Discrete-time feedback control for highly nonlinear hybrid stochastic systems with non-differentiable delays

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Abstract

This paper is concerned with the stabilization problem for a class of nonlinear hybrid stochastic delay systems. Different from most existing results, the system coefficients are highly nonlinear rather than satisfy the conventional linear growth conditions; the time-varying system delays are no longer required to be differentiable and, moreover, feedback control based on discrete-time state and mode observations, which is more practical and costs less, is employed. By using the Lyapunov functional method, we establish the sufficient stabilization criteria in the sense of exponential stability (both the $\bar{q}$th moment stability and the almost sure stability) as well as H_∞ stability and asymptotic stability. Meanwhile, the upper bound on the duration τ between two consecutive state and mode observations is also obtained. Finally, a couple of practical food chain models are discussed to illustrate the theoretical results.

Keywords: highly nonlinear hybrid stochastic systems, non-differentiable delays, feedback control, discrete-time state and mode, Lyapunov functional.

1. Introduction

As an important class of stochastic systems, stochastic differential equations (SDEs) with Markovian switching (also known as hybrid SDEs) have provided a generalized mathematical characterization for many practical systems in branches of science and engineering (see e.g. [1–3]). Accordingly, stochastic systems with Markovian switching have attracted considerable attention, with subsequent emphasis being placed on the analysis of stability and control synthesis, and a great number of remarkable results have been reported, for instance [4–9].

Traditionally, the system coefficients are required to be linear or satisfy the linear growth condition, namely bounded by linear functions. However, many hybrid SDE models in the real world do not satisfy these conditions. In other words, these conditions are too strong and hence restrict the application of the theory. That is why the study on hybrid SDEs without the linear growth condition (namely, highly nonlinear) has recently become more and more popular. Some salient results have been carried out on stability and stabilization of highly nonlinear hybrid stochastic systems, such as [10–13]. The aim of this paper is to develop the stabilization theory further for highly nonlinear hybrid stochastic systems.

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As is well known, time delays exist inevitably in various dynamical systems and very often lead to poor performance, oscillations and instability of control systems [14, 15]. In recent years, many efforts have been devoted to the stability and stabilization problems of hybrid stochastic systems with time delays (see e.g.[16–21]). Limited by the mathematical technique, the time delays in most existing papers have been assumed to be, either constants or functions of time differentiable with the derivative less than 1. However, these conditions really might not be a natural feature of stochastic delay systems in the real world and cannot be satisfied by many important systems. For example, the piecewise constant delays and sawtooth delays frequently arise in sampled-data control and network-based control where delays are commonly referred to as fast varying delays (no assumptions on the delay-derivatives)(see, e.g. [22–24]). A simplest example for the piecewise constant delays is the case when the time delay in a network is larger during business hours than other time. Such a time delay can be described by a piecewise constant function

$$\delta_t = \sum_{k=0}^{\infty} \left(d_1 I_{[k,k+1/3)}(t) + d_2 I_{[k+1/3,k+1)}(t) \right),$$

where $d_1 > d_2$ are two positive numbers, the time unit is one day, $[0, 1/3)$ and $[1/3, 1)$ stand for the business hours and off-business hours respectively. Obviously, such a simple delay function is not differentiable. Therefore, it is necessary and important to avoid requiring the time delay to be a constant or a differentiable function to enable the stability and stabilization theory more extensive scope of applications. This is exactly a part of what we are going to tackle in this paper.

Now that we have known the stability of stochastic systems may encounter degradation or even become unstable because of high nonlinearity, time delays, Markovian switching or some other factors. It is vital to investigate how to make the unstable highly nonlinear hybrid stochastic delay systems return to be stable, which is involved with the stabilization problem of systems. To realize the stabilization of a system, different control schemes have been proposed. For example, a novel least-squares identification was proposed and adaptive control was designed to guarantee the stability in probability of stochastic nonlinear system in [25] and the prescribed-time mean-square stabilization problem was solved by developing a new non-scaling backstepping design scheme in [26]. Delay feedback control for highly nonlinear hybrid stochastic delay systems was considered in [13].

Particularly, Mao [27] initiated a feedback control based on discrete-time state observations $u(x(\eta_t), r(t))$ (where $\eta_t = [t/\tau]\tau$ for $t \geq 0$ and $\tau > 0$) to stabilize continuous-time hybrid stochastic systems. This controller is obviously more practical and costs less. Since then, this new design has attracted increasing interests of researchers [12, 28–32]. It is noted that the feedback controls in the mentioned results are based on discrete-time observations of the state but they still depend on continuous-time observations of the mode. Of course this is perfectly fine if the mode of the system is fully observable at no cost. However, the mode is not obvious in many real-world situations and it costs to identify the current mode of a hybrid stochastic system. So it is more reasonable to design a feedback control $u(x(\eta_t), r(\eta_t))$, which is based on the discrete-time observations of both state and mode.

Very recently, the existence, boundedness and stability for nonlinear hybrid SDEs with non-differential delays have been discussed in [33], but the control problem has not yet been considered. In [34], Dong and Mao have employed a delay feedback control to study the stabilization of highly nonlinear hybrid SDEs with non-differentiable delays. Although the time lag between the time when the state observation is made and the time when the corresponding control reaches the system has been considered, the control function requires continuous-time observations of system state $x(t)$ and system mode $r(t)$, which is not possible in practice and the cost is relatively high. This is why we design the developed controller $u(x(\eta_t), r(\eta_t))$ to stabilize highly nonlinear hybrid SDEs with non-differential delays.

All of the points made above motivate us to investigate the stabilization for highly nonlinear hybrid stochastic systems with non-differentiable delays by feedback control based on discrete-time observations of both state and mode. The key contributions of this paper lie in:

(1) We relax the conditions on system coefficients and time delays. Specifically, we do not require the coefficients to be linear growth and the time delays to be differentiable. Thus our newly established theory would have more extensive scope of applications.

(2) The more reasonable and practical controller $u(x(\eta_t), r(\eta_t))$ is employed to stabilize highly nonlinear hybrid SDEs with non-differential delays. Actually, the factors of high nonlinearity, non-differentiable delays as well as discrete-time state and mode make this study a challenge.

(3) We establish the sufficient stabilization criteria in the sense of H_∞ stability, asymptotic stability, $\bar{q}$th moment exponential stability and almost sure exponential stability.

The remainder of this paper is organised as follows. In Section 2, we present our assumptions and preliminaries. In Section 3, the main results are established one by one. Section 4 covers two examples to illustrate the theoretical results. Finally, this article is concluded in Section 5.

2. Notation and standing hypotheses

Let us first introduce the notation used throughout this paper. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and $|x|$ the Euclidean norm for $x \in R^n$. Let $\mathbb{R}_+ = [0, \infty)$. Let A^T denote the transpose of a vector or matrix A. For a matrix A, we let $|A| = \sqrt{\text{trace}(A^T A)}$ be its trace norm. If A is a symmetric real-valued matrix, denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalue, respectively. By $A \le 0$ and $A < 0$, we mean A is non-positive and negative definite, respectively. Let $h > 0$ and $C([-h, 0]; \mathbb{R}^n)$ denote the family of continuous functions φ from $[-h, 0]$ to R^n with the norm $\|\varphi\| = \sup_{-h \le \theta \le 0} |\varphi(\theta)|$. Denote by $C(\mathbb{R}^n; \mathbb{R}_+)$ the family of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}_+$. If both a, b are real numbers, then $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. If A is a set, I_A stands for its indicator function; that is, $I_A(z) = 1$ if $z \in A$ and 0 otherwise.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \ge 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous with $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets) and $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t), t \ge 0$ represent a right-continuous Markov chain on the same probability space, which is assumed to be independent of the Brownian motion $B(\cdot)$ and take values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator

$\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta)=j|r(t)=i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1+\gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. Denote by $C^b_{\mathcal{F}_0}([-h,0];\mathbb{R}^n)$ the family of all bounded, $\mathcal{F}_0$-measurable $C([-h,0];\mathbb{R}^n)$-valued random variables.

The given unstable system discussed in this paper is described by the nonlinear hybrid stochastic delay differential equation (SDDE)

$$dx(t) = f(x(t), x(t-\delta_t), r(t), t)dt + g(x(t), x(t-\delta_t), r(t), t)dB(t) \tag{2.1}$$

on $t \geq 0$, with initial data

$$\begin{cases} \{x(t) : -h \leq t \leq 0\} = \xi \in C([-h,0];\mathbb{R}^n), \\ r(0) = r_0 \in S, \end{cases} \tag{2.2}$$

where $h > 0$ stands for the upper bound for the time-varying delay, and the coefficients $f : R^n \times R^n \times S \times R_+ \to R^n$, $g : R^n \times R^n \times S \times R_+ \to R^{n\times m}$ are Borel measurable functions.

As mentioned in last section, one of our key features in this paper is that the time delay involved in the underlying system is a non-differentiable function of time. Let us precisely state it as an assumption.

Assumption 2.1. *The time-varying delay δ_t is a Borel measurable function from $\mathbb{R}_+$ to $[h_1, h]$ and has the property that*

$$\bar{h} := \limsup_{\triangle\to 0^+}\left(\sup_{s\geq -h}\frac{\mu(M_{s,\triangle})}{\triangle}\right) < \infty, \tag{2.3}$$

where h_1 and h are both constants with $0 \leq h_1 < h$, $M_{s,\triangle} = \{t \in \mathbb{R}_+ : t - \delta_t \in [s, s+\triangle)\}$ and $\mu(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}_+$.

It should be pointed out that the above assumption is indeed weaker than the traditional condition that the time-varying delay δ_t is differentiable with its derivative being bounded by a positive number less than 1. Moreover, many time-varying delay functions in practice satisfy Assumption 2.1. Besides, under this assumption we always have $\bar{h} \geq 1$ and the following useful lemma. Please see [33, 34] for more details.

Lemma 2.2. *Let Assumption 2.1 hold. Let $T > 0$ and $\varphi : [-h, T-h_1] \to \mathbb{R}_+$ be a continuous function. Then*

$$\int_0^T \varphi(t-\delta_t)dt \leq \bar{h}\int_{-h}^{T-h_1}\varphi(t)dt. \tag{2.4}$$

Another key feature in this paper is that highly nonlinear hybrid SDDEs are considered. That is, we maintain the local Lipschitz condition and impose the polynomial growth condition rather than the linear growth condition on the coefficients. We therefore state the following hypothesis.

Assumption 2.3. *For any positive number k, there exists a positive constant L_k such that*

$$|f(x,y,i,t) - f(\bar{x},\bar{y},i,t)| \vee |g(x,y,i,t) - g(\bar{x},\bar{y},i,t)| \leq L_k(|x-\bar{x}| + |y-\bar{y}|) \tag{2.5}$$

for those $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$ *with* $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq k$ *and all* $(i,t) \in S \times \mathbb{R}_+$. *Moreover, there exist constants* $L > 0, q_1 > 1$ *and* $q_i \geq 1$ $(2 \leq i \leq 4)$ *such that*

$$|f(x,y,i,t)| \leq L(|x| + |y| + |x|^{q_1} + |y|^{q_2}),$$
$$|g(x,y,i,t)| \leq L(|x| + |y| + |x|^{q_3} + |y|^{q_4}) \tag{2.6}$$

for all $(x,y,i,t) \in \mathbb{R}^n \times \mathbb{R}^n \times S \times \mathbb{R}_+$.

The highly nonlinear stochastic systems satisfying the polynomial growth condition exist widely in the real world, such as the Ait-Sahalia interest rate model in finance, the food chain model in ecology and the vibration model in mechanical engineering (see e.g. [35–37]). Specifically, let us see the physical mass-spring-damper model in [38], which is affected by external force involving the environmental noise and abrupt changes in parameters. It can be written as the following 2-dimensional highly nonlinear hybrid stochastic differential delay equation:

$$dx(t) = f(x(t), x(t-\delta_t), r(t), t)dt + g(x(t), x(t-\delta_t), r(t), t)dB(t),$$

where $B(t)$ is a scalar Brownian motion, δ_t is the time-varying delay, $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with its generator $\Gamma = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}$ and the coefficients f, g are defined by

$$f(x,y,1,t) = (x_2, -1.5x_1 - 2.5x_2 - 0.54x_2y_2^2)^T, \quad f(x,y,2,t) = (x_2, -1.5x_1 - 2.5x_2 - 1.5x_2^3y_2^2)^T,$$

$$g(x,y,1,t) = (0, 0.6y_1 + 0.3y_2(1+x_2))^T, \quad g(x,y,2,t) = (0, y_1 + 0.2y_2(1+x_2))^T.$$

It is well known that Assumption 2.3 guarantees the existence and uniqueness of the maximal local solution for the SDDE (2.1), which may explode to infinity at a finite time (see e.g. [2]). To avoid such a possible explosion, we introduce the following Khasminskii-type condition.

