On precedence tests with double sampling

Author 1: Niladri Chakraborty*

Email: niladri.chakraborty30@gmail.com

Affiliation: Department of Mathematical Statistics and Actuarial Science, University of the Free State, 205 Nelson Mandela Dr, Park West, Bloemfontein, South Africa 9301. ORCID ID: 0000-0001-9853-4067.

*Corresponding author.

Author 2: Narayanaswamy Balakrishnan

Email: bala@univmail.cis.mcmaster.ca

Affiliation: Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada.

Author 3: Maxim Finkelstein

Email: FinkelM@ufs.ac.za

Affiliation: 1. Department of Mathematical Statistics and Actuarial Science, University of the Free State, 205 Nelson Mandela Dr, Park West, Bloemfontein, South Africa 9301.

2. Department of Management Science, University of Strathclyde, Glasgow, UK.

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Abstract

A new double sampling-based precedence and weighted precedence tests are introduced and analyzed. The joint distributions of two precedence and weighted precedence statistics are obtained under double sampling framework. Subsequently,the closed-form expressions for the rejection probabilities are derived under the null hypothesis and the Lehmann alternative. The corresponding power comparison is carried out against the Lehmann alternative and the location-scale alternative through Monte-Carlo simulations. Finally, a couple of detailed illustrative example is presented.

Key-words: Precedence test; Weighted precedence test; Life-testing; Lehmann alternative; Level of significance.

1. Introduction

The precedence test is a well-known nonparametric, two-sample life-test that is used to test the equality of two distributions. Suppose, there are two different lifetime distributions $F(x)$ and $G(x)$ and we are interested to test

$$H_0 : F(x) = G(x) \quad ag. \quad H_1 : F(x) > G(x). \tag{1}$$

A precedence test based on few early failures from two samples $\boldsymbol{X}$ and $\boldsymbol{Y}$ is a common choice for this hypothesis testing problem. Precedence test is particularly useful as (i) it provides a reliable decision based on a few early failures from the two samples of a life-test, and (ii) it is beneficial when expensive items are involved so that unused items could be used for other testing purposes.

This test was first introduced by Nelson (1963). Then several authors have considered the precedence-type statistics in online monitoring and retrospective testing problems; e.g., Ilbott and Nadler (1965), Shorack (1967), Nelson (1993), Chakraborti and Van der Laan (1996), van der Laan and Chakraborti (2001), Balakrishnan and Frattina (2000), Ng and Balakrishnan (2004, 2005), Balakrishnan et al. (2008), Balakrishnan et al. (2010), Ng et al.

(2013), Balakrishnan et al. (2015b), Balakrishnan et al. (2015a),Stoimenova and Balakrishnan (2017), Chakraborty et al. (2018), Chakraborty et al. (2022), to name a few. For instance, Ng and Balakrishnan (2005) have proposed the weighted precedence test and the weighted maximal precedence test as extensions to the precedence test (Nelson (1963)) and maximal precedence test (Balakrishnan and Frattina (2000)). They showed that, in many cases the weighted precedence test attains more power than its competitors.

It would be interesting to see if the power properties of precedence-type tests could be retained with a smaller sample size. A double sampling procedure is useful in reducing the sample size. Literature on the double sampling procedures date back to the early works by Cox (1952), Tenenbein (1970), Espeland and Odoroff (1985), among others. Several authors have considered double sampling framework in the online monitoring problems; e.g., Daudin (1992), Carot et al. (2002), Malela-Majika et al. (2021), among others. However, to the best of our knowledge, the power properties of the precedence and weighted precedence tests under a retrospective double sampling testing framework have not been studied so far.

Therefore, in this paper, we introduce and investigate the power properties of the precedence and the weighted precedence tests within the double sampling framework. This is a two-stage testing procedure. On the first stage, a decision is made based on an initial $\boldsymbol{Y}$ sample. If the test statistic from the initial sample falls outside a suitably defined 'warning region', we conclude about H_0. If the test statistic for the initial sample falls within the 'warning region', a decision is made after 'taking' an additional $\boldsymbol{Y}$ sample. Note that the final version of this article will be published in the journal Statistics.

The rest of this article is organized as follows: In Section 2, we discuss the precedence and weighted precedence tests. In Section 3, we introduce the precedence and weighted precedence test under double sampling framework. In Section 4, we study the power properties of the test. An illustrative example is presented in Section 5. Finally, some concluding remarks are made in Section 6.

2. Precedence test and weighted precedence test

Suppose we have a random sample of m units from $F(x)$ and n units from $G(x)$ that are put on a life-testing experiment simultaneously. To test the hypotheses in Eq. (1), we define the precedence statistic as the number of $\boldsymbol{X}$ failures, W_r, preceding the r^{th} $\boldsymbol{Y}$ failure. Therefore, the precedence statistic of order r can be written as

$$W_r = \sum_{i=1}^{m} I(X_i \leq Y_{r:n}), \tag{2}$$

where $I(A)$ is the indicator function taking value 1 if the condition A is true, else 0. If W_r is large, it is reasonable to conclude that the units from $\boldsymbol{Y}$ sample last longer than $\boldsymbol{X}$ sample. Therefore, we can conclude in favour of the alternative hypothesis $H_1 : F(x) > G(x)$.

The weighted precedence statistic (Ng and Balakrishnan (2005)) is obtained as a weighted sum of $\boldsymbol{X}$ failures between every consecutive $\boldsymbol{Y}$ failures. Let m_i be the number of $\boldsymbol{X}$ failures between the i^{th} and the $(i-1)^{th}$ $\boldsymbol{Y}$ failures. Then the r^{th} order weighted precedence statistic (Ng and Balakrishnan (2005)) is obtained by

$$W_r^* = \sum_{i=1}^{r} (n - i + 1) m_i \tag{3}$$

Ng and Balakrishnan (2005) showed that the weighted precedence test can achieve more power than the precedence test and a number of its variations.

Example: Let us consider a life-testing experiment from Example 5.4.3 in Lawless (2011). Two types of electrical cable insulation were subjected to the increasing voltage stress in a laboratory test. Consider $\boldsymbol{X}$ and $\boldsymbol{Y}$ to be the Type I and the Type II insulation, respectively. The voltage levels (in kilovolts per millimeter) at which failures occurred were recorded. We consider $m = n = 10$ specimens from groups $\boldsymbol{X}$ and $\boldsymbol{Y}$ that were placed in a life-testing experiment. Voltages at failures are presented below in Table 1.

For $r = 3$, and $m = n = 10$, the critical value would be $c = 7$ for the precedence test and $c^* = 56$ for the weighted precedence test at 5% level of significance (Ng and Balakrishnan,

Table 1: Voltages at failures for two types of electrical cable insulation.

Type I	32.0	35.4	36.2	39.8	41.2	43.3	45.5	46.0	46.2	46.4
Type II	39.4	45.3	49.2	49.4	51.3	52.0	53.2	53.2	54.9	55.5

2004, 2005). It means that we reject H_0 at 5% level of significance if $W_r \geq 7$ or $W_r^* \geq 56$, respectively. From Table 1, we find that $W_3 = 10$ and $W_3^* = 10 \times 3 + 9 \times 3 + 8 \times 4 = 89$. Therefore, H_0 is rejected by both the precedence test and the weighted precedence test. It would be of interest to see the performance of these test procedures under a variable sampling plan.

3. Precedence test and weighted precedence test under double sampling scheme

In this section, we discuss the precedence test and the weighted precedence test under a double sampling framework. Double sampling from a population is carried out in two ways; i. Nested; ii. Non-nested. In this paper, we consider the non-nested double sampling framework without replacement for the proposed test procedures.

Our general logic to be elaborated in the remaining part of the paper is as follows. Let $X_{(c:d)}$ and $Y_{(c:d)}$ be the c^{th} order statistic from the $\boldsymbol{X}$ and $\boldsymbol{Y}$-sample of size d, respectively. We take a smaller initial $\boldsymbol{Y}$ sample without replacement, and based on the initial decision, we combine the remaining $\boldsymbol{Y}$ sample with the first sample to take the final decision about H_0. Since the precedence-type tests are based on the relative order of the $\boldsymbol{X}$ and $\boldsymbol{Y}$ sample, in the above mentioned procedure, a nested double sampling plan or sampling with replacement result in loss of information. We discuss the precedence and weighted precedence tests under double sampling framework. To obtain the critical limits, we derive the joint distribution of two precedence and weighted precedence statistics obtained from the same sampling frame.

3.1. Double sampling precedence (DPT) test

Let us define 'decision sub-intervals' in $[0, m]$ as $\mathcal{A}_0 = [0, a)$, $\mathcal{A}_1 = [b, m]$, $\mathcal{B} = [a, b)$, $C_0 = [0, c)$, $C_1 = [c, m]$. $\mathcal{B}$ is the *warning* region to decide if a second sample should be drawn.

The steps to carry out the proposed precedence test based on double sampling scheme are as follows:

i. We take $n_1(< n)$ random $\boldsymbol{Y}$-sample from the n sample and calculate the corresponding precedence statistic $W_{r1} = \sum_{i=1}^{m} I(X_i \leq Y_{r:n_1})$, where $I(A)$ is an indicator function as defined before;

ii. If $W_{r1} \in \mathcal{A}_0$, accept H_0, and if $W_{r1} \in \mathcal{A}_1$, we reject H_0;

iii. If $W_{r1} \in \mathcal{B}$, we fuse the remaining $n_2 = n - n_1$ random $\boldsymbol{Y}$-samples with the first n_1 samples and obtain $W_r = \sum_{i=1}^{m} I(X_i \leq Y_{r:n})$ from the pooled sample;

iv. If $W_r \in C_0$, we accept H_0. If $W_r \in C_1$, we reject H_0.

