} > w)}, \; T^* \le t \le T^*_{w_f}$$

Thus, by taking derivative with respect to *t*, the conditional pdf is

$$\frac{\int_0^{w_f} f(t - T^*, w - w_i) g(T^*, w_i) dw_i}{P(W_{T*} \le w_f, W_{T^*_{w_f}} > w)}, \; T^* \le t \le T^*_{w_f} \,.$$

Event 3. An item is preventively replaced at T^* meaning that its degradation at T^* is in $[w_f, w)$. Probability of this event is

$$P(w_f < W_{T*} \le w) = \int_{w_f}^{w} g(T^*, u) du \,.$$

Event 4. An item is preventively replaced at $T^*_{w_f}$. Probability of this event is

$$P(W_{T^*} \le w_f, W_{T^*_{w_f}} < w) = \int_0^{w_f} \int_0^{w-w_i} g(T^*_{w_f} - T^*, u) g(T^*, w_i) du dw_i$$

In accordance with the derived relationships, the expected cost on a cycle is given by

$$\begin{aligned} C(T^*, T^*_{w_f}) &= C_f \left(\int_w^{\infty} g(T^*, w_i) dw_i + \int_0^{w_f} \int_{w-w_i}^{\infty} g(T^*_{w_f} - T^*, u) g(T^*, w_i) du dw_i \right) \\ &+ C_r \left(\int_{w_f}^{w} g(T^*, u) du + \int_0^{w_f} \int_0^{w-w_i} g(T^*_{w_f} - T^*, u) g(T^*, w_i) du dw_i \right) \end{aligned} \tag{26}$$

Denote now by Z the random duration of a cycle that corresponds to the considered events. Its expected value is

$$\begin{aligned} &\mu_{T^*, T^*_{w_f}} \equiv E[Z] \\ &= E[Z \mid W_{T^*} > w] P(W_{T^*} > w) + E[Z \mid W_{T^*} \le w_f, W_{T^*_{w_f}} > w] P(W_{T^*} \le w_f, W_{T^*_{w_f}} > w) \\ &+ E[Z \mid w_f < W_{T^*} \le w] P(w_f < W_{T^*} \le w) + E[Z \mid W_{T^*} \le w_f, W_{T^*_{w_f}} < w] P(W_{T^*} \le w_f, W_{T^*_{w_f}} < w), \end{aligned}$$

where, in accordance with our reasoning and the proposed policy,

$$E[Z \mid W_{T^*} > w] = \frac{\int_0^{T^*} t f(t, w) dt}{P(W_{T^*} > w)},$$

$$E[Z \mid W_{T^*} \le w_f, W_{T^*_{w_f}} > w] = \frac{\int_{T^*}^{T^*_{w_f}} t \int_0^{w_f} f(t - T^*, w - w_i) g(T^*, w_i) dw_i dt}{P(W_{T^*} \le w_f, W_{T^*_{w_f}} > w)}$$

$$E[Z \mid w_f < W_{T^*} \le w] = T^*, \quad E[Z \mid W_{T^*} \le w_f, W_{T^*_{w_f}} < w] = T^*_{w_f}.$$

Therefore, finally

$$\begin{aligned} \mu_{T^*, T^*_{w_f}} &= \int_0^{T^*} t f(t, w) dt + \int_{T^*}^{T^*_{w_f}} t \int_0^{w_f} f(t - T^*, w - w_i) g(T^*, w_i) dw_i dt \\ &+ T^* \int_{w_f}^{w} g(T^*, u) du + T^*_{w_f} \int_0^{w_f} \int_0^{w-w_i} g(T^*_{w_f} - T^*, u) du g(T^*, w_i) dw_i . \end{aligned} \tag{27}$$

Thus, employing the renewal reward theorem (Ross (1996)), as in classical model, for the expected long-run cost rate, and plugging in the obtained relationships, we get

$$c_{w_f}(T^*, T^*_{w_f}) = \frac{C(T^*, T^*_{w_f})}{\mu_{T^*, T^*_{w_f}}} \tag{28}$$

Note, that distinct from (1) and (23), we are not optimizing this function now, as the variables are fixed as $T^*, T^*_{w_f}$ in the described above optimal way. This, as stated in the Introduction, is distinct from Finkelstein et al (2020), where the 'joint' optimization of the time of inspection, time of replacement and of the degradation threshold is performed. However, the optimal policy in Finkelstein et al (2020) fails when inspections are not available at the optimally obtained time T^*, whereas in our model, there is an alternative to replace at T^* without inspection.

