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Stabilization and Destabilization of Hybrid Systems by Periodic Stochastic Controls based on Lévy Noise

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We focus in this paper on determining whether or not a periodic stochastic feedback control based on Lévy noise can stabilize or destabilize a given nonlinear hybrid system. By using the Lyapunov functions and the periodic functions, we establish some sufficient conditions on the stability and instability for nonlinear hybrid systems with Lévy noise. Moreover, we use some numerical examples and simulations to illustrate that an unstable (or stable) nonlinear hybrid system can be stabilized (or destabilized) via periodic stochastic feedback control based on Lévy noise.

Keywords: Stabilize; Destabilize; Nonlinear hybrid systems; Lévy noise, Periodic stochastic feedback control.

1. Introduction

In recent years, the hybrid system driven by a continuous-time Markov chain has received a great deal of attention due to its important applications in economics, control, biology, finance and so on (see, e.g., Deng et al. (2019); Mao (2002); Mao and Yuan (2006); Wang et al. (2020)). One of the important works in these applications is to discuss the stability of a hybrid system. It is well known that noise can be utilized to form stochastic feedback control to stabilize a given unstable system or to destabilize a given stable system. Khasminskii in 1981 was the first to study that a system was stabilized by using two white noise sources. Arnold et al. (1983) showed that a linear system can be stabilized by zero mean stationary parameter noise. In particular, Mao in 1994 pointed out that an unstable nonlinear system can be stabilized by Brownian motion. Since then, there has been extensive literature concerning the stochastic stability of the hybrid system driven by Brownian motion. For example, Fei et al. (2018) discussed the robustness of the exponential stability of the stochastic differential delay equations (SDDEs) with Markovian switching. Hu et al. (2013) studied the robust stability of nonlinear hybrid SDDEs without the linear growth condition. However, most of the works above focus on the stochastic stabilization problems of hybrid systems driven by Brownian motion.

As we all know, Brownian motion is a continuous stochastic process. Nevertheless, many real systems may be subject to some random jump-type instantaneous disturbance (e.g., hurricanes, sandstorms, deluges, and earthquakes). The discontinuous disturbance is inappropriately modeled by Brownian motion. In comparison to the Brownian motion, the Lévy process has been provided the possibility to describe this discontinuous process. To build more realistic models, many scholars incorporated the Lévy process into a hybrid system (Li and Deng (2017); Lu and Ding (2019); Zhou et al. (2020)). Naturally, it is also very important to study the stability of hybrid systems driven by the Lévy process. Applebaum and Siakalli (2010) took the first step in the stochastic stabilization of dynamical systems in which the noise is a more general Lévy process. Following Applebaum and Siakalli's work on stochastic noise stabilization, which employs the Lévy process as a noise source, there has been a significant advancement in stochastic noise stabilization theory (Li and Yang (2022); Liu and Zhou (2015); Wei (2020); Zhu (2018)). Liu and Zhou (2015) gave sufficient conditions of stabilization and destabilization for hybrid differential equations by Lévy noise. Wei (2020) studied the existence and uniqueness of solutions to a nonlinear stochastic system with Markovian switching and Lévy noises, and derived the almost sure exponential stability of the system.

It is worth noting that most existing results on the stability of hybrid systems driven by Lévy noise assume that the drift coefficients, diffusion coefficients and jump coefficients are bounded by time-homogeneous functions, although all the drift coefficients, diffusion coefficients and jump coefficients are time-inhomogeneous. Could we take advantage of the time-inhomogeneous property of all drift, diffusion and jump coefficients to establish alternative criteria on stabilization and destabilization for hybrid systems driven by Lévy noise? Recently, several scholars (Liu et al. (2019); Liu and Chen (2015); Zhang et al. (2018, 2020)) have investigated the stabilization of a class of stochastic differential equations based on time-inhomogeneous functions. It should be mentioned that Li et al. (2021) proved the almost sure exponential stability and instability of hybrid systems by periodic stochastic controls in which the noise is a Brownian motion. To the best of our knowledge, there has been no work on the stabilization and destabilization of nonlinear hybrid systems by periodic stochastic controls in which the noise is a Lévy process. Motivated by the above discussion, the main purpose of this paper is to show a positive answer to the above question, and the contributions of this paper are as follows:

- Lévy noise as the source of noise has been considered in this paper.
- Making use of the time-inhomogeneous property of all coefficients of drift, diffusion and jump to establish alternative criteria for stabilization and destabilization of hybrid systems with Lévy noise.
- Some important processes as special cases in the Lévy processes are provided to illustrate our results, including the Brownian motion, the Poisson process and the Carr-Geman-Madan-Yor (CGMY) process (Carr et al., 2002).

The rest of the paper is arranged as follows. In Section 2, we introduce some related notations and models, as well as concepts of stabilization and destabilization. In Section 3, we establish certain sufficient conditions for stability and instability. In Section 4 and Section 5, we discuss the stochastic stabilization and destabilization of hybrid systems by periodic stochastic feedback controls based on Lévy noise, and present some examples to illustrate our theoretical results. We will finally conclude our paper in Section 6.

2. Model formulation and preliminaries

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space. Suppose $\{\mathfrak{F}_t\}_{t\geq 0}$ is a filtration defined on the probability space, which satisfies the usual conditions (i.e., it is increasing and right continuous with

$\mathfrak{F}_0$ contains all $\mathbb{P}$-null sets). Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. Let $\langle\cdot,\cdot\rangle$ denote the inner product and I denote the $n\times n$ identity matrix. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^TA)}$ while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$. If A is a symmetric matrix, denote by $\lambda_{max}(A)$ and $\lambda_{min}(A)$ its largest and smallest eigenvalue, respectively. Define $\mathbb{R}_+ := [0,\infty)$. For $T > 0$, denote by $\mathscr{K}_T$ the family of bounded periodic functions $\kappa : \mathbb{R}_+ \to \mathbb{R}$ which are right continuous with left limits and have their period T. Let $B(t) = (B_1,\dots,B_d)$ be a d-dimensional Brownian motion and N is an $\mathfrak{F}_t$-adapted Poisson random measure defined on $\mathbb{R}_+\times\mathbb{R}^m\backslash\{0\}$ with compensator $\tilde{N}$ of the form $\tilde{N}(dt,d\sigma) = N(dt,d\sigma)-\nu(d\sigma)dt$, where ν is a Lévy measure. Let $r(t)$, $t\geq 0$, be a right-continuous Markov chain with finite state space $\mathbb{S} = \{1,2,\cdots,S\}$ on the probability space. The generator of $\{r(t)\}_{t\geq 0}$ is defined by $\Gamma = (\gamma_{ij})_{S\times S}$, so that for a sufficiently small $\delta > 0$,

$$\mathbb{P}\{r(t+\delta) = j|r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta) & \text{if } i\neq j, \\ 1+\gamma_{ij}\delta + o(\delta) & \text{if } i = j, \end{cases}$$

where $o(\delta)$ satisfies $\lim_{\delta\to 0}\frac{o(\delta)}{\delta} = 0$. Here γ_{ij} is the transition rate from i to j satisfying $\gamma_{ij} > 0$ if $i\neq j$ while $\gamma_{ii} = -\sum_{i\neq j}\gamma_{ij}$. It is commonly known that almost every sample path of $r(t)$ is a right-continuous step function with the finite number of simple jumps in any finite subinterval of $\mathbb{R}_+$. As a standing hypothesis, we assume in this paper that the Markov chain is irreducible. The algebraic interpretation of irreducibility is rank$(\Gamma) = S-1$. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi = (\pi_1,\pi_2,\cdots,\pi_S)$, which can be determined by solving the following linear equation $\pi\Gamma = 0$ subject to $\sum_{j=1}^S \pi_j = 1$ and $\pi_j > 0$ for all $j\in\mathbb{S}$. We assume that $B(t)$, $\tilde{N}(t,\sigma)$ and $r(t)$ are mutually independent.

Consider a hybrid system that is described by an ordinary differential equation (ODE) with Markovian switching

$$dx(t) = f(x(t),t,r(t))dt, \tag{2.1}$$

where $x(t)$ is in general referred to as the state while $r(t)$ is regarded as the mode.