We follow the two-stage procedure to decide if H_0 in Eq. (1) should be rejected or not. Let $p_{10|H} = P_H[W_{r1} \in \mathcal{A}_0] = P_H[Y_{r:n_1} < X_{a:m}], p_{11|H} = P_H[W_{r1} \in \mathcal{A}_1] = P_H[Y_{r:n_1} \geq X_{b:m}], p_{1w|H} = P_H[W_{r1} \in \mathcal{B}] = P_H[X_{a:m} \leq Y_{r:n_1} < X_{b:m}]$ be the probabilities that W_{r1} would fall in the acceptance region, rejection region, or the warning region, respectively, for $H = H_0$ or H_1. Similarly, $p_{20|H} = P_H[W_r \in C_0] = P_H[Y_{r:n} < X_{c:m}]$ and $p_{21|H} = P_H[W_r \in C_1] = 1 - p_{20|H}$ are the probabilities for W_r to fall in the acceptance or rejection region, for $H = H_0$ or H_1, respectively.

3.1.1. Null distribution

When H_0 is true, for a given α, the probability $(1 - \alpha)$ that H_0 will not be rejected is

$$
\begin{aligned}
1 - \alpha &= P_{H_0}[W_{r1} \in \mathcal{B}, W_r \in C_0] + P_{H_0}[W_{r1} \in \mathcal{A}_0] \\
&= P_{H_0}[Y_{r:n_1} < X_{a:m}] + P_{H_0}[Y_{r:n_1} \leq X_{b:m}, Y_{r:n} < X_{c:m}] - P_{H_0}[Y_{r:n_1} \leq X_{a:m}, Y_{r:n} < X_{c:m}]
\end{aligned} \tag{4}
$$

To obtain the probability in Eq. (4), it is necessary to obtain the joint distribution of $(Y_{r:n_1}, Y_{r:n})$. We follow a mixture approach to derive the joint distribution. Let us divide the $\boldsymbol{Y}$-sample into two mutually exclusive parts: $(Y_1, Y_2, ..., Y_{n_1})$, and $(Y_1^*, Y_2^*, ..., Y_{n_2}^*)$, where $n_1 + n_2 = n$. Then the joint distribution of $(Y_{r:n_1}, Y_{r:n})$ can be obtained following Result 1 and 2.

Result 1. Let D be the number of Y^* observations that are $\leq Y_{r:n_1}$. Then the probability mass function (p.m.f.) of D is

$$P[D = d] = \frac{\binom{r+d-1}{d}\binom{n-r-d}{n-n_1-d}}{\binom{n}{n_1}}, \quad d = 0, 1, 2, ..., n - n_1. \tag{5}$$

The proof of Result 1 is deferred to Appendix A.

Result 2. For $y, y_1 \in R^+$, R^+ being the positive real line, and $n_1 < n$, the joint distribution function of $(Y_{r:n_1}, Y_{r:n})$ is given by

Case I: for $y_1 \leq y$,

$$P[Y_{r:n} \leq y, Y_{r:n_1} \leq y_1] = P[Y_{r:n_1} \leq y_1] = \sum_{i=r}^{n_1} \binom{n_1}{i} G^i(y_1)[1 - G(y_1)]^{(n_1-i)} \tag{6}$$

Case II: for $y_1 > y$,

$$P[Y_{r:n} \leq y, Y_{r:n_1} \leq y_1] = \sum_{d=0}^{n-n_1} P[D = d]P[Y_{r:n} \leq y, Y_{r+d:n} \leq y_1],$$

where,

$$P[Y_{r:n} \leq y, Y_{r+d:n} \leq y_1] = \sum_{k=r}^{n} \sum_{k_1=r+d}^{n} \frac{n!}{k!(k_1 - k)!(n - k_1)!} G^k(y)[G(y_1) - G(y)]^{(k_1-k)}[1 - G(y_1)]^{(n-k_1)}. \tag{7}$$

Proof. Since $n_1 \leq n$, we have $Y_{r:n} \leq Y_{r:n_1}$. Therefore, for Case I, the proof is trivial. For Case II, note that, given $D = d$, $(Y_{r:n}, Y_{r:n_1}) \stackrel{d}{=} (Y_{r:n}, Y_{r+d:n})$ with $P[D = d]$ as in Result 1, where $\stackrel{d}{=}$ implies equality in distribution. Hence, the rest of the proof for Case II is straightforward using Result 1 and the known result on the joint distribution of two order statistics (Arnold et al. (2008)).

Remark: Note that the bivariate vector in (7) $(Y_{r:n}, Y_{r+d:n})$ has a singular part, which is captured by the case $d = 0$.

Note that, for $X_{a:m}, X_{b:m}, X_{c:m}$, there are three possibilities, (i) $X_{c:m} \geq X_{b:m}$; (ii) $X_{a:m} \leq$

$X_{c:m} \leq X_{b:m}$; (iii) $X_{c:m} \leq X_{a:m}$. Under H_0, let us consider the probabilities

$$P_{H_0}[W_{r1} \in \mathcal{A}_0] = 1 - \alpha_0,\ P_{H_0}[W_{r1} \in \mathcal{A}_1] = \alpha_1,\ P_{H_0}[W_r \in C_1] = \alpha_2. \tag{8}$$

Since $1 - P_{H_0}[W_{r1} \in \mathcal{B}, W_r \in C_0] - P_{H_0}[W_{r1} \in \mathcal{A}_0] \geq 1 - P_{H_0}[W_{r1} \in \mathcal{B}] - P_{H_0}[W_{r1} \in \mathcal{A}_0]$, we can write from Eq. (4) and (8),

$$\alpha \geq \alpha_1. \tag{9}$$

Note that, Eq. (9) implies that there is the enhanced protection against the Type-I error in the precedence test under the double sampling framework.

Result 3. If $P_{H_0}[W_{r1} \in \mathcal{B}] > 0$, then

$$P_{H_0}[Y_{r:n_1} < X_{a:m}] > \alpha_2 - \alpha > P_{H_0}[Y_{r:n_1} \leq X_{a:m}] - P_{H_0}[Y_{r:n} \leq X_{c:m}].$$

The proof is deferred to Appendix B.

We consider $\alpha_2 = \alpha$. Then, from Result 3, we get

$$P_{H_0}[Y_{r:n} \leq X_{c:m}] > P_{H_0}[Y_{r:n_1} \leq X_{a:m}] \Rightarrow \alpha_0 > \alpha, \tag{10}$$

when $P_{H_0}[W_{r1} \in \mathcal{B}] > 0$.

Let us consider the two possible cases, (i) $y = x_{c:m} > y_1 = x_{a:m}$; and (ii) $y = x_{c:m} < y_1 = x_{a:m}$ for $X_{a:m} = x_{a:m}, X_{c:m} = x_{c:m}$. For these two cases, we state the probability $P_{H_0}[Y_{r:n_1} \leq X_{a:m}, Y_{r:n} \leq X_{c:m}]$ readily in Results 4 and 5 as they can be easily obtained using Result 2 and known results on order statistics (Arnold et al. (2008)).

Result 4. Under H_0, for $n_1 + n_2 = n$, and $X_{c:m} > X_{a:m}$, the probability $I^{1,0}_{r,a,m,n,n_1} = P_{H_0}[Y_{r:n_1} \leq X_{a:m}, Y_{r:n} \leq X_{c:n}] = P_{H_0}[Y_{r:n_1} \leq X_{a:m}]$ is given by

$$I^{1,0}_{r,a,m,n,n_1} = \sum_{k=r}^{n_1} \frac{\binom{a+k-1}{k}\binom{n_1-k+m-a}{n_1-k}}{\binom{m+n_1}{n_1}} \tag{11}$$

Result 5. Under H_0, for $n_1 + n_2 = n$, and $X_{c:m} < X_{a:m}$, the probability $I^{2,0}_{r,a,c,m,n,n_1} = P_{H_0}[Y_{r:n_1} \leq X_{a:m}, Y_{r:n} \leq X_{c:n}]$ is given by

$$I^{2,0}_{r,a,c,m,n,n_1} = \sum_{z=r}^{n} \sum_{z_1=max(r,z-n_2)}^{min(n_1,z)} \sum_{k=max(0,z-n_2)}^{min(z,z_1)} \sum_{i=0}^{n_2-z+k} (-1)^i \frac{\binom{m+n}{n_2-z+k-i}\binom{n_1-z_1+m-a}{n_1-z_1}\binom{z+i+c-1}{c-1}\binom{z+i}{i}\binom{z}{k}\binom{z_1-k+a-c-1}{a-c-1}}{\binom{m+n}{n}}. \tag{12}$$

Let us consider $n_1 = [0.25n] + 1, [0.5n] + 1, [0.75n] + 1$, and from Eq. (10), $\alpha_0 = 2\alpha, 3\alpha, 4\alpha$. Then the probabilities in Eq. (4) can be obtained by suitably replacing a with b and c. For $\alpha_2 = \alpha = 0.05$, we obtain the values of a and c for different m, n, n_1, and r from Eq. (8). For example, when $m = n = 20$, $r = 3$, $\alpha_0 = 3\alpha$, for $\alpha_2 = \alpha = 0.05$, we have ($a = 7$, $c = 8$), for $n_1 = [0.75n] + 1$, and ($a = 10$, $c = 8$) for $n_1 = [0.5n] + 1$. [x] is the largest integer less than or equals to x.

Once we have the values of a and c, we can obtain the values of b from Eq. (4), using Results 4 and 5 for $X_{c:m} > X_{a:m}$ and $X_{c:m} < X_{a:m}$. The critical limits (a, b, c) for the proposed method and the critical limit a_p for the precedence test are provided in Table 2 along with the corresponding exact level of significance (l.o.s.) α_e.