Remark 4. What happens, if we want to make our decisions on PM postponement based not on the interval for random degradation $[0, w_f)$, but on the 'exact value' of the observed degradation (realization), $w_i \in [0, w_f)$? In this case, the survival function in (18) turns to

$$\bar{F}_{w,\tilde{w}}(t, T^*) = \begin{cases} \bar{F}_w(t),\ 0 \le t < T^* \\ \bar{F}_w(T^*)\bar{F}_{w-w_i}(t - T^*),\ t \ge T^* \end{cases} \tag{29}$$

Based on this relationship, we can proceed in a way similar to what was described in our paper obtaining the optimal $T^*_{w_i} \ge T^*$ for each $w_i \in [0, w_f)$ that decreases with increase of $w \in [0, w_f)$. Then (26)-(28) should be adjusted accordingly via the corresponding mixing with respect to the degradation distribution at T^*. As in this case, we are using the more specific information for obtaining the possible postponement of a replacement, the corresponding long-run cost rate should be smaller than that defined by (28), whereas practical benefits may be questionable. In any case, this topic needs further study.

5. Numerical illustrations and discussion

a. *Poisson process degradation*. We consider this case as supplementary due its better mathematical and intuitive tractability, whereas the gamma process case to follow has a more practical value as it is well-known that it is widely used for modelling degradation of real systems.

Let the threshold $w = N = 10$. Then (6) reads: $\bar{F}_{10}(t) = \exp\{-\lambda t\}\sum_{i=0}^{9}\frac{(\lambda t)^i}{i!}$.

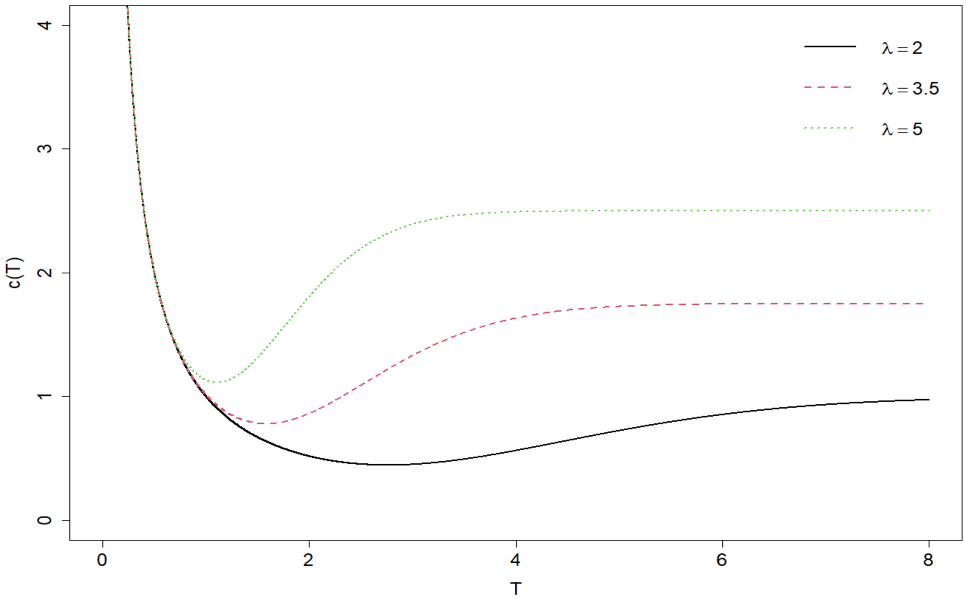

Fig.1. The expected black box long-run cost rate as a function of T, $C_r = 1, C_f = 5$.

Table1. Values of optimal T^* and $c(T^*)$, $C_r = 1, C_f = 5$.

λ	2	3.5	5
T*	2.76	1.58	1.10
C(T*)	0.44	0.78	1.11

As a starting point and for further comparison, the classical expected black box long-run cost rate (1) is plotted in Fig.1 for different values of the rate λ. We see the pronounced minimums. Moreover, as expected, when degradation is increasing (λ is increasing), the optimal PM time is decreasing, whereas optimal cost rate is increasing.