To discuss the stochastic stabilization and destabilization for the hybrid ODE (2.1), let us consider the hybrid SDE with Lévy noise of the form

$$dx(t) = f(x(t),t,r(t))dt + g(x(t),t,r(t))dB(t) + \int_{\mathbb{R}^m\backslash\{0\}} h(t,r(t),\sigma)x(t-)\tilde{N}(dt,d\sigma), \tag{2.2}$$

with the initial value $x(0) = x_0\in\mathbb{R}^n$, where $x(t-) = \lim_{s\uparrow t}x(s)$, $f:\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S}\to\mathbb{R}^n$, $g:\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S}\to\mathbb{R}^{n\times d}$ and $h:\mathbb{R}_+\times\mathbb{S}\times\mathbb{R}^m\backslash\{0\}\to\mathbb{R}^{n\times n}$ satisfy the locally Lipschitz continuous and grows at most linearly, which guarantees the existence and uniqueness of the solution of the given Eq. (2.2) (see, e.g., Yuan and Mao (2010)). Denote the unique solution by $x(t;x_0)$ on $t\geq 0$. For the purpose of stability study in this paper, we also assume that $f(0,t,i)\equiv 0$ and $g(0,t,i)\equiv 0$ for each $i\in\mathbb{S}$. As a result, Eq. (2.2) admits a trivial solution $x(t;0)\equiv 0$.

Remark 1. If we add the left limit to the drift and diffusion terms, then the equation is equivalent to Eq. (2.2). That is, the following equation

$$dx(t) = f(x(t-),t,r(t))dt + g(x(t-),t,r(t))dB(t) + \int_{\mathbb{R}^m\backslash\{0\}} h(t,r(t),\sigma)x(t-)\tilde{N}(dt,d\sigma) \tag{2.3}$$

is equivalent to Eq. (2.2) since the integration of both equations is the same.

Let $C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S})$ denote the family of all nonnegative function $V(x,t,i)$ on $\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S}$ which are continuously twice differentiable in x and once differentiable in t. Given $V\in C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S})$, we define the operator L by

$$\begin{aligned} LV(x,t,i) =& V_t(x,t,i)+V_x(x,t,i)f(x,t,i)+\frac{1}{2}\text{trace}[g^T(x,t,i)V_{xx}(x,t,i)g(x,t,i)] \\ &+\sum_{j}^{S}\gamma_{ij}V(x,t,j)+\int_{\mathbb{R}^m\backslash\{0\}}[V(x+h(t,i,\sigma)x,t,i)-V(x,t,i) \\ &\qquad -V_x(x,t,i)h(t,i,\sigma)x]\nu(d\sigma), \end{aligned} \tag{2.4}$$

where

$$V_x(x,t,i)=\Big(\frac{\partial V(x,t,i)}{\partial x_1},\dots,\frac{\partial V(x,t,i)}{\partial x_n}\Big),\quad V_{xx}(x,t,i)=\Big(\frac{\partial^2 V(x,t,i)}{\partial x_i\partial x_j}\Big)_{n\times n}.$$

To establish the theorems on the almost sure exponential stability and instability of the hybrid system driven by Lévy noise, we need the following definition.

Definition 2.1. (see, e.g., Mao (1994), Mao (2007)) The trivial solution of Eq. (2.2), or simply, Eq. (2.2) is said to be almost surely exponentially stable if for any $x_0\in\mathbb{R}^n$,

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t;x_0)|)<0\quad a.s.$$

It is said to be almost surely exponentially unstable if for any $x_0\neq 0$

$$\liminf_{t\to\infty}\frac{1}{t}\log(|x(t;x_0)|)>0\quad a.s.$$

3. Nonlinear hybrid system driven by Lévy noise

To discuss the stochastic stabilization of the hybrid system driven by Lévy noise, we impose the following assumptions. It is worth noting that the coefficients are all bounded by periodic functions, which are different from the work of others.

Assumption 1. *We require that*

$$\int_{\mathbb{R}^m\backslash\{0\}}\|h(t,i,\sigma)\|^2\vee\|h(t,i,\sigma)\|\nu(d\sigma)<\infty,\ i\in\mathbb{S} \tag{3.1}$$

and that $h(t,i,\sigma)$ does not have any eigenvalue equal to -1. Further we may assume the nondegenerate condition on the jump part:

$$|(h(t,i,\sigma)+I)x|\geq\eta(\sigma)|x|,\quad x\in\mathbb{R}^n, \tag{3.2}$$

for some constant $\eta(\sigma)>0$, ν-a.e. on $\mathbb{R}^m\backslash\{0\}$ with $\int_{\mathbb{R}^m\backslash\{0\}}\frac{1}{\eta(\sigma)}\nu(d\sigma)<\infty$. The condition (3.2) is equivalent to the full rank of $h(t,i,\sigma)+I$, ν-a.e. on $\mathbb{R}^m\backslash\{0\}$.

Assumption 2. *For each $i \in \mathbb{S}$, there are functions $a_i(\cdot)$, $b_i(\cdot)$, $c_i(\cdot)$, $d_i(\cdot)$ and $e_i(\cdot)$ in $\mathscr{K}_T$ such that*

$$x^T f(x,t,i) \le a_i(t)|x|^2, \quad |g(x,t,i)| \le b_i(t)|x|, \quad |x^T g(x,t,i)| \ge c_i(t)|x|^2,$$

$$\|h(t,i,\sigma)\| \le d_i(t)\|H_i(\sigma)\|, \quad \int_{\mathbb{R}^m\setminus\{0\}} x^T h(t,i,\sigma)x\nu(d\sigma) \ge e_i(t)|x|^2,$$

where $H_i(\sigma)$ is a matrix dependent on σ.

We should also point out that all $b_i(\cdot)$, $c_i(\cdot)$ and $d_i(\cdot)$ are nonnegative but $a_i(\cdot)$ and $e_i(\cdot)$ may not. The following lemmas will play an important role in this paper.

Lemma 3.1. *Under Assumption 1, for all $x_0 \neq 0 \in \mathbb{R}^n$*

$$\mathbb{P}\{x(t;x_0) \neq 0 \text{ on } t \ge 0\} = 1. \tag{3.3}$$

That is, almost all the sample paths of any solution of equation (2.2) starting from a nonzero state will never reach the origin.

Proof. The idea of this proof comes from references Applebaum and Siakalli (2009) and Mao and Yuan (2006). Assume that (3.3) is false. This means that there are some $t_0 \ge 0$, $x_0 \neq 0$ and r_0 such that

$$\mathbb{P}\{\rho < \infty\} > 0, \tag{3.4}$$

where ρ is the first time of zero of the solution, that is

$$\rho = \inf\{t \ge t_0 : x(t;x_0) = 0\}.$$

For simplicity, let $x(t;x_0) = x(t)$. Since the paths of $x(t)$ are almost-surely right continuous with left limits, there exist a sufficiently large integer $k > t_0 \vee (1+|x_0|)$ such that $\mathbb{P}(E) > 0$, where

$$E = \{\rho \le k \text{ and } |x(t)| \le k-1 \text{ for all } t_0 \le t \le \rho\}.$$

In fact, the coefficients of the equation are locally Lipschitz continuous, namely, there exists a positive number J_k such that

$$|f(x,t,i)| \vee |g(x,t,i)| \vee |h(t,i,\sigma)x| \le J_k|x|, \quad \text{if } |x| \le k,\ t_0 \le t \le k.$$

