Introduction to Array Processing

Stephan Weiss

Centre for Signal & Image Processing
Department of Electronic & Electrical Engineering
University of Strathclyde, Glasgow, Scotland, UK

UDRC-EURASIP Summer School, Edinburgh, 30 June 2022

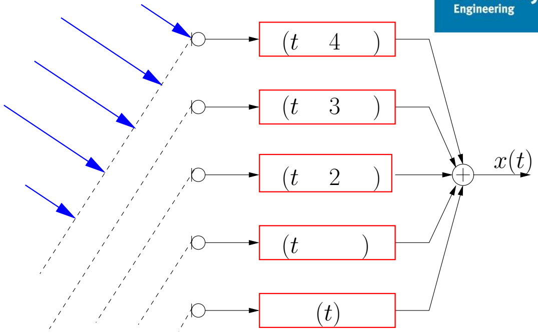
This work was supported by the Engineering and Physical Sciences Research Council (EPSRC) Grant number EP/S000631/1 and the MOD University Defence Research Collaboration in Signal Processing.

Contents:

1.1 Intuitive Beamforming

A far eld wavefront arrives at a linear uniformly spaced sensor array:

due to the direction of arrival (DOA) and nite propagation speed, the wavefront will arrive at di erent sensors with a delay ;



with appropriate processing (beamforming), the sensor signals can be aligned to create constructive interference at the output $x(t)$;

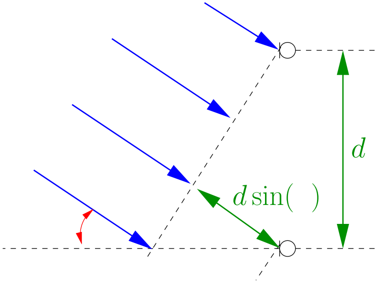
the above is a simple delay-and-sum beamformer [12, 32, 50, 33].

1.2 Spatial Sampling

For unambiguous spatial sampling, we need to take at least two samples per wavelength of the highest frequency component in the array signals [32];

analogy from temporal sampling (Nyquist): take at least two samples per period (relating to the highest frequency component);

Wavelength and frequency f are related by the propagation speed c in the medium:
$= \frac{c}{f}$;



maximum sensor distance

$$d = \frac{\ \ _{\max}}{2} = \frac{c}{2f_{\max}}$$

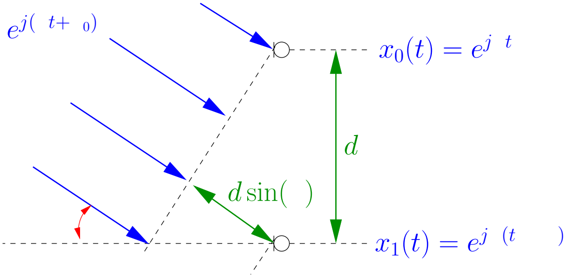
time delay between sensors

$$= \frac{d \sin(\ \ )}{c} = \frac{\sin(\ \ )}{2f_{\max}}$$

Spatial and Temporal Sampling

University of Strathclyde Engineering

Consider the array signals $x_0(t)$ and $x_1(t)$ due to a source $\mathrm{e}^{\mathrm{j}(\ t+ \ _0)}$:



sampling with $t = nT_\mathrm{s}$ leads to

$$x_0[n] = \mathrm{e}^{\mathrm{j} \ nT_\mathrm{s}} \qquad \text{and} \qquad x_1[n] = \mathrm{e}^{\mathrm{j} \ (nT_\mathrm{s} \quad)}$$

with $f_\mathrm{max} = \frac{f_\mathrm{s}}{2} = \frac{1}{2T_\mathrm{s}}$ and normalised angular frequency $\ = \ T_\mathrm{s}$,

$$x_0[n] = \mathrm{e}^{\mathrm{j} \ n} \qquad \text{and} \qquad x_1[n] = \mathrm{e}^{\mathrm{j} \ n} \ \mathrm{e}^{\ \mathrm{j} \quad \sin(\ )}$$

A narrowband source with normalised angular frequency illuminates an M-element linear array of equi-spaced sensors:

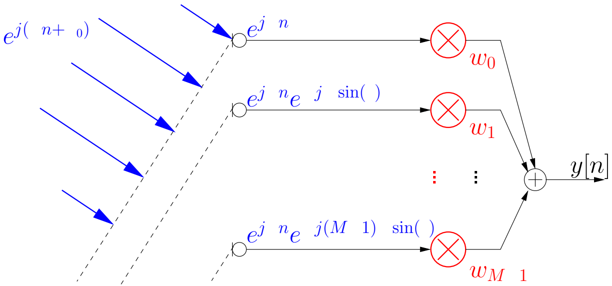
$$\mathbf{x}[n] = \begin{matrix} x_0[n] \\ x_1[n] \\ \vdots \\ x_{M\ \ 1}[n] \end{matrix} = e^{j\ \ n} \begin{matrix} 1 \\ e^{\ j\ \ \sin(\ )} \\ \vdots \\ e^{\ j(M\ \ 1)\ \ \sin(\ )} \end{matrix} = e^{j\ \ n}\ \mathbf{s}$$

the vector $\mathbf{s}$ characterises the phase shifts of waveform with frequency and DOA measured at the array sensors;

since a narrowband signal $e^{j\ \ n}$ only causes phase shifts rather than delays, constructive interference can be accomplished by a set of complex multipliers rather than processors $(t\ \ m\ \ )$, $m = 0\ 1\ \ \ (M\ \ 1)$;

beamforming problem: how to select the set of complex coe cients?

1.4 Data Independent Beamformer



Challenge:
 nd a set of complex multipliers
w_m, $m = 0\ 1\ \ \ (M\ \ 1)$:

to steer the array characteristic towards this source, the output

$$y[n] = [w_0\ \ w_1\ \ \ \ w_{M\ \ 1}]\mathrm{e}^{j\ \ n} \begin{bmatrix} 1 \\ \mathrm{e}^{\ \mathrm{j}\ \ \sin(\ )} \\ \vdots \\ \mathrm{e}^{\ \mathrm{j}(M\ \ 1)\ \ \sin(\ )} \end{bmatrix} = \mathrm{e}^{\mathrm{j}\ \ n}\mathbf{w}^{\mathrm{H}}\mathbf{s}$$

should satisfy $y[n] = \mathrm{e}^{\mathrm{j}\ \ n}$, leading to $\mathbf{w}^{\mathrm{H}}\mathbf{s}\ \ \ = 1$.