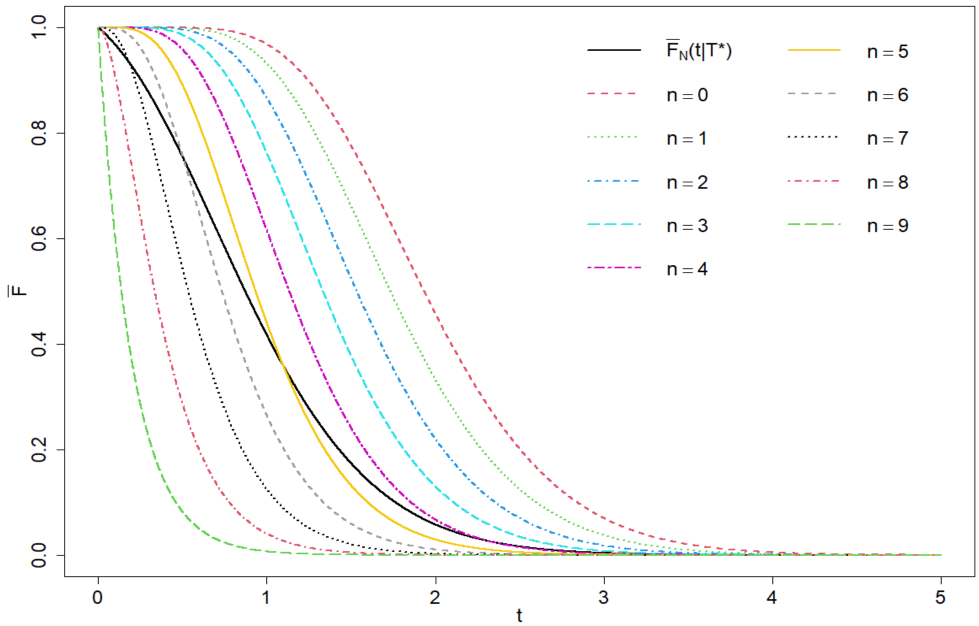

Fig.2. Survival curves, showing that $w_f = 4$ for $C_r = 1, C_f = 5$, λ=5.

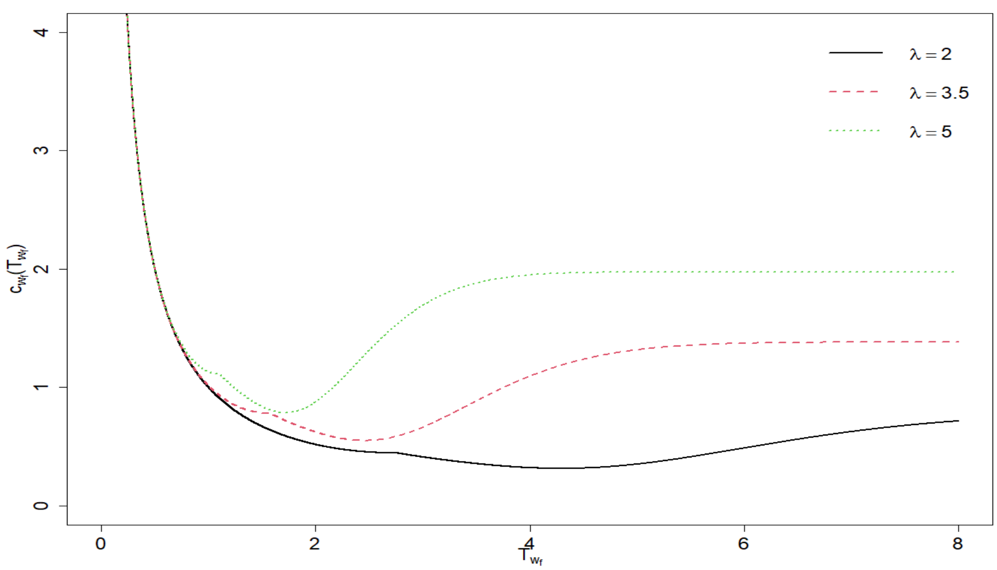

Fig.3. The expected (postponed) long-run cost rate as a function of T_{w_f} , $C_r = 1, C_f = 5$, $w_f = 4$.