3.2. Double sampling weighted precedence (DWPT) test

For the double sampling weighted precedence test, let us define the 'decision sub-intervals' as $\mathcal{A}^*_0 = [0, a^*)$, $\mathcal{A}^*_1 = [b^*, n_1 m]$, $\mathcal{B}^* = [a^*, b^*)$, $C^*_0 = [0, c^*)$, $C^*_1 = [c^*, nm]$. As in the double-sampling precedence test, we define $\mathcal{B}^*$ as the *warning* region. The steps to carry out the proposed double-sampling weighted precedence test are as follows:

i. We take n_1 random $\boldsymbol{Y}$-sample from the n sample and calculate the corresponding weighted precedence statistic $W^*_{r1} = \sum_{i=1}^{r}(n_1 - i + 1)m^1_i$, where $M^1_i = m^1_i$ is the number of $\boldsymbol{X}$-sample between the i^{th} and $(i-1)^{th}$ $\boldsymbol{Y}$-sample of size n_1;

ii. If $W^*_{r1} \in \mathcal{A}^*_0$, accept H_0, and if $W^*_{r1} \in \mathcal{A}^*_1$, we reject H_0;

iii. If $W^*_{r1} \in \mathcal{B}^*$, we merge the remaining $n_2 = n - n_1$ random $\boldsymbol{Y}$-sample with the first n_1 sample and obtain $W^*_r = \sum_{i=1}^{r}(n - i + 1)m_i$, where $M_i = m_i$ is the number of $\boldsymbol{X}$-sample

Table 2: Near 5% critical values (top) and exact l.o.s. α_e (bottom) for the double sampling precedence test (DPT) and the standard precedence test (PT)

			$r = 2$				$r = 3$				$r = 4$				$r = 5$			
			DPT			PT	DPT			PT	DPT			PT	DPT			PT
m	n	n_1	2α	3α	4α	-	2α	3α	4α	-	2α	3α	4α	-	2α	3α	4α	-
10	10	[0.25n]	(10,10,6)	(9,10,6)	(9,10,6)	6	(10,10,7)	(10,10,7)	(10,10,7)	7	-	-	-	8	-	-	-	9
			0.038	0.044	0.045	0.029	0.23	0.23	0.23	0.035	-	-	-	0.035	-	-	-	0.029
		[0.5n]	(8,8,6)	(7,9,6)	(6,9,6)		(9,10,7)	(9,10,7)	(8,10,7)		(10,10,8)	(10,10,8)	(10,10,8)		(10,10,9)	(10,10,9)	(10,10,9)	
			0.045	0.030	0.040		0.032	0.032	0.045		0.10	0.095	0.094		0.33	0.33	0.33	
		[0.75n]	(6,7,6)	(5,7,6)	(5,7,6)		(7,9,7)	(7,9,7)	(6,8,7)		(8,10,8)	(8,10,8)	(7,10,8)		(9,10,9)	(9,10,9)	(9,10,9)	
			0.040	0.040	0.040		0.051	0.051	0.050		0.036	0.036	0.036		0.042	0.042	0.042	
15	15	[0.25n]	(12,13,6)	(11,13,6)	(10,14,6)	6	(14,15,8)	(14,15,8)	(13,15,8)	8	(15,15,9)	(15,15,9)	(15,15,9)	9	-	-	-	10
			0.042	0.046	0.030	0.040	0.036	0.036	0.041	0.025	0.21	0.21	0.21	0.030	-	-	-	0.033
		[0.5n]	(8,9,6)	(7,10,6)	(6,11,6)		(10,11,8)	(9,12,8)	(8,12,8)		(12,13,9)	(11,13,9)	(10,13,9)		(13,14,10)	(13,14,10)	(12,14,10)	
			0.043	0.040	0.047		0.040	0.031	0.037		0.032	0.04	0.046		0.039	0.040	0.048	
		[0.75n]	(6,8,6)	(5,8,6)	(5,8,6)		(8,9,8)	(7,9,8)	(6,9,8)		(9,11,9)	(8,10,9)	(8,10,9)		(10,12,10)	(10,12,10)	(9,12,10)	
			0.046	0.045	0.046		0.040	0.039	0.040		0.051	0.050	0.051		0.042	0.042	0.042	
20	20	[0.25n]	(13,15,6)	(12,16,6)	(11,16,6)	6	(17,18,8)	(16,18,8)	(15,18,8)	8	(19,20,9)	(19,20,9)	(18,20,9)	9	(20,20,10)	(20,20,10)	(20,20,10)	10
			0.048	0.035	0.040	0.046	0.042	0.046	0.049	0.032	0.036	0.036	0.042	0.041	0.20	0.20	0.20	0.048
		[0.5n]	(8,11,6)	(8,11,6)	(7,11,6)		(11,12,8)	(10,13,8)	(9,13,8)		(13,15,9)	(12,15,9)	(11,15,9)		(15,16,10)	(14,17,10)	(13,17,10)	
			0.040	0.040	0.049		0.048	0.040	0.046		0.032	0.038	0.044		0.046	0.036	0.043	
		[0.75n]	(6,10,6)	(6,10,6)	(5,10,6)		(8,10,8)	(7,10,8)	(7,10,8)		(10,12,9)	(9,13,9)	(8,13,9)		(11,13,10)	(10,20,10)	(10,16,10)	
			0.051	0.051	0.050		0.047	0.047	0.047		0.050	0.044	0.044		0.049	0.050	0.050	
30	30	[0.25n]	(14,16,7)	(12,17,7)	(11,17,7)	7	(18,20,8)	(17,21,8)	(15,21,8)	8	(22,23,10)	(20,24,10)	(19,24,10)	10	(25,26,11)	(23,27,11)	(23,27,11)	11
			0.047	0.040	0.043	0.026	0.046	0.039	0.044	0.040	0.037	0.041	0.044	0.029	0.046	0.036	0.036	0.036
		[0.5n]	(9,11,7)	(8,11,7)	(7,12,7)		(11,14,8)	(10,14,8)	(9,15,8)		(14,16,10)	(13,16,10)	(12,16,10)		(16,18,11)	(15,18,11)	(14,19,11)	
			0.040	0.046	0.040		0.042	0.049	0.047		0.038	0.042	0.046		0.041	0.047	0.043	
		[0.75n]	(7,8,7)	(6,8,7)	(5,8,7)		(8,11,8)	(7,11,8)	(7,11,8)		(10,12,10)	(9,12,10)	(8,12,10)		(12,13,11)	(11,14,11)	(10,14,11)	
			0.046	0.047	0.046		0.047	0.048	0.047		0.047	0.048	0.047		0.055	0.047	0.046	
30	50	[0.25n]	(10,12,5)	(9,12,5)	(8,12,5)	5	(13,15,6)	(12,15,6)	(11,15,6)	6	(15,17,7)	(14,17,7)	(13,18,7)	7	(18,20,8)	(17,20,8)	(16,20,8)	8
			0.037	0.039	0.043	0.026	0.037	0.040	0.043	0.029	0.046	0.048	0.039	0.029	0.037	0.039	0.042	0.028
		[0.5n]	(6,8,5)	(5,8,5)	(5,8,5)		(8,9,6)	(7,10,6)	(6,10,6)		(9,11,7)	(9,11,7)	(8,12,7)		(11,13,8)	(10,13,8)	(9,13,8)	
			0.036	0.045	0.045		0.048	0.041	0.047		0.046	0.048	0.040		0.036	0.044	0.046	
		[0.75n]	(5,6,5)	(4,6,5)	(4,6,5)		(6,7,6)	(5,7,6)	(5,7,6)		(7,8,7)	(6,9,7)	(6,9,7)		(8,10,8)	(7,10,8)	(7,10,8)	
			0.037	0.037	0.037		0.048	0.048	0.048		0.036	0.036	0.036		0.039	0.039	0.039	

between the i^{th} and $(i-1)^{th}$ pooled $\boldsymbol{Y}$-sample;

iv. If $W_r^* \in C_0^*$, we accept H_0. If $W_r^* \in C_1^*$, we reject H_0.

Note that $M_i^1 \geq M_i$ as $n_1 \leq n$. Let $p^*_{10|H} = P_H[W^*_{r1} \in \mathcal{A}^*_0]$, $p^*_{11|H} = P_H[W^*_{r1} \in \mathcal{A}^*_1]$, $p^*_{1w|H} = P_H[W^*_{r1} \in \mathcal{B}^*]$ be the probabilities that W^*_{r1} would fall in the acceptance region, rejection region, or the warning region, respectively, when $H = H_0$ or H_1. Similarly, $p^*_{20|H} = P_H[W^*_r \in C^*_0]$ and $p^*_{21|H} = P_H[W^*_r \in C^*_1]$ are the probabilities for W^*_r to fall in the acceptance or rejection region, when $H = H_0$ or H_1, respectively.

3.2.1. Null distribution

Under H_0, for a given α, the probability $(1-\alpha)$ that H_0 will not be rejected is

$$\begin{aligned}1-\alpha &= P_{H_0}[W^*_{r1} \in \mathcal{B}^*, W^*_r \in C^*_0] + P_{H_0}[W^*_{r1} \in \mathcal{A}^*_0] \\ &= P_{H_0}[W^*_{r1} < a^*] + P_{H_0}[W^*_{r1} < b^*, W^*_r < c^*] - P_{H_0}[W^*_{r1} \leq a^*, W^*_r < c^*]. \quad (13)\end{aligned}$$

The probability $P_{H_0}[W^*_{r1} < a^*]$ in Eq.(13) can be obtained from the results of Balakrishnan and Ng (2001) as,

$$P_{H_0}[W^*_{r1} \leq a^*] = \sum_{\substack{m_i^1 (i=1,2,\ldots,r)=0, \\ 0 \leq \sum_{i=1}^r (n_1-i+1)m_i^1 \leq a^*}}^{m} P_{H_0}[M_1^1 = m_1^1, M_2^1 = m_2^1, \ldots, M_r^1 = m_r^1]. \quad (14)$$

where,

$$P_{H_0}[M_1^1 = m_1^1, M_2^1 = m_2^1, \ldots, M_r^1 = m_r^1] = \frac{\binom{m+n_1-\sum_{i=1}^s m_i^1 - s}{n_1-s}}{\binom{m+n_1}{n_1}},$$

To obtain the probability in Eq. (13), it is necessary to obtain the joint distribution of (W^*_{r1}, W^*_r). This is given in Result 6.