Coe cient Vector

For later convenience and compatibility, the Hermitian transpose operator $^{\mathrm{H}}$ is used to denote the coe cient vector

$$\mathbf{w}^{\mathrm{H}} = [w_0 \;\; w_1 \qquad\quad w_{M \;\; 1}] \; ;$$

as a result, the vector $\mathbf{w}$ hold the complex conjugates of the coe cients,

$$\mathbf{w} = \begin{matrix} w_0 \\ w_1 \\ \vdots \\ w_{M \;\; 1} \end{matrix} \qquad ;$$

to access the actual unconjugated coe cients, the beamforming vector $\mathbf{w}$ has to be considered

note that

$$\mathbf{w}^{\mathrm{H}}\mathbf{s} \quad = 1 \qquad\qquad \mathbf{s}^{\mathrm{H}} \; \mathbf{w} = 1$$

Narrowband Beamforming Single Source

The expression $\mathbf{s}^{\mathrm{H}}\ \mathbf{w} = 1$ forms a system with one equation and M unknowns

$$\boxed{\mathbf{s}^{\mathrm{H}}}\ \boxed{\mathbf{w}} = \boxed{1}$$

general solution to an underdetermined system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is the right pseudo-inverse $\mathbf{A}$ [10],

$$\mathbf{x} = \mathbf{A}\ \mathbf{b} = \mathbf{A}^{\mathrm{H}}(\mathbf{A}\mathbf{A}^{\mathrm{H}})\ ^{1}\mathbf{b}\ ;$$

here:

$$\mathbf{w} = (\mathbf{s}^{\mathrm{H}}\ )\ \ 1 = \mathbf{s}\ \ \ (\mathbf{s}^{\mathrm{H}}\ \mathbf{s}\ \ )\ ^{1}\ \ 1 = \frac{\mathbf{s}}{\mathbf{s}\ \ {}^{2}_{2}} = \frac{1}{M}\mathbf{s}\ \ \ ;$$

the complex conjugation for $\mathbf{w}$ inverts and therefore compensates the phase of the steering vector, which could have been foreseen

the formulation via the pseudo-inverse will be powerful for more complicated cases.

Narrowband Beamformer Example

University of Strathclyde Engineering

Source parameters: $= \frac{}{2}$ and $= 30$; array parameter: $M = 5$;
steering vector (with $\sin(\ ) = \frac{1}{4}$):

$$\mathbf{s}^{\mathrm{T}} = [1 \;\; \mathrm{e}^{\mathrm{j}\frac{1}{4}} \qquad \mathrm{e}^{\mathrm{j}\frac{4}{4}}]$$

coe cient vector is given by $\mathbf{w} = (\mathbf{s}^{\mathrm{H}}\ )$;
numerical solution in Matlab;

```
Omega=1/4; theta = pi/6; M=5;
s = exp(-sqrt(-1)*Omega*sin(theta)*(0:(M-1)'));
w = pinv(s');
angle([s conj(w)])/pi
```
yields:

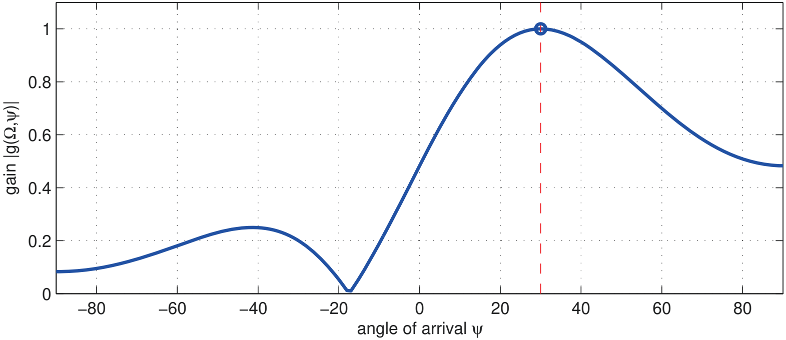
```
-0.00000  0.00000
-0.25000  0.25000
-0.50000  0.50000
-0.75000  0.75000
-1.00000  1.00000
```

Beam Pattern I

- The beamformer has a unit gain towards a source with frequency Ω and DoA θ; what is its gain response towards other angles of arrival?
- the beam pattern measures the response of a beamformer by sweeping the angle ψ of a source with frequency Ω

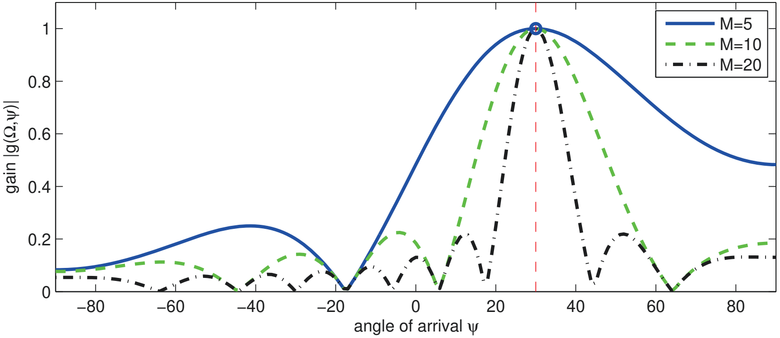
$$g(\Omega, \psi) = \mathbf{w}^{\mathrm{H}} \mathbf{s}_{\Omega,\psi}$$

- beam pattern for the previous example:



Beam Pattern II

- Beam patterns for $\Omega = \frac{\pi}{2}$ and $\vartheta = 30°$ with variable M:



- increasing the sensor number M narrows the main beam, and increases the number of spatial zeros;
- analogous characteristic in the time domain: increased temporal support leads to higher frequency resolution.

Interference

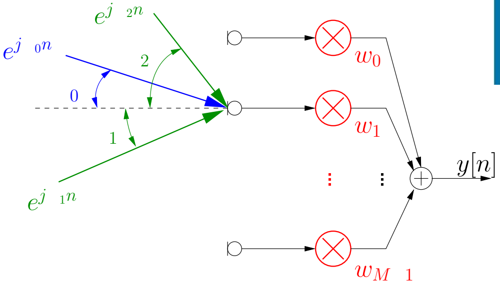
Many scenarios contain a source of interest and a number of interferers:

signal of interest:

0 0

two interferers:

1 1 , 2 2



we would like to control the beampattern to place spatial nulls in the directions of the interfering sources;

Problem formulation and solution :

$$\begin{matrix} \mathbf{s}^{\mathrm{H}}_{0\ \ 0} \\ \mathbf{s}^{\mathrm{H}}_{1\ \ 1} \\ \mathbf{s}^{\mathrm{H}}_{2\ \ 2} \end{matrix} \quad \mathbf{w} = \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \qquad\qquad \mathbf{w} = \begin{matrix} \mathbf{s}^{\mathrm{H}}_{0\ \ 0} \\ \mathbf{s}^{\mathrm{H}}_{1\ \ 1} \\ \mathbf{s}^{\mathrm{H}}_{2\ \ 2} \end{matrix} \quad \begin{matrix} 1 \\ 0 \\ 0 \end{matrix}$$

