Delay Tolerance for Stable Hybrid Stochastic Differential Equations with Lévy Noise Based on Razumikhin Technique

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Abstract

In this article we will study the delay tolerance for stable hybrid stochastic differential equations with Lévy noise (SDEs-LN) under global Lipschitz coefficients. Based on Razumikhin technique, we will show that when the hybrid SDEs-LN without delay is pth moment exponentially stable (p-MES), the system with small delays is still p-MES. We will also obtain explicit delay bounds for p-MES. Finally, an example about neural network will be provided to illustrate the effectiveness and feasibility of theoretical results.

Keywords: Delay tolerance, Lévy noise, pth moment exponentially stable, Razumikhin technique.

1. Introduction

As a class of important mathematical models, hybrid stochastic differential equations (SDEs, also known as stochastic differential equation with Markovian switching) have been widely used in finance, networked systems, mathematical biology and so on. During the long history of hybrid SDEs research, one of the most concerned topics is the asymptotic analysis of stability. Mao [1] studied several kinds of stabilities (e.g., stability in distribution, almost sure stability, exponential stability, etc.) for the following hybrid SDEs

$$dY(t) = f(Y(t), q(t), t)dt + g(Y(t), q(t), t)dB(t), \qquad (1)$$

where the state $Y(t)$ takes values in $\mathbb{R}^n$ and the mode $q(t)$ is a Markov chain taking values in a finite space $\mathbb{S} = \{1, 2, \cdots, M\}$, $B(t)$ is the standard Brownian motion. This form of hybrid SDEs can be used to describe a class of random phenomena, which have the continuous and relatively stable features. Nevertheless, in many practical applications, the systems often reveal discontinuous paths as well as structural changes (see, e.g., [2, 3]). For example, in finance, affected by the market crashes and national policies , the stock price shall have rapid and significant change. In biology, the abrupt and unpredictable environmental disturbances such as typhoon, floods, etc., might have significant effect on population dynamics. Under many real circumstances, it is natural and reasonable to consider hybrid SDEs-LN, which can capture the features of jump discontinuity. There are many literatures studied the stability of hybrid SDEs-LN of the following from different perspectives (see, e.g., [4, 5, 6, 7, 8])

$$\begin{aligned} dY(t) =& f(Y(t), q(t))dt + g(Y(t), q(t))dB(t) \\ &+ \int_{\mathbb{R}_0^n} h(Y(t^-), q(t^-), x)\tilde{N}(dt, dx), \end{aligned} \qquad (2)$$

where $\tilde{N}(dt, dx)$ is a compensated Poisson random measure. The general theory of stochastic differential equations with Poisson's measure is presented in the book (see [9]).

On the other hand, delay is often unavoidable for many applications. For instance, users have to queue up before leaving from the service during the queuing networks scenes [10]; In cellular neural networks, signal propagation is inevitably subjected to traffic congestion caused by the limited switching speed of the amplifiers [11], and so on. To depict the event-driven delay phenomenon, it is imperative to introduce and study hybrid stochastic functional differential equations (SFDEs, including stochastic delay differential equations). As we all known, the existence of time delays may cause instability for hybrid SFDEs (see [12]). There is also well known in literature (see, e.g., [13, 14]) a contrary situation: an unstable system can be stabilized via delays. One key question is: In order to preserve the stability, how much delay can a hybrid SFDEs

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bear? This also can be considered as delay tolerance for stable hybrid SDEs. This question is a region of stirring research. Nguyen and Yin [15] investigated that the stability of SFDEs with regime-switching is preserved under delayed perturbations when the delay is small enough. Song and Mao [16] investigated the more general hybrid SFDEs of the form

$$\begin{aligned} dY(t) =& f(\psi_1(Y_t,t),q(t),t)dt \\ &+ g(\psi_2(Y_t,t),q(t),t)dB(t), \end{aligned} \tag{3}$$

where $B(t)$ is the standard Brownian motion. They proved that if the corresponding SDEs (1) without delay is almost surely exponentially stable, so is the SFDEs (3) provided the time delays are sufficiently small. Yuan and Mao [17] investigated the sufficient conditions for stability of hybrid SFDEs with jumps in the sense of almost sure stability, stability in distribution, and exponential stability in mean square. Zhu [18] obtained the p-MES problem of stochastic delay differential equations driven by Lévy processes.

Although a lot of outstanding works for the stability of SDEs-LN and SFDEs have been investigated up to now, the problem of delay tolerance for hybrid SDEs-LN has not been considered, which is still an interesting and challenging research topic. We will address this gap in this article. The main idea of the article is: if the hybrid SDEs-LN (2) is p-MES ($p \geq 2$), how much delay can tolerate such that the corresponding hybrid SFDEs-LN

$$\begin{aligned} dY(t) =& f(\psi_1(Y_t),q(t))dt + g(\psi_2(Y_t),q(t))dB(t) \\ &+ \int_{\mathbb{R}_0^n} h(\psi_3(Y_{t^-}),q(t^-),x)\tilde{N}(dt,dx), \end{aligned} \tag{4}$$

remains stable, where $f:\mathbb{R}^n\times\mathbb{S}\to\mathbb{R}^n$, $g:\mathbb{R}^n\times\mathbb{S}\to\mathbb{R}^{n\times n}$, $h:\mathbb{R}^n\times\mathbb{S}\times\mathbb{R}_0^n\to\mathbb{R}^n$, $Y_t=\{Y(t+u):u\in[-\tau,0]\}$, $\psi_1,\psi_2,\psi_3: D([-\tau,0],\mathbb{R}^n)\to\mathbb{R}^n$, τ is a positive number. Due to the existence of time delays, it is tough to construct appropriate Lyapunov functional method study the delay tolerance of hybrid SDEs-LN. Meanwhile, distinct from the almost surely continuous Brown process $B(t)$, the sample trajectories of Lévy process are right continuous with left limits almost surely. Hence, we need a useful and powerful method to overcome these issues. It should be pointed out that the Razumikhin technique is one of the most useful methods in the study of stability. The Razumikhin technique is developed to handle the difficulty caused by the non-differentiability and rapid shift of the time delay. Mao firstly applied the Razumikhin method to establish exponential stability for SFDEs [19] and neutral SFDEs [20]. Mao et al. [21] investigated the polynomial stability of hybrid stochastic systems with pantograph delay based on Razumikhin technique. By employing the Razumikhin technique, Li et al. [22] studied the robust stabilization of continuous-time hybrid stochastic systems with time-varying delay. Therefore, inspired from these discussions, Razumikhin method is adopted to study the delay tolerance for hybrid SDEs-LN in this paper.

The key contributions of this paper are in two aspects. First, existing delay tolerance works mainly focus on hybrid SDEs driven by Brownian motion, we incorporate the Lévy noise into the hybrid SDEs, which extends existing models in Song and Mao [16], Zong et al. [23] and Guo et al. [24] and allows more flexibility in modelling. Second, we obtain the explicit delay bounds directly for the moment exponential stability by means of Razumikhin technique. Compared to construct traditional Lyapunov functional method, Razumikhin technique has advantage of easy completing. It should be emphasized that the proof about stability for hybrid SFDEs-LN is not a straightforward generalization of that for hybrid SFDEs. The techniques of analysis are remarkably different.