Result 6. For $0 \leq a^* \leq mn_1$, and $0 \leq c^* \leq mn$, under H_0,

$$
\begin{aligned}
&P_{H_0}[W^*_{r1} \leq a^*, W^*_r \leq c^*] \\
&= \Bigg[\sum_{\substack{m_i(i=1,2,\ldots,r)=0,\\ 0\leq\sum_{i=1}^{r}(n-i+1)m_i\leq c^*}}^{m} \; \sum_{\substack{l_i(i=1,2,\ldots,r)=1,\\ 0\leq n_1 \sum_{i=1}^{l_1} m_i+\sum_{j=2}^{r}(n_1-j+1)\sum_{i=\sum_{k=1}^{j-1} l_k+1}^{\sum_{k=1}^{j} l_k} m_i\leq a^*\\ r\leq\sum_{k=1}^{r} l_k\leq n}}^{n} P_{H_0}[M_1 = m_1, M_2 = m_2, \ldots, M_{\sum_{k=1}^r l_k} = m_{\sum_{k=1}^r l_k}]\Bigg] \\
&\qquad \Bigg[\sum_{\substack{m_i^1(i=1,2,\ldots,r)=0,\\ 0\leq\sum_{i=1}^{r}(n_1-i+1)m_i^1\leq a^*}}^{m} P_{H_0}[M_1^1 = m_1^1, M_2^1 = m_2^1, \ldots, M_r^1 = m_r^1]\Bigg], \qquad (15)
\end{aligned}
$$

where

$$
P_{H_0}[V_1 = v_1, V_2 = v_2, \ldots, V_s = v_s] = \frac{\binom{m+n'-\sum_{i=1}^{s} v_i - s}{n_1 - s}}{\binom{m+n'}{n'}},
$$

from Balakrishnan and Ng (2001), for $V_i = M_i^1, M_i$, $n' = n_1, n$, and $s = \sum_{k=1}^r l_k, r$, for the first sample and the pooled sample, respectively.

The proof of Result 6 is deferred to Appendix C. Let us consider now the probabilities

$$
P_{H_0}[W^*_{r1} \in \mathcal{A}^*_0] = 1 - \alpha^*_0,\; P_{H_0}[W^*_{r1} \in \mathcal{A}^*_1] = \alpha^*_1,\; P_{H_0}[W^*_r \in C^*_1] = \alpha^*_2. \qquad (16)
$$

Using similar arguments as in Eq.(10), we get

$$
\alpha \geq \alpha^*_1. \qquad (17)
$$

This implies that, as in the double sampling precedence test, the enhanced protection against the Type-I error is obtained for the double sampling weighted precedence test as well.

Result 7. If $P_{H_0}[W^*_{r1} \in \mathcal{B}^*] > 0$, then

$$
P_{H_0}[W^*_{r1} \in \mathcal{A}^*_0] > \alpha^*_2 - \alpha > P_{H_0}[W^*_{r1} \in \mathcal{A}^*_0] - P_{H_0}[W^*_r \in C^*_0]. \qquad (18)
$$

The proof of this result is similar to that of Result 3. Taking $\alpha^*_2 = \alpha$, from Result 7, we get

$$P_{H_0}[W_r^* \in C_0^*] > P_{H_0}[W_{r1}^* \in \mathcal{A}_0^*] \Rightarrow \alpha_0^* > \alpha, \tag{19}$$

when $P_{H_0}[W_{r1}^* \in \mathcal{B}^*] > 0$.

In case of the double sampling precedence test, no significant differences in the critical limits are observed for $\alpha_0 = 2\alpha, 3\alpha, 4\alpha$. Therefore, to obtain the critical limits for the double sampling weighted precedence test, we consider $n_1 = [0.5n] + 1, [0.75n] + 1$, and $\alpha_0^* = 2\alpha$. For $\alpha = 0.05$, taking $\alpha_2^* = \alpha$ and $\alpha_0^* = 2\alpha$, we obtain the values of a^* and c^* for different m, n, n_1, and r from Eq.(16). Then from Eq. (13), we can obtain the values of b^*. The critical limits (a^*, b^*, c^*) for the proposed method and the critical limit a_p for the standard precedence test are provided in Table 3 along with the corresponding exact levels of significance (l.o.s.) α_e^*.

Table 3: Near 5% critical values (top) and exact l.o.s. α_e^* (bottom) for the double sampling weighted precedence test (DWPT) and the standard precedence test(PT).

			$r = 2$		$r = 3$		$r = 4$		$r = 5$	
m	n	n_1	DWPT	PT	DWPT	PT	DWPT	PT	DWPT	PT
10	10	$[0.5n]$	29,39,47	6	34,40,56	7	35,41,62	8	36,41,66	9
			0.048	0.029	0.046	0.035	0.048	0.035	0.049	0.029
		$[0.75n]$	37,80,47		43,80,56		48,80,62		52,62,66	
			0.057		0.053		0.051		0.049	
15	15	$[0.5n]$	52,68,74	6	62,76,95	8	70,83,107	9	75,86,118	10
			0.049	0.040	0.048	0.025	0.047	0.030	0.051	0.033
		$[0.75n]$	58,180,74		74,180,92		85,179,106		94,169,117	
			0.053		0.051		0.051		0.048	
20	20	$[0.5n]$	70,94,100	6	88,110,132	8	101,121,152	9	111,130,170	10
			0.049	0.046	0.053	0.032	0.049	0.041	0.050	0.048
		$[0.75n]$	75,300,100		101,190,133		119,168,152		134,174,170	
			0.053		0.046		0.049		0.048	
30	30	$[0.5n]$	115,152,174	7	145,185,205	8	172,212,247	10	194,232,280	11
			0.053	0.026	0.049	0.040	0.047	0.029	0.046	0.036
		$[0.75n]$	132,690,174		160,690,205		196,277,249		224,298,279	
			0.052		0.045		0.048		0.050	

4. Power comparison

In this section, we compare the power of the proposed test procedures with the precedence test (PT), maximal precedence test (MPT), weighted precedence test (WPT), and weighted maximal precedence test (WMPT). The power values for the competing precedence-type test procedures are obtained from Ng and Balakrishnan (2005).

4.1. Power comparison under Lehmann alternative

We obtain the closed-form expression under Lehmann alternative for the power function of the precedence test and the weighted precedence test under the double sampling framework. Lehmann alternative $H_1 : G(x) = F^{\gamma}(x)$ for some γ was first proposed by Lehmann (1953). It is a subclass of the alternative $H_1 : F(x) > G(x)$ for $\gamma > 1$. For more details about Lehmann alternative, please refer to Lehmann (1975), Wolfe (2012), Hollander et al. (2013).

To derive the power function under the Lehmann alternative for the double sampling precedence test, we consider the cases (i) $y = x_{c:m} > y_1 = x_{a:m}$; and (ii) $y = x_{c:m} \leq y_1 = x_{a:m}$ for $X_{a:m} = x_{a:m}, X_{c:m} = x_{c:m}$. The probability $P_{H_1}[Y_{r:n_1} \leq X_{a:m}, Y_{r:n} \leq X_{c:m}]$ is readily stated in Results 8 and 9 for case (i) and (ii), respectively, as they can be easily obtained using Result 2 and known results on order statistics (Arnold et al. (2008)).

Result 8. Under $H_1 : G(x) = F^{\gamma}(x)$, for $n_1 + n_2 = n$, and $X_{c:m} > X_{a:m}$, the probability $I^{1,1}_{r,a,m,n,n_1} = P_{H_1}[Y_{r:n_1} \leq X_{a:m}, Y_{r:n} \leq X_{c:m}] = P_{H_1}[Y_{r:n_1} \leq X_{a:m}]$ is given by

$$I^{1,1}_{r,a,m,n,n_1} = \sum_{k=r}^{n_1} \sum_{j=0}^{n_1-k} (-1)^j \frac{\binom{n_1}{k\ j}\binom{\gamma(k+j)+a-1}{a-1}}{\binom{\gamma(k+j)+m}{m}}. \tag{20}$$

Result 9. Under $H_1 : G(x) = F^{\gamma}(x)$, for $n_1 + n_2 = n$, and $X_{c:m} < X_{a:m}$, the probability $I^{2,1}_{r,a,c,m,n,n_1} = P_{H_1}[Y_{r:n_1} \leq X_{a:m}, Y_{r:n} \leq X_{c:m}]$ is given by

$$I^{2,1}_{r,a,c,m,n,n_1} = \sum_{z=r}^{n} \sum_{z_1=max(r,z-n_2)}^{min(n_1,z)} \sum_{k=max(0,z-n_2)}^{min(z_1,z)} \sum_{i_1=0}^{z_1-k} \sum_{i_2=0}^{n_2-z+k} \sum_{i_3=0}^{n_1-z_1} (-1)^{(i_1+i_2+i_3)}$$
$$\frac{\binom{\gamma(i_1+i_2)+\gamma z+c-1}{c-1}\binom{\gamma(z+z_1)+\gamma(i_2+i_3)-\gamma k+a-1}{\gamma(z+z_1)+\gamma(i_2+i_3)-\gamma k}\binom{n}{(n_1-z_1-i_3)(n_2-z+k-i_2)}\binom{z+z_1+i_2+i_3-k}{z-k}\binom{z_1+i_2+i_3}{i_2\ i_3}\binom{z_1}{k\ i_1}}{\binom{\gamma(z+z_1)+\gamma(i_2+i_3)-\gamma k+m}{m}\binom{\gamma(i_1+i_2)+\gamma z+a-1}{a-1}}. \tag{21}$$