Narrowband BF Example   Multiple Sources

The signal of interest illuminates an $M = 5$ element array at a frequency $_0 = \frac{}{2}$ with a DoA $_0 = 30$

two interferers at $_1 = \quad _2 = \quad _0$ are present with DoA $_1 = \quad 45$ and $_2 = 60$

results via right pseudo-inverse of steering vectors

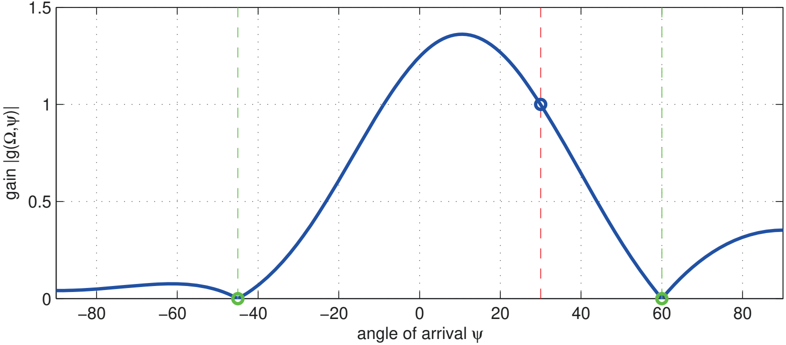
$\mathbf{s}\ _{0\ \ 0}$	$\mathbf{s}\ _{1\ \ 1}$	$\mathbf{s}\ _{2\ \ 2}$	$\mathbf{w}$	$\mathbf{w}$
0.00	0.00	0.00	-42.81	0.3172
45.00	63.64	-77.94	-105.01	0.3004
90.00	127.28	-155.89	-90.00	0.2343
135.00	-169.08	126.17	-74.99	0.3004
180.00	-105.44	48.23	-137.19	0.3172

the angle of $\mathbf{w}$ is no longer intuitive; also note that the coe cients in $\mathbf{w}$ no longer have the same modulus

amongst a manifold of solutions, the right pseudo-inverse provides the minimum norm solution.

Multiple Source Example — Beampattern

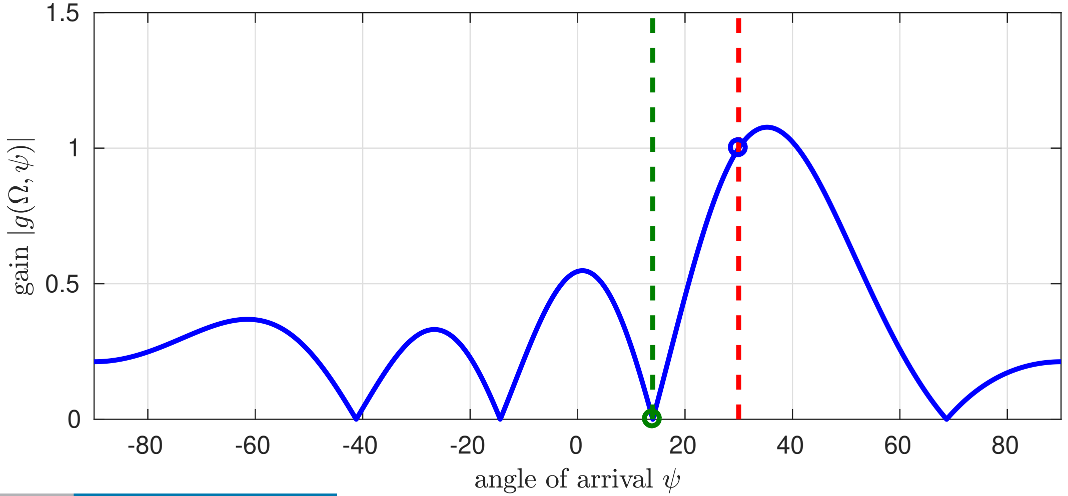
- Beam pattern one source of interest and two interferers:



- the pseudo-inverse is the minimum norm solution, keeping the general gain response as low as possible;
- the minimum norm property protects against spatially white noise.

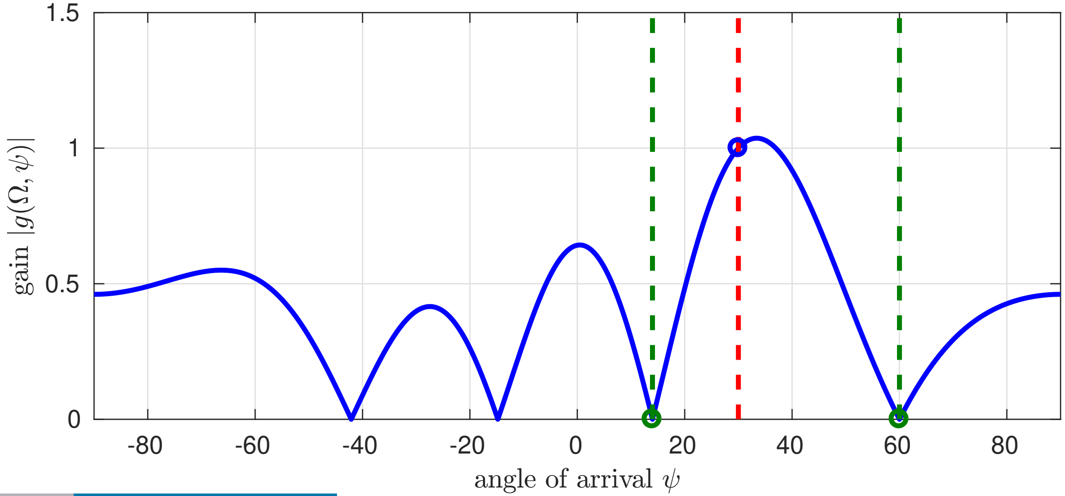
Beamforming Example   Variable Interferer I

$M = 5$ sensors, source of interest towards $_0 = 30$, interferer variable:



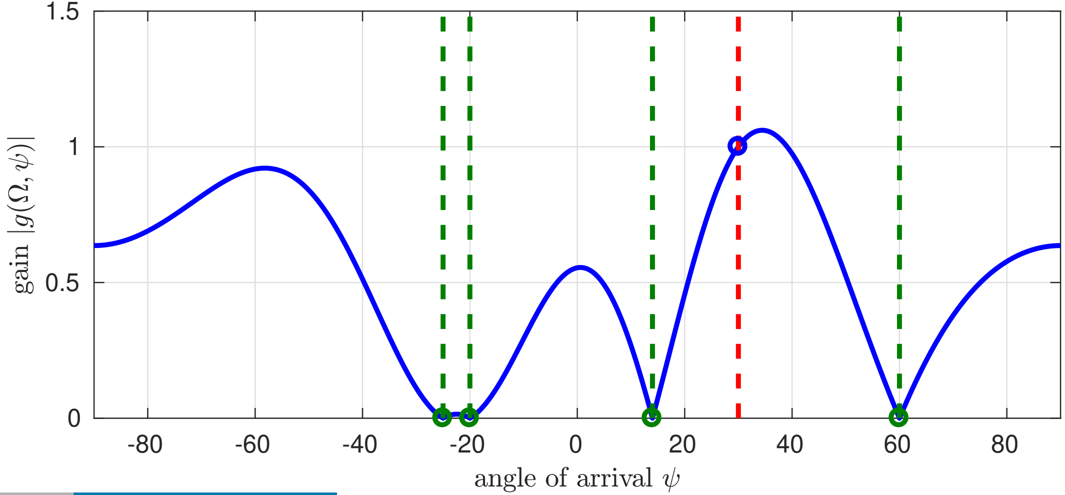
Beamforming Example Variable Interferer II