Let $V(x,t,i) = |x|^{-1}$. Then, for $0 \le |x| \le k$, it follows from Assumption 1 that

$$\begin{aligned}
LV(x,t,i) =& -|x|^{-3}x^T f(x,t,i) + \frac{1}{2}\Big(-|x|^{-3}|g(x,t,i)|^2 + 3|x|^{-5}|x^T g(x,t,i)|^2\Big) \\
&+ \int_{\mathbb{R}^m\setminus\{0\}} \Big[\frac{1}{|x+h(t,i,\sigma)x|} - \frac{1}{|x|} + |x|^{-3}x^T h(t,i,\sigma)x\Big]\nu(d\sigma) \\
\le& |x|^{-2}|f(x,t,i)| + |x|^{-3}|g(x,t,i)|^2 + \int_{\mathbb{R}^m\setminus\{0\}} \frac{2|h(t,i,\sigma)x|}{|x|^2}\frac{|h(t,i,\sigma)x|+|x|}{|h(t,i,\sigma)x+x|}\nu(d\sigma) \\
\le& |x|^{-1}J_k + |x|^{-1}J_k^2 + \int_{\mathbb{R}^m\setminus\{0\}} \frac{2J_k}{|x|}\frac{J_k+1}{\eta(\sigma)}\nu(d\sigma) \\
:=& C_{J_k}|x|^{-1} = C_{J_k}V(x,t,i),
\end{aligned} \tag{3.5}$$

where C_{J_k} is a positive constant dependent on J_k. For each $\theta \in (0, |x_0|)$, define a stopping time

$$\rho_\theta = \inf\{t \geq t_0 : |x(t)| \notin (\theta, k)\}.$$

By the Itô formula,

$$\begin{aligned}
&\mathbb{E}\Big[e^{-C_{J_k}(\rho_\theta \wedge k)}|x(\rho_\theta \wedge k)|^{-1}\Big] \\
=&|x_0|^{-1}e^{-C_{J_k}t_0} + \mathbb{E}\int_{t_0}^{\rho_\theta \wedge k} e^{-C_{J_k}s}\Big[-C_{J_k}|x(s)|^{-1} + LV(x(s), s, r(s))\Big]ds \\
\leq&|x_0|^{-1}e^{-C_{J_k}t_0}.
\end{aligned} \tag{3.6}$$

If $\omega \in E$, we obtain $\rho_\theta \leq k$ and $|x(\rho_\theta)| = \theta$. Then,

$$\mathbb{E}\Big[e^{-C_{J_k}k}\theta^{-1}1_E\Big] \leq |x_0|^{-1}e^{-C_{J_k}t_0}, \tag{3.7}$$

where 1_E denotes the indicator function of the set E. So, we have

$$\mathbb{P}(E) \leq \theta|x_0|^{-1}e^{C_{J_k}(k-t_0)}.$$

Letting $\theta \to 0$ yield $\mathbb{P}(E) = 0$, which is in contradicts with $\mathbb{P}(E) > 0$. Thus, the proof is complete. □

Lemma 3.2. *(Li et al. (2021)) Let $\psi_i(\cdot) \in \mathscr{K}_T$ for $i \in \mathbb{S}$. Then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \psi_{r(s)}(s)ds = \sum_{i\in\mathbb{S}} \frac{\pi_i}{T}\int_0^T \psi_i(s)ds \quad a.s. \tag{3.8}$$

Remark 2. Most existing results assume the coefficients are bounded by time-homogeneous functions. Such conditions do not make use of the time-inhomogeneous property of the coefficients. It should be pointed out that Li et al. (2021) made full use of the periodicity of the coefficients and the ergodic property of the Markov chain to obtain this new result.

Let us now begin to prove our main results.

Theorem 3.3. *Under Assumption 2, the solution of equation (2.2) satisfies*

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t}\log(|x(t;x_0)|) \leq \sum_{i\in\mathbb{S}} \frac{\pi_i}{T}\int_0^T \Big(a_i(s) + 0.5b_i^2(s) - c_i^2(s) - e_i(s) \\
+ d_i(s)\int_{\mathbb{R}^m\setminus\{0\}} \|H_i(\sigma)\|\nu(d\sigma)\Big)ds \quad a.s.
\end{aligned} \tag{3.9}$$

In particular, the nonlinear hybrid system driven by Lévy noise is almost surely exponentially stable, if

$$\sum_{i\in\mathbb{S}} \frac{\pi_i}{T}\int_0^T \Big(a_i(s) + 0.5b_i^2(s) - c_i^2(s) - e_i(s) + d_i(s)\int_{\mathbb{R}^m\setminus\{0\}} \|H_i(\sigma)\|\nu(d\sigma)\Big)ds < 0.$$

Proof. If $x_0 = 0$, the solution $x(t;0) \equiv 0$ and hence assertion (3.9) holds. Fix any $x_0 \neq 0$ and write $x(t;x_0) = x(t)$. Recalling that this solution $x(t)$ will never reach zero with probability one, it follows from the generalised Itô formula (Mao and Yuan (2006); Yang et al. (2015); Yuan and Mao (2010)) that

$$\begin{aligned}
&\log(|x(t)|^2)\\
=&\log(|x_0|^2) + \int_0^t \frac{2x^T(s)}{|x(s)|^2} f(x(s),s,r(s))ds + \int_0^t \frac{2x^T(s)}{|x(s)|^2} g(x(s),s,r(s))dB(s)\\
&+\int_0^t \Big(\frac{|g(x(s),s,r(s))|^2}{|x(s)|^2} - \frac{2|x^T(s)g(x(s),s,r(s))|^2}{|x(s)|^4}\Big)ds\\
&+\int_0^t \int_{\mathbb{R}^m\setminus\{0\}} \big(\log(|x(s-)+h(s,r(s),\sigma)x(s-)|^2) - \log(|x(s-)|^2)\big)\tilde{N}(ds,d\sigma)\\
&+\int_0^t \int_{\mathbb{R}^m\setminus\{0\}} \big(\log(|x(s-)+h(s,r(s),\sigma)x(s-)|^2) - \log(|x(s-)|^2)\\
&-\frac{2x^T(s-)}{|x(s-)|^2} h(s,r(s),\sigma)x(s-)\big)\nu(d\sigma)ds\\
:=&I_1 + I_2 + I_3,
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
I_1 =&\log(|x_0|^2) + \int_0^t \frac{2x^T(s)}{|x(s)|^2} f(x(s),s,r(s))ds + \int_0^t \frac{2x^T(s)}{|x(s)|^2} g(x(s),s,r(s))dB(s)\\
&+\int_0^t \Big(\frac{|g(x(s),s,r(s))|^2}{|x(s)|^2} - \frac{2|x^T(s)g(x(s),s,r(s))|^2}{|x(s)|^4}\Big)ds,\\
I_2 =&\int_0^t \int_{\mathbb{R}^m\setminus\{0\}} \big(\log(|x(s-)+h(s,r(s),\sigma)x(s-)|^2) - \log(|x(s-)|^2)\big)\tilde{N}(ds,d\sigma),\\
I_3 =&\int_0^t \int_{\mathbb{R}^m\setminus\{0\}} \big(\log(|x(s-)+h(s,r(s),\sigma)x(s-)|^2) - \log(|x(s-)|^2)\\
&-\frac{2x^T(s-)}{|x(s-)|^2} h(s,r(s),\sigma)x(s-)\big)\nu(d\sigma)ds.
\end{aligned}$$

By Assumption 2, we obtain that

$$I_1 \leq \log(|x_0|^2) + \int_0^t \big(2a_{r(s)}(s) + b_{r(s)}^2(s) - 2c_{r(s)}^2(s)\big)ds + M(t), \tag{3.11}$$

and

$$M(t) = \int_0^t \frac{2x^T(s)}{|x(s)|^2} g(x(s),s,r(s))dB(s),$$

which is a continuous martingale vanishing at $t=0$. The quadratic variation of the martingale is given by

$$\langle M(t), M(t)\rangle = \int_0^t \frac{4|x^T(s)g(x(s),s,r(s))|^2}{|x(s)|^4} ds \leq 4t \max_{i\in\mathbb{S}} \sup_{0\leq s\leq T} b_i^2(s).$$