Assumption 2.4. *Assume that there exist positive constants* $p, q, \alpha_1, \alpha_2, \alpha_3$ *with*

$$q > (p + q_1 - 1) \vee [2(q_1 \vee q_2 \vee q_3 \vee q_4)], \quad p \geq 2(q_1 \vee q_2 \vee q_3 \vee q_4) - q_1 + 1 \tag{2.7}$$

such that

$$x^T f(x,y,i,t) + \frac{q-1}{2}|g(x,y,i,t)|^2 \leq \alpha_1(|x|^2 + |y|^2) - \alpha_2|x|^p + \alpha_3|y|^p \tag{2.8}$$

for all $(x,y,i,t) \in \mathbb{R}^n \times \mathbb{R}^n \times S \times \mathbb{R}_+$. *Moreover, we assume that* $\alpha_2 > \alpha_3\bar{h}$. *Letting*

$$\beta_1 = q\alpha_2 - \frac{\alpha_3 q(q-2)}{p+q-2}, \quad \beta_2 = \frac{\alpha_3 pq}{p+q-2},$$

we have $\beta_1 > \beta_2\bar{h}$.

From Assumptions 2.1, 2.3 and 2.4, it follows that system (2.1) has a unique global solution $x(t)$ with initial value (2.2) such that $\sup_{-h \leq t < \infty} \mathbb{E}|x(t)|^q < \infty$ (see e.g. [34]). But the boundedness can not educe the stability. So our aim here is to design a feedback control $u(x(\eta_t), r(\eta_t), t)$ so that the controlled hybrid SDDE

$$dx(t) = \big(f(x(t), x(t-\delta_t), r(t), t) + u(x(\eta_t), r(\eta_t), t)\big)dt + g(x(t), x(t-\delta_t), r(t), t)dB(t), \quad t \geq 0, \tag{2.9}$$

will become stable in some certain sense, where $u : \mathbb{R}^n \times S \times \mathbb{R}_+ \to \mathbb{R}^n$ is Borel measurable and $\eta_t = [t/\tau]\tau$, in which $[t/\tau]$ is the integer part of t/τ and hence $\tau > 0$ means the duration between two consecutive observations. We can observe that the feedback control $u(x(\eta_t), r(\eta_t), t)$ is designed based on the discrete-time state observations $x(0), x(\tau), x(2\tau), \cdots$ and discrete-time mode observations $r(0), r(\tau), r(2\tau), \cdots$ as well, though the given unstable hybrid SDDE (2.1) is of continuous-time, which is also an important feature of this paper. We will design the control function to satisfy the following assumption.

Assumption 2.5. *There exists a positive constant κ such that*

$$|u(x, i, t) - u(y, i, t)| \leq \kappa |x - y| \tag{2.10}$$

for all $(x, y, i, t) \in \mathbb{R}^n \times \mathbb{R}^n \times S \times \mathbb{R}_+$. Moreover, assume that $u(0, i, t) = 0$ for all $(i, t) \in S \times \mathbb{R}_+$.

Let us close this section by introducing a useful lemma, which will play an important role in coping with the discrete-time Markov chain. For its explanation and proof details, we refer the reader to [39, 40].

Lemma 2.6. *For any $t \geq 0$, $v > 0$ and $i \in S$, we have*

$$\mathbb{P}\big(r(s) \neq i \ \textit{for some} \ s \in [t, t+v] | r(t) = i\big) \leq 1 - e^{-\bar{\gamma} v},$$

in which

$$\bar{\gamma} = \max_{i \in S}(-\gamma_{ii}).$$

3. Analysis for the controlled system

3.1. Boundedness

As pointed out in last section, the qth moment of the solution of SDDE (2.1) is bounded. The following theorem shows that the controlled system (2.9) preserves this nice property, which will be a foundation of this paper.

Theorem 3.1. *Under Assumptions 2.1, 2.3, 2.4 and 2.5, the controlled system (2.9) with the initial data (2.2) has a unique global solution $x(t)$ on $[-h, \infty)$ which satisfies*

$$\sup_{-h \leq t < \infty} \mathbb{E}|x(t)|^q < \infty. \tag{3.1}$$

Proof. We divide the proof into three steps to make it more understandable.

Step1. In fact, if we define a bounded function $\nu : \mathbb{R}_+ \to [0, \tau]$ by

$$\nu(t) = t - k\tau \quad \textit{for} \quad k\tau \leq t < (k+1)\tau, \quad k = 0, 1, 2 \cdots,$$

then the controlled system (2.9) can be written as

$$dx(t) = \big[f(x(t), x(t-\delta_t), r(t), t) + u(x(t-\nu(t)), r(t-\nu(t)), t)\big]dt + g(x(t), x(t-\delta_t), r(t), t)dB(t) \tag{3.2}$$

on $t \geq 0$ with the initial data (2.2). Let $\bar{U}(x) = |x|^q$, then it follows from the Itô formula that

$$\begin{aligned} d\bar{U}(x(t)) =& L\bar{U}(x(t), x(t-\delta_t), x(t-\nu(t)), r(t), r(t-\nu(t)), t)dt \\ &+ q|x|^{q-2}x^T(t)g(x(t), x(t-\delta_t), r(t), t)dB(t), \end{aligned} \tag{3.3}$$

where the operator $L\bar{U} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times S \times S \times \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$\begin{aligned} L\bar{U}(x, y, z, i, \hat{i}, t) =& q|x|^{q-2}x^T[f(x, y, i, t) + u(z, \hat{i}, t)] + \frac{q}{2}|x|^{q-2}|g(x, y, i, t)|^2 \\ &+ \frac{q(q-2)}{2}|x|^{q-4}|x^T g(x, y, i, t)|^2 \\ \leq& q|x|^{q-2}\big(x^T[f(x, y, i, t) + u(z, \hat{i}, t)] + \frac{q-1}{2}|g(x, y, i, t)|^2\big). \end{aligned}$$

By Assumptions 2.4 and 2.5, we can derive that

$$L\bar{U}(x, y, z, i, \hat{i}, t) \leq q\kappa|x|^{q-1}|z| + \alpha_1 q|x|^q + \alpha_1 q|x|^{q-2}|y|^2 - \alpha_2 q|x|^{p+q-2} + \alpha_3 q|x|^{q-2}|y|^p. \tag{3.4}$$

According to Assumption 2.4, we can choose a constant ε_0 for

$$0 < \varepsilon_0 < \beta_1 - \beta_2\bar{h}, \quad 2 - \frac{1}{h}\ln\frac{\beta_1 - \varepsilon_0}{\beta_2\bar{h}} > 0$$

and then choose another constant ε satisfying

$$0 < \varepsilon < \min\{\frac{2}{\bar{h}}, \frac{2 - \frac{1}{h}\ln\frac{\beta_1-\varepsilon_0}{\beta_2\bar{h}}}{1 + \frac{\beta_1-\varepsilon_0}{\beta_2}}\}.$$

By the Young inequality, we can obtain

$$\begin{aligned} q\kappa|x|^{q-1}|z| &\leq c|x|^q + \varepsilon|z|^q, \\ \alpha_1 q|x|^{q-2}|y|^2 &\leq c|x|^q + \varepsilon|y|^q, \\ |x|^{q-2}|y|^p &\leq \frac{q-2}{p+q-2}|x|^{p+q-2} + \frac{p}{p+q-2}|y|^{p+q-2}, \end{aligned}$$

where, here and in the remaining part, c stands for a positive constant that may change from line to line but its special form is of no use. Hence we have

$$L\bar{U}(x, y, z, i, \hat{i}, t) \leq C - 2|x|^q + \varepsilon|y|^q + \varepsilon|z|^q - (\beta_1 - \varepsilon_0)|x|^{p+q-2} + \beta_2|y|^{p+q-2}, \tag{3.5}$$

where

$$C := \sup_{u\geq 0}[(2c+2)|u|^q - \varepsilon_0|u|^{p+q-2}].$$

Step2. Next let us prove the existence and uniqueness of the global solution of the hybrid SDDE (2.9) on $t \geq -h$. Since the coefficients of the controlled system (2.9) are locally Lipschitz continuous, for any given initial data (2.2), there is a unique maximal local solution $x(t)$ on $[-h, e_\infty)$, where e_∞ is the explosion time (see e.g. [2]). Let $k_0 > 0$ be sufficiently large for $k_0 \geq \|\xi\|$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0, e_\infty) : |x(t)| \geq k\},$$

where, throughout this paper, we set $\inf \emptyset = \infty$ (in which $\emptyset$ denotes the empty set as usual). Obviously, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \le e_\infty$ a.s.. If we show $\tau_\infty = \infty$ a.s., then $e_\infty = \infty$ a.s. and the globality of the unique solution $x(t)$ follows.

By the stopping time technique, we can derive from (3.3) and (3.5) that

$$\begin{aligned}\mathbb{E}|x(t\wedge\tau_k)|^q \le& |\xi(0)|^q + \mathbb{E}\int_0^{t\wedge\tau_k} [C - 2|x(s)|^q + \varepsilon|x(s-\delta_s)|^q + \varepsilon|x(s-\nu(s))|^q \\ &\qquad - (\beta_1-\varepsilon_0)|x(s)|^{p+q-2} + \beta_2|x(s-\delta_s)|^{p+q-2}]ds \\ \le& |\xi(0)|^q + Ct - 2\mathbb{E}\int_0^{t\wedge\tau_k}|x(s)|^q ds + \varepsilon\mathbb{E}\int_0^{t\wedge\tau_k}|x(s-\delta_s)|^q ds + \varepsilon\mathbb{E}\int_0^{t\wedge\tau_k}|x(s-\nu(s))|^q ds \\ &- (\beta_1-\varepsilon_0)\mathbb{E}\int_0^{t\wedge\tau_k}|x(s)|^{p+q-2}ds + \beta_2\mathbb{E}\int_0^{t\wedge\tau_k}|x(s-\delta_s)|^{p+q-2}ds.\end{aligned} \tag{3.6}$$

By Lemma 2.2, we can show that

$$\mathbb{E}\int_0^{t\wedge\tau_k}|x(s-\delta_s)|^q ds \le \bar{h}\big(\mathbb{E}\int_{-h}^0|x(s)|^q ds + \mathbb{E}\int_0^{t\wedge\tau_k}|x(s)|^q ds\big),$$

$$\mathbb{E}\int_0^{t\wedge\tau_k}|x(s-\delta_s)|^{p+q-2} ds \le \bar{h}\big(\mathbb{E}\int_{-h}^0|x(s)|^{p+q-2} ds + \mathbb{E}\int_0^{t\wedge\tau_k}|x(s)|^{p+q-2} ds\big).$$

Substituting these into (3.6) and recalling the choosing conditions on ε_0 and ε, we have

$$\begin{aligned}\mathbb{E}|x(t\wedge\tau_k)|^q &\le C_0 + Ct + \varepsilon\mathbb{E}\int_0^{t\wedge\tau_k}|x(s-\nu(s))|^q ds \\ &= C_0 + Ct + \varepsilon\int_0^t \mathbb{E}(|x(s-\nu(s))|^q I_{[0,\tau_k]}(s))ds,\end{aligned}$$

where $C_0 = |\xi(0)|^q + \varepsilon h\bar{h}\|\xi\|^q + \beta_2 h\bar{h}\|\xi\|^{p+q-2}$. From the definition of function $\nu(\cdot)$, we observe that $0 \le s - \nu(s) \le s$ for all $s \ge 0$. Then we have

$$\mathbb{E}(|x(s-\nu(s))|^q I_{[0,\tau_k]}(s)) \le \sup_{0\le u\le s}\mathbb{E}(|x(u)|^q I_{[0,\tau_k]}(s)) \le \sup_{0\le u\le s}\mathbb{E}|x(u\wedge\tau_k)|^q.$$

Thus,

$$\mathbb{E}|x(t\wedge\tau_k)|^q \le C_0 + Ct + \varepsilon\int_0^t \sup_{0\le u\le s}\mathbb{E}|x(u\wedge\tau_k)|^q ds. \tag{3.7}$$

Having in mind that the right-hand side of (3.7) is increasing in t, we can further get

$$\sup_{0\le u\le t}\mathbb{E}|x(u\wedge\tau_k)|^q \le C_0 + Ct + \varepsilon\int_0^t \sup_{0\le u\le s}\mathbb{E}|x(u\wedge\tau_k)|^q ds.$$

An application of the Gronwall inequality yields

$$\sup_{0\le u\le t}\mathbb{E}|x(u\wedge\tau_k)|^q \le (C_0 + Ct)e^{\varepsilon t}.$$

Consequently,

$$k^q\mathbb{P}(\tau_k \le t) \le \mathbb{E}|x(t\wedge\tau_k)|^q \le (C_0 + Ct)e^{\varepsilon t}.$$

Letting $k \to \infty$ yields $\mathbb{P}(\tau_\infty \le t) = 0$, namely $\mathbb{P}(\tau_\infty > t) = 1$. Since $t \ge 0$ is arbitrary, we must have $\mathbb{P}(\tau_\infty = \infty) = 1$.