2. Notation and Assumption

Let $|\cdot|$ denote the Euclidean norm or the matric trace norm, respectively. $\mathbb{R}^n$ denote the n-dimensional Euclidean space. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A|=\sqrt{\text{trace}(A^TA)}$ while its operator norm is denoted by $\|A\|=\sup|Ax|:|x|=1$. If D is a set, we use I_D denote the indicator function of D.

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete probability space satisfying the usual condition on which is defined an n-dimensional standard $\mathcal{F}_t$-adapted Brownian motions. Let $\{\eta(\cdot)\}$ be an $\mathcal{F}_t$-adapted Lévy process with Lévy measure $\nu(\cdot)$. Denote by $N(\cdot,\cdot)$ the $\mathcal{F}_t$-adapted Poisson random measure defined on $\mathbb{R}_+\times\mathbb{R}_0^n$:

$$N(t,U) := \sum_{0<s\leq t} I_U(\Delta\eta_s) = \sum_{0<s\leq t} I_U(\eta(s)-\eta(s^-)),$$

where U is a Borel subset of $\mathbb{R}_0^n=\mathbb{R}^n-\{0\}$. The compensator $\tilde{N}$ of N is given by $\tilde{N}(dt,dx)=N(dt,dx)-\nu(dx)dt$ with $\nu\neq 0$ and $\nu(\mathbb{R}_0^n)<\infty$. Let $q(t)$, $t\geq 0$, be a right-continuous Markov chain with finite state space $\mathbb{S}=\{1,2,\cdots,M\}$ on the probability space. The generator of $\{q(t)\}_{t\geq 0}$ is defined by $\Gamma=(\gamma_{il})_{M\times M}$, so that for a sufficiently small $\delta>0$,

$$\mathbb{P}\{q(t+\delta)=l|q(t)=i\} = \begin{cases} \gamma_{il}\delta+o(\delta) & \text{if } i\neq l, \\ 1+\gamma_{il}\delta+o(\delta) & \text{if } i=l, \end{cases}$$

where $o(\delta)$ satisfies $\lim_{\delta\to 0}\frac{o(\delta)}{\delta}=0$. Here γ_{il} is the transition rate from i to l satisfying $\gamma_{il}>0$ if $i\neq l$ while $\gamma_{ii}=-\sum_{i\neq l}\gamma_{il}$. As a standing hypothesis we assume in this paper that the Markov chain is irreducible. We also assume that $B(\cdot)$, $N(\cdot,\cdot)$ and $q(\cdot)$ are mutually independent.

Denote by $D([-\tau,0];\mathbb{R}^n)$ the family of all càdlàg (i.e., right continuous with left limits) function $\vartheta:[-\tau,0]\to\mathbb{R}^n$ endowed with the $\|\vartheta\|=\sup_{-\tau\le u\le 0}|\vartheta(u)|$. For $\vartheta\in D([-\tau,0];\mathbb{R}^n)$, define

$$\mathbb{D}(\vartheta)=\sup_{-\tau\le u\le 0}|\vartheta(u)-\vartheta(0)|.$$

Let $D^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $D([-\tau,0];\mathbb{R}^n)$-valued random variables $\xi=\{\xi(\theta):-\tau\le\theta\le 0\}$. For $t\ge 0$, denote by $\mathbb{L}^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)$ the family of all $\mathcal{F}_t$-measurable $D([-\tau,0];\mathbb{R}^n)$-valued random variables $\vartheta=\{\vartheta(\theta):-\tau\le\theta\le 0\}$ such that $\sup_{-\tau\le\theta\le 0}\mathbb{E}|\vartheta(\theta)|^p<\infty$.

In order to investigate the delay tolerance for stable hybrid SDEs-LN, we need the following assumptions.

Assumption 2.1. *Assume that $f(0,i)=g(0,i)=h(0,i,x)=0$ and there exist three nonnegative constants K_1, K_2 and K_3 such that*

$$|f(y,i)-f(z,i)|\le K_1|y-z|,$$
$$|g(y,i)-g(z,i)|\le K_2|y-z|,$$
$$\int_{\mathbb{R}^n_0}|h(y,i,x)-h(z,i,x)|^p\nu(dx)\le K_3|y-z|^p$$

for all $y,z\in\mathbb{R}^n$, $i\in\mathbb{S}$ and $p\ge 2$.

Assumption 2.2. *Assume that for $j=1,2,3$,*

$$|\psi_j(\vartheta)-\psi_j(\phi)|\le\|\vartheta-\phi\| \text{ and } |\psi_{\mathrm{j}}(\vartheta)-\vartheta(0)|\le\mathbb{D}(\vartheta),$$

where $\vartheta,\phi\in D([-\tau,0];\mathbb{R}^n)$.

We notice that $\psi_1(0)=\psi_2(0)=\psi_3(0)=0,$ for all t ≥ 0. These assumptions imply that

$$|f(\psi_1(\vartheta),i)-f(\psi_1(\phi),i)|\le K_1\|\vartheta-\phi\|,$$
$$|g(\psi_2(\vartheta),i)-g(\psi_2(\phi),i)|\le K_2\|\vartheta-\phi\|,$$
$$\int_{\mathbb{R}^n_0}|h(\psi_3(\vartheta),i,x)-h(\psi_3(\phi),i,x)|^p\nu(dx)\le K_3\|\vartheta-\phi\|^p$$

for all $\vartheta,\phi\in D([-\tau,0];\mathbb{R}^n)$. It is well known (see, e.g., [25]) that under these assumptions, the hybrid SFDEs-LN (4) has a unique solution on $t\ge-\tau$.

Let $C^2(\mathbb{R}^n\times\mathbb{S};\mathbb{R}_+)$ denote the family of all nonnegative continuous functions $V(y,i)$ on $\mathbb{R}^n$ which are twice continuously differentiable in y. Define an operator L, associated with hybrid SDEs-LN (2), acting on $V\in C^2(\mathbb{R}^n\times\mathbb{S};\mathbb{R}_+)$ by

$$\begin{aligned}LV(y,i)&=V_y(y,i)f(y,i)\\&\quad+\frac{1}{2}\text{trace}[(g(y,i))^TV_{yy}(y,i)g(y,i)]\\&\quad+\int_{\mathbb{R}^n_0}[V(y+h(y,i,x),i)-V(y,i)\\&\quad-V_y(y,i)h(y,i,x)]\nu(dx)+\sum_{l=1}^M\gamma_{il}V(y,l),\end{aligned}\tag{5}$$

where $V_y(y,i)=\left(\frac{\partial V(y,i)}{\partial y_1},\frac{\partial V(y,i)}{\partial y_2},\dots,\frac{\partial V(y,i)}{\partial y_n}\right)$, $V_{yy}(y,i)=\left(\frac{\partial^2V(y,i)}{\partial y_k\partial y_j}\right)_{n\times n}$.

Before presenting the delay bound for the stability of the hybrid SFDEs-LN, we present an important Lemma.

The following Lemma provides a sufficient condition for moment stability in terms of a Lyapunov function.