By the strong law of the large numbers for local martingale,

$$\lim_{t\to\infty}\frac{M(t)}{t}=0 \quad a.s. \tag{3.12}$$

Furthermore, I_2 is a continuous local martingale, using logarithmic inequalities $(\log(x)\le x-1)$ we obtain that

$$\begin{aligned}\langle I_2, I_2\rangle &= \int_0^t\int_{\mathbb{R}^m\setminus\{0\}}\big[\log\big(\frac{|x(s-)+h(s,r(s),\sigma)x(s-)|^2}{|x(s-)|^2}\big)\big]^2\nu(d\sigma)ds\\ &\le 4\int_0^t\int_{\mathbb{R}^m\setminus\{0\}}\big[\log\big(1+\|h(s,r(s),\sigma)\|\big)\big]^2\nu(d\sigma)ds\\ &\le 4t\int_{\mathbb{R}^m\setminus\{0\}}\|h(s,r(s),\sigma)\|^2\nu(d\sigma).\end{aligned}$$

By the strong law of the large numbers for local martingale again,

$$\lim_{t\to\infty}\frac{I_2}{t}=0 \quad a.s. \tag{3.13}$$

Making use of Assumption 2, we obtain

$$\begin{aligned}I_3 &\le \int_0^t\int_{\mathbb{R}^m\setminus\{0\}}\big(\log\frac{|x(s-)+h(s,r(s),\sigma)x(s-)|^2}{|x(s-)|^2}\big)\nu(d\sigma)ds-2\int_0^t e_{r(s)}(s)ds\\ &\le 2\int_0^t\int_{\mathbb{R}^m\setminus\{0\}}\log(1+\|h(s,r(s),\sigma)\|)\nu(d\sigma)ds-2\int_0^t e_{r(s)}(s)ds\\ &\le 2\int_0^t\int_{\mathbb{R}^m\setminus\{0\}}d_{r(s)}(s)\|H_{r(s)}(\sigma)\|\nu(d\sigma)ds-2\int_0^t e_{r(s)}(s)ds.\end{aligned} \tag{3.14}$$

Substituting (3.11)-(3.14) into (3.10) we get

$$\begin{aligned}\log(|x(t)|^2)\le\log(|x_0|^2)&+\int_0^t\big(2a_{r(s)}(s)+b^2_{r(s)}(s)-2c^2_{r(s)}(s)-2e_{r(s)}(s)\\ &+2d_{r(s)}(s)\int_{\mathbb{R}^m\setminus\{0\}}\|H_{r(s)}(\sigma)\|\nu(d\sigma)\big)ds+M(t)+I_2.\end{aligned} \tag{3.15}$$

Dividing both sides of (3.15) by $2t$, letting $t\to\infty$ and according to the Lemma 3.2, we have

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|)\le\sum_{i\in\mathbb{S}}\frac{\pi_i}{T}\int_0^T&\big(a_i(s)+0.5b_i^2(s)-c_i^2(s)-e_i(s)\\ &+d_i(s)\int_{\mathbb{R}^m\setminus\{0\}}\|H_i(\sigma)\|\nu(d\sigma)\big)ds \quad a.s.\end{aligned} \tag{3.16}$$

and if

$$\sum_{i\in\mathbb{S}}\frac{\pi_i}{T}\int_0^T\big(a_i(s)+0.5b_i^2(s)-c_i^2(s)-e_i(s)+d_i(s)\int_{\mathbb{R}^m\setminus\{0\}}\|H_i(\sigma)\|\nu(d\sigma)\big)ds<0,$$

the hybrid system driven by Lévy noise is almost surely exponentially stable. The proof is therefore complete. □

To discuss the instability, we impose the following assumption.

Assumption 3. *For each $i \in \mathbb{S}$, there are functions $a_i(\cdot)$, $b_i(\cdot)$, $c_i(\cdot)$, $d_i(\cdot)$ and $e_i(\cdot)$ in $\mathscr{K}_T$ such that*

$$x^T f(x,t,i) \geq a_i(t)|x|^2, \quad |g(x,t,i)| \geq b_i(t)|x|, \quad |x^T g(x,t,i)| \leq c_i(t)|x|^2,$$

$$\|h(t,i,\sigma)\| \geq d_i(t)\|H_i(\sigma)\|, \quad \int_{\mathbb{R}^m\setminus\{0\}} x^T h(t,i,\sigma) x \nu(d\sigma) \leq e_i(t)|x|^2,$$

where $H_i(\sigma)$ is a matrix dependent on σ, and all $b_i(\cdot)$, $c_i(\cdot)$ and $d_i(\cdot)$ are nonnegative but $a_i(\cdot)$ and $e_i(\cdot)$ may not.

Theorem 3.4. *Under Assumption 3, the solution of equation (2.2) satisfies*

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{t}\log(|x(t;x_0)|) \geq \sum_{i\in\mathbb{S}} \frac{\pi_i}{T}\int_0^T \big(a_i(s) + 0.5b_i^2(s) - c_i^2(s) - e_i(s) \\ + \int_{\mathbb{R}^m\setminus\{0\}} \log(d_i(s)\|H_i(\sigma)\|)\nu(d\sigma)\big)ds \quad a.s.\end{aligned} \tag{3.17}$$

In particular, the nonlinear hybrid system driven by Lévy noise is almost surely exponentially unstable, if

$$\sum_{i\in\mathbb{S}} \frac{\pi_i}{T}\int_0^T \big(a_i(s) + 0.5b_i^2(s) - c_i^2(s) - e_i(s) + \int_{\mathbb{R}^m\setminus\{0\}} \log(d_i(s)\|H_i(\sigma)\|)\nu(d\sigma)\big)ds > 0.$$

Proof. Fix any $x_0 \neq 0$ and write $x(t;x_0) = x(t)$ again. By Assumption 3, we can show from (3.10) that

$$\begin{aligned}\log(|x(t)|^2) \geq \log(|x_0|^2) + \int_0^t \big(2a_{r(s)}(s) + b_{r(s)}^2(s) - 2c_{r(s)}^2(s)\big)ds \\ + M(t) + I_2 + I_3,\end{aligned} \tag{3.18}$$

where $M(t)$, I_2 and I_3 as defined in the proof of Theorem 3.3 but there quadratic variation are now estimated as

$$\langle M(t), M(t)\rangle = \int_0^t \frac{4|x^T(s)g(x(s),s,r(s))|^2}{|x(s)|^4} ds \leq 4t \max_{i\in\mathbb{S}} \sup_{0\leq s\leq T} c_i^2(s).$$

By the strong law of the large numbers for local martingale,

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \quad a.s. \tag{3.19}$$

Furthermore,

$$\langle I_2, I_2\rangle \leq 4t \int_{\mathbb{R}^m\setminus\{0\}} \|h(s,r(s),\sigma)\|^2 \nu(d\sigma).$$

By the strong law of the large numbers for local martingale again,

$$\lim_{t\to\infty}\frac{I_2}{t}=0 \quad a.s. \tag{3.20}$$

For I_3, using Assumption 3, we obtain

$$\begin{aligned} I_3 \geq & \int_0^t\int_{\mathbb{R}^m\setminus\{0\}}\Big(\log\frac{|x(s-)+h(s,r(s),\sigma)x(s-)|^2}{|x(s-)|^2} \\ & \qquad -\frac{2x^T(s-)}{|x(s-)|^2}h(s,r(s),\sigma)x(s-)\Big)\nu(d\sigma)ds \\ \geq & 2\int_0^t\int_{\mathbb{R}^m\setminus\{0\}}\log(d_{r(s)}(s)\|H_{r(s)}(\sigma)\|)\nu(d\sigma)ds-2\int_0^t e_{r(s)}(s)ds, \end{aligned} \tag{3.21}$$

Note that $\log|\psi_i(t)|\in\mathscr{K}_T$ whenever $\psi_i(t)\in\mathscr{K}_T$. Dividing both sides of (3.18) by $2t$, letting $t\to\infty$ and according to the Lemma 3.2, we have

$$\begin{aligned} \liminf_{t\to\infty}\frac{1}{t}\log(|x(t)|)\geq & \sum_{i\in\mathbb{S}}\frac{\pi_i}{T}\int_0^T\Big(a_i(s)+0.5b_i^2(s)-c_i^2(s)-e_i(s) \\ & +\int_{\mathbb{R}^m\setminus\{0\}}\log(d_i(s)\|H_i(\sigma)\|)\nu(d\sigma)\Big)ds \quad a.s. \end{aligned} \tag{3.22}$$

and if

$$\sum_{i\in\mathbb{S}}\frac{\pi_i}{T}\int_0^T\Big(a_i(s)+0.5b_i^2(s)-c_i^2(s)-e_i(s)+\int_{\mathbb{R}^m\setminus\{0\}}\log(d_i(s)\|H_i(\sigma)\|)\nu(d\sigma)\Big)ds>0,$$

the hybrid system driven by Lévy noise is almost surely exponentially unstable. The proof is therefore complete. □