Step3. Finally, we will show the boundedness of the qth moment of the solution. Let $\varepsilon_1 > 0$ be the unique root to equation

$$\beta_1 - \varepsilon_0 = \beta_2 \bar{h} e^{\varepsilon_1 h}. \tag{3.8}$$

By the Itô formula and (3.5), we have

$$\begin{aligned} e^{\varepsilon_1 t}\mathbb{E}|x(t)|^q \leq & |\xi(0)|^q + \mathbb{E}\int_0^t e^{\varepsilon_1 s}\big(\varepsilon_1|x(s)|^q + C - 2|x(s)|^q + \varepsilon|x(s-\delta_s)|^q + \varepsilon|x(s-\nu(s))|^q \\ & - (\beta_1 - \varepsilon_0)|x(s)|^{p+q-2} + \beta_2|x(s-\delta_s)|^{p+q-2}\big)ds \end{aligned} \tag{3.9}$$

We can derive from Lemma 2.2 that

$$\begin{aligned} \mathbb{E}\int_0^t e^{\varepsilon_1 s}|x(s-\delta_s)|^q ds \leq & e^{\varepsilon_1 h}\mathbb{E}\int_0^t e^{\varepsilon_1 (s-\delta_s)}|x(s-\delta_s)|^q ds \\ \leq & \bar{h}e^{\varepsilon_1 h}\Big(\mathbb{E}\int_{-h}^0 e^{\varepsilon_1 s}|x(s)|^q ds + \mathbb{E}\int_0^t e^{\varepsilon_1 s}|x(s)|^q ds\Big) \\ \leq & \bar{h}e^{\varepsilon_1 h}\mathbb{E}\int_{-h}^0 e^{\varepsilon_1 s}|x(s)|^q ds + \bar{h}e^{\varepsilon_1 h}\mathbb{E}\int_0^t e^{\varepsilon_1 s}|x(s)|^q ds, \end{aligned}$$

$$\mathbb{E}\int_0^t e^{\varepsilon_1 s}|x(s-\delta_s)|^{p+q-2} ds \leq \bar{h}e^{\varepsilon_1 h}\mathbb{E}\int_{-h}^0 e^{\varepsilon_1 s}|x(s)|^{p+q-2} ds + \bar{h}e^{\varepsilon_1 h}\mathbb{E}\int_0^t e^{\varepsilon_1 s}|x(s)|^{p+q-2} ds.$$

Substituting these into (3.9) and recalling (3.8), we get

$$e^{\varepsilon_1 t}\mathbb{E}|x(t)|^q \leq \hat{C}_0 + \frac{C}{\varepsilon_1}e^{\varepsilon_1 t} + \mathbb{E}\int_0^t e^{\varepsilon_1 s}[(\varepsilon_1 + \varepsilon\bar{h}e^{\varepsilon_1 h} - 2)|x(s)|^q + \varepsilon|x(s-\nu(s))|^q]ds, \tag{3.10}$$

where $\hat{C}_0 = |\xi(0)|^q + \varepsilon\bar{h}e^{\varepsilon_1 h}\mathbb{E}\int_{-h}^0 e^{\varepsilon_1 s}|x(s)|^q ds + \beta_2\bar{h}e^{\varepsilon_1 h}\mathbb{E}\int_{-h}^0 e^{\varepsilon_1 s}|x(s)|^{p+q-2}ds$. In a similar way as we did in Step 2, we can derive that

$$\sup_{0\leq u\leq t} e^{\varepsilon_1 u}\mathbb{E}|x(u)|^q \leq \hat{C}_0 + \frac{C}{\varepsilon_1}e^{\varepsilon_1 t} + \int_0^t (\varepsilon_1 + \varepsilon\bar{h}e^{\varepsilon_1 h} - 2 + \varepsilon)\sup_{0\leq u\leq s} e^{\varepsilon_1 u}\mathbb{E}|x(u)|^q ds.$$

It follows from the Gronwall inequality that

$$\sup_{0\leq u\leq t} e^{\varepsilon_1 u}\mathbb{E}|x(u)|^q \leq (\hat{C}_0 + \frac{C}{\varepsilon_1}e^{\varepsilon_1 t})e^{(\varepsilon_1+\varepsilon\bar{h}e^{\varepsilon_1 h}-2+\varepsilon)t}.$$

Therefore,

$$\mathbb{E}|x(t)|^q \leq \hat{C}_0 e^{(\varepsilon\bar{h}e^{\varepsilon_1 h}-2+\varepsilon)t} + \frac{C}{\varepsilon_1}e^{(\varepsilon_1+\varepsilon\bar{h}e^{\varepsilon_1 h}-2+\varepsilon)t}$$

for all $t \geq 0$. Recalling (3.8) and the definition of ε, we find

$$\varepsilon_1 + \varepsilon\bar{h}e^{\varepsilon_1 h} - 2 + \varepsilon < 0.$$

Hence we have

$$\sup_{-h\leq t<\infty} \mathbb{E}|x(t)|^q < \infty.$$

We complete the proof.

It should be pointed out that Theorem 3.1 along with conditions (2.6) and (2.7) ensures that for any $t \geq 0$, $f(x(t), x(t-\delta_t), r(t), t)$ and $g(x(t), x(t-\delta_t), r(t), t)$ are bounded in L^2 and $x(t)$ is bounded in $L^{\bar{q}}$ for any $\bar{q} \in (0, q]$, which are fundamental when we discuss the stabilization problem in the following subsections.

3.2. Asymptotic stabilization

The asymptotic stability of the controlled system (2.9) will be discussed in this subsection. Although we have just shown that the controlled system preserves the boundedness of the original system (2.1) as long as the control function satisfies Assumption 2.5, such a control function may not be able to stabilize the given system. So we need to propose some new hypotheses.

Assumption 3.2. *Design the control function $u : \mathbb{R}^n \times S \times \mathbb{R}_+ \to \mathbb{R}^n$ so that we can find real numbers $a_i, \bar{a}_i$, positive numbers $c_i, \bar{c}_i$ and nonnegative numbers $b_i, \bar{b}_i, d_i, \bar{d}_i$ $(i \in S)$ for both*

$$x^T[f(x,y,i,t) + u(x,i,t)] + \frac{1}{2}|g(x,y,i,t)|^2 \le a_i|x|^2 + b_i|y|^2 - c_i|x|^p + d_i|y|^p \tag{3.11}$$

and

$$x^T[f(x,y,i,t) + u(x,i,t)] + \frac{q_1}{2}|g(x,y,i,t)|^2 \le \bar{a}_i|x|^2 + \bar{b}_i|y|^2 - \bar{c}_i|x|^p + \bar{d}_i|y|^p \tag{3.12}$$

to hold for all $(x,y,i,t) \in \mathbb{R}^n \times \mathbb{R}^n \times S \times \mathbb{R}_+$; while both

$$\mathcal{A}_1 := -2\text{diag}(a_1, a_2, \cdots, a_N) - \Gamma \quad \text{and} \quad \mathcal{A}_2 := -(q_1+1)\text{diag}(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_N) - \Gamma \tag{3.13}$$

are nonsingular M-matrices. Moreover,

$$\lambda_1 \bar{h} < 1, \quad \lambda_3 \bar{h} < \lambda_2, \quad \frac{\lambda_4(q_1 - 1 + 2\bar{h})}{q_1+1} < 1 \quad \text{and} \quad \frac{\lambda_6(q_1 - 1 + p\bar{h})}{p + q_1 - 1} < \lambda_5, \tag{3.14}$$

where

$$\begin{aligned}
&\lambda_1 = \max_{i\in S} 2\theta_i b_i, \qquad \lambda_2 = \min_{i\in S} 2\theta_i c_i, \qquad \lambda_3 = \max_{i\in S} 2\theta_i d_i, \\
&\lambda_4 = \max_{i\in S}(q_1+1)\bar{\theta}_i\bar{b}_i, \quad \lambda_5 = \min_{i\in S}(q_1+1)\bar{\theta}_i\bar{c}_i, \quad \lambda_6 = \max_{i\in S}(q_1+1)\bar{\theta}_i\bar{d}_i, \\
&(\theta_1, \cdots, \theta_N)^T = \mathcal{A}_1^{-1}(1, \cdots, 1)^T, \quad (\bar{\theta}_1, \cdots, \bar{\theta}_N)^T = \mathcal{A}_2^{-1}(1, \cdots, 1)^T.
\end{aligned} \tag{3.15}$$

By the theory of M-matrices (see e.g. [2] for more details), we see that all θ_i and $\bar{\theta}_i$ defined in (3.15) are positive due to the nonsingularity of M-matrices $\mathcal{A}_1$ and $\mathcal{A}_2$.

It is important and useful to point out that under Assumption 2.4, there is a rich class of control functions which satisfy Assumptions 2.5 and 3.2. For example, the linear functions of the form $u(x,i,t) = -A_i x$, where A_i are symmetric positive-definite matrices such that $\lambda_{\min}(A) \ge (\upsilon+1)\alpha_1$ with $\upsilon > \bar{h} \vee (q_1 - 1 + 2\bar{h})/(q_1+1)$. Clearly, Assumption 2.5 holds for such a control function. Note

$$x^T u(x,i,t) \le -\lambda_{\min}(A_i)|x|^2 \le -(\upsilon+1)\alpha_1|x|^2.$$

This along with Assumption 2.4 implies

$$\begin{aligned}
&x^T[f(x,y,i,t) + u(x,i,t)] + \frac{1}{2}|g(x,y,i,t)|^2 \\
<&x^T[f(x,y,i,t) + u(x,i,t)] + \frac{q_1}{2}|g(x,y,i,t)|^2 \\
\le& -\upsilon\alpha_1|x|^2 + \alpha_1|y|^2 - \alpha_2|x|^p + \alpha_3|y|^p.
\end{aligned}$$

This means $a_i = \bar{a}_i = -\upsilon\alpha_1$, $b_i = \bar{b}_i = \alpha_1$, $c_i = \bar{c}_i = \alpha_2$, $d_i = \bar{d}_i = \alpha_3$ for (3.11) and (3.12). Then by the theory of M-matrices, both

$$\mathcal{A}_1 = 2\upsilon\text{diag}(\alpha_1, \cdots, \alpha_1) - \Gamma$$
$$\mathcal{A}_2 = \upsilon(q_1+1)\text{diag}(\alpha_1, \cdots, \alpha_1) - \Gamma$$

are nonsingular M-matrices. When υ is sufficiently large, $\theta_i \approx 1/(2\upsilon\alpha_1)$ and $\bar{\theta}_i \approx 1/[\upsilon\alpha_1(q_1+1)]$. To explain this, let us choose $N = 2$ as an example, namely the system mode r has two states. Let the generator matrix

$$\Gamma = \begin{pmatrix} -\gamma_{12} & \gamma_{12} \\ \gamma_{21} & -\gamma_{21} \end{pmatrix}.$$