Lemma 2.3. *(see [8]) Let Assumption 2.1 hold. Let a_1, a_2, β, p be positive numbers. There exist a function $V:\mathbb{R}^n\times\mathbb{S}\to\mathbb{R}_+$ such that $V(\cdot,i)\in C^2(\mathbb{R}^n)$ for each $i\in\mathbb{S}$ satisfying*

$$\begin{aligned}&a_1|y|^p\le V(y,i)\le a_2|y|^p,\\&LV(y,i)\le-\beta V(y,i)\end{aligned}\tag{6}$$

for all $(y,i)\in\mathbb{R}^n\times\mathbb{S}$. Then the hybrid SDEs-LN (2) is p-MES, i.e.

$$\mathbb{E}|Y(t)|^p\le\frac{a_2}{a_1}|Y_0|^pe^{-\beta t}.\tag{7}$$

3. Main results

In this section, we will establish the Razumikhin technique to study the p-MES for hybrid SFDEs-LN, which plays a key role in the proof of our main results. For hybrid SFDEs-LN (4), let $V\in C^2(\mathbb{R}^n\times\mathbb{S};\mathbb{R}_+)$, define an operator $\mathcal{L}V$ from $D([-\tau,0];\mathbb{R}^n)\times\mathbb{S}$ to $\mathbb{R}$ by

$$\begin{aligned}&\mathcal{L}V(\vartheta,i)=V_y(\vartheta(0),i)f(\psi_1(\vartheta),i)\\&+\frac{1}{2}\text{trace}[(g(\psi_2(\vartheta),i))^TV_{yy}(\vartheta(0),i)g(\psi_2(\vartheta),i)]\\&+\int_{\mathbb{R}^n_0}[V(\vartheta(0)+h(\psi_3(\vartheta),i,x),i)-V(\vartheta(0),i)\\&-V_y(\vartheta(0),i)h(\psi_3(\vartheta),i,x)]\nu(dx)+\sum_{l=1}^M\gamma_{il}V(\vartheta(0),l).\end{aligned}\tag{8}$$

Theorem 3.1. *Let γ, p, a_1, a_2, be all positive numbers and $q > 1$. Assume that there exist function $V(y,i) \in C^2(\mathbb{R}^n \times \mathbb{S}; \mathbb{R}_+)$ such that*

$$a_1|y|^p \le V(y,i) \le a_2|y|^p, \ \forall (y,i) \in \mathbb{R}^n \times \mathbb{S}, \tag{9}$$

and also for all $t \ge 0$

$$\mathbb{E}\Big[\max_{1\le i\le M} \mathcal{L}V(\vartheta,i)\Big] \le -\gamma \mathbb{E}\Big[\max_{1\le i\le M} V(\vartheta(0),i)\Big] \tag{10}$$

provided $\vartheta = \{\vartheta(\theta) : -\tau \le \theta \le 0\} \in \mathbb{L}^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)$ satisfying

$$\mathbb{E}\Big[\min_{1\le i\le M} V(\vartheta(\theta),i)\Big] < q\mathbb{E}\Big[\max_{1\le i\le M} V(\vartheta(0),i)\Big] \tag{11}$$

for all $\theta \in [-\tau,0]$. Then, for all $\xi \in D^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$

$$\mathbb{E}|Y(t;\xi)|^p \le \frac{a_2}{a_1}\mathbb{E}\|\xi\|^p e^{-\lambda t} \quad \text{on t} \ge 0, \tag{12}$$

where $\lambda = \min\{\gamma, \log(q)/\tau\}$.

Its proof is a generalization of Theorem 8.9 in [1]. Different from Mao's works, we focus on hybrid SFDEs-LN, which the sample paths are right continuous with left limits. The proofs can be found in Appendix.

We now apply the Razumikhin Theorem 3.1 to deal with the delay tolerance for stable hybrid SDEs-LN.

Theorem 3.2. *Let Assumption 2.1, 2.2 and the conditions of Lemma 2.3 are valid. Moreover, there exist positive numbers $p \ge 2$, $q > 1$, a_3, a_4 and a function $V(y,i) \in C^2(\mathbb{R}^n \times \mathbb{S}; \mathbb{R}_+)$ such that*

$$\begin{aligned} &|V_y(y,i)| \le a_3|y|^{p-1}, \\ &\|V_{yy}(y,i)\| \le a_4|y|^{p-2} \end{aligned} \tag{13}$$

for all $y \in \mathbb{R}^n$. Assume also that

$$\mathbb{E}|\psi_j(\vartheta)|^p \le \sup_{-\tau\le u\le 0} \mathbb{E}|\vartheta(u)|^p \tag{14}$$

for all $t \ge \tau$ and those $\vartheta = \{\vartheta(\theta) : -2\tau \le \theta \le 0\} \in \mathbb{L}^p_{\mathcal{F}_t}([-2\tau,0];\mathbb{R}^n)$ satisfying

$$\mathbb{E}\Big[\min_{1\le i\le M} V(\vartheta(\theta),i)\Big] < q\mathbb{E}\Big[\max_{1\le i\le M} V(\vartheta(0),i)\Big] \tag{15}$$

for all $\theta \in [-2\tau,0]$. If

$$\begin{aligned} a_1\beta > &\frac{a_2}{a_1}\big[K_1 a_3 (K_\tau)^{\frac{1}{p}} + K_2^2 a_4 (K_\tau)^{\frac{1}{p}} \\ &+ a_4 a_5 2^{p-4} (K_3 K_\tau)^{\frac{2}{p}} / (K_3 + J)^{(\frac{2}{p}-1)} \\ &+ a_3 (K_3 K_\tau)^{\frac{1}{p}} / J^{(\frac{1}{p}-1)} + \frac{a_4 a_5}{2} K_3 K_\tau \\ &+ a_3 2^{p-2+\frac{1}{p}} (K_3 K_\tau)^{\frac{1}{p}} / (J+K_3)^{(\frac{1}{p}-1)}\big], \end{aligned} \tag{16}$$

where $J = \nu(\mathbb{R}^n_0) < \infty$, $a_5 = 2^{p-3} \vee 1$,

$$\begin{aligned} K_\tau = &\tau\big[(3\tau)^{p-1}K_1^p + 3^{p-1}\tau^{\frac{p}{2}-1}K_2^p p_0 \\ &+ (3^{p-1}p_1 K_3 \tau^{\frac{p}{2}-1} + 3^{p-1}p_2 K_3)\big], \end{aligned}$$

$$p_0 = [\frac{p^{p+1}}{2(p-1)^{p-1}}]^{\frac{p}{2}}, \ p_2 = (\frac{p}{p-1})^p p(p-1)2^{p-3},$$

and

$$p_1 = (\frac{p}{p-1})^{\frac{p^2}{2}}(p-1)^{\frac{p}{2}}(p-2)^{\frac{p}{2}-1}2^{\frac{(p-3)p}{2}},$$

then the solution of (4) is p-MES.

Remark 3.3. *The condition (14) is not empty, for example: $\psi_j(Y_t) = Y(t-\tau(t))$, $\psi_j(Y_t) = \int_{-\tau}^0 Y(t+\theta)d\theta$.*

Before proving this main theorem, we first give the following useful lemma.