$M = 5$ sensors, SOI $_0 = 30$, one xed and one variable interferer:



Beamforming Example   Variable Interferer III

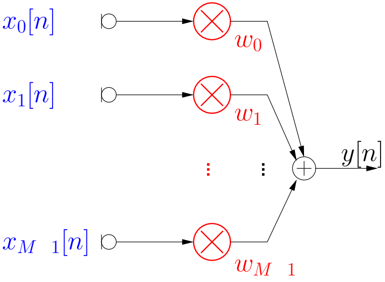
$M = 5$ sensors, SOI $_0 = 30$, one variable and three xed interferers:



Data Independent Beamforming

- Previous beamformer designs were based on the knowledge of DoA of the signal of interest and of interfering sources;
- remaining degrees of freedom are invested to suppress spatially white noise;
- using the analogy between spatial and temporal processing, classical lter design techniques can be invoked to design arrays with a bandpass-type angular response;
- beamformers based on source parameters (frequency, DoA) rather the actual received waveforms are termed data independent beamformers;
- this is in contrast to statistically optimum beamformers, which take the received signal statistics into account.

1.8 Statistically Optimum Beamforming



Statistically optimum beamformer minimise e.g. the signal power of the beamformer output, $y[n]$;

to avoid the trivial solution $\mathbf{w} = \mathbf{0}$, the signal of interest needs to be protected by constraints;

this results in e.g. the following constrained optimisation problem

$$\min_{\mathbf{w}} \mathcal{E}\{|y[n]|^2\} \quad \text{subject to} \quad \mathbf{s}^{\mathrm{H}} \mathbf{w} = 1 \; ;$$

the solution to this specific statistically optimum beamformer is known as the minimum variance distortionless response (MVDR) [33].

MVDR Beamformer

Solving the MVDR problem: minimise the power of $y[n] = \mathbf{w}^{\mathrm{H}}\mathbf{x}$ subject to the contraint $\mathbf{w}^{\mathrm{H}}\mathbf{s}_{\ _0\ _0} = 1$;

Formulation using a Lagrange multiplier :

$$\frac{}{\mathbf{w}}\ \mathbf{w}^{\mathrm{H}}\mathcal{E}\ \ \mathbf{x}\mathbf{x}^{\mathrm{H}}\ \ \mathbf{w}\qquad (\mathbf{w}^{\mathrm{H}}\mathbf{s}_{\ _0\ _0}\quad 1)\ = \mathbf{R}_{xx}\mathbf{w}\qquad \mathbf{s}_{\ _0\ _0} = \mathbf{0}$$

the solution $\mathbf{w} = \ \mathbf{R}_{xx}^{\ 1}\mathbf{s}_{\ _0\ _0}$ is inserted into the constraint equation to determine :

$$\mathbf{s}^{\mathrm{H}}_{\ _0\ _0}\mathbf{R}_{xx}^{\ 1}\mathbf{s}_{\ _0\ _0} = 1$$

therefore

$$\mathbf{w}_{\mathrm{MVDR}} = \ \mathbf{s}^{\mathrm{H}}_{\ _0\ _0}\mathbf{R}_{xx}^{\ 1}\mathbf{s}_{\ _0\ _0}\ ^{\ 1}\mathbf{R}_{xx}^{\ 1}\mathbf{s}_{\ _0\ _0}$$

this statistically optimum beamformer has various other names, e.g. Capon beamformer [7, 32].

MVDR Beamformer   Simple Case

In the case of spatially white noise input, such that

$$\mathbf{R}_{xx} = \ {}^{2}_{xx}\mathbf{I} \qquad\qquad \mathbf{R}_{xx}^{\ 1} = \ {}_{xx}^{\ 2}\mathbf{I}$$

the MVDR solution reduces to

$$\mathbf{w}_{\mathrm{MVDR}} = \frac{\mathbf{s}\ \ _{0}\ \ _{0}}{\mathbf{s}\ \ _{0}\ \ _{0}\ {}^{2}_{2}} = \frac{\mathbf{s}\ \ _{0}\ \ _{0}}{M} \quad ;$$

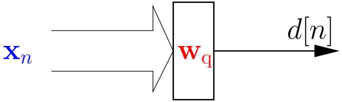
this is identical to the data independent beamformer in the absence of interference (i.e. no spatially structured noise);

Generalised Sidelobe Canceller (GSC)

The generalised sidelobe canceller is a speci c implementation of the MVDR beamformer; it transforms the constrained MVDR problem into an unconstrained optimisation problem;

a rst guess at the solution is performed by the quiescent beamformer $\mathbf{w}_\mathrm{q}$, which is identical to the previously de ned data independent beamformer, obtained by inverting the constraint equation

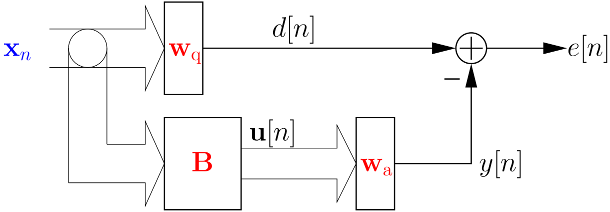
$$\mathbf{C}^\mathrm{H}\mathbf{w}_\mathrm{q} = \mathbf{f} \qquad\qquad \mathbf{w}_\mathrm{q} = \;\; \mathbf{C}^\mathrm{H} \;\; \mathbf{f}$$



the quiescent beamformer eliminates interferers captured by $\mathbf{C}$ and $\mathbf{f}$, but passes the signal of interest, any interferers unaccounted for in the constraints, and spatially distributed noise.

GSC Idea

GSE idea: produce array signals that are free of any contribution from the signal of interest, and use the resulting signal vector $\mathbf{u}[n]$ to eliminate remaining interference from the quiescent output:



the blocking matrix $\mathbf{B}$ eliminates the signal of interest and any interferers captured by the constraints;

the vector $\mathbf{w}_\mathrm{a}$ will be based on the statistics of $\mathbf{u}[n]$ and $d[n]$ to minimise the beamformer output variance $\mathcal{E}\{|e[n]|^2\}$.