Then

$$\mathcal{A}_1 = \begin{pmatrix} 2\upsilon\alpha_1 + \gamma_{12} & -\gamma_{12} \\ -\gamma_{21} & 2\upsilon\alpha_1 + \gamma_{21} \end{pmatrix} \text{ and } \mathcal{A}_2 = \begin{pmatrix} (q_1+1)\upsilon\alpha_1 + \gamma_{12} & -\gamma_{12} \\ -\gamma_{21} & (q_1+1)\upsilon\alpha_1 + \gamma_{21} \end{pmatrix},$$

and the inverse matrices are

$$\mathcal{A}_1^{-1} = \frac{1}{4\upsilon^2\alpha_1^2 + 2\upsilon\alpha_1(\gamma_{12}+\gamma_{21})} \begin{pmatrix} 2\upsilon\alpha_1 + \gamma_{21} & \gamma_{12} \\ \gamma_{21} & 2\upsilon\alpha_1 + \gamma_{12} \end{pmatrix}$$

$$\mathcal{A}_2^{-1} = \frac{1}{(q_1+1)^2\upsilon^2\alpha_1^2 + (q_1+1)\upsilon\alpha_1(\gamma_{12}+\gamma_{21})} \begin{pmatrix} (q_1+1)\upsilon\alpha_1 + \gamma_{21} & \gamma_{12} \\ \gamma_{21} & (q_1+1)\upsilon\alpha_1 + \gamma_{12} \end{pmatrix}.$$

So

$$\theta_i = \frac{2\upsilon\alpha_1 + \gamma_{12} + \gamma_{21}}{4\upsilon^2\alpha_1^2 + 2\upsilon\alpha_1(\gamma_{12}+\gamma_{21})},$$

when υ is sufficiently large,

$$\theta_i \approx \frac{1}{2\upsilon\alpha_1}.$$

Similarly,

$$\bar{\theta}_i = \frac{(q_1+1)\upsilon\alpha_1 + \gamma_{12} + \gamma_{21}}{(q_1+1)^2\upsilon^2\alpha_1^2 + (q_1+1)\upsilon\alpha_1(\gamma_{12}+\gamma_{21})},$$

when υ is sufficiently large,

$$\bar{\theta}_i \approx \frac{1}{(q_1+1)\upsilon\alpha_1}.$$

Therefore, in (3.15), $\lambda_1 = 2\alpha_1\theta_i \approx 1/\upsilon$, we have $\lambda_1\bar{h} < 1$ as $\upsilon > \bar{h}$. The condition $\lambda_3\bar{h} < \lambda_2$ can be guaranteed by $\alpha_3\bar{h} < \alpha_2$ in Assumption 2.4. Similarly, $\lambda_4 = (q_1+1)\alpha_1\bar{\theta}_i \approx 1/\upsilon$. Since $\upsilon > (q_1 - 1 + 2\bar{h})/(q_1+1)$, then $[\lambda_4(q_1 - 1 + 2\bar{h})]/(q_1+1) < 1$. The condition $[\lambda_6(q_1 - 1 + p\bar{h})]/(p + q_1 - 1) < \lambda_5$ can be guaranteed as $q_1 > 1$, $\bar{h} > 1$ and $\alpha_2 > \alpha_3\bar{h}$. Consequently, Assumption 3.2 is satisfied as long as υ is sufficiently large and the control function has $\lambda_{\min}(A_i) > (\upsilon+1)\alpha_1$. Please note this is a general sufficient condition. When the specific form of the system is known, all the information will be considered in the calculation, to obtain a better control as needed.

Define a function $U : \mathbb{R}^n \times S \to \mathbb{R}_+$ by

$$U(x,i) = \theta_i|x|^2 + \bar{\theta}_i|x|^{q_1+1} \tag{3.16}$$

and define an operator $\mathcal{L}U : \mathbb{R}^n \times \mathbb{R}^n \times S \times \mathbb{R}_+ \to \mathbb{R}$ by

$$
\begin{aligned}
&\mathcal{L}U(x,y,i,t)\\
=&2\theta_i\Big(x^T[f(x,y,i,t)+u(x,i,t)]+\frac{1}{2}|g(x,y,i,t)|^2\Big)\\
&+(q_1+1)\bar{\theta}_i\Big(|x|^{q_1-1}x^T[f(x,y,i,t)+u(x,i,t)]+\frac{1}{2}|x|^{q_1-1}|g(x,y,i,t)|^2+\frac{1}{2}(q_1-1)|x|^{q_1-3}|x^Tg(x,y,i,t)|^2\Big)\\
&+\sum_{j=1}^N\gamma_{ij}(\theta_j|x|^2+\bar{\theta}_j|x|^{q_1+1}).
\end{aligned}
\tag{3.17}
$$

From (3.11), (3.12), the Young inequality and (3.15), we can derive that

$$
\begin{aligned}
\mathcal{L}U(x,y,i,t)\le&-|x|^2+\lambda_1|y|^2-\lambda_2|x|^p+\lambda_3|y|^p-\Big(1-\frac{\lambda_4(q_1-1)}{q_1+1}\Big)|x|^{q_1+1}+\frac{2\lambda_4}{q_1+1}|y|^{q_1+1}\\
&-\Big(\lambda_5-\frac{\lambda_6(q_1-1)}{p+q_1-1}\Big)|x|^{p+q_1-1}+\frac{\lambda_6 p}{p+q_1-1}|y|^{p+q_1-1}.
\end{aligned}
\tag{3.18}
$$

This observation makes it possible to give the following assumption.

Assumption 3.3. *Find eight positive constants $\rho_j(1\le j\le 8)$ with $\rho_4>\rho_5\bar{h}$ and $\rho_6\in(0,1/\bar{h})$ and a function $W\in C(\mathbb{R}^n;\mathbb{R}_+)$ such that*

$$
\begin{aligned}
&\mathcal{L}U(x,y,i,t)+\rho_1\left(2\theta_i|x|+(q_1+1)\bar{\theta}_i|x|^{q_1}\right)^2+\rho_2|f(x,y,i,t)|^2+\rho_3|g(x,y,i,t)|^2\\
\le&-\rho_4|x|^2+\rho_5|y|^2-W(x)+\rho_6W(y)
\end{aligned}
\tag{3.19}
$$

and

$$
\rho_7|x|^{p+q_1-1}\le W(x)\le\rho_8(1+|x|^{p+q_1-1})
\tag{3.20}
$$

for all $(x,y,i,t)\in\mathbb{R}^n\times\mathbb{R}^n\times S\times\mathbb{R}_+$.

To realize our stabilization purpose, we will use the method of Lyapunov functional. For this purpose, let us define two segments $\hat{x}_t:=\{x(t+s):-2h\le s\le 0\}$ and $\hat{r}_t:=\{r(t+s):-2h\le s\le 0\}$ for $t\ge 0$. For $\hat{x}_t$ and $\hat{r}_t$ to be well defined for $0\le t<2h$, we set $x(s)=\xi(-h)$ for $s\in[-2h,-h)$ and $r(s)=r_0$ for $s\in[-2h,0)$. The Lyapunov functional used in this paper will be of the form

$$
\begin{aligned}
V(\hat{x}_t,\hat{r}_t,t)=U(x(t),r(t))+\theta\int_{-\tau}^0\int_{t+s}^t\Big[&\tau|f(x(v),x(v-\delta_v),r(v),v)+u(x(\eta_v),r(\eta_v),v)|^2\\
&+|g(x(v),x(v-\delta_v),r(v),v)|^2\Big]dvds
\end{aligned}
\tag{3.21}
$$

for $t\ge 0$, where U has been defined by (3.16) and θ is a positive constant to be determined later. We set

$$
f(x,y,i,v)=f(x,y,i,0),\quad g(x,y,i,v)=g(x,y,i,0),\quad u(x,i,v)=u(x,i,0)
$$

for $(x,y,i,v)\in\mathbb{R}^n\times\mathbb{R}^n\times S\times[-2h,0)$. By the generalized Itô formula (see e.g. [2]), we can obtain

$$
dU(x(t),r(t))=LU(x(t),x(t-\delta_t),x(\eta_t),r(t),r(\eta_t),t)dt+dM(t)
\tag{3.22}
$$

for $t \geq 0$, where $M(t)$ is a continuous local martingale with $M(0) = 0$ and $LU : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times S \times S \times \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$
\begin{aligned}
&LU(x, y, z, i, \hat{i}, t)\\
=&2\theta_i\Big(x^T[f(x, y, i, t) + u(z, \hat{i}, t)] + \frac{1}{2}|g(x, y, i, t)|^2\Big)\\
&+ (q_1 + 1)\bar{\theta}_i|x|^{q_1-1}\Big(x^T[f(x, y, i, t) + u(z, \hat{i}, t)] + \frac{1}{2}|g(x, y, i, t)|^2\Big)\\
&+ \frac{(q_1 + 1)(q_1 - 1)}{2}\bar{\theta}_i|x|^{q_1-3}|x^T g(x, y, i, t)|^2 + \sum_{j=1}^{N}\gamma_{ij}(\theta_j|x|^2 + \bar{\theta}_j|x|^{q_1+1})\\
=&\mathcal{L}U(x, y, i, t) - [2\theta_i + (q_1 + 1)\bar{\theta}_i|x|^{q_1-1}]x^T[u(x, i, t) - u(z, \hat{i}, t)]
\end{aligned}
$$

On the other hand, the fundamental theory of calculus shows

$$
\begin{aligned}
&d\Big(\theta\int_{-\tau}^{0}\int_{t+s}^{t}\big[\tau|f(x(v), x(v - \delta_v), r(v), v) + u(x(\eta_v), r(\eta_v), v)|^2 + |g(x(v), x(v - \delta_v), r(v), v)|^2\big]\,dvds\Big)\\
=&\Big(\theta\tau\big[\tau|f(x(t), x(t - \delta_t), r(t), t) + u(x(\eta_t), r(\eta_t), t)|^2 + |g(x(t), x(t - \delta_t), r(t), t)|^2\big]\\
&- \theta\int_{t-\tau}^{t}\big[\tau|f(x(v), x(v - \delta_v), r(v), v) + u(x(\eta_v), r(\eta_v), v)|^2 + |g(x(v), x(v - \delta_v), r(v), v)|^2\big]dv\Big)dt. \quad (3.23)
\end{aligned}
$$

Combining (3.22), (3.23) and recalling the definition of the Lyapunov functional, we can get

$$
dV(\hat{x}_t, \hat{r}_t, t) \leq \mathbb{L}V(\hat{x}_t, \hat{r}_t, t)dt + dM(t) \quad (3.24)
$$

where

$$
\begin{aligned}
&\mathbb{L}V(\hat{x}_t, \hat{r}_t, t)\\
=&\mathcal{L}U(x(t), x(t - \delta_t), r(t), t) + \rho_1\big(2\theta_{r(t)}|x(t)| + (q_1 + 1)\bar{\theta}_{r(t)}|x(t)|^{q_1}\big)^2 + \frac{1}{4\rho_1}|u(x(t), r(t), t) - u(x(\eta_t), r(\eta_t), t)|^2\\
&+ \theta\tau\big[\tau|f(x(t), x(t - \delta_t), r(t), t) + u(x(\eta_t), r(\eta_t), t)|^2 + |g(x(t), x(t - \delta_t), r(t), t)|^2\big]\\
&- \theta\int_{t-\tau}^{t}\big[\tau|f(x(v), x(v - \delta_v), r(v), v) + u(x(\eta_v), r(\eta_v), v)|^2 + |g(x(v), x(v - \delta_v), r(v), v)|^2\big]dv. \quad (3.25)
\end{aligned}
$$

By Assumptions 2.3-2.5 and 3.3 as well as Theorem 3.1, it is straightforward to see that

$$
\mathbb{E}|\mathbb{L}V(\hat{x}_t, \hat{r}_t, t)| < \infty, \quad \forall t \geq 0. \quad (3.26)
$$

We can now state our first stabilization result.