Table 2. Values of optimal $T^*_{w_f}$, $c_{w_f}(T^*_{w_f})$, $c_{w_f}(T^*, T^*_{w_f})$, $C_r = 1, C_f = 5, w_f = 4$.

λ	2	3.5	5
T_{wf}*	4.30	2.45	1.72
$C_{wf}(T_{wf}*)$	0.31	0.55	0.79
$C_{wf}(T*,T_{wf}*)$	0.41	0.71	1.02

Fig.2 describes obtaining w_f from (15). It is the maximal value of the observed at T^* degradation for which inequality (15) still holds. The bold curve plots $\bar{F}_w(t \mid T^*)$, whereas other curves plot $\bar{F}_{w-w_i}(t)$ for different values of $w_i = 0,1,2,.....9$. It can be seen from the graph and was verified by the corresponding numerical values that $w_f = 4$. This was done for the rate λ=5. For other considered values of the rate of the Poisson process ($\lambda = 2,3$), the result is the same. Fig. 3 plots the curves that describe the expected long-run cost rate (23) when all cycles are conditional meaning that degradation at T^* is smaller than $w_f = 4$. Comparing the values in Table 2 (with the corresponding values in Table 1), we see that relationship (24) ($c_{w_f}(T^*_{w_f}) \le c(T^*)$) holds and the decrease in the expected long-run cost rate is substantial. For instance, for the case $\lambda = 5$, we have $c(T^*) = 1.11$, whereas $c_{w_f}(T^*_{w_f}) = 0.79$. When not all cycles are conditional (which is the case for the real-life operation of the repairable item) with other cycles exhibiting degradation at T^* larger than $w_f = 4$, the obtained in accordance with our methodology minimal expected long-run cost $c_{w_f}(T^*, T^*_{w_f})$ defined in (28) is also reasonably smaller (1.02). It should be noted also that, as can be seen from Fig.3, the cost rates have discontinuities in their derivatives. This follows from the observation that the survival function (18), obviously, has a discontinuity in its derivative at $t = T^*$. The latter leads to the jump in the corresponding failure rate, as was already mentioned.

b. *Gamma process of degradation.* We follow the same steps and omit similar to the Poisson case explanations. Fig.4 plots the classical curve (1). We vary parameter α as the one that defines the rate of the process, whereas parameter β that responsible for the size of the jumps is fixed.

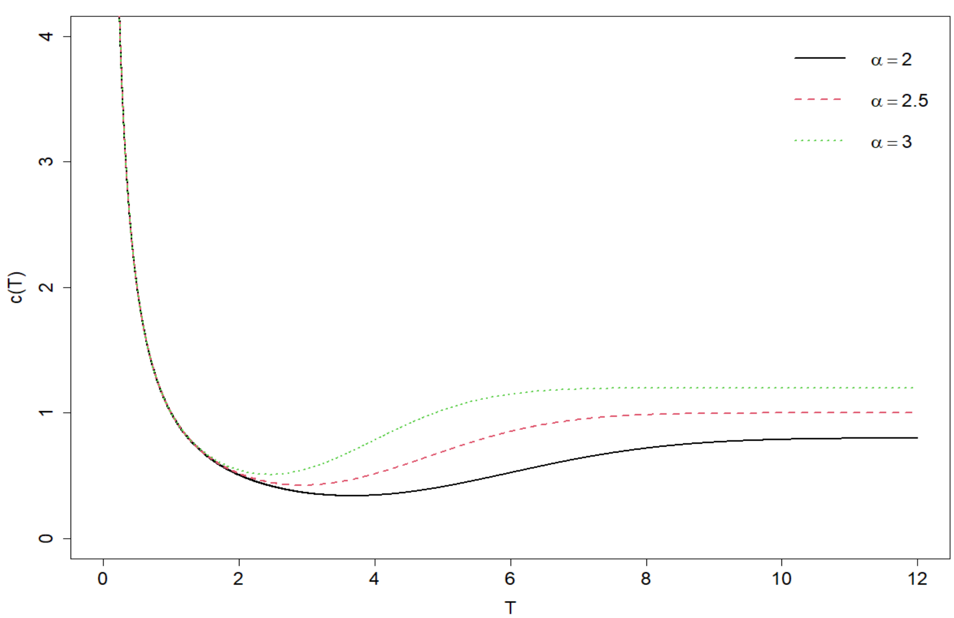

Fig.4. The expected black box long-run cost rate as a function of *T*, $C_r = 1, C_f = 5, w = 1, \beta = 12$.