Then the power function for the double sampling precedence test can be obtained as

$$\beta = 1 - P_{H_1}[Y_{r:n_1} < X_{a:m}] - P_{H_1}[Y_{r:n_1} < X_{b:m}, Y_{r:n} < X_{c:m}] + P_{H_1}[Y_{r:n_1} < X_{a:m}, Y_{r:n} < X_{c:m}]. \tag{22}$$

For the double sampling weighted precedence test, we can obtain $P_{H_1}[W^*_{r1} \leq a^*, W^*_r \leq c^*]$ under the Lehmann alternative from Result 6 by replacing the joint probability

$$P_{H_1}[V_1 = v_1, V_2 = v_2, \ldots, V_r = v_r] = \frac{m!n^{'}!\gamma^r}{v_1!(n^{'} - r)!}\left\{\prod_{j=1}^{r-1} \frac{\Gamma(\sum_{i=1}^{j} v_i + j\gamma)}{\Gamma(\sum_{i=1}^{j+1} v_i + j\gamma + 1)}\right\}$$
$$\sum_{k=0}^{n^{'}-r}\binom{n^{'} - r}{k}(-1)^k \times \frac{\Gamma(\sum_{i=1}^{r} v_i + (r + k)\gamma)}{\Gamma(m + (r + k)\gamma + 1)}, \quad (23)$$

from Balakrishnan and Ng (2001), for $V_i = M^1_i, M_i$, $n^{'} = n_1, n$, for the first sample and the pooled sample, respectively. The power function for the double sampling weighted precedence test can be obtained as

$$\beta^* = 1 - P_{H_1}[W^*_{r1} \leq a^*] - P_{H_1}[W^*_{r1} \leq b^*, W^*_r \leq c^*] + P_{H_1}[W^*_{r1} \leq a^*, W^*_r \leq c^*]. \quad (24)$$

It is to be noted from the above derivations that the double sampling precedence and weighted precedence tests are distribution-free under the Lehmann alternative. Though we have an explicit expression for the power function under the Lehmann alternative for the proposed test procedures, we cannot have the same for location shift alternative. To be consistent, we estimate the power values by 20000 Monte-Carlo simulations.

Note that, in case of double sampling precedence and weighted precedence test, for $W = W_{r1}, W^*_{r1}$,

$$n = n_1 I[W \in \mathcal{B}^c] + (n_1 + n_2)I[W \in \mathcal{B}]. \quad (25)$$

This implies,

$$\tilde{n} = E_H(n) = n_1 + n_2 P_H[W \in \mathcal{B}] \leq n, \quad (26)$$

for $H = H_0, H_1$.

Eq. (26) shows that the expected sampled size in double sampling framework is always smaller than the single sampling framework. Under H_1, when $P_{H_1}[W \in \mathcal{B}]$ increases, for

$W = W_{r1}, W^*_{r1}$, the expected sample size $\tilde{n}$ also increases. Therefore, the double sampling precedence and weighted precedence test eliminate the burden of unnecessary sampling and only collect additional $\boldsymbol{Y}$ sample depending on the amount of departure from H_0.

For $m = n = 10$, $r = 2, 3$, $\gamma = 2, .., 6$, we estimate the power values for the double sampling precedence and weighted precedence test. For the double sampling precedence test, we take $n_1 = [0.75n] + 1, [0.5n] + 1$, and for the double sampling weighted precedence test, we take $n_1 = [0.75n] + 1$. The power comparison under the Lehmann alternative is presented in Table 4. For the competing precedence-type tests in Table 4, power values are obtained from Ng and Balakrishnan (2005).

In Table 4, it is observed that, while all the competing tests have similar size, the double sampling precedence test attains more power than the precedence test, maximal precedence test and weighted maximal precedence test in many cases. The double sampling weighted precedence test furnish the best performance among its competitors under the Lehmann alternative. It is clearly seen from Table 4 that the double sampling precedence and weighted precedence tests can achieve better power with a reduced sample size.

Table 4: Power comparison under Lehmann alternative for $m = n = 10$.

r = 2															
n	n_1	Test	Crit. Lim.	$E(n)$	$\gamma = 1$	$E(n)$	$\gamma = 2$	$E(n)$	$\gamma = 3$	$E(n)$	$\gamma = 4$	$E(n)$	$\gamma = 5$	$E(n)$	$\gamma = 6$
10	-	PT	6	10	0.029	10	0.240	10	0.490	10	0.673	10	0.790	10	0.862
10	-	MPT	4	10	0.032	10	0.220	10	0.453	10	0.636	10	0.760	10	0.841
10	-	WPT	48	10	0.049	10	0.357	10	0.644	10	0.810	10	0.896	10	0.941
10	-	WMPT	40	10	0.059	10	0.357	10	0.630	10	0.794	10	0.884	10	0.933
10	8	DPT	6,7,6	8.07	0.040	8.28	0.278	8.38	0.541	8.36	0.710	8.31	0.820	8.26	0.880
10	5	DPT	8,8,6	5	0.045	5	0.203	5	0.365	5	0.511	5	0.620	5	0.705
10	8	DWPT	37,80,47	8.20	0.055	8.95	0.381	9.44	0.660	9.70	0.823	9.80	0.911	9.84	0.947
r=3															
10	-	PT	7	10	0.035	10	0.246	10	0.481	10	0.652	10	0.764	10	0.837
10	-	MPT	4	10	0.049	10	0.242	10	0.468	10	0.646	10	0.766	10	0.845
10	-	WPT	56	10	0.051	10	0.351	10	0.631	10	0.796	10	0.884	10	0.931
10	-	WMPT	45	10	0.038	10	0.228	10	0.458	10	0.639	10	0.762	10	0.842
10	8	DPT	7,9,7	8.13	0.051	8.55	0.257	8.82	0.501	8.88	0.677	8.88	0.785	8.82	0.848
10	5	DPT	9,10,7	5.268	0.032	5.78	0.161	6.20	0.321	6.48	0.448	6.52	0.549	6.63	0.640
10	8	DWPT	43,80,56	8.23	0.051	8.96	0.359	9.46	0.638	9.69	0.798	9.80	0.892	9.84	0.941

4.2. Power comparison under location shift

To assess the performance of the proposed precedence-type tests under location shift, we consider the alternative $H_1 : G(x) = F(x + \theta)$ for $\theta > 0$. We consider the location shift $\theta = 0.5, 1$ for the following distributions: i. standard normal distribution; ii. standard exponential distribution; iii. gamma distribution with shape parameter $\beta = 2, 10$, standardized by mean β and standard deviation $\sqrt{\beta}$; iv. lognormal distribution with shape parameter $\xi = 0.1, 0.5$, standardized by mean $e^{\xi^2/2}$ and standard deviation $\sqrt{e^{\xi^2}(e^{\xi^2} - 1)}$. More details about these distributions could be found in Rohatgi and Saleh (2015). For both double sampling precedence and weighted precedence tests, we set $\alpha_0 = 2\alpha$, where $\alpha = 0.05$.

For $m = n = 10, 20$, $r = 2, 3, 4, 5, 6$, the power values for the double sampling precedence and weighted precedence tests are reported in Table 5 - 8. For the competing precedence-type tests, i.e., the precedence test, the weighted precedence test, maximal precedence test, and the weighted maximal precedence test, the power values are obtained from Ng and Balakrishnan (2005). We also include the exact level of significance for all competing test procedures. For the double sampling precedence and the weighted precedence test, we set $n_1 = [0.75n] + 1, [0.5n] + 1$.

Table 5: Power values, and expected sample size $\tilde{n}$, for m = 10, θ =0.5, α_0 = 2α.

r	Dist	PT	MPT	WPT	WMPT	DPT	DPT	DWPT	DWPT
		n=10	n=10	n=10	n=10	n=10,n_1=8	n = 10, n_1 = 5	n = 10, n_1 = 8	n = 10, n_1 = 5
2	Exact l.o.s.	0.029	0.032	0.049	0.059	0.040,8.07	0.045,5	0.055, 8.20	0.048,5.50
	N(0,1)	0.150	0.142	0.228	0.231	0.196,8.20	-	0.248,8.65	0.213,6.21
	Exp(1)	0.392	0.492	0.614	0.720	0.407,8.40	-	0.771,12.41	0.493,6.92
	Gamma(2)	0.252	0.247	0.408	0.434	0.280,8.22	-	0.523,11.77	0.337,6.57
	Gamma(10)	0.173	0.161	0.271	0.274	0.208,8.10	-	0.319,11.20	0.240,6.31
	LN(0.1)	0.160	0.148	0.238	0.246	0.194,8.22	-	0.273,11.11	0.223,6.26
	LN(0.5)	0.269	0.254	0.421	0.430	0.300,8.32	-	0.511,11.76	0.350,6.58
3	Exact l.o.s.	0.035	0.049	0.051	0.038	0.051,8.09	0.032,5.28	0.053, 8.23	0.046,5.36
	N(0,1)	0.178	0.175	0.240	0.154	0.223,8.26	0.140,5.72	0.248,8.74	0.213,5.96
	Exp(1)	0.307	0.495	0.512	0.489	0.320,8.64	0.161,5.73	0.509,9.19	0.384,6.28
	Gamma(2)	0.234	0.258	0.371	0.248	0.269,8.24	0.144,5.55	0.374,8.96	0.285,6.12
	Gamma(10)	0.189	0.187	0.272	0.173	0.233,8.15	0.133,5.39	0.294,8.74	0.225,5.98
	LN(0.1)	0.183	0.178	0.252	0.156	0.231,8.27	0.141,5.73	0.266,8.69	0.218,5.98
	LN(0.5)	0.257	0.268	0.384	0.256	0.274,8.57	0.156,5.76	0.396,9.02	0.301,6.15
4	Exact l.o.s.	0.035	0.065	0.051	0.045	0.036,8.17	0.1	0.051,8.23	0.048,5.45
	N(0,1)	0.177	0.199	0.249	0.172	0.193,8.53	-	0.252,8.78	0.229,6.15
	Exp(1)	0.225	0.497	0.421	0.485	0.232,8.60	-	0.421,8.23	0.378,6.49
	Gamma(2)	0.194	0.265	0.336	0.253	0.229,8.26	-	0.337,8.93	0.299,6.32
	Gamma(10)	0.176	0.204	0.268	0.179	0.185,8.26	-	0.274,8.80	0.241,6.18
	LN(0.1)	0.178	0.199	0.256	0.171	0.219,8.31	-	0.262,8.78	0.231,6.14
	LN(0.5)	0.215	0.277	0.355	0.267	0.224,8.58	-	0.361,8.98	0.311,6.36
5	Exact l.o.s.	0.027	0.027	0.051	0.045	0.042,8.13	0.331	0.049,8.16	0.049,5.39
	N(0,1)	0.147	0.090	0.256	0.164	0.188,8.37	-	0.263,8.50	0.232,6.02
	Exp(1)	0.145	0.268	0.425	0.486	0.171,8.97	-	0.438,8.62	0.386,6.37
	Gamma(2)	0.138	0.120	0.337	0.251	0.166,8.27	-	0.341,8.57	0.299,6.20
	Gamma(10)	0.139	0.090	0.267	0.174	0.170,8.19	-	0.273,8.51	0.242,6.04
	LN(0.1)	0.144	0.090	0.261	0.168	0.182,8.36	-	0.266,8.50	0.232,6.02
	LN(0.5)	0.153	0.130	0.354	0.270	0.180,8.36	-	0.357,8.59	0.312,6.22
6	Exact l.o.s.	0.065	0.032	0.049	0.045	0.08	-	0.052, 8.22	
	N(0,1)	0.266	0.094	0.259	0.172	-	-	0.278,8.82	-
	Exp(1)	0.239	0.269	0.410	0.484	-	-	0.435,9.14	-
	Gamma(2)	0.234	0.121	0.326	0.252	-	-	0.339,8.96	-
	Gamma(10)	0.244	0.092	0.279	0.179	-	-	0.285,8.84	-
	LN(0.1)	0.257	0.094	0.260	0.171	-	-	0.281,8.82	-
	LN(0.5)	0.253	0.131	0.347	0.267	-	-	0.371,8.99	-