Blocking Matrix

In order to project away from the constraints,

$$\mathbf{B}\quad \mathbf{C} = \mathbf{B}\quad \mathbf{s}\ {}_{0}\ {}_{0}\ \mathbf{s}\ {}_{1}\ {}_{1}\qquad \mathbf{s}\ {}_{r\ \ 1}\ {}_{r\ \ 1}\quad = \quad \mathbf{0}$$

assuming that the r constraints are linearly independent, the singular value decomposition of the constraint matrix yields

$$\mathbf{B}\quad \mathbf{U}_0\ \ \mathbf{U}_0\quad \left[\begin{array}{c|c} \begin{matrix} 0 & & \\ & \ddots & \\ & & r\ \ 1 \end{matrix} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right] \quad \mathbf{V}^{\mathrm{H}} \quad = \quad \mathbf{0}$$

the matrix $\mathbf{U}_0 \quad \mathbb{C}^{M \quad (M \quad r)}$ spans the nullspace of $\mathbf{C}^{\mathrm{H}}$, and

$$\mathbf{B} = (\mathbf{U}_0\ )^{\mathrm{H}} \quad \mathbb{C}^{(M \quad r) \quad M}$$

has the required property, as $(\mathbf{U}_0\ )^{\mathrm{H}} \quad \mathbf{U}_0\ \ \mathbf{U}_0 \qquad = [\mathbf{0}\ \ \mathbf{I}] \qquad = \mathbf{0}.$

GSC Unconstrained Optimisation

The beamforming vector $\mathbf{w}_{\mathrm{a}}$ is adjusted to minimise the output power; the MMSE or Wiener solution is given by

$$\mathbf{w}_{\mathrm{a}} = \mathbf{R}_{uu}^{\;1}\;\;\mathbf{p} = \frac{\mathbf{B}\mathbf{R}_{xx}(\mathbf{C}^{\mathrm{H}})\;\;\mathbf{f}}{\mathbf{B}\mathbf{R}_{xx}\mathbf{B}^{\mathrm{H}}}$$

with the covariance matrix

$$\mathbf{R}_{uu} = \mathcal{E}\;\;\mathbf{u}[n]\;\;\mathbf{u}^{\mathrm{H}}[n]\;\;= \mathbf{B}\,\mathcal{E}\;\;\mathbf{x}[n]\;\;\mathbf{x}^{\mathrm{H}}[n]\;\;\mathbf{B}^{\mathrm{H}} = \mathbf{B}\mathbf{R}_{xx}\mathbf{B}^{\mathrm{H}}$$

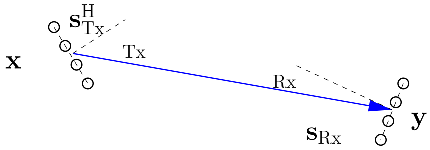
and the cross-correlation vector

$$\mathbf{p} = \mathcal{E}\;\;\mathbf{u}[n]\;\;d\;\;[n]\;\;= \mathbf{B}\mathbf{R}_{xx}\mathbf{w}_{\mathrm{q}}$$

iterative optimisation schemes, such as the least mean squares (LMS) algorithm [12, 50] may be used to approximate the MMSE solution.

1.9 Beamforming and MIMO Processing

Assume a transmission scenario with an M-element transmit (Tx) antenna array and an N-element receive (Rx) array;



in the absence of scatterers and any attenuation, the far eld transmission from the transmit antenna is characterised by a steering vector $\mathbf{s}_{\mathrm{Tx}}^{\mathrm{H}}$;

the incoming waveform at the Rx device is described by another steering vector $\mathbf{s}_{\mathrm{Rx}}$;

the overall MIMO system between a Tx vector $\mathbf{x} \quad \mathbb{C}^M$ and an Rx vector $\mathbf{y} \quad \mathbb{C}^N$ is described as

$$\mathbf{y} = \mathbf{s}_{\mathrm{Rx}} \quad \mathbf{s}_{\mathrm{Tx}}^{\mathrm{H}} \quad \mathbf{x} \quad = \quad \mathbf{Hx}$$

the MIMO system matrix $\mathbf{H} = \mathbf{s}_{\mathrm{Rx}} \quad \mathbf{s}_{\mathrm{Tx}}^{\mathrm{H}}$ is rank one only.

MIMO Requirements

- The far eld assumption is convenient for beamforming, but leads to a rank one MIMO system matrix which is incompatible with the desire to extract multiple independent subchannels or with to achieve diversity;
- rich scattering in connection with MIMO usually implies multiple re ections of signals;
- together with a su ciently large antenna spacing means that the far eld assumption is invalid and the MIMO system matrix is not rank de cient;
- some suggestions of su ciently large spacing imply an antenna element distance of $d > 10$;
- recall spatial sampling requires $d < \frac{1}{2}$!

Beamforming with Spatial Aliasing

For a exible spatial sampling with $d = \quad , 0 < \quad \mathbb{R}$, the steering vector for a waveform with normalised angular frequency and DoA is

$$\mathbf{y} = e^{j \quad n} \begin{bmatrix} 1 \\ e^{j2 \quad \sin(\;)} \\ \vdots \\ e^{j2 \; (M \quad 1) \; \sin(\;)} \end{bmatrix} = \mathbf{s}_2 \qquad e^{j}$$

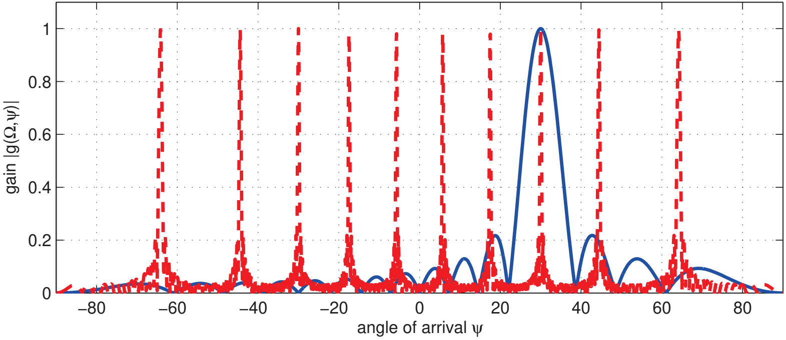
inspecting $\mathbf{s}_2$ the steering vector is aliased to a di erent frequency 2 ;

although the correct frequency can be retrieved unambigiously from temporal sampling of any array element, at various di erent angles could provide the same steering vector $\mathbf{s}_2$;

the array performs spatial undersampling, resulting in spatial aliasing.