Theorem 3.4. *Under Assumptions 2.1, 2.3 and 2.4, we can design a control function u to satisfy Assumptions 2.5, 3.2 and then find eight positive constants $\rho_j (1 \leq j \leq 8)$ and a function $W \in C(\mathbb{R}^n; \mathbb{R}_+)$ to meet Assumption 3.3. Set*

$$
\theta = \frac{2\kappa^2}{\rho_1}\big(1 + 8(1 - e^{-\bar{\gamma}/4\kappa})\big). \quad (3.27)
$$

If $\tau > 0$ is sufficiently small for

$$
\tau \leq \sqrt{\frac{\rho_2}{2\theta}} \wedge \frac{\rho_3}{\theta} \wedge \frac{1}{4\kappa} \quad (3.28)
$$

and

$$\rho_4 - \rho_5 \bar{h} - 4\theta\tau^2\kappa^2 - \frac{4\kappa^2}{\rho_1}(1 - e^{-\bar{\gamma}\tau}) > 0, \tag{3.29}$$

then the solution of the controlled system (2.9) with initial data (2.2) has the property that

$$\int_0^\infty \mathbb{E}|x(t)|^{\bar{q}} dt < \infty, \quad \forall \bar{q} \in [2, p + q_1 - 1]. \tag{3.30}$$

That is, the controlled system (2.9) is H_∞-stable in $L^{\bar{q}}$ for any $\bar{q} \in [2, p + q_1 - 1]$.

Proof. Fix the initial value ξ arbitrarily. Let $k_0 > 0$ be sufficiently large for $k_0 > \|\xi\|$. For each integer $k \geq k_0$, define the stopping time

$$\zeta_k = \inf\{t \geq 0 : |x(t)| \geq k\}.$$

By Theorem 3.1, we see that ζ_k is increasing to infinity a.s. as $k \to \infty$. By the generalized Itô formula, it follows from (3.24) that

$$\mathbb{E}V(\hat{x}_{t\wedge\zeta_k}, \hat{r}_{t\wedge\zeta_k}, t \wedge \zeta_k) \leq V(\hat{x}_0, \hat{r}_0, 0) + \mathbb{E}\int_0^{t\wedge\zeta_k} \mathbb{L}V(\hat{x}_s, \hat{r}_s, s)ds$$

for any $t \geq 0$ and $k \geq k_0$. Recalling (3.26), we can let $k \to \infty$ and then apply the dominated convergence theorem as well as the Fubini theorem to obtain

$$0 \leq \mathbb{E}V(\hat{x}_t, \hat{r}_t, t) \leq V(\hat{x}_0, \hat{r}_0, 0) + \int_0^t \mathbb{E}\big(\mathbb{L}V(\hat{x}_s, \hat{r}_s, s)\big)ds \tag{3.31}$$

for any $t \geq 0$. We can derive from (3.25), (3.28), Assumptions 2.5 and 3.3 that

$$\begin{aligned}
&\mathbb{E}\big(\mathbb{L}V(\hat{x}_t, \hat{r}_t, t)\big) \\
\leq& -\rho_4\mathbb{E}|x(t)|^2 + \rho_5\mathbb{E}|x(t-\delta_t)|^2 - \mathbb{E}W(x(t)) + \rho_6\mathbb{E}W(x(t-\delta_t)) + 2\theta\tau^2\kappa^2\mathbb{E}|x(\eta_t)|^2 \\
&+ \frac{1}{4\rho_1}\mathbb{E}|u(x(t), r(t), t) - u(x(\eta_t), r(t), t) + u(x(\eta_t), r(t), t) - u(x(\eta_t), r(\eta_t), t)|^2 \\
&- \theta\mathbb{E}\int_{t-\tau}^t \big[\tau|f(x(v), x(v-\delta_v), r(v), v) + u(x(\eta_v), r(\eta_v), v)|^2 + |g(x(v), x(v-\delta_v), r(v), v)|^2\big]dv \\
\leq& -\rho_4\mathbb{E}|x(t)|^2 + \rho_5\mathbb{E}|x(t-\delta_t)|^2 - \mathbb{E}W(x(t)) + \rho_6\mathbb{E}W(x(t-\delta_t)) \\
&+ 4\theta\tau^2\kappa^2\mathbb{E}|x(t) - x(\eta_t)|^2 + 4\theta\tau^2\kappa^2\mathbb{E}|x(t)|^2 + \frac{\kappa^2}{2\rho_1}\mathbb{E}|x(t) - x(\eta_t)|^2 \\
&- \theta\mathbb{E}\int_{t-\tau}^t \big[\tau|f(x(v), x(v-\delta_v), r(v), v) + u(x(\eta_v), r(\eta_v), v)|^2 + |g(x(v), x(v-\delta_v), r(v), v)|^2\big]dv \\
&+ \frac{1}{2\rho_1}\mathbb{E}|u(x(\eta_t), r(t), t) - u(x(\eta_t), r(\eta_t), t)|^2.
\end{aligned} \tag{3.32}$$

Moreover, by Assumption 2.5 and Lemma 2.6, we can derive that

$$\begin{aligned}
&\mathbb{E}|u(x(\eta_t),r(t),t)-u(x(\eta_t),r(\eta_t),t)|^2\\
=&\mathbb{E}\left[\mathbb{E}|u(x(\eta_t),r(t),t)-u(x(\eta_t),r(\eta_t),t)|^2|\mathcal{F}_{\eta_t}\right]\\
\le&\mathbb{E}\left[4\kappa^2|x(\eta_t)|^2\mathbb{E}\big(\mathbf{1}_{\{r(\eta_t)\neq r(t)\}}|\mathcal{F}_{\eta_t}\big)\right]\\
=&\mathbb{E}\left[4\kappa^2|x(\eta_t)|^2\mathbb{E}\Big(\sum_{i\in S}\mathbf{1}_{\{r(\eta_t)=i\}}\mathbf{1}_{\{r(t)\neq i\}}|\mathcal{F}_{\eta_t}\Big)\right]\\
=&\mathbb{E}\left[4\kappa^2|x(\eta_t)|^2\sum_{i\in S}\mathbf{1}_{\{r(\eta_t)=i\}}\times\mathbb{P}(r(t)\neq i|r(\eta_t)=i)\right]\\
\le&\mathbb{E}\left[4\kappa^2|x(\eta_t)|^2(1-e^{-\bar{\gamma}\tau})\right]\\
=&4\kappa^2(1-e^{-\bar{\gamma}\tau})\mathbb{E}|x(\eta_t)|^2\\
\le&8\kappa^2(1-e^{-\bar{\gamma}\tau})\mathbb{E}|x(t)-x(\eta_t)|^2+8\kappa^2(1-e^{-\bar{\gamma}\tau})\mathbb{E}|x(t)|^2.
\end{aligned}\tag{3.33}$$

On the other hand, considering $t-\eta_t\le\tau$ for all $t\ge 0$, we can prove from equation (2.9) that

$$\begin{aligned}
&\mathbb{E}|x(t)-x(\eta_t)|^2\\
\le&2\mathbb{E}\int_{t-\tau}^{t}\left[\tau|f(x(s),x(s-\delta_s),r(s),s)+u(x(\eta_s),r(\eta_s),s)|^2+|g(x(s),x(s-\delta_s),r(s),s)|^2\right]ds.
\end{aligned}\tag{3.34}$$

Substituting (3.33) and (3.34) into (3.32) and recalling (3.27) and (3.28), we can show that

$$\begin{aligned}
\mathbb{E}V(\hat{x}_t,\hat{r}_t,t)\le&V(\hat{x}_0,\hat{r}_0,0)-[\rho_4-4\theta\tau^2\kappa^2-\frac{4\kappa^2}{\rho_1}(1-e^{-\bar{\gamma}\tau})]\mathbb{E}\int_0^t|x(s)|^2ds-\mathbb{E}\int_0^t W(x(s))ds\\
&+\rho_5\mathbb{E}\int_0^t|x(s-\delta_s)|^2ds+\rho_6\mathbb{E}\int_0^t W(x(s-\delta_s))ds
\end{aligned}$$

By Lemma 2.2, we have

$$\mathbb{E}\int_0^t|x(s-\delta_s)|^2ds\le\bar{h}\mathbb{E}\int_{-h}^{t-h_1}|x(s)|^2ds\le\bar{h}\left(\int_{-h}^{0}|x(s)|^2ds+\mathbb{E}\int_0^t|x(s)|^2ds\right)$$

and

$$\mathbb{E}\int_0^t W(x(s-\delta_s))ds\le\bar{h}\mathbb{E}\int_{-h}^{t-h_1}W(x(s))ds\le\bar{h}\left(\int_{-h}^{0}W(x(s))ds+\mathbb{E}\int_0^t W(x(s))ds\right)$$

Hence, we can further obtain

$$\mathbb{E}V(\hat{x}_t,\hat{r}_t,t)\le C_1-[\rho_4-4\theta\tau^2\kappa^2-\frac{4\kappa^2}{\rho_1}(1-e^{-\bar{\gamma}\tau})-\rho_5\bar{h}]\int_0^t\mathbb{E}|x(s)|^2ds-(1-\rho_6\bar{h})\int_0^t\mathbb{E}W(x(s))ds$$

where $C_1=V(\hat{x}_0,\hat{r}_0,0)+h\bar{h}\sup_{-h\le s\le 0}[\rho_5\mathbb{E}|x(s)|^2+\rho_6\mathbb{E}W(x(s))]<\infty$. By (3.29) and $0<\rho_6<1/\bar{h}$, we have

$$\int_0^t\mathbb{E}|x(s)|^2ds\le\frac{C_1}{\rho_4-4\theta\tau^2\kappa^2-\rho_5\bar{h}-\frac{4\kappa^2}{\rho_1}(1-e^{-\bar{\gamma}\tau})},\quad\int_0^t\mathbb{E}W(x(s))ds\le\frac{C_1}{1-\rho_6\bar{h}}.$$

Letting $t\to\infty$ and recalling (3.20) yield

$$\int_0^\infty\mathbb{E}|x(s)|^2ds<\infty,\quad\int_0^\infty\mathbb{E}|x(s)|^{p+q_1-1}ds<\infty,$$

which implies the required assertion (3.30) as $\mathbb{E}|x(s)|^{\bar{q}}\le\mathbb{E}|x(s)|^2+\mathbb{E}|x(s)|^{p+q_1-1}$ for any $\bar{q}\in[2,p+q_1-1]$.

Theorem 3.5. *Under the same conditions of Theorem 3.4, the solution of the controlled system (2.9) with initial data (2.2) has the property that*

$$\lim_{t\to\infty} \mathbb{E}|x(t)|^{\bar q} = 0 \tag{3.35}$$

for any $\bar q \in [2, q)$. That is, the controlled system (2.9) is asymptotically stable in $L^{\bar q}$.

Proof. By Theorem 3.1,

$$C_2 := \sup_{-h\le t<\infty} \mathbb{E}|x(t)|^q < \infty.$$

For any $0 \le t_1 < t_2 < \infty$, the Itô formula shows

$$\begin{aligned}
&\mathbb{E}|x(t_2)|^2 - \mathbb{E}|x(t_1)|^2\\
=&\mathbb{E}\int_{t_1}^{t_2} \Big(2x^T(t)\big[|f(x(t), x(t-\delta_t), r(t), t) + u(x(\eta_t), r(\eta_t), t)\big] + |g(x(t), x(t-\delta_t), r(t), t)|^2\Big) dt.
\end{aligned}$$

By conditions (2.6), (2.7) and (2.10), we have

$$\begin{aligned}
\big|\mathbb{E}|x(t_2)|^2 - \mathbb{E}|x(t_1)|^2\big| \le& \mathbb{E}\int_{t_1}^{t_2} \Big(2L|x(t)|\big(|x(t)| + |x(t-\delta_t)| + |x(t)|^{q_1} + |x(t-\delta_t)|^{q_2}\big) + 2\kappa|x(t)|\cdot|x(\eta_t)|\\
&+ L^2\big(|x(t)| + |x(t-\delta_t)| + |x(t)|^{q_3} + |x(t-\delta_t)|^{q_4}\big)^2\Big) dt\\
\le& \int_{t_1}^{t_2} C_3\left(1 + \mathbb{E}|x(t)|^q + \mathbb{E}|x(t-\delta_t)|^q + \mathbb{E}|x(\eta_t)|^q\right) dt\\
\le& C_3(1 + 3C_2)(t_2 - t_1),
\end{aligned}$$

where C_3 is a constant independent of t_1 and t_2. Thus, $\mathbb{E}|x(t)^2$ is uniformly continuous in t on $\mathbb{R}_+$. This together with (3.30) implies that

$$\lim_{t\to\infty} \mathbb{E}|x(t)|^2 = 0. \tag{3.36}$$

That is, the assertion (3.35) holds when $\bar q = 2$. Now we fix any $\bar q \in (2, q)$. For a constant $0 < \varepsilon < 1$, we can derive from the Hölder inequality that

$$\mathbb{E}|x(t)|^{\bar q} \le \left(\mathbb{E}|x(t)|^2\right)^{\varepsilon} \left(\mathbb{E}|x(t)|^{(\bar q-2\varepsilon)/(1-\varepsilon)}\right)^{1-\varepsilon}.$$

Choosing $\varepsilon = \frac{q-\bar q}{q-2}$, we get

$$\mathbb{E}|x(t)|^{\bar q} \le \left(\mathbb{E}|x(t)|^2\right)^{(q-\bar q)/(q-2)} \left(\mathbb{E}|x(t)|^q\right)^{(\bar q-2)/(q-2)} \le C_2^{(\bar q-2)/(q-2)} \left(\mathbb{E}|x(t)|^2\right)^{(q-\bar q)/(q-2)}. \tag{3.37}$$

Then it follows from (3.36) that

$$\lim_{t\to\infty} \mathbb{E}|x(t)|^{\bar q} = 0, \quad \forall \bar q \in (2, q).$$

The proof is therefore complete.