Table 6: Power values, and expected sample size $\tilde{n}$, for $m = 10$, θ =1.0, $\alpha_0 = 2\alpha$.

r	Dist	PT	MPT	WPT	WMPT	DPT	DPT	DWPT	DWPT
		n=10	n=10	n=10	n=10	n=10,n_1=8	n=10,n_1=5	n=10,n_1=8	n=10,n_1=5
2	Exact l.o.s.	0.029	0.032	0.049	0.059	0.040,8.07	0.045,5	0.055, 8.20	0.048,5.50
	N(0,1)	0.424	0.377	0.528	0.507	0.486,8.31	-	0.566,9.28	0.527,6.67
	Exp(1)	0.838	0.918	0.947	0.978	0.846,8.37	-	0.951,9.86	0.886,6.97
	Gamma(2)	0.688	0.730	0.852	0.888	0.716,8.40	-	0.858,9.74	0.778,7.09
	Gamma(10)	0.506	0.457	0.649	0.631	0.552,8.16	-	0.668,9.46	0.596,6.82
	LN(0.1)	0.457	0.408	0.580	0.554	0.509,8.34	-	0.613,9.36	0.556,6.76
	LN(0.5)	0.725	0.733	0.873	0.880	0.746,8.39	-	0.882,9.77	0.800,7.01
3	Exact l.o.s.	0.035	0.049	0.051	0.038	0.051,8.09	0.032,5.28	0.053, 8.23	0.046,5.36
	N(0,1)	0.481	0.417	0.584	0.394	0.554,8.37	0.409,6.25	0.588,9.41	0.542,6.33
	Exp(1)	0.724	0.918	0.882	0.916	0.732,8.98	0.427,6.40	0.894,9.78	0.790,6.44
	Gamma(2)	0.615	0.733	0.799	0.734	0.650,8.41	0.395,5.95	0.803,9.68	0.691,6.47
	Gamma(10)	0.518	0.476	0.654	0.461	0.574,8.21	0.489,5.52	0.661,9.50	0.577,6.44
	LN(0.1)	0.496	0.440	0.609	0.414	0.558,8.39	0.391,6.26	0.618,9.45	0.556,6.36
	LN(0.5)	0.668	0.736	0.822	0.729	0.678,8.98	0.424,6.40	0.834,9.74	0.728,6.43
4	Exact l.o.s.	0.035	0.065	0.051	0.045	0.036,8.17	0.1	0.051,8.23	0.048,5.45
	N(0,1)	0.480	0.436	0.608	0.398	0.506,8.92	-	0.612,9.47	0.573,6.54
	Exp(1)	0.578	0.918	0.792	0.916	0.578,9.05	-	0.804,9.74	0.774,6.60
	Gamma(2)	0.511	0.734	0.717	0.744	0.554,8.43	-	0.733,9.64	0.695,6.62
	Gamma(10)	0.476	0.485	0.644	0.471	0.489,8.37	-	0.649,9.51	0.601,6.68
	LN(0.1)	0.478	0.454	0.616	0.417	0.537,8.45	-	0.628,9.49	0.582,6.58
	LN(0.5)	0.565	0.738	0.776	0.738	0.573,9.05	-	0.779,9.71	0.732,6.62
5	Exact l.o.s.	0.027	0.027	0.051	0.045	0.042,8.13	0.331	0.049,8.16	0.049,5.39
	N(0,1)	0.424	0.248	0.633	0.405	0.486,8.58	-	0.639,8.63	0.578,6.37
	Exp(1)	0.396	0.779	0.819	0.911	0.418,8.60	-	0.831,8.56	0.788,6.36
	Gamma(2)	0.372	0.514	0.737	0.741	0.406,8.45	-	0.751,8.60	0.706,6.43
	Gamma(10)	0.383	0.283	0.656	0.468	0.432,8.26	-	0.664,8.64	0.608,6.50
	LN(0.1)	0.404	0.259	0.633	0.426	0.458,8.59	-	0.645,8.64	0.587,6.39
	LN(0.5)	0.414	0.528	0.782	0.746	0.447,8.63	-	0.787,8.60	0.740,6.42
6	Exact l.o.s.	0.065	0.032	0.049	0.045	0.08	-	0.052, 8.22	
	N(0,1)	0.583	0.249	0.637	0.398	-	-	0.661,9.54	-
	Exp(1)	0.499	0.779	0.796	0.916	-	-	0.817,9.73	-
	Gamma(2)	0.495	0.514	0.720	0.745	-	-	0.745,9.66	-
	Gamma(10)	0.527	0.284	0.664	0.471	-	-	0.681,9.55	-
	LN(0.1)	0.554	0.260	0.638	0.417	-	-	0.664,9.54	-
	LN(0.5)	0.531	0.529	0.762	0.738	-	-	0.784,9.71	-

Table 7: Power values, and expected sample size $\tilde{n}$, for m = 20, θ =0.5, α_0 = 2α.

r	Dist.	PT	MPT	WPT	WMPT	DPT	DPT	DWPT	DWPT
		n=20	n=20	n=20	n=20	n=20,n_1=15	n=20,n_1=10	n=20,n_1=15	n=20,n_1=10
2	Exact l.o.s.	0.046	0.047	0.053	0.047	0.051,15.45	0.040,10.79	0.053,15.54	0.049,10.74
	N(0,1)	0.266	0.243	0.291	0.24	0.279,16.50	0.254,12.30	0.280,17.04	0.289,11.99
	Exp(1)	0.936	0.963	0.97	0.964	0.934,17.38	0.835,12.62	0.969,19.87	0.588,13.12
	Gamma(2)	0.671	0.658	0.735	0.656	0.675,17.16	0.565,12.08	0.733,18.95	0.614,12.96
	Gamma(10)	0.364	0.324	0.393	0.324	0.387,15.82	0.324,11.28	0.399,17.58	0.366,12.22
	LN(0.1)	0.304	0.274	0.32	0.266	0.321,16.69	0.287,12.37	0.324,17.24	0.319,12.10
	LN(0.5)	0.645	0.598	0.692	0.605	0.657,17.50	0.550,13.45	0.696.18.86	0.603,12.78
3	Exact l.o.s.	0.064	0.03	0.05	0.044	0.047,15.30	0.048,10.29	0.046,15.50	0.053,10.77
	N(0,1)	0.349	0.181	0.31	0.233	0.302,15.95	0.289,10.86	0.300,17.04	0.316,12.06
	Exp(1)	0.903	0.906	0.902	0.963	0.815,16.30	0.584,11.45	0.904,18.88	0.788,13.28
	Gamma(2)	0.677	0.484	0.674	0.649	0.571,16.16	0.421,10.92	0.660,18.42	0.559,12.86
	Gamma(10)	0.439	0.233	0.41	0.317	0.379,15.51	0.319,10.51	0.392,17.41	0.374,12.37
	LN(0.1)	0.383	0.201	0.349	0.27	0.327,16.01	0.293,10.92	0.337,17.20	0.341,12.18
	LN(0.5)	0.684	0.447	0.677	0.588	0.593,16.32	0.423,12.29	0.661,18.33	0.581,12.83
4	Exact l.o.s.	0.041	0.04	0.05	0.054	0.050,15.15	0.032,10.52	0.049,15.48	0.049,10.82
	N(0,1)	0.295	0.214	0.33	0.258	0.318,15.50	0.257,11.78	0.325,16.61	0.335,12.11
	Exp(1)	0.75	0.905	0.858	0.962	0.687,15.71	0.444,12.55	0.830,17.30	0.709,12.85
	Gamma(2)	0.532	0.495	0.645	0.66	0.503,15.56	0.340,11.66	0.610,17.14	0.527,12.66
	Gamma(10)	0.358	0.255	0.423	0.344	0.366,15.27	0.313,10.52	0.406,16.79	0.381,12.31
	LN(0.1)	0.322	0.227	0.37	0.296	0.310,15.98	0.305,10.96	0.352,16.68	0.356,12.20
	LN(0.5)	0.561	0.457	0.664	0.599	0.539,15.68	0.381,12.32	0.617,17.15	0.554,12.77
5	Exact l.o.s.	0.048	0.05	0.05	0.044	0.049,15.31	0.046,10.32	0.048,15.45	0.050,10.76
	N(0,1)	0.332	0.236	0.35	0.225	0.347,16.04	0.276,11.02	0.351,16.49	0.347,12.22
	Exp(1)	0.69	0.906	0.804	0.904	0.630,16.49	0.337,11.26	0.778,16.87	0.666,12.93
	Gamma(2)	0.517	0.5	0.598	0.497	0.480,16.05	0.297,10.87	0.580,16.88	0.504,12.70
	Gamma(10)	0.381	0.271	0.419	0.263	0.354,15.72	0.273,10.56	0.420,16.62	0.380,12.43
	LN(0.1)	0.353	0.247	0.382	0.247	0.333,16.49	0.273,11.03	0.376,16.55	0.360,12.31
	LN(0.5)	0.556	0.464	0.625	0.457	0.523,16.33	0.327,11.18	0.610,16.87	0.530,12.77
6	Exact l.o.s.	0.053	0.06	0.05	0.049	0.041,15.15	0.030,10.30	0.050,15.40	0.048,10.71
	N(0,1)	0.357	0.255	0.36	0.225	0.301,15.53	0.212,11.13	0.366,16.37	0.358,12.15
	Exp(1)	0.629	0.906	0.75	0.902	0.447,15.74	0.205,11.11	0.723,16.68	0.634,12.58
	Gamma(2)	0.497	0.504	0.568	0.497	0.368,15.50	0.186,10.79	0.552,16.66	0.490,12.47
	Gamma(10)	0.392	0.283	0.421	0.267	0.316,15.28	0.198,10.48	0.422,16.50	0.381,12.26
	LN(0.1)	0.371	0.263	0.378	0.243	0.306,15.55	0.198,11.11	0.385,16.41	0.367,12.17
	LN(0.5)	0.537	0.469	0.598	0.458	0.413,15.68	0.209,11.19	0.588,16.66	0.518,12.52