Lemma 3.4. *For all $t \ge \tau$, we have*

$$\mathbb{E}|\mathbb{D}(Y_t)|^p \le K_\tau \sup_{-2\tau\le\theta\le 0} \mathbb{E}|Y(t+\theta)|^p. \tag{17}$$

Proof: By Hölder's inequality, Burkhölder-Davis-Gundy's inequality and Kunita's estimates, we can derive from (4) that

$$\begin{aligned} &\mathbb{E}|\mathbb{D}(Y_t)|^p \\ \le&(3\tau)^{p-1}K_1^p \int_{t-\tau}^t \mathbb{E}|\psi_1(Y_s)|^p ds \\ &+ 3^{p-1}\tau^{\frac{p}{2}-1}K_2^p p_0 \int_{t-\tau}^t \mathbb{E}|\psi_2(Y_s)|^p ds \\ &+ 3^{p-1}p_1\mathbb{E}\big[(\int_{t-\tau}^t \int_{\mathbb{R}^n_0} |h(\psi_3(Y_{s^-}), q(s^-), x)|^2 \nu(dx)ds)^{\frac{p}{2}}\big] \\ &+ 3^{p-1}p_2\mathbb{E}\big[\int_{t-\tau}^t \int_{\mathbb{R}^n_0} |h(\psi_3(Y_{s^-}), q(s^-), x)|^p \nu(dx)ds\big] \\ \le&(3\tau)^{p-1}K_1^p \int_{t-\tau}^t \mathbb{E}|\psi_1(Y_s)|^p ds \\ &+ 3^{p-1}\tau^{\frac{p}{2}-1}K_2^p p_0 \int_{t-\tau}^t \mathbb{E}|\psi_2(Y_s)|^p ds \\ &+ (3^{p-1}p_1 K_3 \tau^{\frac{p}{2}-1} + 3^{p-1}p_2K_3) \int_{t-\tau}^t \mathbb{E}|\psi_3(Y_s)|^p ds \\ \le&\big[(3\tau)^{p-1}K_1^p + 3^{p-1}\tau^{\frac{p}{2}-1}K_2^p p_0 + (3^{p-1}p_1K_3\tau^{\frac{p}{2}-1} \\ &+ 3^{p-1}p_2K_3)\big] \int_{t-\tau}^t \sup_{-\tau\le u\le 0} \mathbb{E}|Y(s+u)|^p ds \\ \le&K_\tau \sup_{-2\tau\le\theta\le 0} \mathbb{E}|Y(t+\theta)|^p. \end{aligned}$$

Thus, the proof is complete. □

We can now begin to prove Theorem 3.2. The proof is an application of the Theorem 3.1.

Proof of Theorem 3.2. Obviously,

$$a_1|y|^p \le V(y,i) \le a_2|y|^p.$$

In the next, we need to show that

$$\mathbb{E}\Big[\max_{1\le i\le M} \mathcal{L}V(\vartheta,i)\Big] \le -\gamma\mathbb{E}\Big[\max_{1\le i\le M} V(\vartheta(0),i)\Big] \tag{18}$$

for all $t \ge \tau$ and those $\vartheta = \{\vartheta(\theta) : -2\tau \le \theta \le 0\} \in \mathbb{L}^p_{\mathcal{F}_t}([-2\tau,0];\mathbb{R}^n)$ satisfying

$$\mathbb{E}\Big[\min_{1\le i\le M} V(\vartheta(\theta),i)\Big] < q\mathbb{E}\Big[\max_{1\le i\le M} V(\vartheta(0),i)\Big] \tag{19}$$

for all $\theta \in [-2\tau,0]$.

To show (18) under (19), we compute $\mathcal{L}V(\vartheta,i)$ as follows

$$\begin{aligned}
&\mathcal{L}V(\vartheta,i)\\
=&V_y(\vartheta(0),i)f(\psi_1(\vartheta),i) + \sum_{l=1}^{M}\gamma_{il}V(\vartheta(0),l)\\
&+ \frac{1}{2}\mathrm{trace}\big[\big(g(\psi_2(\vartheta),i)\big)^T V_{yy}(\vartheta(0),i)g(\psi_2(\vartheta),i)\big]\\
&+ \int_{\mathbb{R}^n_0}\big[V(\vartheta(0)+h(\psi_3(\vartheta),i,x),i) - V(\vartheta(0),i)\\
&- V_y(\vartheta(0),i)h(\psi_3(\vartheta),i,x)\big]\nu(dx).
\end{aligned} \tag{20}$$

Note that

$$\begin{aligned}
V_y(\vartheta(0),i)f(\psi_1(\vartheta),i) =& V_y(\vartheta(0),i)f(\vartheta(0),i)\\
&+ V_y(\vartheta(0),i)\big[f(\psi_1(\vartheta),i) - f(\vartheta(0),i)\big],
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&\big(g(\psi_2(\vartheta),i)\big)^T V_{yy}(\vartheta(0),i)g(\psi_2(\vartheta),i)\\
=&\big(g(\vartheta(0),i)\big)^T V_{yy}(\vartheta(0),i)g(\vartheta(0),i)\\
&+ \big(g(\psi_2(\vartheta),i)\big)^T V_{yy}(\vartheta(0),i)\\
&\times \big(g(\psi_2(\vartheta),i) - g(\vartheta(0),i)\big)\\
&+ \big(g(\psi_2(\vartheta),i) - g(\vartheta(0),i)\big)^T\\
&\times V_{yy}(\vartheta(0),i)g(\vartheta(0),i).
\end{aligned} \tag{22}$$

Meanwhile, using Taylor's formula, we can derive that

$$\begin{aligned}
&V(\vartheta(0)+h(\psi_3(\vartheta),i,x),i) - V(\vartheta(0),i)\\
&- V_y(\vartheta(0),i)h(\psi_3(\vartheta),i,x)\\
=&V(\vartheta(0)+h(\vartheta(0),i,x),i) - V(\vartheta(0),i)\\
&- V_y(\vartheta(0),i)h(\vartheta(0),i,x)\\
&+ V(\vartheta(0)+h(\psi_3(\vartheta),i,x),i)\\
&- V(\vartheta(0)+h(\vartheta(0),i,x),i)\\
&- V_y(\vartheta(0)+h(\vartheta(0),i,x),i)\\
&\times \big(h(\psi_3(\vartheta),i,x) - h(\vartheta(0),i,x)\big)\\
&+ V_y(\vartheta(0),i)\big(h(\vartheta(0),i,x) - h(\psi_3(\vartheta),i,x)\big)\\
&+ V_y(\vartheta(0)+h(\vartheta(0),i,x),i)\\
&\times \big(h(\psi_3(\vartheta),i,x) - h(\vartheta(0),i,x)\big)\\
=&V(\vartheta(0)+h(\vartheta(0),i,x),i) - V(\vartheta(0),i)\\
&- V_y(\vartheta(0),i)h(\vartheta(0),i,x)\\
&+ \frac{1}{2}\big(h(\psi_3(\vartheta),i,x) - h(\vartheta(0),i,x)\big)^T\\
&\times V_{yy}(\vartheta(0)+h(\vartheta(0),i,x)\\
&+ \theta(h(\psi_3(\vartheta),i,x) - h(\vartheta(0),i,x)),i)\\
&\times \big(h(\psi_3(\vartheta),i,x) - h(\vartheta(0),i,x)\big)\\
&+ V_y(\vartheta(0),i)\big(h(\vartheta(0),i,x) - h(\psi_3(\vartheta),i,x)\big)\\
&+ V_y(\vartheta(0)+h(\vartheta(0),i,x),i)\\
&\times \big(h(\psi_3(\vartheta),i,x) - h(\vartheta(0),i,x)\big),
\end{aligned} \tag{23}$$

where $0 \le \theta \le 1$.