Theorem 3.6. *Under the same conditions of Theorem 3.4, the solution of the controlled system (2.9) is asymptotically stable almost surely, that is,*

$$\lim_{t\to\infty} |x(t)| = 0 \quad a.s. \tag{3.38}$$

for any initial value ξ and r_0.

Proof. The proof follows the original method developed in [41] to establish the stochastic LaSalle-type theorem. However, comparing with the constant delay there, the non-differential time-varying delay and the highly nonlinear coefficients here bring us new difficulties and increase the complication. To state it more clearly, we divide the proof into three steps.

Step 1. Let us fix any initial data ξ and r_0. It follows from Theorem 3.5 and the Fubini theorem that

$$\mathbb{E}\int_0^\infty |x(t)|^2 dt < \infty. \tag{3.39}$$

This implies

$$\int_0^\infty |x(t)|^2 dt < \infty \quad a.s.$$

and further

$$\liminf_{t\to\infty} |x(t)| = 0 \quad a.s. \tag{3.40}$$

We now claim that

$$\lim_{t\to\infty} |x(t)| = 0 \quad a.s. \tag{3.41}$$

If this was not true, then there exists a sufficiently small $\varepsilon > 0$ such that

$$\mathbb{P}(\Omega_1) \geq 3\varepsilon, \tag{3.42}$$

where

$$\Omega_1 = \Big\{\omega \in \Omega : \limsup_{t\to\infty} |x(t,\omega)| > 2\varepsilon\Big\}.$$

Let us define a sequence of stopping times:

$$\begin{aligned}
\sigma_1 &= \inf\{t \geq 0 : |x(t)|^2 \geq 2\varepsilon\}, \\
\sigma_{2i} &= \inf\{t \geq \sigma_{2i-1} : |x(t)|^2 \leq \varepsilon\}, \quad i = 1, 2, \cdots, \\
\sigma_{2i+1} &= \inf\{t \geq \sigma_{2i} : |x(t)|^2 \geq 2\varepsilon\}, \quad i = 1, 2, \cdots.
\end{aligned}$$

By (3.40) and the definition of Ω_1, we see that $\sigma_i(\omega) < \infty$ for all $i \geq 1$ whenever $\omega \in \Omega_1$.

Step 2. We choose a number $l > \|\xi\|$ and define the stopping time

$$\beta_l = \inf\{t \geq 0 : |x(t)| \geq l\}.$$

Then we can derive from the Itô formula that

$$\begin{aligned}
&\mathbb{E}|x(t\wedge\beta_l)|^2 \\
=&|\xi(0)|^2 + \mathbb{E}\int_0^{t\wedge\beta_l} \Big(2x^T(s)[f(x(s), x(s-\delta_s), r(s), s) + u(x(\eta_s), r(\eta_s), s)] + |g(x(s), x(s-\delta_s), r(s), s)|^2\Big) ds
\end{aligned}$$

for all $t \geq 0$. It follows from Assumptions 2.3 and 2.5 as well as Theorem 3.1 that

$$\mathbb{E}|x(t\wedge\beta_l)|^2 \leq C_4$$

and hence

$$l^2\mathbb{P}(\beta_l \leq t) \leq C_4,$$

where C_4 denotes a positive number whose special form is of no use. Letting $t \to \infty$ and then choosing l sufficiently large, we can get

$$\mathbb{P}(\beta_l < \infty) \le \frac{C_4}{l^2} < \varepsilon.$$

This implies

$$\mathbb{P}(\Omega_2) \ge 1 - \varepsilon, \tag{3.43}$$

where

$$\Omega_2 = \{\omega \in \Omega : |x(t,\omega)| < l \text{ for all } 0 \le t < \infty\}.$$

Then by (3.42) and (3.43), we have

$$\mathbb{P}(\Omega_1 \cap \Omega_2) \ge 2\varepsilon. \tag{3.44}$$

Step 3. With the above notations, we can now derive from (3.39) that

$$\begin{aligned}\infty > \mathbb{E}\int_0^\infty |x(t)|^2 dt &\ge \sum_{i=1}^\infty \mathbb{E}\Big(I_{\{\sigma_{2i-1}<\infty,\sigma_{2i}<\infty,\beta_l=\infty\}} \int_{\sigma_{2i-1}}^{\sigma_{2i}} |x(t)|^2 dt\Big)\\ &\ge \varepsilon \sum_{i=1}^\infty \mathbb{E}\Big(I_{\{\sigma_{2i-1}<\infty,\beta_l=\infty\}}(\sigma_{2i} - \sigma_{2i-1})\Big),\end{aligned} \tag{3.45}$$

where we have noted from (3.40) that $\sigma_{2i} < \infty$ whenever $\sigma_{2i-1} < \infty$. Now, to make the notations concise, we set

$$F(t) = f(x(t), x(t-\delta_t), r(t), t) + u(x(\eta_t), r(\eta_t), t) \quad \text{and} \quad G(t) = g(x(t), x(t-\delta_t), r(t), t)$$

for $t \ge 0$. In view of Assumptions 2.3 and 2.5, we see

$$|F(t)|^2 \vee |G(t)|^2 \le K_l$$

if $t \le \beta_l$, where K_l is a positive constant. Also, set

$$A_i = \{\beta_l \wedge \sigma_{2i-1} < \infty\} \quad \text{for} \quad i \ge 1.$$

From (2.7), we have $q > 2q_3 \vee 2q_4 \ge 2$. Recalling (2.6) and (3.1), we can deduce $G \in \mathcal{M}^2([0,T];\mathbb{R})$ for any $T > 0$. Then according to [37], $\int_{\beta_l\wedge\sigma_{2i-1}}^{\beta_l\wedge(\sigma_{2i-1}+t)} G(s)dB(s)$ is a square-integrable martingale for $0 \le t \le T$. By the Hölder inequality and the Doob martingale inequality, we can further derive that

$$\begin{aligned}&\mathbb{E}\Big(I_{A_i} \sup_{0\le t\le T} |x(\beta_l \wedge (\sigma_{2i-1}+t)) - x(\beta_l\wedge\sigma_{2i-1})|^2\Big)\\ \le& 2\mathbb{E}\Big(I_{A_i} \sup_{0\le t\le T}\Big|\int_{\beta_l\wedge\sigma_{2i-1}}^{\beta_l\wedge(\sigma_{2i-1}+t)} F(s)ds\Big|^2\Big) + 2\mathbb{E}\Big(I_{A_i} \sup_{0\le t\le T}\Big|\int_{\beta_l\wedge\sigma_{2i-1}}^{\beta_l\wedge(\sigma_{2i-1}+t)} G(s)dB(s)\Big|^2\Big)\\ \le& 2T\mathbb{E}\Big(I_{A_i}\int_{\beta_l\wedge\sigma_{2i-1}}^{\beta_l\wedge(\sigma_{2i-1}+T)} |F(s)|^2 ds\Big) + 8\mathbb{E}\Big(I_{A_i}\int_{\beta_l\wedge\sigma_{2i-1}}^{\beta_l\wedge(\sigma_{2i-1}+T)} |G(s)|^2 ds\Big)\\ \le& 2K_l T(T+4).\end{aligned} \tag{3.46}$$

Let $\theta = \varepsilon/(2l)$. It is easy to see that

$$\big||x|^2 - |y|^2\big| < \varepsilon \ \text{ whenever } |x-y| < \theta,\ |x| \vee |y| \le l. \tag{3.47}$$

Choose T sufficiently small for

$$\frac{2K_l T(T+4)}{\theta^2} < \varepsilon.$$

It then follows from (3.46) and the Chebyshev inequality that

$$\mathbb{P}\Big(A_i \cap \{\sup_{0\le t\le T} |x(\beta_l \wedge (\sigma_{2i-1}+t)) - x(\beta_l \wedge \sigma_{2i-1})| \ge \theta\}\Big) \le \frac{2K_l T(T+4)}{\theta^2} < \varepsilon.$$

Consequently,

$$\begin{aligned}
&\mathbb{P}\Big(\{\sigma_{2i-1} < \infty, \beta_l = \infty\} \cap \{\sup_{0\le t\le T} |x(\sigma_{2i-1}+t) - x(\sigma_{2i-1})| \ge \theta\}\Big)\\
=&\mathbb{P}\Big(\{\sigma_{2i-1}\wedge\beta_l < \infty, \beta_l = \infty\} \cap \{\sup_{0\le t\le T} |x(\beta_l \wedge (\sigma_{2i-1}+t)) - x(\beta_l \wedge \sigma_{2i-1})| \ge \theta\}\Big)\\
\le&\mathbb{P}\Big(A_i \cap \{\sup_{0\le t\le T} |x(\beta_l \wedge (\sigma_{2i-1}+t)) - x(\beta_l \wedge \sigma_{2i-1})| \ge \theta\}\Big) < \varepsilon.
\end{aligned}$$

Using (3.44), we can further derive

$$\begin{aligned}
&\mathbb{P}\Big(\{\sigma_{2i-1} < \infty, \beta_l = \infty\} \cap \{\sup_{0\le t\le T} |x(\sigma_{2i-1}+t) - x(\sigma_{2i-1})| < \theta\}\Big)\\
=&\mathbb{P}(\{\sigma_{2i-1} < \infty, \beta_l = \infty\}) - \mathbb{P}\Big(\{\sigma_{2i-1} < \infty, \beta_l = \infty\} \cap \{\sup_{0\le t\le T} |x(\sigma_{2i-1}+t) - x(\sigma_{2i-1})| \ge \theta\}\Big)\\
\ge&\mathbb{P}(\Omega_1 \cap \Omega_2) - \varepsilon \ge \varepsilon.
\end{aligned}$$

Set

$$\bar{\Omega}_i = \{\sup_{0\le t\le T} \big||x(\sigma_{2i-1}+t)|^2 - |x(\sigma_{2i-1})|^2\big| < \varepsilon\}.$$

According to (3.47), we have

$$\begin{aligned}
&\mathbb{P}\Big(\{\sigma_{2i-1} < \infty, \beta_l = \infty\} \cap \bar{\Omega}_i\Big)\\
\ge&\mathbb{P}\Big(\{\sigma_{2i-1} < \infty, \beta_l = \infty\} \cap \{\sup_{0\le t\le T} |x(\sigma_{2i-1}+t) - x(\sigma_{2i-1})| < \theta\}\Big) \ge \varepsilon.
\end{aligned} \tag{3.48}$$

Observing

$$\sigma_{2i}(\omega) - \sigma_{2i-1}(\omega) \ge T \quad \text{if} \quad \omega \in \{\sigma_{2i-1} < \infty, \beta_l = \infty\} \cap \bar{\Omega}_i,$$

we can finally derive from (3.45) and (3.48) that

$$\begin{aligned}
\infty > \varepsilon \sum_{i=1}^{\infty} \mathbb{E}\Big[I_{\{\sigma_{2i-1}<\infty, \beta_l=\infty\}}(\sigma_{2i} - \sigma_{2i-1})\Big] &\ge \varepsilon \sum_{i=1}^{\infty} \mathbb{E}\Big[I_{\{\sigma_{2i-1}<\infty, \beta_l=\infty\}\cap\bar{\Omega}_i}(\sigma_{2i} - \sigma_{2i-1})\Big]\\
\ge \varepsilon T \sum_{i=1}^{\infty} \mathbb{P}(\{\sigma_{2i-1} < \infty, \beta_l = \infty\} \cap \bar{\Omega}_i) &\ge \varepsilon T \sum_{i=1}^{\infty} \varepsilon = \infty,
\end{aligned}$$

which is a contradiction. So (3.41) must hold. The proof is complete.