Spatial Undersampling Example

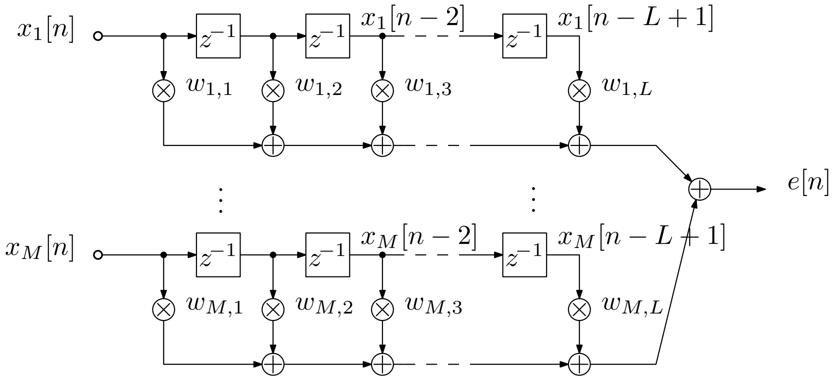
- Beamforming parameters: signal of interest with $\Omega = \frac{\pi}{2}$, direction of arrival $\vartheta = 30°$, $M = 32$ array elements;
- data independent beamformer design with correct spatial sampling $(d = \lambda/2)$ and incorrect spatial sampling $(d = 10\lambda)$:



- MIMO systems perform beamforming, but may dissipate energy into aliased directions.

1.10 Broadband MVDR Beamformer

Each sensor is followed by a tap delay line of dimension L, giving a total of ML coe cients in a vector $\mathbf{v}$ $\mathbb{C}^{ML}$ [5]



Broadband MVDR Beamformer Constraints

University of Strathclyde Engineering

- A larger input vector $\mathbf{x}_n \quad \mathbb{C}^{ML}$ is generated, also including lags;
- the general approach is similar to the narrowband system, minimising the power of $e[n] = \mathbf{v}^{\mathrm{H}}\mathbf{x}_n$;
- however, we require several constraint equations to protect the signal of interest, e.g.

$$\mathbf{C} = [\mathbf{s}(\ _{\mathrm{s}} \quad _0) \quad \mathbf{s}(\ _{\mathrm{s}} \quad _1) \quad\quad \mathbf{s}(\ _{\mathrm{s}} \quad _{L\ \ 1})] \tag{1}$$

- these L constraints pin down the response to unit gain at L separate points in frequency:

$$\mathbf{C}^{\mathrm{H}}\mathbf{v} = \mathbf{1}\ ; \tag{2}$$

- generally $\mathbf{C} \quad \mathbb{C}^{ML \quad L}$, but simpli cations can be applied if the look direction is towards broadside.

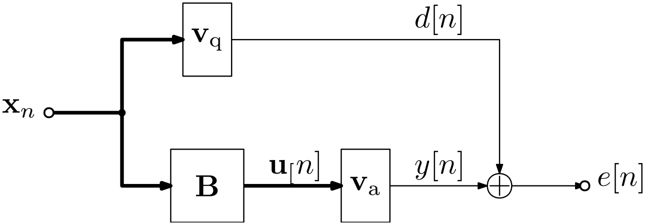
Broadband Generalised Sidelobe Canceller

A quiescent beamformer $\mathbf{v}_\mathrm{q} = \ \mathbf{C}^\mathrm{H} \quad \mathbf{1} \quad \mathbb{C}^{ML}$ picks the signal of interest;

the quiescent beamformer is optimal for AWGN but generally passes structured interference;

the output of the blocking matrix $\mathbf{B}$ contains interference only, which requires $[\mathbf{BC}]$ to be unitary; hence $\mathbf{B} \quad \mathbb{C}^{ML \quad (M \quad 1)L}$;

an adaptive noise canceller $\mathbf{v}_\mathrm{a} \quad \mathbb{C}^{(M \quad 1)L}$ aims to remove the residual interference:

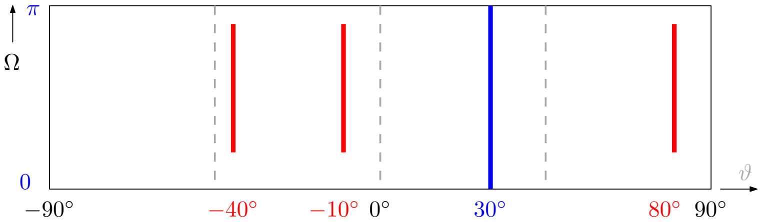


note: all dimensions are determined by $\ M \ \ L \ $.

Broadband Beamformer Example

We assume a signal of interest from $= 30$;

three interferers with angles $_i$ 40 10 80 active over the frequency range $= 2$ $[0\ 1;\ 0\ 45]$ at signal to interference ratio of -40 dB;



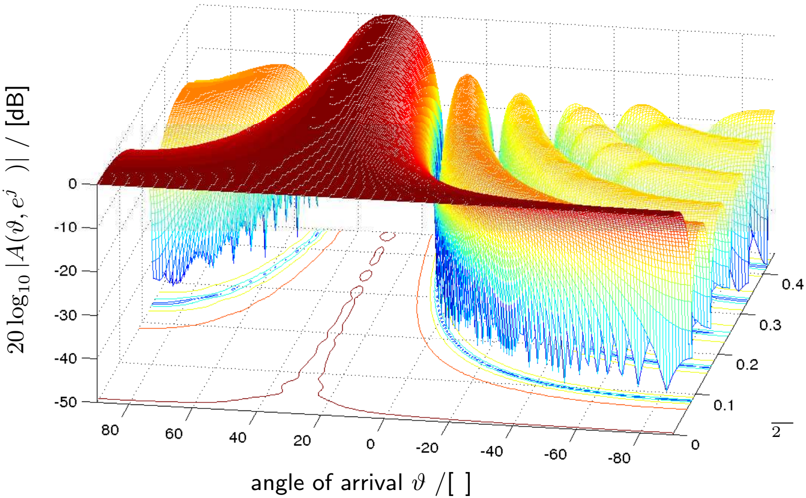
$M = 8$ element linear uniform array is also corrupted by spatially and temporally white additive Gaussian noise at 20 dB SNR;

tap-delay-line length: $L = 150$;

cost per iteration: approx. 2 MMACs (standard), can be reduced to 10 kMACs when e ciently implemented.