Substituting (21), (22) and (23) into (20), one yields

$$\begin{aligned}
&\mathcal{L}V(\vartheta,i)\\
=&LV(\vartheta(0),i) + V_y(\vartheta(0),i)\big[f(\psi_1(\vartheta),i) - f(\vartheta(0),i)\big]\\
&+ \frac{1}{2}\mathrm{trace}\big[\big(g(\psi_2(\vartheta),i)\big)^T V_{yy}(\vartheta(0),i)\\
&\times \big(g(\psi_2(\vartheta),i) - g(\vartheta(0),i)\big)\big]\\
&+ \frac{1}{2}\mathrm{trace}\big[\big(g(\psi_2(\vartheta),i) - g(\varphi(0),i)\big)^T\\
&\times V_{yy}(\vartheta(0),i)g(\vartheta(0),i)\big]\\
&+ \int_{\mathbb{R}^n_0}\Big[\frac{1}{2}\big(h(\psi_3(\vartheta),i,x) - h(\vartheta(0),i,x)\big)^T\\
&\times V_{yy}(\vartheta(0)+h(\vartheta(0),i,x)\\
&+ \theta(h(\psi_3(\vartheta),i,x) - h(\vartheta(0),i,x)),i)\\
&\times \big(h(\psi_3(\vartheta),i,x) - h(\vartheta(0),i,x)\big)\\
&+ V_y(\vartheta(0),i)\big(h(\vartheta(0),i,x) - h(\psi_3(\vartheta),i,x)\big)\\
&+ V_y(\vartheta(0)+h(\vartheta(0),i,x),i)\\
&\quad\times \big(h(\psi_3(\vartheta),i,x) - h(\vartheta(0),i,x)\big)\Big]\nu(dx).
\end{aligned} \tag{24}$$

In view of Assumption 2.2, (13) and $ab^{p-2}c \le \frac{1}{p}\varepsilon(a^p + (p-2)b^p) + \frac{1}{p\varepsilon^{p-1}}c^p$, for all $a,b,c,\varepsilon > 0$, $p \ge 2$, we have

$$\begin{aligned}
&V_y(\vartheta(0),i)\big[f(\psi_1(\vartheta),i) - f(\vartheta(0),i)\big]\\
&+ \frac{1}{2}\mathrm{trace}\big[\big(g(\psi_2(\vartheta),i)\big)^T V_{yy}(\vartheta(0),i)\\
&\times \big(g(\psi_2(\vartheta),i) - g(\vartheta(0),i)\big)\big]\\
&+ \frac{1}{2}\mathrm{trace}\big[\big(g(\psi_2(\vartheta),i) - g(\vartheta(0),i)\big)^T\\
&\times V_{yy}(\vartheta(0),i)g(\vartheta(0),i)\big]\\
\le&K_1a_3\Big(\varepsilon_1\frac{p-1}{p}|\vartheta(0)|^p + \frac{1}{p\varepsilon_1^{p-1}}\big|\mathbb{D}(\vartheta)\big|^p\Big)\\
&+ \frac{K_2^2a_4}{2}\Big(\varepsilon_2\frac{1}{p}|\psi_2(\vartheta)|^p + \varepsilon_2\frac{p-2}{p}|\vartheta(0)|^p\\
&+ \frac{1}{p\varepsilon_2^{p-1}}\big|\mathbb{D}(\vartheta)\big|^p + \varepsilon_2\frac{p-1}{p}|\vartheta(0)|^p\\
&+ \frac{1}{p\varepsilon_2^{p-1}}\big|\mathbb{D}(\vartheta)\big|^p\Big),
\end{aligned} \tag{25}$$

and

$$
\begin{aligned}
&\frac{1}{2}\big(h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)\big)^T V_{yy}(\vartheta(0)\\
&+h(\vartheta(0),i,x)+\theta(h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)),i)\\
&\times\big(h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)\big)\\
&+V_y(\vartheta(0),i)\big(h(\vartheta(0),i,x)-h(\psi_3(\vartheta),i,x)\big)\\
&+V_y(\vartheta(0)+h(\vartheta(0),i,x),i)\\
&\times\big(h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)\big)\\
\le&\frac{1}{2}|h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)|^2\\
&\times\|V_{yy}(\vartheta(0)+h(\vartheta(0),i,x)\\
&+\theta(h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x))\,i)\|\\
&+|V_y(\vartheta(0),i)||h(\vartheta(0),i,x)-h(\psi_3(\vartheta),i,x)|\\
&+|V_y(\vartheta(0)+h(\vartheta(0),i,x),i)|\\
&\times|h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)|\\
\le&\frac{a_4}{2}|h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)|^2|\vartheta(0)\\
&+h(\vartheta(0),i,x)+\theta(h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x))|^{p-2}\\
&+a_3|\vartheta(0)|^{p-1}|h(\vartheta(0),i,x)-h(\psi_3(\vartheta),i,x)|\\
&+a_3|\vartheta(0)+h(\vartheta(0),i,x)|^{p-1}\\
&\times|h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)|\\
\le&\frac{a_4}{2}[a_5\frac{1}{p\varepsilon_3^{(p-2)/2}}|h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)|^p\\
&+\varepsilon_3\frac{p-2}{p}a_5|\vartheta(0)+h(\vartheta(0),i,x)|^p\\
&+a_5|h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)|^p]\\
&+a_3[\varepsilon_4\frac{p-1}{p}|\vartheta(0)|^p\\
&+\frac{1}{p\varepsilon_4^{p-1}}|h(\vartheta(0),i,x)-h(\psi_3(\vartheta),i,x)|^p]\\
&+a_3[\varepsilon_5\frac{p-1}{p}|\vartheta(0)+h(\vartheta(0),i,x)|^p\\
&+\frac{1}{p\varepsilon_5^{p-1}}|h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)|^p]\\
\le&\frac{a_4}{2}[a_5\frac{1}{p\varepsilon_3^{(p-2)/2}}|h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)|^p\\
&+a_5\varepsilon_3\frac{p-2}{p}2^{p-1}|\vartheta(0)|^p\\
&+a_5\varepsilon_3\frac{p-2}{p}2^{p-1}|h(\vartheta(0),i,x)|^p\\
&+a_5|h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)|^p]\\
&+a_3[\varepsilon_4\frac{p-1}{p}|\vartheta(0)|^p \qquad (26)\\
&+\frac{1}{p\varepsilon_4^{p-1}}|h(\vartheta(0),i,x)-h(\psi_3(\vartheta),i,x)|^p]\\
&+a_3[\varepsilon_5\frac{p-1}{p}2^{p-1}|\vartheta(0)|^p\\
&+\varepsilon_5\frac{p-1}{p}2^{p-1}|h(\vartheta(0),i,x)|^p\\
&+\frac{1}{p\varepsilon_5^{p-1}}|h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)|^p],
\end{aligned}
$$

where ε_1, ε_2, ε_3, ε_4 and ε_5 are all positive numbers to be chosen.