Table3. Values of optimal T^* and $c(T^*)$, $C_r = 1, C_f = 5, w = 1, \beta = 12$.

A	2	2.5	3
T*	3.66	2.93	2.44
C(T*)	0.34	0.43	0.51

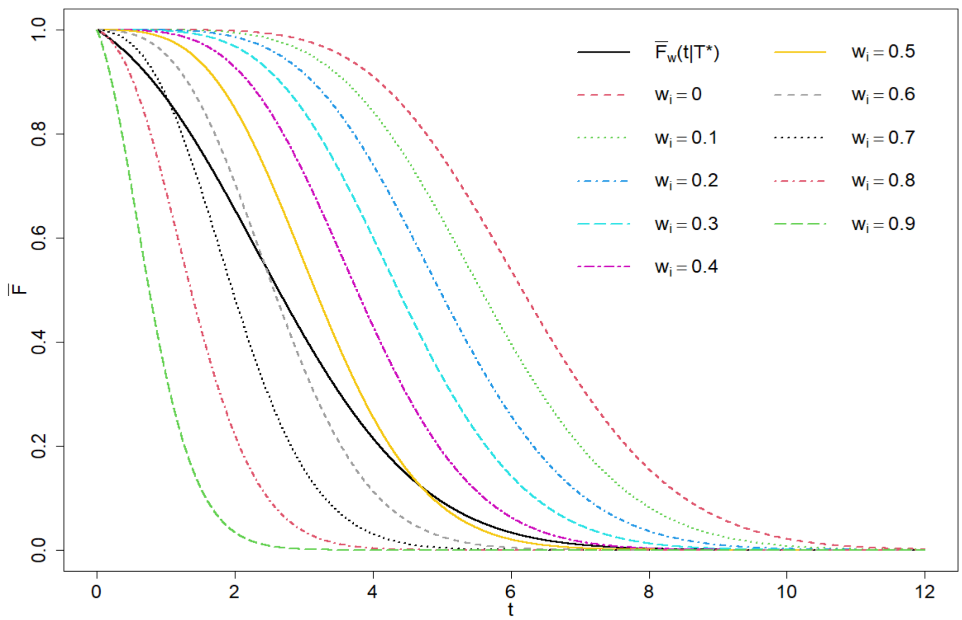

Fig.5. Survival curves, showing that $w_f = 0.4$ for $C_r = 1, C_f = 5, w = 1, \alpha = 2, \beta = 12$

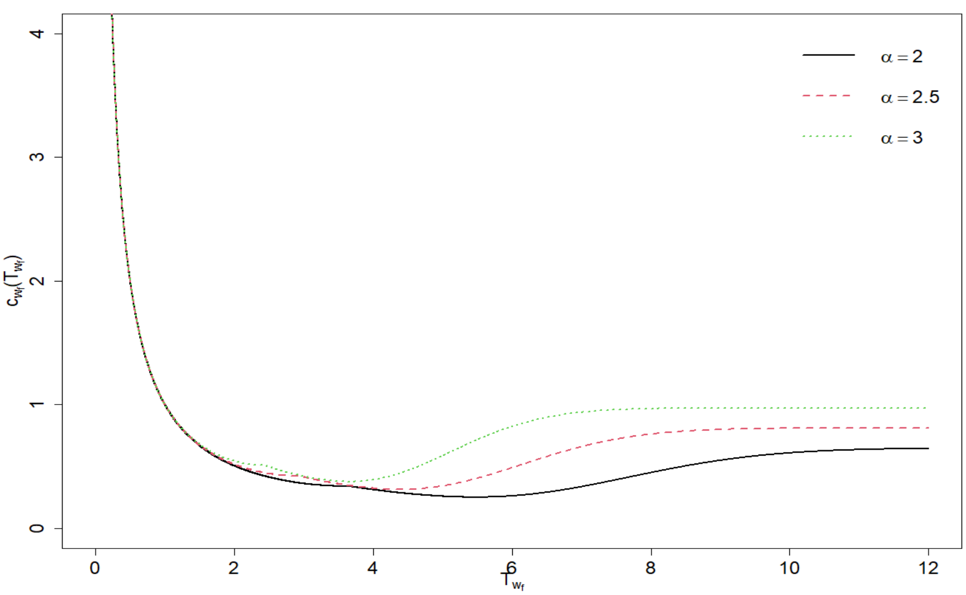

Fig.6. The expected (postponed) long-run cost rate as a function of T_{w_f}, $C_r = 1, C_f = 5, w = 1, \alpha = 2, \beta = 12$, $w_f = 0.4$.