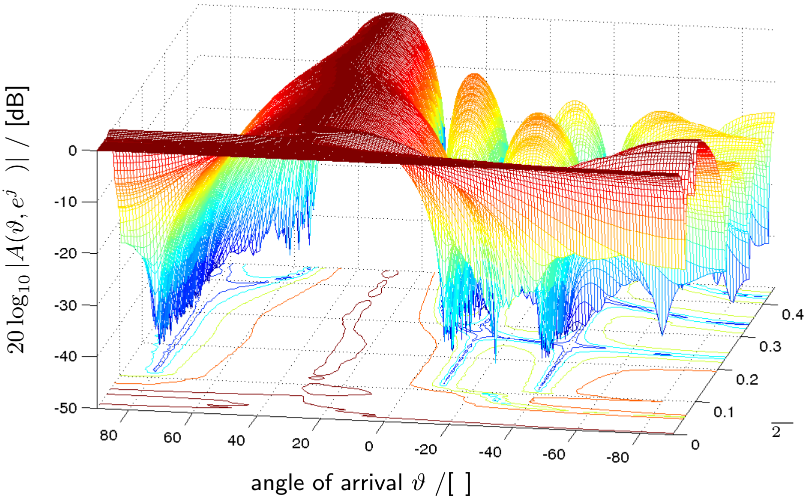
Broadband Quiescent Beamformer

Directivity pattern of quiescent standard broadband beamformer:



Optimised Broadband Beamformer

Directivity pattern of the broadband beamformer:



1.11 Summary

- Spatial sampling by an array of sensors (e.g. antenna elements) has been explored;
- the spatial data window of a narrowband source is characterised by the steering vector;
- appropriate data independent beamformers can be designed based on the steering vectors of desired sources and interferers;
- statistically optimum beamformers are based on the signal statistics;
- a speci c statistically optimum beamformer, the generalised sidelobe canceller, has been reviewed it uses signal statistics to improve the performance of a data independent beamformer derived from the constraint equations;
- some similarities and di erences between beamforming and MIMO systems have been highlighted;
- broadband beamforming requires the inclusion of tap delay lines.

Related Broadband Beamforming Work

University of Strathclyde Engineering

General wideband beamforming: [19];

time domain adaptive broadband beamforming: [5, 6, 11, 14, 23, 29, 33];

discrete Fourier transform domain processing: [17, 30, 44]

subband domain beamforming [21, 34, 46, 47, 48, 49, 44];

frequency-invariant broadband beamforming [18, 22, 23, 38];

polynomial matrix-based beamforming [1, 2, 3, 4, 8, 9, 15, 16, 24, 25, 28, 27, 26, 31, 35, 36, 40, 41, 42, 45, 37, 43].

References I

[1] M. Alrmah, J. Corr, A. Alzin, K. Thompson, and S. Weiss.
Polynomial subspace decomposition for broadband angle of arrival estimation.
In *Sensor Signal Processing for Defence*, pages 1 5, Edinburgh, Scotland, Sept 2014.

[2] M. Alrmah and S. Weiss.
Filter bank based fractional delay lter implementation for widely accurate broadband steering vectors.
In *5th IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, Saint Martin, December 2013.

[3] M. Alrmah, S. Weiss, and S. Lambotharan.
An extension of the MUSIC algorithm to broadband scenarios using polynomial eigenvalue decomposition.
In *19th European Signal Processing Conference*, pages 629 633, Barcelona, Spain, August 2011.

[4] A. Alzin, F. Coutts, J. Corr, S. Weiss, I. K. Proudler, and J. A. Chambers.
Adaptive broadband beamforming with arbitrary array geometry.
In *IET/EURASIP Intelligent Signal Processing*, London, UK, December 2015.

[5] K. M. Buckley.
Spatial/Spectral Filtering with Linearly Constrained Minimum Variance Beamformers.
IEEE Transactions on Acoustics, Speech, and Signal Processing, ASSP-35(3):249 266, March 1987.

[6] K. M. Buckley and L. J. Gri th.
An Adaptive Generalized Sidelobe Canceller with Derivative Constraints.
IEEE Transactions on Antennas and Propagation, 34(3):311 319, March 1986.

[7] J. Capon.
High-resolution frequency-wavenumber spectrum analysis.
Proceedings of the IEEE, 57(8):1408 1418, Aug 1969.

[8] F. Coutts, K. Thompson, S. Weiss, and I. Proudler.
Impact of fast-converging PEVD algorithms on broadband AoA estimation.
In *Sensor Signal Processing for Defence Conference*, pages 1 5, London, UK, December 2017.

References II

[9] F. K. Coutts, I. K. Proudler, and S. Weiss.
E cient implementation of iterative polynomial matrix evd algorithms exploiting structural redundancy and parallelisation.
IEEE Transactions on Circuits and Systems I: Regular Papers, 66(12):4753 4766, Dec. 2019.

[10] G. H. Golub and C. F. Van Loan.
Matrix Computations.
John Hopkins University Press, Baltimore, Maryland, 3rd edition, 1996.

[11] K. M. Gri th, L. J.and Buckley.
Quiescent Pattern Control in Linearly Constrained Adaptive Arrays.
IEEE Transactions on Acoustics, Speech, and Signal Processing, 35(7):917 926, July 1987.

[12] S. Haykin.
Adaptive Filter Theory.
Prentice Hall, Englewood Cli s, 2nd edition, 1991.

[13] S. Haykin and K. R. Liu.
Handbook on Array Processing and Sensor Networks.
Wiley-IEEE Press, 2010.

[14] M. W. Ho man and K. M. Buckley.
Robust Time-Domain Processing of Broadband Microphone Array Data.
IEEE Transactions on Speech and Audio Processing, 3(3):193 203, May 1995.

[15] A. Hogg, V. Neo, S. Weiss, C. Evers, and P. Naylor.
A polynomial eigenvalue decomposition music approach for broadband sound source localization.
In *Proc. IEEE Workshop on Applications of Signal Processing to Audio and Acoustics*, New Paltz, NY, Oct. 2021.

References III

[16] C. L. Koh, S. Redif, and S. Weiss.
Broadband GSC beamformer with spatial and temporal decorrelation.
In *17th European Signal Processing Conference*, pages 889 893, Glasgow, Scotland, August 2009.

[17] C. L. Koh and S. Weiss.
Overlap-Save Broadband GSC Beamforming Algorithms Using Alternative Constraints.
In *International ITG/IEEE Workshop on Smart Antennas*, Duisburg, Germany, April 4-5 2005.

[18] W. Liu and S. Weiss.
New Class of Broadband Arrays with Frequency-Invariant Beam Patterns.
In *IEEE International Conference on Acoustics, Speech, and Signal Processing*, volume II, pages 185 188, Montreal, Canada, May 17 21 2004.

[19] W. Liu and S. Weiss.
Wideband Beamforming — Concepts and Techniques.
Wiley, 2010.

[20] W. Liu, S. Weiss, and L. Hanzo.
A Novel Method for Partially Adaptive Broadband Beamforming.
In *IEEE Workshop on Signal Processing Systems*, pages 361 372, Antwerp, September 2001.

[21] W. Liu, S. Weiss, and L. Hanzo.
Subband Adaptive Generalized Sidelobe Canceller for Broadband Beamforming.
In *11th IEEE Signal Processing Workshop on Statistical Signal Processing*, volume 1, pages 591 594, Singapore, August 2001.

[22] W. Liu, S. Weiss, and L. Hanzo.
A generalized sidelobe canceller employing two-dimensional frequency Invariant lters.
IEEE Transactions on Antennas and Propagation, 53(7):2339 2343, July 2005.

References IV

[23] W. Liu, S. Weiss, and C.-L. Koh.
Constrained Adaptive Broadband Beamforming Algorithm in the Frequency Domain.
In *Sensor Array and Multichannel Signal Processing Workshop*, volume 1, pages 94 98, Barcelona,Spain, July 18 21 2004.