Then, by Assumption 2.1 we have

$$
\begin{aligned}
&\int_{\mathbb{R}_0^n}\big[\frac{1}{2}\big(h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)\big)^T V_{yy}(\vartheta(0)\\
&+h(\vartheta(0),i,x)+\theta(h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)),i)\\
&\times\big(h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)\big)\\
&+V_y(\vartheta(0),i)\big(h(\vartheta(0),i,x)-h(\psi_3(\vartheta),i,x)\big)\\
&+V_y(\vartheta(0)+h(\vartheta(0),i,x),i)\\
&\times\big(h(\psi_3(\vartheta),i,x)-h(\vartheta(0),i,x)\big)\big]\nu(dx)\\
\le&\frac{a_4}{2}\big[a_5K_3\frac{1}{p\varepsilon_3^{(p-2)/2}}\big|\mathbb{D}(\vartheta)\big|^p+Ja_5\varepsilon_3\frac{p-2}{p}2^{p-1}|\vartheta(0)|^p\\
&+a_5K_3\varepsilon_3\frac{p-2}{p}2^{p-1}|\vartheta(0)|^p+a_5K_3\big|\mathbb{D}(\vartheta)\big|^p\big]\\
&+a_3\big[J\varepsilon_4\frac{p-1}{p}|\vartheta(0)|^p+K_3\frac{1}{p\varepsilon_4^{p-1}}\big|\mathbb{D}(\vartheta)\big|^p\big] \qquad (27)\\
&+a_3\big[J\varepsilon_5\frac{p-1}{p}2^{p-1}|\vartheta(0)|^p+K_3\varepsilon_5\frac{p-1}{p}2^{p-1}|\vartheta(0)|^p\\
&+K_3\frac{1}{p\varepsilon_5^{p-1}}\big|\mathbb{D}(\vartheta)\big|^p\big].
\end{aligned}
$$

Substituting above inequalities into (24) we obtain

$$
\begin{aligned}
&\mathcal{L}V(\vartheta,i)\\
\le&LV(\vartheta(0),i)+K_1a_3\Big(\varepsilon_1\frac{p-1}{p}|\vartheta(0)|^p+\frac{1}{p\varepsilon_1^{p-1}}\big|\mathbb{D}(\vartheta)\big|^p\Big)\\
&+\frac{K_2^2a_4}{2}\Big(\varepsilon_2\frac{1}{p}|\psi_2(\vartheta)|^p+\varepsilon_2\frac{p-2}{p}|\vartheta(0)|^p+\frac{1}{p\varepsilon_2^{p-1}}\big|\mathbb{D}(\vartheta)\big|^p\\
&\qquad+\varepsilon_2\frac{p-1}{p}|\vartheta(0)|^p+\frac{1}{p\varepsilon_2^{p-1}}\big|\mathbb{D}(\vartheta)\big|^p\Big)\\
&+\frac{a_4}{2}\Big(a_5K_3\frac{1}{p\varepsilon_3^{(p-2)/2}}\big|\mathbb{D}(\vartheta)\big|^p+Ja_5\varepsilon_3\frac{p-2}{p}2^{p-1}|\vartheta(0)|^p\\
&+a_5K_3\varepsilon_3\frac{p-2}{p}2^{p-1}|\vartheta(0)|^p\Big)+\frac{a_4}{2}a_5K_3\big|\mathbb{D}(\vartheta)\big|^p\\
&+a_3\Big(J\varepsilon_4\frac{p-1}{p}|\vartheta(0)|^p+K_3\frac{1}{p\varepsilon_4^{p-1}}\big|\mathbb{D}(\vartheta)\big|^p\Big)\\
&+a_3\Big(J\varepsilon_5\frac{p-1}{p}2^{p-1}|\vartheta(0)|^p+K_3\varepsilon_5\frac{p-1}{p}2^{p-1}|\vartheta(0)|^p\\
&\qquad+K_3\frac{1}{p\varepsilon_5^{p-1}}\big|\mathbb{D}(\vartheta)\big|^p\Big)\\
:=&LV(\vartheta(0),i)+\sum_{j=1}^{5}\Theta_j(\vartheta,i)+\frac{a_4a_5}{2}K_3\big|\mathbb{D}(\vartheta)\big|^p. \qquad (28)
\end{aligned}
$$

Choosing $\varepsilon_1=(K_\tau)^{\frac{1}{p}}$, $\varepsilon_2=(K_\tau)^{\frac{1}{p}}$, $\varepsilon_3=(\frac{K_3K_\tau}{2^p(J+K_3)})^{\frac{2}{p}}$, $\varepsilon_4=(\frac{K_3K_\tau}{J})^{\frac{1}{p}}$ and $\varepsilon_5=(\frac{K_3K_\tau}{2^{p-1}(J+K_3)})^{\frac{1}{p}}$, applying Lemma

3.4 and (14) yields

$$\mathbb{E}\Theta_1(\vartheta,i) \le K_1 a_3 \frac{1}{p}\Big(\varepsilon_1(p-1)+\frac{K_\tau}{\varepsilon_1^{p-1}}\Big)\sup_{-2\tau\le\theta\le0}\mathbb{E}|\vartheta(\theta)|^p = K_1 a_3 (K_\tau)^{\frac{1}{p}}\sup_{-2\tau\le\theta\le0}\mathbb{E}|\vartheta(\theta)|^p, \quad (29)$$

$$\begin{aligned}\mathbb{E}\Theta_2(\vartheta,i) \le& \frac{K_2^2 a_4}{2}\frac{1}{p}\Big(\varepsilon_2+\varepsilon_2(p-2)+\frac{K_\tau}{\varepsilon_2^{p-1}}\\ &+\varepsilon_2(p-1)+\frac{K_\tau}{\varepsilon_2^{p-1}}\Big)\sup_{-2\tau\le\theta\le0}\mathbb{E}|\vartheta(\theta)|^p\\ =&K_2^2 a_4(K_\tau)^{\frac{1}{p}}\sup_{-2\tau\le\theta\le0}\mathbb{E}|\vartheta(\theta)|^p, \end{aligned}\quad (30)$$

$$\begin{aligned}\mathbb{E}\Theta_3(\vartheta,i) \le& \frac{a_4 a_5}{2p}[K_3\frac{K_\tau}{\varepsilon_3^{(p-2)/2}}+J\varepsilon_3(p-2)2^{p-1}\\ &+K_3\varepsilon_3(p-2)2^{p-1}]\sup_{-2\tau\le\theta\le0}\mathbb{E}|\vartheta(\theta)|^p\\ =&\frac{a_4a_5 2^{p-4}(K_3K_\tau)^{\frac{2}{p}}}{(K_3+J)^{(\frac{2}{p}-1)}}\sup_{-2\tau\le\theta\le0}\mathbb{E}|\vartheta(\theta)|^p, \end{aligned}\quad (31)$$