4. Stochastic stabilization

Let us now begin with the discussion of the stochastic stabilization for the hybrid ODE

$$dx(t)=f(x(t),t,r(t))dt. \tag{4.1}$$

As previously stated, f satisfies the local Lipschitz condition and

$$x^Tf(x,t,i)\leq a_i(t)|x|^2. \tag{4.2}$$

Assume that this given hybrid ODE is not stable, and we are required to use stochastic feedback control to make the following stochastic system

$$dx(t)=f(x(t),t,r(t))dt+u_1(x(t),t,r(t))dB(t)+\int_{\mathbb{R}^m\setminus\{0\}}u_2(x(t-),t,r(t),\sigma)\tilde{N}(dt,d\sigma) \tag{4.3}$$

becomes almost surely exponentially stable, where $u_1 : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{n\times d}$ and $u_2 : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \times \mathbb{R}^m \to \mathbb{R}^n$. In this paper we only consider the periodic linear feedback control of the form

$$\begin{aligned} u_1(x,t,i) &= \beta_i(t)(A_{1,i}x, A_{2,i}x, \cdots, A_{d,i}x), \\ u_2(x,t,i,\sigma) &= \rho_i(t)M_i(\sigma)x, \end{aligned} \tag{4.4}$$

where $(A_{k,i} : k = 1,2,\dots,d) \in \mathbb{R}^{n\times n}$, $M_i(\sigma) \in \mathbb{R}^{n\times n}$, $\beta_i(\cdot) \in \mathscr{K}_T$ and $\rho_i(\cdot) \in \mathscr{K}_T$ for $i \in \mathbb{S}$. Thus the controlled system (4.3) becomes

$$\begin{aligned} dx(t) = f(x(t),t,r(t))dt &+ \sum_{k=1}^{d} \beta_{r(t)}(t)A_{k,r(t)}x(t)dB_k(t) \\ &+ \int_{\mathbb{R}^m\setminus\{0\}} \rho_{r(t)}(t)M_{r(t)}(\sigma)x(t-)\tilde{N}(dt,d\sigma). \end{aligned} \tag{4.5}$$

Theorem 4.1. *Let* (4.2) *hold. Assume that for each $i \in \mathbb{S}$, the matrices $A_{k,i}$ and $M_i(\sigma)$ in the controller have the property*

$$\sum_{k=1}^{d} |A_{k,i}x|^2 \le b_i|x|^2, \quad \sum_{k=1}^{d} |x^T A_{k,i}x|^2 \ge c_i|x|^4, \quad \int_{\mathbb{R}^m\setminus\{0\}} x^T M_i(\sigma)x\nu(d\sigma) \ge e_i|x|^2,$$

where b_i, c_i are some nonnegative constants and e_i is constant. Choose $\beta_i(\cdot) \in \mathscr{K}_T$ and $\rho_i(\cdot) \in \mathscr{K}_T$, then the solution of (4.5) *satisfies*

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{t}\log|x(t;x_0)| \le \sum_{i\in\mathbb{S}} \frac{\pi_i}{T}\Big[\int_0^T a_i(s)ds + (0.5b_i - c_i)\int_0^T (\beta_i(s))^2 ds \\ - e_i\int_0^T \rho_i(s)ds + \int_{\mathbb{R}^m\setminus\{0\}} \|M_i(\sigma)\|\nu(d\sigma)\int_0^T |\rho_i(s)|ds\Big] \qquad a.s. \end{aligned} \tag{4.6}$$

for any $x_0 \in \mathbb{R}^n$. In particular, if

$$\begin{aligned} -\sum_{i\in\mathbb{S}} \frac{\pi_i}{T}\int_0^T a_i(s)ds > \sum_{i\in\mathbb{S}} \frac{\pi_i}{T}\Big[(0.5b_i - c_i)\int_0^T (\beta_i(s))^2 ds - e_i\int_0^T \rho_i(s)ds \\ + \int_{\mathbb{R}^m\setminus\{0\}} \|M_i(\sigma)\|\nu(d\sigma)\int_0^T |\rho_i(s)|ds\Big], \end{aligned} \tag{4.7}$$

then the controlled system (4.5) *is almost surely exponentially stable.*

The proof is a simple application of Theorem 3.3 so is omitted. Theorem 4.1 ensures that there are many choices for the matrices $A_{k,i}$ and $M_i(\sigma)$ in order to stabilize the given hybrid system (4.5).

Let us discuss two examples to compare our new results with existing ones (Liu and Zhou (2015); Yin and Xi (2010)). We will use two special Lévy processes in order to avoid unnecessary calculations but the superiority of our results will be explained clearly. In particular, we omit the case of the hybrid system driven by Brownian motion as it is very similar to the Brownian motion case presented in Li et al. (2021).

In the following, we give an example that is motivated by modeling stock price movements in financial mathematics.

Example 4.1. Consider a 1-dimensional hybrid ODE

$$dx(t) = f(x(t),t,r(t))dt, \tag{4.8}$$

where the Markov chain $r(t)$ be a right-continuous Markov chain taking values $\mathbb{S} = \{1,2\}$ with generator

$$\Gamma = \begin{pmatrix} -13 & 13 \\ 7 & -7 \end{pmatrix},$$

while

$$f(x,t,i) = \begin{cases} (0.5+0.5\cos(t))sin(x), & \text{if } i = 1, \\ (\cos(\frac{t}{2})\sin(\frac{t}{2})+0.5)x, & \text{if } i = 2, \end{cases}$$

for $(x,t,i) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}$. Use a CGMY process as a source of Lévy noise so that the controlled system has the form

$$dx(t) = f(x(t),t,r(t))dt + \int_{\mathbb{R}\setminus\{0\}} \beta_{r(t)}\sigma x(t-)\tilde{N}(dt,d\sigma), \tag{4.9}$$

where the CGMY process is a pure jump process with Lévy measure

$$\nu(d\sigma) = \begin{cases} C'\frac{\exp(-Q|\sigma|)}{|\sigma|^{1+\alpha}} & \text{for } \sigma < 0, \\ C\frac{\exp(-M|\sigma|)}{|\sigma|^{1+\alpha}} & \text{for } \sigma > 0, \end{cases} \tag{4.10}$$

where $C > 0$, $C' > 0$, $Q > 0$, $M > 0$ and $0 \le \alpha < 2$. The stationary distribution of the Markov chain is $\pi_1 = 0.35$ and $\pi_2 = 0.65$. Let $C = C' = 1$, $M = 4$, $Q = 2$, $\alpha = 0$, $\beta_1 = 1$ and $\beta_2 = \sqrt{2}$. To apply Theorem 3.1 in Liu and Zhou (2015), we note

$$x^T f(x,t,i) \le a_i|x|^2,$$

where $a_1 = 1$, $a_2 = 1$. Theorem 3.1 in Liu and Zhou (2015) needs

$$\sum_{i=1}^{2}\pi_i\big(a_i - \beta_i\int_{\mathbb{R}\setminus\{0\}}\sigma\nu(d\sigma) + \max_{i\in\mathbb{S}}\beta_i^2\int_{\mathbb{R}\setminus\{0\}}|\sigma|^2\nu(d\sigma)\big) < 0, \tag{4.11}$$

but the left-hand-side term is equal to 0.3875, so the condition does not hold and we can not apply Theorem 3.1 in Liu and Zhou (2015) to conclude that the controlled SDE (4.9) is almost surely exponentially stable. On the other hand, we can apply our Theorem 4.1 to demonstrate that it is. In fact, $f(x,t,i)$ is a periodic function of t with period 2π. Observe that

$$x^T f(x,t,i) \le a_i(t)|x|^2,$$

where $a_1(t) = 0.5\cos(t)+0.5$ and $a_2(t) = \cos(t/2)\sin(t/2)+0.5$. It is easy to see that

$$\sum_{i=1}^{2}\frac{\pi_i}{2\pi}\int_0^{2\pi}a_i(s)ds + \sum_{i=1}^{2}\pi_i(-\beta_i\int_{\mathbb{R}\setminus\{0\}}\sigma\nu(d\sigma) + \beta_i\int_{\mathbb{R}\setminus\{0\}}|\sigma|\nu(d\sigma)) = -0.7692 < 0.$$

By Theorem 4.1, we can conclude that the controlled system is almost surely exponentially stable.