Table 8: Power values, and expected sample size $\tilde{n}$, for m = 20, θ =1.0, α_0 = 2α.

r	Dist.	PT	MPT	WPT	WMPT	DPT	DPT	DWPT	DWPT
		n=20	n=20	n=20	n=20	n=20,n_1=15	n=20,n1=10	n=20,n_1=15	n=20,n_1=10
2	Exact l.o.s.	0.046	0.047	0.053	0.047	0.051,15.45	0.040,10.79	0.053,15.54	0.049,10.74
	N(0,1)	0.653	0.599	0.678	0.599	0.682,16.93	0.656,12.64	0.679,18.84	0.683,12.02
	Exp(1)	1.000	1.000	1.000	1.000	0.999,15.17	0.998,10.21	1.000,19.99	0.999,10.22
	Gamma(2)	0.991	0.993	0.996	0.993	0.992,17.28	0.971,12.26	1.00,19.98	0.983,11.03
	Gamma(10)	0.837	0.789	0.86	0.791	0.852,16.28	0.795,11.98	0.866,19.52	0.827,11.96
	LN(0.1)	0.734	0.679	0.758	0.674	0.755,16.94	0.715,12.70	0.764,19.16	0.744,12.08
	LN(0.5)	0.99	0.988	0.994	0.989	0.991,15.78	0.974,11.60	0.999,19.98	0.982,10.88
3	Exact l.o.s.	0.064	0.03	0.05	0.044	0.047,15.30	0.048,10.29	0.046,15.50	0.053,10.77
	N(0,1)	0.762	0.525	0.736	0.609	0.730,15.89	0.694,10.90	0.736,17.95	0.744,11.99
	Exp(1)	0.999	1.000	0.999	1.000	0.997,15.10	0.967,10.45	1.000,16.18	0.996,10.53
	Gamma(2)	0.989	0.979	0.989	0.991	0.974,16.13	0.898,11.26	0.997,17.30	0.965,11.36
	Gamma(10)	0.889	0.688	0.882	0.794	0.851,15.81	0.762,10.72	0.874,18.11	0.847,11.95
	LN(0.1)	0.821	0.589	0.805	0.687	1.000,15	1.000,10	0.800,18.06	0.788,11.99
	LN(0.5)	0.991	0.964	0.991	0.986	1.000,15.01	0.995,10.41	0.996,17.06	0.974,11.05
4	Exact l.o.s.	0.041	0.04	0.05	0.054	0.050,15.15	0.032,10.52	0.049,15.48	0.049,10.82
	N(0,1)	0.735	0.561	0.78	0.638	0.761,15.43	0.690,12.05	0.779,16.60	0.778,11.84
	Exp(1)	0.993	1.000	0.999	1.000	0.988,15.12	0.898,12.34	0.999,15.42	0.982,10.83
	Gamma(2)	0.962	0.979	0.983	0.992	0.943,15.60	0.823,12.74	0.988,15.91	0.948,11.36
	Gamma(10)	0.842	0.703	0.89	0.792	0.838,15.42	0.750,10.76	0.884,16.45	0.854,11.82
	LN(0.1)	0.788	0.617	0.83	0.716	0.771,15.96	0.731,10.99	0.829,16.56	0.809,11.86
	LN(0.5)	0.972	0.966	0.988	0.987	0.962,15.21	0.870,12.29	0.991,15.72	0.963,11.11
5	Exact l.o.s.	0.048	0.05	0.05	0.044	0.049,15.31	0.046,10.32	0.048,15.45	0.050,10.76
	N(0,1)	0.779	0.583	0.800	0.564	0.796,15.90	0.700,11.15	0.813,16.27	0.802,11.88
	Exp(1)	0.986	1.000	0.996	0.999	0.975,15.52	0.764,11.57	1.000,15.32	0.973,10.95
	Gamma(2)	0.949	0.979	0.975	0.977	0.934,16.32	0.722,11.44	0.976,15.70	0.940,11.46
	Gamma(10)	0.853	0.712	0.885	0.707	0.834,16.09	0.685,10.78	0.887,16.17	0.857,11.86
	LN(0.1)	0.816	0.633	0.84	0.632	0.807,16.52	0.693,11.26	0.849,16.24	0.824,11.88
	LN(0.5)	0.965	0.967	0.984	0.964	0.953,15.61	0.772,11.42	0.987,15.55	0.955,11.23
6	Exact l.o.s.	0.053	0.06	0.05	0.049	0.041,15.15	0.030,10.30	0.050,15.40	0.048,10.71
	N(0,1)	0.806	0.598	0.821	0.573	0.758,15.51	0.617,11.62	0.838,16.07	0.818,11.70
	Exp(1)	0.973	1.000	0.991	1.000	0.904,15.44	0.564,11.97	0.995,15.42	0.967,10.93
	Gamma(2)	0.932	0.98	0.964	0.975	0.843,15.73	0.549,11.50	0.962,15.64	0.932,11.32
	Gamma(10)	0.856	0.717	0.881	0.702	0.781,15.40	0.569,10.74	0.888,15.99	0.854,11.69
	LN(0.1)	0.83	0.642	0.860	0.63	0.769,15.51	0.587,11.70	0.863,16.05	0.834,11.71
	LN(0.5)	0.954	0.967	0.977	0.967	0.888,15.44	0.598,11.95	0.981,15.51	0.949,11.16

Note that, under the location shift and Lehmann alternative, for the smaller sample $m = n = 10$, and $n_1 = [0.5n] + 1$, in Table 4-8, the double sampling precedence and the weighted precedence test may not have a reasonable size under H_0, or these tests might become single sample tests having no *warning* region. For example, the critical limits for the double sampling precedence test for $m = n = 10$ and $n_1 = [0.5n] + 1$ are $a = b = 8$ resulting in $\tilde{n} = 5$. The power performance is also not satisfactory. Therefore, in case of double sampling precedence and weighted precedence tests, $n_1 = [0.5n] + 1$ is not recommended.

From Table 5-8, we observe that for smaller sample $m = n = 10$, the double sampling precedence test with $n_1 = [0.75n] + 1$ attains more power than the precedence, the maximal precedence and the weighted maximal precedence tests, while the double sampling weighted precedence test has overall better power than others. For $m = n = 20$, the smaller shift $\theta = 0.5$, the weighted precedence test attains better power than others. For a larger shift $\theta = 1$, the weighted precedence and its double sampling counterpart have close performance while the latter having better power in many cases. However, in case of the proposed double sampling tests, we achieve a close or better performance with a smaller average sample than the single sampling counterparts.

In Table 9, power values and the expected sample sizes are presented for $m = n = 100$, $\theta = 0.5$, $\alpha_0 = 2\alpha$, $r = 2, 3$, for the weighted precedence and the double sampling weighted precedence test. We consider the initial sample size as $n_1 = 25$. It can be observed that, with a much smaller sample size, the double sampling weighted precedence test achieves a close performance to the weighted precedence test, specially for skewed distributions.

Table 9: Power values, and expected sample size $\tilde{n}$, for $m = n = 100$, $\theta = 0.5$, $\alpha_0 = 2\alpha$.