$$\begin{aligned}\mathbb{E}\Theta_4(\vartheta,i) \le& \frac{a_3}{p}[J\varepsilon_4(p-1)+K_3\frac{K_\tau}{\varepsilon_4^{p-1}}]\sup_{-2\tau\le\theta\le0}\mathbb{E}|\vartheta(\theta)|^p\\ =&a_3(K_3K_\tau)^{\frac{1}{p}}/J^{(\frac{1}{p}-1)}\sup_{-2\tau\le\theta\le0}\mathbb{E}|\vartheta(\theta)|^p, \end{aligned}\quad (32)$$

and

$$\begin{aligned}\mathbb{E}\Theta_5(\vartheta,i) \le& \frac{a_3}{p}[J\varepsilon_5(p-1)2^{p-1}+K_3\varepsilon_5(p-1)2^{p-1}\\ &+K_3\frac{K_\tau}{\varepsilon_5^{p-1}}]\sup_{-2\tau\le\theta\le0}\mathbb{E}|\vartheta(\theta)|^p\\ =&\frac{a_3 2^{p-2+\frac{1}{p}}(K_3K_\tau)^{\frac{1}{p}}}{(J+K_3)^{(\frac{1}{p}-1)}}\sup_{-2\tau\le\theta\le0}\mathbb{E}|\vartheta(\theta)|^p. \end{aligned}\quad (33)$$

In view of (19), we have

$$\mathbb{E}|\vartheta(\theta)|^p < \frac{a_2 q}{a_1}\mathbb{E}|\vartheta(0)|^p, \forall\theta\in[-2\tau,0]. \quad (34)$$

Combining (29)-(34) with (28) yield

$$\begin{aligned}&\mathbb{E}\Big[\max_{1\le i\le N}\mathcal{L}V(\vartheta,i)\Big]\\ \le&\Big[-a_1\beta+\frac{a_2q}{a_1}\big[K_1a_3(K_\tau)^{\frac{1}{p}}+K_2^2a_4(K_\tau)^{\frac{1}{p}}\\ &+a_4a_5 2^{p-4}(K_3K_\tau)^{\frac{2}{p}}/(K_3+J)^{(\frac{2}{p}-1)}\\ &+a_3 2^{p-2+\frac{1}{p}}(K_3K_\tau)^{\frac{1}{p}}/(J+K_3)^{(\frac{1}{p}-1)}\\ &+a_3(K_3K_\tau)^{\frac{1}{p}}/J^{(\frac{1}{p}-1)}+\frac{a_4a_5}{2}K_3K_\tau\big]\Big]\mathbb{E}|\vartheta(0)|^p. \end{aligned}\quad (35)$$

By (16), one can choose $q>1$ such that

$$\begin{aligned}a_1\beta >& \frac{a_2q}{a_1}\big[K_1a_3(K_\tau)^{\frac{1}{p}}+K_2^2a_4(K_\tau)^{\frac{1}{p}}\\ &+a_4a_5 2^{p-4}(K_3K_\tau)^{\frac{2}{p}}/(K_3+J)^{(\frac{2}{p}-1)}\\ &+a_3(K_3K_\tau)^{\frac{1}{p}}/J^{(\frac{1}{p}-1)}+\frac{a_4a_5}{2}K_3K_\tau\\ &+a_3 2^{p-2+\frac{1}{p}}(K_3K_\tau)^{\frac{1}{p}}/(J+K_3)^{(\frac{1}{p}-1)}\big]. \end{aligned}\quad (36)$$

Therefore, (35) implies

$$\begin{aligned}&\mathbb{E}\Big[\max_{1\le i\le N}\mathcal{L}V(\vartheta,i)\Big]\\ \le&-\frac{1}{a_2}\Big[a_1\beta-\frac{a_2q}{a_1}\big[K_1a_3(K_\tau)^{\frac{1}{p}}+K_2^2a_4(K_\tau)^{\frac{1}{p}}\\ &+\frac{a_4a_5}{2}K_3K_\tau+a_4a_5 2^{p-4}(K_3K_\tau)^{\frac{2}{p}}/(K_3+J)^{(\frac{2}{p}-1)}\\ &+a_3(K_3K_\tau)^{\frac{1}{p}}/J^{(\frac{1}{p}-1)}+a_3 2^{p-2+\frac{1}{p}}\\ &\times(K_3K_\tau)^{\frac{1}{p}}/(J+K_3)^{(\frac{1}{p}-1)}\big]\Big]\mathbb{E}\Big[\max_{1\le i\le M}V(\vartheta(0),i)\Big]\\ =&-\gamma\mathbb{E}\Big[\max_{1\le i\le M}V(\vartheta(0),i)\Big],\end{aligned}$$

which is the required inequality (18). Thus, the proof is complete. □

4. Examples

In this section, we use a neural networks example to illustrate our results. Consider a two-neuron neural network with Markovian switching and Lévy noises of the form:

$$\begin{aligned}dY(t)=&[-F(q(t))Y(t)+G(q(t))s(Y(t))]dt\\ &+g(Y(t),q(t))dB(t)\\ &+h(Y(t^-),q(t^-))d\tilde{N}(t),\end{aligned}\quad (37)$$

where s stand for the neuron activation function with $s(0)=0$, $B(t)$ is a scalar Brownian motion, $\tilde{N}(t)=N(t)-\lambda dt$, $\tilde{N}(t)$ is a compensated Poisson random measure which means that $N(t)$ is a scalar Poisson process with intensify λ, $q(t)$ is a Markov chain on the state space $\mathbb{S}=\{1,2\}$ with the generator

$$\Gamma=\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}.$$

We set $s(\cdot)=\tanh(\cdot)$ as the neuron activation function and $\lambda=1$. The other parameters concerning the system (37) are as follows.

$$F(1)=\begin{bmatrix}0.5 & 0\\ 0 & 0.4\end{bmatrix},\ G(1)=\begin{bmatrix}-1 & 0.1\\ 1 & -0.2\end{bmatrix},$$

$$g(y,1)=\frac{1}{4}y,\quad h(y,1)=-\frac{1}{4}y,$$

$$F(2)=\begin{bmatrix}1 & 0\\ 0 & 0.7\end{bmatrix},\ G(2)=\begin{bmatrix}1 & 0\\ 1.5 & 0.1\end{bmatrix},$$

$$g(y,2)=-\frac{1}{2}y,\quad h(y,2)=\frac{1}{2}y.$$

Obviously, the coefficients gave by (37) satisfy Assumption 2.1 with $K_1=0.46458$, $K_2=0.5$ and $K_3=0.5^2$. Let $V(y,i)=|y|^2$, here $a_1=1=a_2$. We compute

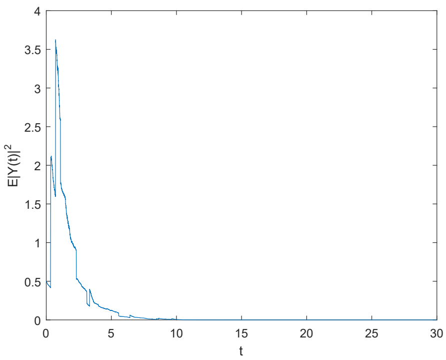
$$\begin{aligned}LV(y,i)=&2y^T(-F(i))y+2y^TG(i)s(y)+|g(y,i)|^2\\ &+\lambda\big[|y+h(y,i)|^2-|y|^2-2y^Th(y,i)\big]\\ \le&-0.20834|y|^2=-0.20834V(y,i).\end{aligned}\quad (38)$$

By Lemma 2.3, we can conclude that the solution of the the neural network (37) is mean square exponentially stable (MSES).