3.3. Exponential stabilization

In this subsection, we will take a further step to show the discrete-time feedback control can stabilize the given SDDE (2.1) exponentially fast.

Theorem 3.7. *Under Assumptions 2.1, 2.3 and 2.4, we can design a control function u to satisfy Assumptions 2.5, 3.2 and then find eight positive constants $\rho_j (1 \le j \le 8)$ and a function $W \in C(\mathbb{R}^n; \mathbb{R}_+)$ to meet Assumption 3.3. Recall that*

$$\theta = \frac{2\kappa^2}{\rho_1}\big(1 + 8(1 - e^{-\bar{\gamma}/4\kappa})\big). \tag{3.49}$$

If $\tau > 0$ is sufficiently small for

$$\tau \le \sqrt{\frac{\rho_2}{2\theta}} \wedge \frac{\rho_3}{\theta} \wedge \frac{1}{4\sqrt{2}\kappa}, \quad \rho_4 - \rho_5 \bar{h} - 4\theta\tau^2\kappa^2 - \frac{4\kappa^2}{\rho_1}(1 - e^{-\bar{\gamma}\tau}) > 0, \tag{3.50}$$

then the solution of the controlled system (2.9) satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^{\bar{q}}) < 0 \tag{3.51}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t)|) < 0 \quad a.s. \tag{3.52}$$

for any $\bar{q} \in [2, q)$ and any initial value ξ.

Proof. Fix the initial value ξ arbitrarily. Let γ be a sufficiently small positive number to be determined later. In a similar way as we did in the proof of Theorem 3.4, we can show that for $t \ge 0$

$$e^{\gamma t}\mathbb{E}V(\hat{x}_t, \hat{r}_t, t) \le V(\hat{x}_0, \hat{r}_0, 0) + \int_0^t e^{\gamma s}\mathbb{E}\big(\gamma V(\hat{x}_s, \hat{r}_s, s) + \mathbb{L}V(\hat{x}_s, \hat{r}_s, s)\big)ds.$$

Recalling the structure of V, we have

$$\begin{aligned} &\eta_1 e^{\gamma t}\mathbb{E}|x(t)|^2 \\ \le &V(\hat{x}_0, \hat{r}_0, 0) + \int_0^t e^{\gamma s}\big(\gamma\eta_2\mathbb{E}|x(s)|^2 + \gamma\eta_3\mathbb{E}|x(s)|^{q_1+1}\big)ds + \gamma\theta J_1(t) + \int_0^t e^{\gamma s}\mathbb{E}\big(\mathbb{L}V(\hat{x}_s, \hat{r}_s, s)\big)ds, \end{aligned} \tag{3.53}$$

where $\eta_1 = \min_{i\in S}\theta_i, \eta_2 = \max_{i\in S}\theta_i, \eta_3 = \max_{i\in S}\bar{\theta}_i$ and

$$\begin{aligned} J_1(t) = \mathbb{E}\int_0^t e^{\gamma s}\Big(\int_{-\tau}^0 \int_{s+u}^s &\big[\tau|f(x(v), x(v - \delta_v), r(v), v) + u(x(\eta_v), r(\eta_v), v)|^2 \\ &+ |g(x(v), x(v - \delta_v), r(v), v)|^2\big]dvdu\Big)ds. \end{aligned}$$

As we did in the proof of Theorem 3.4, we can show that

$$\begin{aligned} &\mathbb{E}\big(\mathbb{L}V(\hat{x}_s, \hat{r}_s, s)\big) \\ \le &-\left(\rho_4 - 4\theta\tau^2\kappa^2 - \frac{4\kappa^2}{\rho_1}(1 - e^{-\bar{\gamma}\tau})\right)\mathbb{E}|x(s)|^2 + \rho_5\mathbb{E}|x(s - \delta_s)|^2 - \mathbb{E}W(x(s)) + \rho_6\mathbb{E}W(x(s - \delta_s)) \\ &-\left(\theta - 2[4\theta\tau^2\kappa^2 + \frac{\kappa^2}{2\rho_1} + \frac{4\kappa^2}{\rho_1}(1 - e^{-\bar{\gamma}\tau})]\right)\mathbb{E}\int_{s-\tau}^s \big[\tau|f(x(v), x(v - \delta_v), r(v), v) + u(x(\eta_v), r(\eta_v), v)|^2 \\ &+ |g(x(v), x(v - \delta_v), r(v), v)|^2\big]dv \end{aligned} \tag{3.54}$$

Moreover, we obviously have

$$\mathbb{E}|x(s)|^{q_1+1} \le \mathbb{E}|x(s)|^2 + \mathbb{E}|x(s)|^{p+q_1-1} \le \mathbb{E}|x(s)|^2 + \rho_7^{-1}\mathbb{E}W(x(s)). \tag{3.55}$$

Substituting (3.54) and (3.55) into (3.53) while noting by Lemma 2.2 that

$$\int_0^t e^{\gamma s}\mathbb{E}|x(s-\delta_s)|^2 ds \le \bar{h}e^{\gamma h}\int_{-h}^{t-h_1} e^{\gamma s}\mathbb{E}|x(s)|^2 ds \le \bar{h}e^{\gamma h}\left(\int_{-h}^{0} e^{\gamma s}\mathbb{E}|x(s)|^2 ds + \int_0^t e^{\gamma s}\mathbb{E}|x(s)|^2 ds\right),$$

$$\int_0^t e^{\gamma s}\mathbb{E}W(x(s-\delta_s)) ds \le \bar{h}e^{\gamma h}\int_{-h}^{t-h_1} e^{\gamma s}\mathbb{E}W(x(s)) ds \le \bar{h}e^{\gamma h}\left(\int_{-h}^{0} e^{\gamma s}\mathbb{E}W(x(s)) ds + \int_0^t e^{\gamma s}\mathbb{E}W(x(s)) ds\right),$$

we can obtain

$$\begin{aligned}
\eta_1 e^{\gamma t}\mathbb{E}|x(t)|^2 \le & C_5 - \left(\rho_4 - 4\theta\tau^2\kappa^2 - \frac{4\kappa^2}{\rho_1}(1-e^{-\bar{\gamma}\tau}) - \gamma\eta_2 - \gamma\eta_3 - \rho_5\bar{h}e^{\gamma h}\right)\int_0^t e^{\gamma s}\mathbb{E}|x(s)|^2 ds \\
& - \left(1 - \gamma\eta_3\rho_7^{-1} - \rho_6\bar{h}e^{\gamma h}\right)\int_0^t e^{\gamma s}\mathbb{E}W(x(s))ds + \gamma\theta J_1(t) \\
& - \left(\theta - 2[4\theta\tau^2\kappa^2 + \frac{\kappa^2}{2\rho_1} + \frac{4\kappa^2}{\rho_1}(1-e^{-\bar{\gamma}\tau})]\right) J_2(t),
\end{aligned} \tag{3.56}$$

where

$$C_5 = V(\hat{x}_0, \hat{r}_0, 0) + \bar{h}e^{\gamma h}\left(\rho_5\int_{-h}^0 e^{\gamma s}\mathbb{E}|x(s)|^2 ds + \rho_6\int_{-h}^0 e^{\gamma s}\mathbb{E}W(x(s))ds\right)$$

and

$$J_2(t) = \mathbb{E}\int_0^t e^{\gamma s}\int_{s-\tau}^s \left[\tau|f(x(v),x(v-\delta_v),r(v),v) + u(x(\eta_v),r(\eta_v),v)|^2 + |g(x(v),x(v-\delta_v),r(v),v)|^2\right]dvds.$$

On the other hand, it is easy to observe that

$$J_1(t) \le \tau J_2(t).$$

We can now choose $\gamma > 0$ small enough such that

$$\gamma\tau \le \frac{1}{4}, \quad \rho_4 - 4\theta\tau^2\kappa^2 - \frac{4\kappa^2}{\rho_1}(1-e^{-\bar{\gamma}\tau}) - \gamma\eta_2 - \gamma\eta_3 - \rho_5\bar{h}e^{\gamma h} \ge 0, \quad 1 - \gamma\eta_3\rho_7^{-1} - \rho_6\bar{h}e^{\gamma h} \ge 0.$$

And further recalling (3.49) and $\tau \le \frac{1}{4\sqrt{2}\kappa}$, it follows from (3.56) immediately that

$$\mathbb{E}|x(t)|^2 \le \frac{C_5}{\eta_1}e^{-\gamma t}, \quad \forall t \ge 0. \tag{3.57}$$

Finally, for any $\bar{q} \in (2, q)$, it follows from (3.37) and (3.57) that

$$\mathbb{E}|x(t)|^{\bar{q}} \le C_2^{(\bar{q}-2)/(q-2)}\left(\frac{C_5}{\eta_1}\right)^{(q-\bar{q})/(q-2)} e^{-\gamma t(q-\bar{q})/(q-2)}. \tag{3.58}$$

Therefore, the required assertion (3.51) holds for any $\bar{q} \in [2, q)$.

Let $t_k = k\tau$ for $k = 0, 1, 2, \cdots$. By the Hölder inequality and the Doob martingale inequality, we can show that

$$\begin{aligned}
\mathbb{E}\Big(\sup_{t_k \le t \le t_{k+1}} |x(t)|^2\Big) \le 3\mathbb{E}|x(t_k)|^2 + 3\tau\mathbb{E}\int_{t_k}^{t_{k+1}} |f(x(t),x(t-\delta_t),r(t),t) + u(x(\eta_t),r(\eta_t),t)|^2 dt \\
+ 12\mathbb{E}\int_{t_k}^{t_{k+1}} |g(x(t),x(t-\delta_t),r(t),t)|^2 dt.
\end{aligned}$$

It then follows from Assumptions 2.3 and 2.5 that

$$\mathbb{E}\Big(\sup_{t_k \le t \le t_{k+1}} |x(t)|^2\Big) \le 3\mathbb{E}|x(t_k)|^2 + C_6\int_{t_k}^{t_{k+1}} \mathbb{E}\Big(|x(t)|^2 + |x(t-\delta_t)|^2 + |x(\eta_t)|^2 + |x(t)|^{\bar{q}} + |x(t-\delta_t)|^{\bar{q}}\Big)dt,$$

where $\bar{q} = 2(q_1 \vee q_2 \vee q_3 \vee q_4)$ and C_6 is a positive constant. By Assumption 2.4, we see $\bar{q} \in [2, q)$. We can apply (3.57) and (3.58) to get

$$\mathbb{E}\Big(\sup_{t_k \le t \le t_{k+1}} |x(t)|^2\Big) \le C_7 e^{-\hat{\varepsilon} t_k},$$

where $\hat{\varepsilon} = \varepsilon(q - \bar{q})/(q-2)$ and C_7 is another positive constant. Then

$$\sum_{k=0}^{\infty} \mathbb{P}\Big(\sup_{t_k \le t \le t_{k+1}} |x(t)| > e^{-0.25\hat{\varepsilon} t_k}\Big) \le \sum_{k=0}^{\infty} C_7 e^{-0.5\hat{\varepsilon} t_k} < \infty.$$

The Borel-Cantelli lemma (see e.g. [2]) shows that for almost all $\omega \in \Omega$, there exists a positive integer $k_0 = k_0(\omega)$ such that

$$\sup_{t_k \le t \le t_{k+1}} |x(t)| \le e^{-0.25\hat{\varepsilon} t_k}, \quad k \ge k_0.$$

Therefore, for almost all $\omega \in \Omega$,

$$\frac{1}{t}\log(|x(t)|) \le -\frac{0.25\hat{\varepsilon}\tau k}{\tau(k+1)}, \quad t \in [t_k, t_{k+1}], \quad k \ge k_0.$$

This implies

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) \le -0.25\hat{\varepsilon} < 0 \quad a.s.$$

which is the required assertion (3.52). The proof is therefore complete.

4. Examples

Now we illustrate our theoretical results with examples.