Dist	WPT r=2	WPT r=3	DWPT($n_1 = 25$) r=2	DWPT($n_1 = 25$) r=3
Exact l.o.s	0.047	0.049	0.056,29.53	0.48, 29.07
N(0,1)	0.421	0.501	0.423,36.68	0.491,37.04
Exp(1)	1.000	1.000	1.000,25	1.000,25
Gamma(2)	1.000	0.997	0.992,28.06	0.992,29.02
Gamma(10)	0.751	0.803	0.642,37.95	0.693,37.11
LN(0.1)	0.534	0.618	0.506,37.46	0.562,37.20
LN(0.5)	0.997	0.998	0.972,30.14	0.965,29.95

4.3. Remarks about the choice of initial sample size

Choice of the initial sample size n_1 is essential. We intend to find if there is any optimal choice of n_1 in terms of the power of the proposed test. For $m = n = 20$, $n_1 = [pn] + 1$ where $p = 0.3, 0.4, 0.5, 0.6, 0.7, 0.75, 0.8, 0.85, 0.9$, and $r = 2, 3$, we have obtained the power of the double sampling precedence and weighted precedence test under the Lehmann alternative and location shift alternative. For the Lehmann alternative, we consider $\gamma = 2(1)6$. For the location shift, we consider $\theta = 0.2, 0.5, 1.0$, for the symmetric and skewed distributions considered in Section 4.2.

It is observed that the power of the double sampling precedence and weighted precedence test do not increase significantly after $p = 0.75$ or 0.8, for $n_1 = [pn] + 1$. Sometimes the power decreases after $p = 0.75$ or 0.8. For example, let us consider the power for the double sampling weighted precedence test for $m = n = 20$, and $r = 3$ for shift $\theta = 0.5$. For the standard exponential distribution, the maximum power is 0.904 that is attained for $n_1 = [0.75n] + 1$. For the gamma distribution with shape parameter $\beta = 10$, the maximum power is 0.416 that is attained for $n_1 = [0.8n] + 1$. A similar power performance is observed for both the double sampling precedence and weighted precedence tests for other shift values $\theta = 0.2, 1.0$ for other distributions considered. Under the Lehmann alternative, we observe an optimal power for $n_1 = [0.75n] + 1$ or [0.8n]+1 for all $\gamma = 2(1)6$. Hence, an optimal split of the $\boldsymbol{Y}$ sample can be taken as $n_1 = [0.75n] + 1$ or [0.8n]+1.

5. Real life application

Two life-testing experiments are presented to illustrate the application of the proposed tests.

Example 1: We consider a data from the Problem 5.4, Chapter 5 in Nelson (2003) about oil breakdown voltage for electrodes which is presented in Table 10. An insulating oil was tested between a pair of parallel disk electrodes under increasing voltage over time. The oil breakdown voltage was measured with two types of electrodes, 60 times each. To test for any significant difference between the voltage data for two types of electrodes, we consider $r = 3$ and $n_1 = [0.8n] + 1$ at 5% level of significance. Note that, $m = n = 60$ and $n_1 = 49$.

The 3-inch-diameter electrodes are taken as $\boldsymbol{X}$ sample and the 1-inch-diameter electrodes are as $\boldsymbol{Y}$ sample. We obtain $W^*_{r1} = 1568$ for the first 49 breakdown voltages for the 1-inch-diameter electrodes that falls within the warning region $\mathcal{B}^* = (334,2880)$. Therefore, 11 more breakdown voltages are observed to compute the pooled statistic $W^*_r = 1912 > c^* = 418$. Similarly, for the double sampling precedence test, the initial test statistic $W_{r1} = 32$ falls within the warning region $\mathcal{B} = (8,45)$. Therefore, we calculate the pooled test statistic $W_r = 32 > c = 8$. Hence, both test procedures suggest rejection of the H_0 at 5% level of significance that there is no significant difference between the voltage data for two types of electrodes.

Table 10: The oil breakdown voltage measured with two sizes of electrodes.

1-inch diameter									
57	59	56	56	58	64	58	55	58	54
65	61	64	65	65	52	53	60	58	63
60	62	54	63	60	52	62	50	60	57
68	57	57	58	52	67	52	62	56	59
55	65	63	57	67	64	62	58	66	60
57	64	66	52	65	57	58	62	60	59
3-inch diameter									
57	49	49	41	52	40	48	48	43	45
57	54	49	49	52	53	51	46	55	54
49	51	50	49	51	49	47	55	49	51
51	50	50	55	46	55	57	53	54	54
54	41	60	50	55	54	53	54	53	46
55	50	59	58	60	55	55	56	59	51

Example 2: We consider another data from the Table 6.1, Chapter 3 in Nelson (2003). The data presented in Table 11 is about the times to failure of specimens of a new Class H electrical insulation at different temperatures. The hours to failure data at two different temperatures, viz. at 240°, taken as the $\boldsymbol{X}$ sample, and at 190°, taken as the $\boldsymbol{Y}$ sample. We consider $r = 3$ and $n_1 = [0.75n] + 1$ for the proposed tests at 5% level of significance. Note that $m = n = 10$ and $n_1 = 8$.

The test statistics $W_{r1}^* = 80$ and $W_r = 10$ are obtained for the initial sample of the first 8 failure times for temperature 190°. Note that, in both cases, the initial test statistic falls on or beyond the critical values $b^* = 80$ and $b = 8$, respectively. Hence, we reject H_0 with a smaller sample at 5% level of significance that there is no significant failure time differences at two different temperatures.

Table 11: Class-H Insulation Life Data

	Hours to failure									
High temp.	1175	1175	1521	1569	1617	1665	1665	1713	1761	1953
Low temp.	7228	7228	7228	8448	9167	9167	9167	9167	10511	10511

6. Summary and conclusion

In this paper, we propose new precedence and weighted precedence tests under a double sampling framework. We have obtained the joint distribution of two precedence and weighted precedence statistics under the non-nested double sampling without replacement. Explicit expressions for the power function under the null and the Lehmann alternative are also obtained. With extensive Monte-Carlo simulation, we find that, with a smaller average sample size, the proposed double sampling precedence and weighted precedence tests perform close or better than their single sampling counterparts. Extending the proposed approach to other types of nonparametric tests could be an interesting topic for future research.

7. Declaration of interest

The authors report there are no competing interests to declare.

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8. Appendix

A. Proof of Result 1.

Proof. Conditioning on $Y_{r:n_1} = y$, we have the conditional probability of D = d as

$$P[D = d|Y_{r:n_1} = y] = \binom{n - n_1}{d} G^d(y)[1 - G(y)]^{(n-n_1-d)}$$

Hence,

$$\begin{aligned} P[D = d] &= \int_{y=0}^{\infty} \binom{n - n_1}{d} G^d(y)[1 - G(y)]^{(n-n_1-d)} \frac{n_1!}{(r-1)!(n_1 - r)!} G^{(r-1)}(y)[1 - G(y)]^{(n_1-r)} dG \\ &= \frac{\binom{r+d-1}{d}\binom{n-r-d}{n-n_1-d}}{\binom{n}{n_1}}. \end{aligned}$$

B. Proof of Result 3. Let us consider that $P_{H_0}[W_{r1} \in \mathcal{B}] > 0$. Then $P_{H_0}[X_{a:m} \leq Y_{r:n_1} <$

$X_{b:m}, Y_{r:n} \leq X_{c:m}] < P_{H_0}[Y_{r:n} < X_{c:m}]$. This implies from Eq. (4) that

$$P_{H_0}[Y_{r:n_1} < X_{a:m}] > \alpha_2 - \alpha. \tag{*}$$

Then we consider

$$\alpha_2 = 1 - P_{H_0}[W_r \in C_0] = 1 - P_{H_0}[W_r \in C_0, W_{r1} \in \mathcal{B}] - P_{H_0}[W_r \in C_0, W_{r1} \in \mathcal{B}^c]$$

$$\Rightarrow \alpha_2 + P_{H_0}[W_r \in C_0, W_{r1} \in \mathcal{B}^c] = 1 - P_{H_0}[W_r \in C_0, W_{r1} \in \mathcal{B}]$$

$$\Rightarrow \alpha_2 + P_{H_0}[W_r \in C_0, W_{r1} \in \mathcal{B}^c] - P_{H_0}[W_{r1} \in \mathcal{A}_0] = \alpha$$

$$\Rightarrow \alpha_2 - \alpha = P_{H_0}[W_{r1} \in \mathcal{A}_0] - P_{H_0}[W_r \in C_0, W_{r1} \in \mathcal{B}^c]$$

$$\Rightarrow \alpha_2 - \alpha > P_{H_0}[Y_{r:n_1} < X_{a:m}] - P_{H_0}[Y_{r:n} < X_{c:m}]. \tag{**}$$

By combining (*) and (**), we get the proof.

C. Proof of Result 6.

For given a and c, we can write

$$P_{H_0}[W_{r1}^* \leq c, W_r^* \leq a] = P_{H_0}[W_r^* \leq a | W_{r1}^* \leq c] P_{H_0}[W_{r1}^* \leq c].$$

First, we obtain the probability $P_{H_0}[W_r^* \leq a | W_{r1}^* \leq c]$. Note that $M_i^* \geq M_i$ for $i = 1, 2, \ldots, r$. Let us write $M_1^1 = \sum_{i=1}^{l_1} M_i$, and $M_j^1 = \sum_{i=\sum_{k=1}^{j-1} l_k + 1}^{\sum_{k=1}^{j} l_k} M_i$, $j = 2, 3, \ldots, r$. For given m, n, n_1, and $1 \leq l_i \leq n$, the conditional probability $P_{H_0}[W_r^* \leq a | W_{r1}^* \leq c]$ can be obtained by adding the joint probabilities over all possible $m_i, i = 1, 2, \ldots, \sum_{k=1}^{r} l_k$, given $0 \leq n_1 \sum_{i=1}^{l_1} m_i + \sum_{j=2}^{r} (n_1 - j + 1) \sum_{i=\sum_{k=1}^{j-1} l_k + 1}^{\sum_{k=1}^{j} l_k} m_i \leq a$. Then, from the results of Balakrishnan and Ng (2001) about the joint distribution of $M_1^1, M_2^1, \ldots, M_r^1$ and $M_i, i = 1, 2, \ldots, \sum_{k=1}^{r} l_k$ under H_0, we prove Result 6.