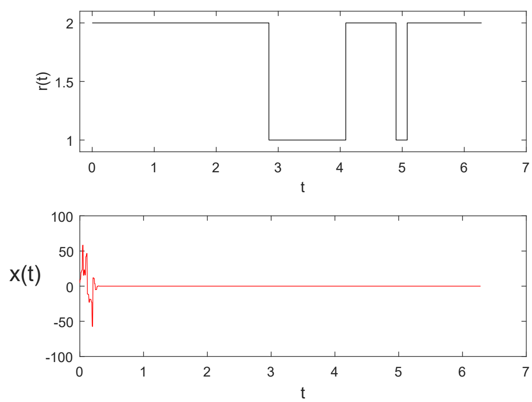

FIG. 1. The computer simulation of the sample paths of the Markov chain and the solution of equation (4.9) for one period of time.

For illustration, we perform a computer simulation and use the Monte Carlo method (see Ballotta and Kyriakou (2014)). We set the initial data $x(0) = 1$ and $r(0) = 2$. FIG. 1 shows the sample paths of the Markov chain and the state, which clearly support our theoretical results.

Let's further consider a 2-dimensional example in which the state $x(t)$ is only observable in mode 2 but not in mode 1, so the stochastic control could only be made in mode 2.

Example 4.2. Consider a 2-dimensional hybrid ODE

$$dx(t) = f(x(t), t, r(t))dt, \tag{4.12}$$

where the Markov chain $r(t)$ be a right-continuous Markov chain taking values $\mathbb{S} = \{1, 2\}$ with generator

$$\Gamma = \begin{pmatrix} -7 & 7 \\ 13 & -13 \end{pmatrix},$$

while

$$f(x,t,i) = \begin{cases} (1+\cos^2(t))F_1 x, & \text{if } i = 1, \\ (\cos(t)\sin(t)+0.5)F_2 x, & \text{if } i = 2, \end{cases}$$

where $F_1 = \begin{pmatrix} -0.1 & 0.3 \\ -0.8 & 1 \end{pmatrix}$, $F_2 = \begin{pmatrix} -1 & -0.1 \\ -0.4 & 0.1 \end{pmatrix}$ and $(x,t,i) \in \mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{S}$. Let us introduce the linear diffusion driven by a Brownian motion with $u_1(x,t,i) = \beta_i(t)A_i x$ and the linear Poisson process with $u_2(x,t,i,\sigma) = \rho_i(t)D_i x$ as the stochastic feedback to stabilize the original hybrid system. This is equivalent to considering the stability of the following nonlinear hybrid system driven by Lévy noise

$$dx(t) = f(x(t),t,r(t))dt + \beta_{r(t)}(t)A_{r(t)}x(t)dB(t) + \rho_{r(t)}(t)D_{r(t)}x(t-)d\tilde{N}(t), \tag{4.13}$$

where $B(t)$ is a one-dimensional Brownian motion and $\tilde{N}(t)$ is the compensated Poisson process with $\tilde{N}(t) = N(t) - \lambda t$, where λ is the intensity of the Poisson process $N(t)$. We only consider the case where the state $x(t)$ is only controlled by Brownian motion in mode 1 and only by the scalar Poisson process in mode 2. In terms of mathematics, we have to set $A_1 = 0 = D_1$ and $\beta_1(t) = 0 = \rho_1(t)$. Moreover, we set $A_2 = diag(\sqrt{b_1}, \sqrt{b_1})$, $D_2 = diag(e_2, e_2)$, $\beta_2(t) = 1$ and $\rho_2(t) = 1$. The stationary distribution of the Markov chain is $\pi_1 = 0.65$ and $\pi_2 = 0.35$. To apply Theorem 4.1 in Yin and Xi (2010), we note

$$x^T f(x,t,i) \le a_i |x|^2,$$

where $a_1 = 1.4$, $a_2 = 0.1325$. Take $\lambda = 1$, $b_1 = c_1 = 7$, $e_2 = 1$. Theorem 4.1 in Yin and Xi (2010) needs

$$\sum_{i=1}^{2} \pi_i \big(a_i + 0.5b_i - c_i + \lambda e_i\big) < 0, \tag{4.14}$$

but the left-hand-side term is equal to 0.081375, so the condition does not hold and we can not apply Theorem 4.1 in Yin and Xi (2010) to conclude that the controlled SDE (4.13) is almost surely exponentially stable. On the other hand, we can apply our Theorem 4.1 to show it is. In fact, $f(x,t,i)$ is a periodic function of t with period π. Observe that

$$x^T f(x,t,i) \le a_i(t) |x|^2,$$

where $a_1(t) = 0.7(1+\cos^2(t))$ and $a_2(t) = 0.1325(0.5+\cos(t)\sin(t))$. It is easy to see that

$$\sum_{i=1}^{2} \frac{\pi_i}{\pi} \int_0^{\pi} a_i(s)ds + \sum_{i=1}^{2} \pi_i (0.5b_i - c_i + \lambda e_i) = -0.16932 < 0. \tag{4.15}$$

By Theorem 4.1, we can conclude that the controlled system is almost surely exponentially stable. For

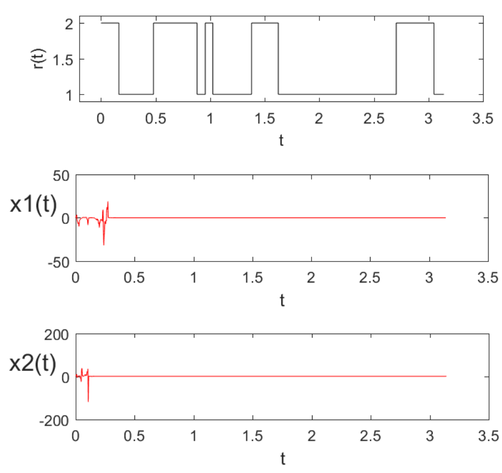

FIG. 2. The computer simulation of the sample paths of the Markov chain and the solution of equation (4.13) for one period of time.

illustration, we perform a computer simulation. We set the initial data $x(0) = 2$ and $r(0) = 2$, and use the Euler-Maruyama method (see Kühn and Schilling (2019); Zou and Wang (2014)) with the step size $10^{-3}\pi$. FIG. 2 shows the sample paths of the Markov chain and the state, which clearly support our theoretical results.

5. Stochastic destabilization

Let us now turn to consider the opposite problem—stochastic destabilization. More precisely, given a nonlinear stable hybrid system (4.1), can we design a linear controller $u_1(x,t,i)$, $u_2(x,t,i,\sigma)$ of the form (4.5) so that the controlled system (4.1) becomes unstable? To answer this question positively, let us state a result that follows from Theorem 5.1 directly.