[24] N. Moret, A. Tonello, and S. Weiss.
Mimo precoding for lter bank modulation systems based on psvd.
In *IEEE 73rd Vehicular Technology Conference*, pages 1 5, May 2011.
(best paper award).

[25] V. Neo, C. Evers, S. Weiss, and P. Naylor.
Signal compaction using polynomial EVD for spherical array processing with applications.
IEEE Transactions on Audio, Speech, and Language Processing, submitted December 2022.

[26] V. W. Neo, E. d Olne, A. H. Moore, and P. A. Naylor.
Fixed beamformer design using polynomial eigenvalue decomposition.
In *2022 International Workshop on Acoustic Signal Enhancement (IWAENC)*, pages 1 5, Sep. 2022.

[27] V. W. Neo, S. Weiss, S. W. McKnight, A. O. T. Hogg, and P. A. Naylor.
Polynomial eigenvalue decomposition-based target speaker voice activity detection in the presence of competing talkers.
In *17th International Workshop on Acoustic Signal Enhancement*, Bamberg, Germany, Sept. 2022.

[28] V. W. Neo, S. Weiss, and P. A. Naylor.
A polynomial subspace projection approach for the detection of weak voice activity.
In *Sensor Signal Processing for Defence Conference*, pages 1 5, London, UK, Sept. 2022.

[29] D. Nunn.
Performance Assessments of a Time-Domain Adaptive Processor in a Broad-Band Environment.
IEE Proc. F: Radar and Signal Processing, 130(1):139 146, January 1983.

References V

[30] D. Nunn.
Suboptimal Frequency-Domain Adaptive Antenna Processing Algorithm for Broad-Band Environments.
IEE Proc. F: Radar and Signal Processing, 134(4):341 351, July 1987.

[31] S. Redif, S. Weiss, and J. G. McWhirter.
An approximate polynomial matrix eigenvalue decomposition algorithm for para-hermitian matrices.
In *11th IEEE International Symposium on Signal Processing and Information Technology*, pages 421 425, Bilbao, Spain, December 2011.

[32] H. L. Van Trees.
Detection, Estimation and Modulation Theory: Optimum Array Processing.
Wiley, New York, 2002.

[33] B. D. Van Veen and K. M. Buckley.
Beamforming: A Versatile Approach to Spatial Filtering.
IEEE Acoustics, Speech, and Signal Processing Magazine, 5(2):4 24, April 1988.

[34] S. Weiss.
Analysis and Fast Implementation of Oversampled Modulated Filter Banks.
In J. G. McWhirter and I. K. Proudler, editors, *Mathematics in Signal Processing V*, chapter 23, pages 263 274. Oxford University Press, March 2002.

[35] S. Weiss, M. Alrmah, S. Lambotharan, J. McWhirter, and M. Kaveh.
Broadband angle of arrival estimation methods in a polynomial matrix decomposition framework.
In *IEEE 5th International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, pages 109 112, Dec 2013.

[36] S. Weiss, S. Bendoukha, A. Alzin, F. Coutts, I. Proudler, and J. Chambers.
MVDR broadband beamforming using polynomial matrix techniques.
In *23rd European Signal Processing Conference*, pages 839 843, Nice, France, September 2015.

References VI

[37] S. Weiss, C. Delaosa, J. Matthews, I. Proudler, and B. Jackson.
Detection of weak transient signals using a broadband subspace approach.
In *International Conference on Sensor Signal Processing for Defence*, pages 65 69, Edinburgh, Scotland, Sept. 2021.

[38] S. Weiss, M. Hadley, and J. Wilcox.
Implementation of a exible frequency-invariant broadband beamformer based on fourier properties.
In *Sensor Signal Processing for Defence Conference*, pages 1 5, London, UK, December 2017.

[39] S. Weiss, W. Liu, R. W. Stewart, and I. K. Proudler.
A Generalised Sidelobe Canceller Architecture based on Oversampled Subband Decompositions.
In *5th International Conference on Mathematics in Signal Processing*, Warwick, December 2000.

[40] S. Weiss, J. Pestana, and I. K. Proudler.
On the existence and uniqueness of the eigenvalue decomposition of a parahermitian matrix.
IEEE Transactions on Signal Processing, 66(10):2659 2672, May 2018.

[41] S. Weiss, J. Pestana, I. K. Proudler, and F. K. Coutts.
Corrections to on the existence and uniqueness of the eigenvalue decomposition of a parahermitian matrix .
IEEE Transactions on Signal Processing, 66(23):6325 6327, Dec 2018.

[42] S. Weiss, I. Proudler, F. Coutts, and J. Deeks.
Extraction of analytic eigenvectors from a parahermitian matrix.
In *International Conference on Sensor Signal Processing or Defence*, Edinburgh, UK, 2020.

[43] S. Weiss, I. Proudler, F. Coutts, and F. Khattak.
Eigenvalue decomposition of a parahermitian matrix: Extraction of analytic eigenvectors.
IEEE Transactions on Signal Processing, submitted February 2023.

References VII

[44] S. Weiss and I. K. Proudler.
Comparing E cient Broadband Beamforming Architectures and Their Performance Trade-O s.
In *14th International Conference on Digital Signal Processing*, volume I, pages 417 422, Santorini, Greece, July 1 3 2002. (invited paper).

[45] S. Weiss, I. K. Proudler, and F. K. Coutts.
Eigenvalue decomposition of a parahermitian matrix: Extraction of analytic eigenvalues.
IEEE Transactions on Signal Processing, 69:722 737, 2021.

[46] S. Weiss, A. Stenger, R. Stewart, and R. Rabenstein.
Steady-State Performance Limitations of Subband Adaptive Filters.
IEEE Transactions on Signal Processing, 49(9):1982 1991, September 2001.

[47] S. Weiss, R. W. Stewart, and W. Liu.
A Broadband Adaptive Beamforming Structure with Scaled Aperture.
In *Thirty-Sixth Asilomar Conference on Signals, Systems and Computers*, volume 2, pages 1298 1302, Paci c Grove, CA, November 3 6 2002.

[48] S. Weiss, R. W. Stewart, M. Schabert, I. K. Proudler, and M. W. Ho man.
An E cient Scheme for Broadband Adaptive Beamforming.
In *33rd Asilomar Conference on Signals, Systems, and Computers*, volume I, pages 496 500, Paci c Grove, CA, November 1999.

[49] S. Weiss, R. W. Stewart, A. Stenger, and R. Rabenstein.
Performance Limitations of Subband Adaptive Filters.
In *9th European Signal Processing Conference*, volume III, pages 1245 1248, Rodos, Greece, September 1998.

[50] B. Widrow and S. D. Stearns.
Adaptive Signal Processing.
Prentice Hall, Englewood Cli s, New York, 1985.