Table 4. Optimal $T^*_{w_f}$, $c_{w_f}(T^*_{w_f}), c_{w_f}(T^*, T^*_{w_f})$, $C_r = 1, C_f = 5, w = 1, \alpha = 2, \beta = 12, w_f = 0.4$.

A	2	2.5	3
T_{wf}*	5.47	4.38	3.65
$C_{wf}(T_{wf}*)$	0.25	0.31	0.38
$C_{wf}(T*,T_{wf}*)$	0.31	0.39	0.47

It follows from Fig. 5 that $w_f = 0.4$ when $\alpha = 2$. Numerical experiments for $\alpha = 3.5; 5$ show the same value of w_f. Thus, the conditional cycles should have the observed at $T*$ value of degradation less than 0.4 for the failure threshold defined as $w = 1$. The cost-wise effect of the proposed policy, as compared with a classical one, is also noticeable comparing tables 3 and 4. Also optimal replacement time is significantly larger, which can be very convenient in practice. It will be more pronounced, if $w_f > 0.5$ ($w_f > 5$ in the Poisson case). However, it is not formally clear how this level is depends on parameters of the degradation processes. This needs further study.

6. Concluding remarks

The described innovative approach improves the classical optimal replacement modeling by considering the value of degradation at the obtained optimal PM time $T*$. If $w_i \in [0, w_f)$, the PM can be postponed to the predetermined time $T*_{w_f} \geq T*$ with the smaller long-run cost rate.

The proposed policy assumes availability of inspections at the predetermined (in accordance with the classical, non-degradation-wise) policy. However, at some instances, due to logistics, failure of inspection equipment and other reasons, the necessary inspection with observation of degradation cannot be executed or not relied on at that time. In this case, a replacement is carried out at the black-box optimal time, which is a convenient and well-justified in practice 'back-up' option.

Our decisions are based on observing degradation in the obtained interval. Therefore, the postponement will be the same for all realization $w_i \in [0, w_f)$. Remark 3 defines the possibility of considering the postponement for each realization. This suggestion has pros and contras, which should be explored in the future research. Another possible direction for further studies of this model is considering the case when the failure threshold is random.

We are considering *degrading* items when degradation is observed only at inspections. An important assumption/limitation that the stochastic process of degradation is monotonically increasing. A generalization to the non-monotonically increasing process (e.g., Wiener process) can be considered in the future research.

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References

Alaswad S, Xiang Y (2017) A review on condition-based maintenance optimization models for stochastically deteriorating system. Reliability Engineering and System Safety 157:54-63.

Barlow RE and Hunter L (1960). Optimum preventive maintenance policies. Operations Research 1960; 8: 90-100.

Badıa FG, Berrade MD, Campos CA (2002) Optimal inspection and preventive maintenance of units with revealed and unrevealed failures. Reliability Engineering & System Safety 78: 157-163.

Castanier B, Berenguer C, Grall A (2003). A sequential condition-based repair/replacement policy with non-periodic inspections for a system subject to continuous wear. Applied Stochastic Models in Business and Industry, 19:327-347.

Castro IT (2013) An age-based maintenance strategy for a degradation-threshold shock-model for a system subjected to multiple defects. Asia-Pacific Journal of Operation Research 30: 1350016–29.

Castro IT, Caballe NC, Perez CJ (2015). A condition-based maintenance for a system subject to multiple degradation processes and external shocks. Int. J. Syst. Sci., 46:1692-1704.

Cha JH, Finkelstein M (2018) Point processes for reliability analysis: Shocks and repairable systems. Springer, London.