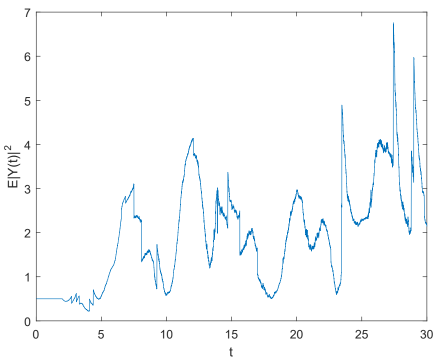
In neural networks, the finite switching speed of amplifiers and communication time will occur time delays in the interaction between neurons, which may lead to some instabilities [26]. Then, the question is under what conditions the following two-neuron delay neural network with Markovian switching and Lévy noises of the form:

$$\begin{aligned} dY(t) =&[-F(q(t))Y(t-\tau(t)) + G(q(t))s(Y(t-\tau(t)))]dt \\ &+ g(Y(t-\tau(t)), q(t))dB(t) \\ &+ h(Y((t-\tau(t))^-), q(t^-))d\tilde{N}(t), \end{aligned} \tag{39}$$

is still MSES.



(a) without delay: $\tau(t) = 0$



(b) with delay: $\tau(t) = 1.5 - 0.5\sin(t)$

Figure 1: The computer simulation of the second moment of the solution of (37) and (39) using the Euler-Maruyama method with sample size 100, respectively.

Before applying our new theory, we consider two special cases: $\tau(t) = 0$ and $\tau(t) = 1.5 - 0.5\sin(t)$. We perform a computer simulation with the initial values $Y(0) = 0.5$ and $q(0) = 1$. For the delay-free case, the MSES is plotted in Fig 1. (a) by taking 100 samples to approximate $E|Y(t)|^2$. In the case of $\tau(t) = 1.5 - 0.5\sin(t)$, we perform a computer simulation for the solution of the delay neural network (39). The second moment of the solution of (39) is simulated in Fig 1. (b), from which we see that the delay neural network (39) is not stable.

Through these simulation results, we can know that when the delay is getting smaller and smaller, the delay neural network (39) tends to be stable. Our Theorem 3.2 will be able to show a bound for the delay. Namely, the solution of the the delay neural network (39) is MSES if

$$\begin{aligned} a_1\beta >& \frac{a_2}{a_1}\big[K_1 a_3 (K_\tau)^{\frac{1}{2}} + K_2^2 a_4 (K_\tau)^{\frac{1}{2}} + a_4 2^{-2} K_3 K_\tau \\ &+ a_3 (K_3 K_\tau)^{\frac{1}{2}} \lambda^{\frac{1}{2}} + a_3 2^{\frac{1}{2}} (K_3 K_\tau)^{\frac{1}{2}} (\lambda + K_3)^{\frac{1}{2}} \\ &+ \frac{a_4}{2} K_3 K_\tau\big], \end{aligned} \tag{40}$$

where $K_\tau = \tau(3\tau K_1^2 + 12K_2^2 + 12K_3)$, $a_3 = 2$, $a_4 = 2$. That is, the MSES neural network (37) can tolerate a delay $\tau \leq 0.0004455$ such that the delay neural network (39) is still MSES.

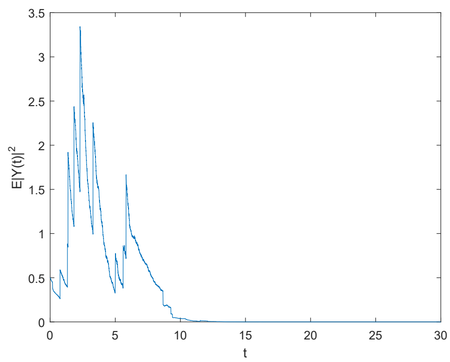


Figure 2: The computer simulation of the second moment of the solution of (39) using the Euler-Maruyama method with $\tau(t) = 2 * 10^{-4} - 2 * 10^{-4}\sin(t)$.

We perform a computer simulation with $\tau(t) = 2 * 10^{-4} - 2 * 10^{-4}\sin(t)$ and $r(0) = 1$. Similarly, taking 100 samples to approximate $E|Y(t)|^2$ will produce Fig 2, which depicts the MSES.

5. Conclusion

This article establishes delay tolerance for stable hybrid SDEs-LN. Based on Razumikhin technique, we show that if the p-MES for hybrid SDEs-LN without delay, a delay is allowed for the hybrid SFDEs-LN to be p-MES. Another advantage of our results is that a bound on τ is given for p-MES.

Appendix

Proof of Theorem 3.1. Fix any initial data $\xi \in D^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ and write $Y(t;\xi) = Y(t)$. Recalling the facts that $Y(t)$ is right continuous with left limit, $q(t)$ is right continuous and $\mathbb{E}(\sup_{-\tau\le s\le t}|Y(s)|^p) < \infty$ for all $t \ge 0$, we see that $\mathbb{E}V(Y(t), q(t))$ is right continuous with left limit. Let $\epsilon \in (0, \lambda)$ be arbitrary and set $\bar{\lambda} = \lambda - \epsilon$. Define

$$L(t) = \sup_{-\tau\le\theta\le 0}\left[e^{\bar{\lambda}(t+\theta)}\mathbb{E}V(Y(t+\theta), q(t+\theta))\right] \quad \text{for all t} \ge 0.$$

Due to the right continuity of $\mathbb{E}V(Y(\cdot), q(\cdot))$ and $f(0, i) = 0$, $g(0, i) = 0$, $h(0, i, x) = 0$. Similar to the proof of Theorem 8.9 in [1] , we can get

$$D_+L(t) := \limsup_{\delta\to 0+}\frac{L(t+\delta) - L(t)}{\delta} \le 0 \quad \text{for all t} \ge 0.$$

This implies that

$$L(t) \le L(0) \quad \text{for all t} \ge 0.$$

By condition (9), we obtain

$$\mathbb{E}|Y(t)|^p \le \frac{a_2}{a_1}\mathbb{E}\|\xi\|^p e^{-\bar{\lambda}t} = \frac{a_2}{a_1}\mathbb{E}\|\xi\|^p e^{-(\lambda-\epsilon)t}. \tag{41}$$

Since $\epsilon \in (0, \lambda)$ is arbitrary, the required inequality (12) must hold. Thus, the proof is complete. □

6. Acknowledgements

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