Theorem 5.1. *Let $x^T f(x,t,i) \geq a_i(t)|x|^2$. Assume that for each $i \in \mathbb{S}$, the matrices $A_{k,i}$ and $M_i(\sigma)$ in the controller have the property*

$$\sum_{k=1}^{d} |A_{k,i}x|^2 \geq b_i|x|^2, \quad \sum_{k=1}^{d} |x^T A_{k,i}x|^2 \leq c_i|x|^4, \quad \int_{\mathbb{R}^m\setminus\{0\}} x^T M_i(\sigma)x\nu(d\sigma) \leq e_i|x|^2,$$

where b_i, c_i are some nonnegative constants and e_i is constant. Choose $\beta_i(\cdot) \in \mathscr{K}_T$ and $\rho_i(\cdot) \in \mathscr{K}_T$, then the solution of (4.5) *satisfies*

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{t} \log|x(t;x_0)| \geq \sum_{i\in\mathbb{S}} \frac{\pi_i}{T} \Big[\int_0^T a_i(s)ds + (0.5b_i - c_i) \int_0^T (\beta_i(s))^2 ds - e_i \int_0^T \rho_i(s)ds \\ + \int_0^T \int_{\mathbb{R}^m\setminus\{0\}} \log(|\,\rho_i(s)|\|M_i(\sigma)\|)\nu(d\sigma)ds \Big] \; a.s.\end{aligned} \tag{5.1}$$

for any $x_0 \in \mathbb{R}^n$. In particular, if

$$\begin{aligned}-\sum_{i\in\mathbb{S}} \frac{\pi_i}{T} \int_0^T a_i(s)ds < \sum_{i\in\mathbb{S}} \frac{\pi_i}{T} \Big[(0.5b_i - c_i) \int_0^T (\beta_i(s))^2 ds - e_i \int_0^T \rho_i(s)ds \\ + \int_0^T \int_{\mathbb{R}^m\setminus\{0\}} \log(|\rho_i(s)|\|M_i(\sigma)\|)\nu(d\sigma)ds \Big],\end{aligned} \tag{5.2}$$

then the controlled system (4.5) *is almost surely exponentially unstable.*

The proof is a simple application of Theorem 3.4, so it is omitted. The question now becomes: can we find matrices $A_{k,i}$ and $M_i(\sigma)$ so that (5.2) hold? We shall show that this is possible if the dimension of the state space is greater than or equal to 2.

First, let the dimension n of the state space be an even number. For each $i \in \mathbb{S}$, define

$$A_{1,i} = \begin{pmatrix} 0 & \sqrt{b_i} & & & \\ -\sqrt{b_i} & 0 & & & \\ & & \ddots & & \\ & & & 0 & \sqrt{b_i} \\ & & & -\sqrt{b_i} & 0 \end{pmatrix} \quad M_i(\sigma) = \begin{pmatrix} 0 & e_i\sigma & & & \\ -e_i\sigma & 0 & & & \\ & & \ddots & & \\ & & & 0 & e_i\sigma \\ & & & -e_i\sigma & 0 \end{pmatrix}$$

but set $A_{k,i} = 0$ for $2 \leq k \leq d$. The controlled system(4.5) becomes

$$dx(t) = f(x(t),t,r(t))dt + \beta_{r(t)}(t)\sqrt{b_{r(t)}}y(t)dB_1(t) + \int_{\mathbb{R}^m\setminus\{0\}} \rho_i(t)e_i\sigma y(t-)\tilde{N}(dt,d\sigma). \tag{5.3}$$

where $y(t)=(x_2(t),-x_1(t),\cdots,x_n(t),-x_{n-1}(t))$. Note that for each $i\in\mathbb{S}$,

$$\sum_{k=1}^{d}|A_{k,i}x|^2=b_i|x|^2,\quad \sum_{k=1}^{d}|x^TA_{k,i}x|^2=0,\quad \int_{\mathbb{R}^m\setminus\{0\}}x^TM_i(\sigma)x\nu(d\sigma)=0.$$

By Theorem 5.1, the solution of (5.3) satisfies

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\log|x(t;x_0)|\geq\sum_{i\in\mathbb{S}}\frac{\pi_i}{T}\Big[\int_0^T a_i(s)ds+0.5b_i\int_0^T(\beta_i(s))^2ds\\ +\int_0^T\int_{\mathbb{R}^m\setminus\{0\}}\log(|e_i\rho_i(s)||\sigma|)\nu(d\sigma)ds\Big]\qquad a.s.\end{aligned}\tag{5.4}$$

Clearly we can choose b_i, e_i, $\beta_i(s)$ and $\rho_i(s)$ such that

$$-\sum_{i\in\mathbb{S}}\frac{\pi_i}{T}\int_0^T a_i(s)ds\leq\sum_{i\in\mathbb{S}}\frac{\pi_i}{T}\Big[0.5b_i\int_0^T(\beta_i(s))^2ds+\int_0^T\int_{\mathbb{R}^m\setminus\{0\}}\log(|e_i\rho_i(s)||\sigma|)\nu(d\sigma)ds\Big]\ a.s.$$

So that the controlled system (5.3) becomes unstable.

Let us discuss an example to illustrate our new theory on the destabilization.

Example 5.1. Consider a 2-dimensional hybrid SDE driven by CGMY process

$$dx(t)=f(x(t),t,r(t))dt+\int_{\mathbb{R}\setminus\{0\}}\beta_{r(t)}G_{r(t)}x(t-)\sigma\tilde{N}(dt,d\sigma),\tag{5.5}$$

where the $r(t)$ and Lévy measure are the same as in equation (4.9) but

$$f(x,t,i)=\begin{cases}(0.5+\cos(t))F_1x, & \text{if } i=1,\\ (\cos(\frac{t}{2})\sin(\frac{t}{2})+0.5)F_2x, & \text{if } i=2.\end{cases}$$

where $F_1=\begin{pmatrix}-1 & 0.2\\ 0.1 & -2\end{pmatrix}$, $F_2=\begin{pmatrix}-2 & 0.3\\ 0.2 & -1\end{pmatrix}$, $G_1=0$ and $G_2=\begin{pmatrix}0 & \sqrt{\gamma_2}\\ -\sqrt{\gamma_2} & 0\end{pmatrix}$.
As a result, the controlled system has the form of

$$dx(t)=f(x(t),t,r(t))dt+\int_{\mathbb{R}\setminus\{0\}}\beta_{r(t)}(t)\sqrt{\gamma_2}z(t-)\sigma\tilde{N}(dt,d\sigma),\tag{5.6}$$

where $z(t-)^T=(x_2(t-),-x_1(t-))$, $\beta_1(t)=0$ and $\beta_2(t)=\Sigma_{q=0}^{\infty}I_{[2q\pi,2q\pi+\delta_2]}(t)$. To destabilize it stochastically, we observe that

$$x^Tf(x,t,i)\geq\begin{cases}-2.02202(0.5+\cos(t))|x|^2, & \text{if } i=1,\\ -2.05912(\cos(\frac{t}{2})\sin(\frac{t}{2})+0.5)|x|^2, & \text{if } i=2,\end{cases}$$

where

$$a_1(t)=-2.02202(0.5+\cos(t))\quad\text{and}\quad a_2(t)=-2.05912(\cos(\frac{t}{2})\sin(\frac{t}{2})+0.5).$$

We can apply our Theorem 3.2, and observe that

$$\sum_{i=1}^{2}\frac{\pi_i}{2\pi}\int_0^{2\pi}\Big(a_i(s)+\int_{\mathbb{R}\backslash\{0\}}\log|\beta_i\gamma_i\sigma|\nu(d\sigma)\Big)ds\geq 0. \tag{5.7}$$

Therefore, we can conclude that if we choose $\log|\beta_i\gamma_i|>-\int_{\mathbb{R}\backslash\{0\}}\log|\sigma|\nu(d\sigma))+1.0230675$, then controlled system (5.6) is almost surely exponentially unstable.

For computer simulations, we choose $\delta_2=1.8\pi$, $\gamma_2=0.31$ while set $x_1(0)=x_2(0)=0.01$ and $r(0)=1$. The following computer simulations (FIG. 3) support our theoretical results clearly.

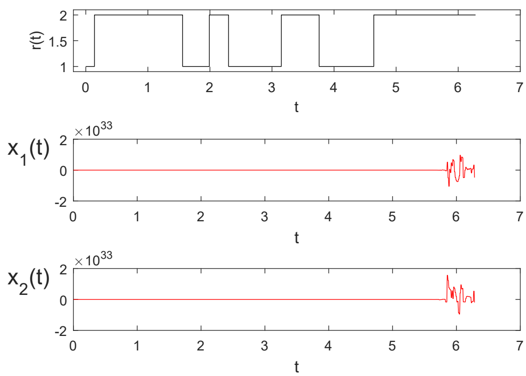

FIG. 3. The computer simulation of the sample paths of the Markov chain and the solution of equation (5.6) for one period of time.