Cha JH, Finkelstein M, Levitin G (2017). On preventive maintenance of systems with lifetimes dependent on a random shock process. Reliability Engineering & System Safety,168:90-97.

Cha JH, Finkelstein M, Levitin G (2018). Bivariate preventive maintenance of systems with lifetimes dependent on a random shock process. European Journal of Operational Research, 266,122-134.

Cha JH, Finkelstein M (2020). On optimal life extension for degrading systems. Journal of Risk and Reliability, 234: 487–495.

Çinlar E (1980) On a generalization of gamma process. Journal of Applied Probability 17: 467–480.

Finkelstein M, Cha JH (2013) Stochastic modelling for reliability: Shocks, burn-in and heterogeneous populations. Springer, London.

Finkelstein M, Cha JH, Levitin G (2020) On a new age-replacement policy for items with observed stochastic degradation. Quality and Reliability Engineering International 36: 1132-1144.

Finkelstein M, Shafiee M, Kotchap AN. (2016) Classical optimal replacement strategies revisited. IEEE Transaction on Reliability 2016;65:540-546.

Finkelstein M., Gertsbakh I. On 'time-free' preventive maintenance of systems with structures described by signatures. Applied Stochastic Models in Business and Industry, 2015, 31:836-845.

Gertsbakh I (2005) Reliability theory with applications to preventive maintenance. Springer, Berlin.

Grall A, Dieulle D, Berenguer C, Roussignol M (2002) Continuous-time predictive maintenance scheduling for a deteriorating system. IEEE Transactions on Reliability 51:141-150.

Guo C, Liang, Z (2022). A predictive Markov decision process for optimizing inspection and maintenance strategies of partially observable multi-state systems Reliability Engineering & System Safety, 226, 108683.

Hamidi M, Szidarovsky F, Szidarovszky M. (2016) New one cycle criteria for optimizing preventive replacement policies. Reliability Engineering & System Safety,154, 42-48.

Jardine AKS, Lin D, Banjevic D (2006). A review on machinery diagnostics and prognostics implementing condition-based maintenance. *Mechanical Systems and Signal Processing,* 20:1483-1510.

Liao H, Elsayed EA, Chan LY (2006) Maintenance of continuously monitored degrading systems. European Journal of Operational Research 175:821-835.

Nakagawa T (2005). Maintenance theory of reliability. London: Springer, 2005.

Ochella S, Shafiee M, Sansom C (2022). An RUL-informed approach for life extension of high-value assets. Computers and Industrial Engineering, 171, 108332

Pan Z, Balakrishnan N (2011) Reliability modeling of degradation of products with multiple performance characteristics based on gamma processes. Reliability Engineering & System Safety 96:949–957.

Pedersen TI, Vatn J (2022). Optimizing a condition-based maintenance policy by taking the preferences of a risk-averse decision maker into account. Reliability Engineering & System Safety, 228, 108775.

Ross S. *Stochastic Processes*. New York: John Wiley; 1996.

Shaked M, Shanthikumar JG (2007). S*tochastic Orders*. New York: Springer.

Tambe P, Kulkarni M.S (2022). A reliability based integrated model of maintenance planning with quality control and production decision for improving operational performance. Reliability Engineering & System Safety, 226, 108681.

van Noortwijk JM (2009). A survey of the application of gamma processes in maintenance. Reliability Engineering and System Safety 94:2–21.

Zhang F, Shen J, Liao, H, Ma, Y (2021). Optimal preventive maintenance policy for a system subject to two-phase imperfect inspections. Reliability Engineering & System Safety, 205, 107254.

Wang J, Miao Y (2021). Optimal preventive maintenance policy of the balanced system under the semi-Markov model. Reliability Engineering & System Safety, 213, 107690.

Wang J, Qiu Q, Wang H (2021a). , Joint optimization of condition-based and age-based replacement policy and inventory policy for a two-unit series system. Reliability Engineering and System Safety 205 107251.

Wang J, Qiu Q, Wang H, Lin C (2021b). Optimal condition-based preventive maintenance policy for balanced systems. Reliability Engineering & System Safety 211, 107606.

Wang H. (2002). A survey of maintenance policies of deteriorating systems. European Journal of Operational Research, 226,469-489.

Wang H, Pham H (2006). Reliability and Optimal Main*tenance*. London: Springer; 2006.