Example 4.1. Our new results can be applied to the example in [34], which is a simple version of food chain model (see e.g. [36, 37]). Keeping the same original system, we can use the discrete-time feedback control, instead of the delay feedback control, for system stabilization. As long as the observation interval $\tau \le 0.00047$, our results guarantee the controlled system of the form

$$dx(t) = \left(f(x(t), x(t-\delta_t), r(t)) + u(x(\eta_t), r(\eta_t))\right)dt + g(x(t), x(t-\delta_t), r(t))dB(t)$$

is $\bar{q}$th moment exponential stable for any $\bar{q} \in [2, 6]$ and almost surely exponential stable. The numerical simulation in Figure 4.1 supports our theoretical results.

Example 4.2. Consider the scalar hybrid SDDE

$$dx(t) = f(x(t), x(t-\delta_t), r(t))dt + g(x(t), x(t-\delta_t), r(t))dB(t) \tag{4.1}$$

on $t \ge 0$ with $x(t) = 1 + sin(t)$ for $t \in [-0.2, 0]$. Here $B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with generator matrix

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

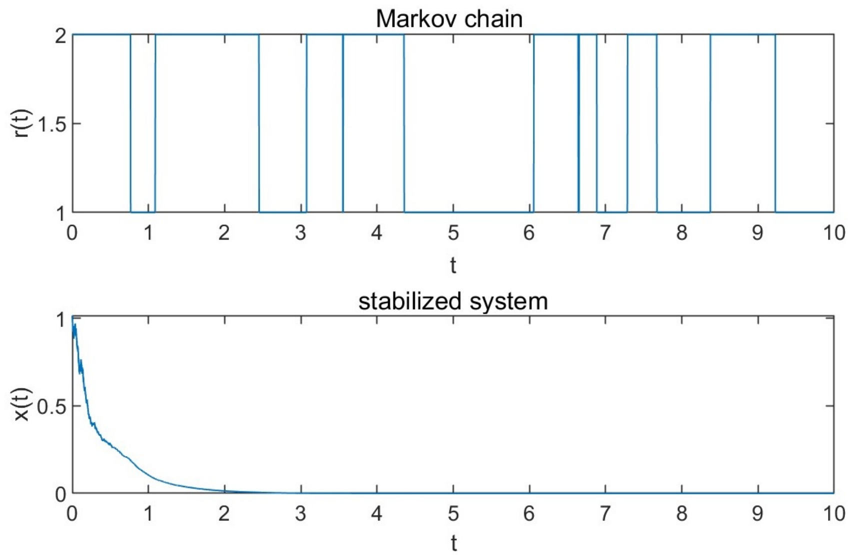

Figure 4.1: Simulated path of Markov chain $r(t)$ and state $x(t)$, by the Euler–Maruyama method with step size 10^{-6}, observation interval $\tau = 0.00045$ and random initial values.

the time-varying delay

$$\delta_t = \sum_{k=0}^{\infty} \big[\big(0.1 + 0.1(t-2k)\big) I_{[2k,2k+1)}(t) + \big(0.2 - 0.1(t-2k-1)\big) I_{[2k+1,2(k+1))}(t)\big].$$

and the system coefficients are

$$f(x,y,1) = -1.2x^3 + xy, \quad g(x,y,1) = 0.2xy,$$
$$f(x,y,2) = -x^3 + 1.2xy, \quad g(x,y,2) = 0.1xy.$$

Figure 4.2 shows the simulated paths, and obviously the system (4.1) in the middle plot is not exponentially stable in mean square. To stabilize it, we use the control function

$$u(x,1) = -2(|x| \wedge 1.8)x/|x|, \quad u(x,2) = -2.5(|x| \wedge 2)x/|x|,$$

and the controlled system has the form

$$dx(t) = \big(f(x(t), x(t-\delta_t), r(t)) + u(x(\eta_t), r(\eta_t))\big)dt + g(x(t), x(t-\delta_t), r(t))dB(t). \tag{4.2}$$

Now let us check that all assumptions hold. Assumption 2.1 holds with $h_1 = 0.1$, $h = 0.2$ and $\bar{h} = 1.1111$. Assumption 2.3 holds with $q_1 = 3$, $q_2 = q_3 = q_4 = 2$. Assumption 2.4 holds with $p = 4$, $\alpha_1 = 1$, $\alpha_2 = 0.6775$, $\alpha_3 = 0.5125$, $\beta_1 = 2.7494$, $\beta_2 = 1.5944$, $q > 6$, then we might as well choose $q = 7$, and $\beta_1 > \beta_2 \bar{h}$ is satisfied. Assumption 2.5 holds with $\kappa = 2.5$. By Theorem 3.1, the controlled system (4.2) has unique global solution which is qth moment bounded. Since

$$xu(x,i) \le 0.095x^4 - (2I_1(i) + 2.5I_2(i))x^2,$$

then for $(x,y,i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}$, we have

$$x[f(x,y,i) + u(x,i)] + \frac{1}{2}|g(x,y,i)|^2$$

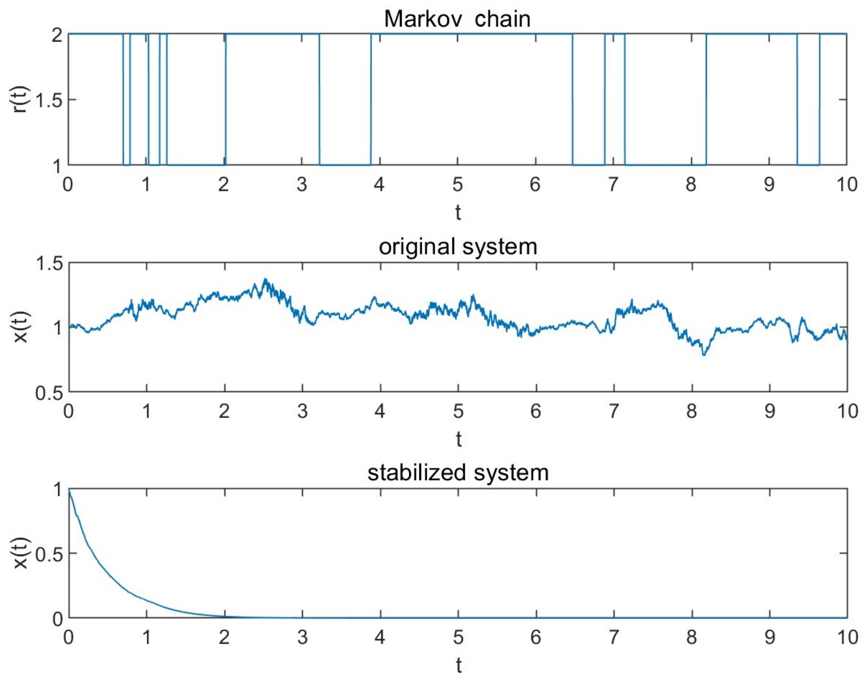

Figure 4.2: Simulated paths of the system mode and state. The upper plot shows the Markov chain $r(t)$, the middle and lower plots respectively show the state $x(t)$ of the original system and the stabilized system. The Euler–Maruyama method with step size 10^{-6} and random initial values have been used.

$$\leq \begin{cases} -2x^2 + y^2 - 0.845x^4 + 0.01y^4, & i = 1, \\ -2.5x^2 + 1.49y^2 - 0.59x^4 + 0.0025y^4, & i = 2, \end{cases}$$

and

$$x[f(x,y,i) + u(x,i)] + \frac{q_1}{2}|g(x,y,i)|^2$$

$$\leq \begin{cases} -2x^2 + y^2 - 0.825x^4 + 0.03y^4, & i = 1, \\ -2.5x^2 + 1.49y^2 - 0.585x^4 + 0.0075y^4, & i = 2. \end{cases}$$

So Assumption 3.2 holds with

$$\begin{aligned} &a_1 = -2, && b_1 = 1, && c_1 = 0.845, && d_1 = 0.01, \\ &a_2 = -2.5, && b_2 = 1.49, && c_2 = 0.59, && d_2 = 0.0025, \\ &\bar{a}_1 = -2, && \bar{b}_1 = 1, && \bar{c}_1 = 0.825, && \bar{d}_1 = 0.03, \\ &\bar{a}_2 = -2.5, && \bar{b}_2 = 1.49, && \bar{c}_2 = 0.585, && \bar{d}_2 = 0.0075. \end{aligned}$$

Therefore,

$$\mathcal{A}_1 = \begin{pmatrix} 5 & -1 \\ -1 & 6 \end{pmatrix} \text{ and } \mathcal{A}_2 = \begin{pmatrix} 9 & -1 \\ -1 & 11 \end{pmatrix},$$

which are both M-matrices. Then we have

$$\theta_1 = 0.2414, \ \ \theta_2 = 0.2069, \ \ \bar{\theta}_1 = 0.1333, \ \ \bar{\theta}_2 = 0.1111,$$

and

$$\lambda_1 = 0.6166, \ \ \lambda_2 = 0.2441, \ \ \lambda_3 = 0.0048, \ \ \lambda_4 = 0.6622, \ \ \lambda_5 = 0.26, \ \ \lambda_6 = 0.0133.$$

That is, all conditions in Assumption 3.2 hold. To verify Assumption 3.3, we note that the Lyapunov function defined by (3.16) has the form

$$U(x,i) = \begin{cases} 0.2414x^2 + 0.1333x^4, & i = 1, \\ 0.2069x^2 + 0.1111x^4, & i = 2. \end{cases}$$

We can then derive that

$$\mathcal{L}U(x,y,i) \le -x^2 + 0.6166y^2 - 0.913x^4 + 0.3359y^4 - 0.2556x^6 + 0.0089y^6.$$

It's easy to find that

$$\begin{aligned} (2\theta_i|x| + (q_1+1)\bar{\theta}_i|x|^{q_1})^2 \le& 0.2331x^2 + 0.5149x^4 + 0.2844x^6, \\ |f(x,y,i)|^2 \le& 1.44(x^4 + y^4 + 2x^6), \\ |g(x,y,i)|^2 \le& 0.02(x^4 + y^4). \end{aligned}$$

To satisfy Assumption 3.3, we might as well choose $\rho_1 = 0.5$, $\rho_2 = 0.035$ and $\rho_3 = 1$, then we have

$$\begin{aligned} &\mathcal{L}U(x,y,i) + \rho_1(2\theta_i|x| + (q_1+1)\bar{\theta}_i|x|^{q_1})^2 + \rho_2|f(x,y,i)|^2 + \rho_3|g(x,y,i)|^2 \\ \le& -0.8835x^2 + 0.6166y^2 - 0.5851x^4 + 0.4063y^4 - 0.0125x^6 + 0.0089y^6 \\ \le& -\rho_4x^2 + \rho_5 0.6166y^2 - W(x) + \rho_6W(y), \end{aligned}$$

where $W(x) = 0.58512x^4 + 0.0125x^6$. Then Assumption 3.3 holds for

$$\rho_4 = 0.8835, \ \ \rho_5 = 0.6166, \ \ \rho_6 = 0.75, \ \ \rho_7 = 0.1254, \ \ \rho_8 = 0.5977.$$

According to Theorems 3.4 and 3.7, we have $\theta = 44.0325$ and $\tau \le 0.0036$. Hence, we can conclude by Theorems 3.4, 3.5 and 3.7 that the controlled system (4.2) is H_∞-stable, asymptotically stable and exponentially stable in $L^{\bar{q}}$ for any $\bar{q} \in [2,6]$. Moreover, the controlled system (4.2) is also almost surely asymptotically stable and almost surely exponentially stable by Theorems 3.6 and 3.7. The simulation result shown in lower plot of Figure 4.2 supports our theoretical conclusions.

5. Conclusion

This paper reports the stabilization problem of highly nonlinear hybrid stochastic systems with non-differentiable time-varying delays, by feedback control based on discrete-time observations of system state and mode. The theorems on existence, uniqueness and boundedness of the solution of controlled system have been firstly established in this paper. By employing the Lyapunov functional method, we have developed sufficient stability criteria for the controlled system, including H_∞ stability in $L^{\bar{q}}$, asymptotic stability in $\bar{q}$th moment, almost surely asymptotic stability, $\bar{q}$th moment exponential stability and almost surely exponential stability. Moreover, we also provide a sufficient upper bound on the duration τ between two consecutive observations. A couple of examples and computer simulations have been taken to illustrate our theory. In future research, the conditions on system coefficients, time delays and control functions would be further relaxed.

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