Remark 3. We validated our theoretical results with three examples. Without loss of generality, from Example 4.1, we can further observe that to study the stabilization of the system (4.3) using Theorem 3.1 in Liu and Zhou (2015), we need not only $x^T f(x,t,i)\leq a_i|x|^2$ but also $\sum_{i=1}^{2}\pi_i\big(a_i-\beta_i\int_{\mathbb{R}\backslash\{0\}}\sigma\nu(d\sigma)+\max_{i\in\mathbb{S}}\beta_i^{\,2}\int_{\mathbb{R}\backslash\{0\}}|\sigma|^2\nu(d\sigma)\big)<0$ for the given drift coefficient $f(x,t,i)$. By computing, $f(x,t,i)$ in Example 4.1 cannot satisfy both conditions, so we cannot use Theorem 3.1 of Liu and Zhou (2015) to study the stability of the system. In this paper, we make full use of the time-inhomogeneous property of the function $f(x,t,i)$ to give the new condition (4.2), and then apply Theorem 4.1 to show that the system is stable. Similarly, in the case of 2-dimensional systems, Example 4.2 and Example 5.1 can be used to demonstrate that we have provided new sufficient conditions in this paper. These adequately show the difference between the sufficient conditions given in this paper and those in the existing literature.

6. Conclusion

In this paper, we have introduced a new method and established stochastic stabilization and destabilization for hybrid systems by periodic stochastic feedback control based on Lévy noise. Comparing with the previous results, we pointed out that the existing results on the stochastic stability and instability for hybrid differential equations driven by Lévy noise do not take the time-inhomogeneous property into account, whereas our methods make use of the time-inhomogeneous properties. In fact, we consider the effect of the time-inhomogeneous periodic property and replace the previous time-homogeneous functions. We then successfully established alternative criteria on the stochastic stability and instability for hybrid differential equations driven by Lévy noise. We have also discussed some examples plus some computer simulations to illustrate the advantages of our new results.

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References

Applebaum, D. & Siakalli, M. (2009) Asymptotic stability of stochastic differential equations driven by Lévy noise. *J. Appl. Probab.*, **46**, 1116–1129.

Applebaum, D. & Siakalli, M. (2010) Stochastic stabilization of dynamical systems using Lévy noise. *Stoch. Dyn.*, **10**, 509–527.

Arnold, L., Crauel, H. & Wihstutz, V. (1983) Stabilization of linear systems by noise. *SIAM J. Control Optim.*, **21**, 451–461.

Ballotta, L. & Kyriakou, I. (2014) Monte Carlo simulation of the CGMY process and option pricing. *J. Futures Mark.*, **34**, 1095–1121.

Carr, P. P., Geman, H., Madan, D. & Yor, M. (2002) The fine structure of asset returns: An empirical investigation. *J. Bus.*, **75**, 305–333.

Deng, S. N., Fei, W. Y., Liang, Y. & Mao, X. R. (2019) Convergence of the split-step theta-method for stochastic age-dependent population equations with Markovian switching and variable delay. *Appl. Numer. Math.*, **139**, 15–37.

Fei, W. Y., Hu, L. J., Mao, X. R. & Shen, M. X. (2018) Structured robust stability and boundedness of nonlinear hybrid delay systems. *SIAM J. Control Optim.*, **56**, 2662–2689.

Hu, L. J., Mao, X. R. & Zhang, L. G. (2013) Robust stability and boundedness of nonlinear hybrid stochastic differential delay equations. *IEEE Trans. Autom. Control.*, **58**, 2319–2332.

Khasminskii, R. (1981) *Stochastic Stability of Differential Equations*. *Sijthoff and Noordhoff.*

Kühn, F. & Schilling, R. L. (2019) Strong convergence of the Euler-Maruyama approximation for a class of Lévy-driven SDEs. *Stoch. Process. Their Appl.*, **129**, 2654–2680.

Li, G. J. & Yang, Q. G. (2020) Stabilization of hybrid stochastic systems with Lévy noise by discrete-time feedback control. *Int. J. Control.*, **94**, 1–16.

Li, M. L. & Deng, F. Q. (2017) Almost sure stability with general decay rate of neutral stochastic delayed hybrid systems with Lévy noise. *Nonlinear Anal. Hybrid Syst.*, **24**, 171–185.

Li, X. Y., Liu, W., Mao, X. R. & Zhao, J. S. (2021) Stabilization and destabilization of hybrid systems by periodic stochastic controls. *Syst. Control Lett.*, **152**, 104929.

Liu, D. Z. & Zhou, G. L. (2015) Stochastically partial stabilization of hybrid differential equations by Lévy noise. *Sci. China-Math.*, **45**, 713–722.

Liu, L., Matjaz, P. & Cao, J. D. (2019) Aperiodically intermittent stochastic stabilization via discrete time or delay feedback control. *Sci. China-Math.*, **62**, 183–195.

Liu, X. W. & Chen, T. P. (2015) Synchronization of linearly coupled networks with delays via aperiodically intermittent pinning control. *IEEE Trans. Neural Netw. Learn. Syst.*, **26**, 2396–2407.

Lu, C. & Ding, X. H. (2019) Dynamical behavior of stochastic delay Lotka-Volterra competitive model with general Lévy jumps. *Physica A.*, **531**, 121730.

Mao, X. R. (1994) Stochastic stabilization and destabilization. *Syst. Control Lett.*, **23**, 279–290.

Mao, X. R. (1994) Exponential Stability of Stochastic Differential Equations. *New York: Marcel Dekker.*

Mao, X. R. (2002) Exponential stability of stochastic delay interval systems with Markovian switching. *IEEE Trans. Autom. Control.*, **47**, 1604–1612.

Mao, X. R. (2007) Stochastic Differential Equations and Their Applications. *New York: 2nd edition, Elsevie.*

Mao, X. R. & Yuan, C. G. (2006) Stochastic Differential Equations with Markovian Switching. *London: Imperial College Press.*

Wang, L., Dong, Y. P., Xie, D. & Cao, J. D. (2020) Robust passivity analysis of Markov-type Lotka-Volterra model with time-varying delay and uncertain mode transition rates. *Math Method Appl Sci.*, **43**, 6976–6984.

Wei, C. (2020) Existence, uniqueness, and almost sure exponential stability of solutions to nonlinear stochastic system with Markovian switching and Lévy noises. *Complexity.*, **2020**.

Yang, J., Zhou, W. N., Yang, X. Q., Dai, A. D., Liu, H. S. & Fang, J. A. (2015) pth moment asymptotic stability of stochastic delayed hybrid systems with Lévy noise. *Int. J. Control.*, **88**, 1726–1734.

Yin, G. & Xi, F. B. (2010) Stability of regime-switching jump diffusions. *SIAM J. Control Optim.*, **48**, 4525–4549.

Yuan, C. G. & Mao, X. R. (2010) Stability of stochastic delay hybrid systems with jumps. *Eur. J. Control.*, **16**, 595–608.

Zhang, B., Deng, F. Q., Peng, S. G. & Xie, S. L. (2018) Stabilization and destabilization of nonlinear systems via intermittent stochastic noise with application to memristor-based system. *J. Franklin Inst.*, **355**, 3829–3852.

Zhang, B., Lim, C. C., Shi, P., Xie, S. L. & Deng, F. Q. (2020) Stabilization of a class of nonlinear systems with random disturbance via intermittent stochastic noise. *IEEE Trans. Autom. Control.*, **65**, 1318–1324.

Zhou, H., Li, Y. Y. & Li, W. X. (2020) Synchronization for stochastic hybrid coupled controlled systems with Lévy noise. *Math Method Appl Sci.*, **43**, 9557–9581.

Zhu, Q. X. (2018) Stability analysis of stochastic delay differential equations with Lévy noise. *Syst. Control Lett.*, **118**, 62–68.

Zou, X. L. & Wang, K. (2014) Numerical simulations and modeling for stochastic biological systems with jumps. *Commun. Nonlinear Sci. Numer. Simul.*, **19**, 1557–1568.