Estimates for the Weyl coefficient of a two-dimensional canonical system

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Abstract. For a two-dimensional canonical system $y'(t) = zJH(t)y(t)$ on some interval (a, b) whose Hamiltonian H is a.e. positive semi-definite and which is regular at a and in the limit point case at b, denote by q_H its Weyl coefficient. De Branges' inverse spectral theorem states that the assignment $H \mapsto q_H$ is a bijection between Hamiltonians (suitably normalised) and Nevanlinna functions.

We give upper and lower bounds for $|q_H(z)|$ and $\operatorname{Im} q_H(z)$ when z tends to $i\infty$ non-tangentially. These bounds depend on the Hamiltonian H near the left endpoint a and determine $|q_H(z)|$ up to universal multiplicative constants. We obtain that the growth of $|q_H(z)|$ is independent of the off-diagonal entries of H and depends monotonically on the diagonal entries in a natural way. The imaginary part is, in general, not fully determined by our bounds (in forthcoming work we shall prove that for "most" Hamiltonians also $\operatorname{Im} q_H(z)$ is fully determined).

We translate the asymptotic behaviour of q_H to the behaviour of the spectral measure μ_H of H by means of Abelian–Tauberian results and obtain conditions for membership of growth classes defined by weighted integrability condition (Kac classes) or by boundedness of tails at $\pm\infty$ w.r.t. a weight function. Moreover, we apply our results to Krein strings and Sturm–Liouville equations.

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1 Introduction

We study two-dimensional canonical systems

$$y'(t) = zJH(t)y(t), \qquad t \in [a, b), \tag{1.1}$$

where $-\infty < a < b \leq \infty$, $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $z \in \mathbb{C}$, and where the *Hamiltonian* H is assumed to satisfy

▷ $H \in L^1_{\mathrm{loc}}\big([a, b), \mathbb{R}^{2\times 2}\big)$ and $\{t \in [a, b)\colon\ H(t) = 0\}$ has measure 0;

▷ $H(t) \geq 0$, $t \in [a, b)$ a.e. (in the sense of positive semi-definiteness);

▷ H is in the limit point case at b, i.e.

$$\int_a^b \operatorname{tr} H(t)\, \mathrm{d}t = \infty. \tag{1.2}$$

Differential equations of this form appear frequently in theory and applications. They can be shown to be a unifying framework for classical equations like Schrödinger equations, Krein strings, Dirac systems, and others; see, e.g. [34, 23, 37, 36]. For the relevance (and origins) of canonical systems in natural sciences we refer to [1, 8, 10, 33, 38].

In the spectral theory of equation (1.1) the notion of the *Weyl coefficient* q_H associated with a Hamiltonian H plays a crucial role. The construction of q_H goes back to H. Weyl [40] and is based on a nested discs argument; see (2.2) below for the definition. The Weyl coefficient is a Nevanlinna function, i.e. it is analytic in the open upper and lower half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$, it is symmetric in the sense that $q_H(\overline{z}) = \overline{q_H(z)}$, and $\operatorname{Im} q_H(z) \geq 0$ for $z \in \mathbb{C}^+$. Being a Nevanlinna function, q_H admits the Herglotz integral representation

$$q_H(z) = \alpha_H + \beta_H z + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, \mathrm{d}\mu_H(t), \qquad z \in \mathbb{C} \setminus \mathbb{R}, \tag{1.3}$$

where $\alpha_H \in \mathbb{R}$, $\beta_H \geq 0$ and μ_H is a Borel measure on $\mathbb{R}$ that satisfies $\int_{\mathbb{R}} \frac{\mathrm{d}\mu_H(t)}{1+t^2} < \infty$. The differential operator corresponding to (1.1) is unitarily equivalent to the multiplication operator with the independent variable in the space $L^2(\mu_H)$, and the unitary equivalence is established by a natural integral operator (if $\beta_H > 0$, then the differential operator is multi-valued and a point mass at infinity has to be added). For this reason μ_H is called the *spectral measure* of H. The famous inverse spectral theorem of L. de Branges [5] states that for every Nevanlinna function q there exists an essentially unique Hamiltonian H such that $q = q_H$. Up-to-date references for the spectral theory of canonical systems are [13, 35, 36].

Having the — essentially one-to-one — correspondence between Hamiltonians on the one side and their Weyl coefficients or spectral measures on the other side, it is a natural task to relate properties of the one to properties of the other. Both directions in the de Branges correspondence involve limiting processes. This makes the correspondence difficult to handle, but also intriguing to investigate.

There are indeed several properties which can — fully or partially — be translated. Some of them instantiate the following principle.

> *The behaviour of the distribution function of the spectral measure towards $\pm\infty$ corresponds to the behaviour of the Hamiltonian locally at the left endpoint a.*

Usually it is not difficult to find Abelian–Tauberian theorems which allow to translate the behaviour of tails of μ_H at $\pm\infty$ to the behaviour of its Herglotz integral q_H at $i\infty$ (also called the high-energy behaviour of q_H). Thus, when seeking theorems which instantiate the quoted principle, the essence is to translate properties of q_H locally at $i\infty$ to properties of H locally at a. In the present paper we contribute to this family of results. Namely, we prove upper and lower estimates for the modulus and the imaginary part of q_H locally at $i\infty$ in terms of integrals of the Hamiltonian in a neighbourhood of the left endpoint a.

For Sturm–Liouville equations such estimates go back to, at least, [15], [2] and [3]; for Krein strings estimates for the principal Titchmarsh–Weyl coefficient were proved in [24] and [25]; for Jacobi operators in [16]; for canonical systems some results were obtained in [44]. Estimates of the distribution function of the spectral measure are studied in, e.g. [30] for Sturm–Liouville equations and [20] for strings. A detailed discussion of related work is given in Section 6.

The following theorem is our main result. We establish an explicit quantitative relation between q_H and H.

1.1 Theorem. *Let H be a Hamiltonian defined on some interval $[a,b)$ as at the beginning of the introduction, and write*

$$H(t) = \begin{pmatrix} h_1(t) & h_3(t) \\ h_3(t) & h_2(t) \end{pmatrix}, \quad M(t) = \begin{pmatrix} m_1(t) & m_3(t) \\ m_3(t) & m_2(t) \end{pmatrix} := \int_a^t H(s)\, \mathrm{d}s. \tag{1.4}$$

Assume that neither $h_1 = 0$ a.e. nor $h_2 = 0$ a.e. Fix a parameter $\eta \in \big(0, 1 - \frac{1}{\sqrt{2}}\big)$ and set $\sigma := \frac{1}{(1-\eta)^2} - 1 \in (0,1)$. For $r > 0$, let $\mathring{t}(r) \in (a,b)$ be the unique number that satisfies

$$(m_1 m_2)\big(\mathring{t}(r)\big) = \frac{\eta^2}{4r^2}\,. \tag{1.5}$$

Further, set

$$A(r) := \sqrt{\frac{m_1(\mathring{t}(r))}{m_2(\mathring{t}(r))}}\,, \qquad L(r) := A(r) \cdot \frac{\det M(\mathring{t}(r))}{(m_1 m_2)(\mathring{t}(r))}\,.$$

Then the Weyl coefficient q_H associated with the Hamiltonian H satisfies, for each $\vartheta \in (0,\pi)$ and $r>0$,

$$\left(\frac{1+\sigma+\frac{2}{\eta\sin\vartheta}}{1-\sigma}\right)^{-1}\cdot A(r) \le \big|q_H\big(re^{i\vartheta}\big)\big| \le \frac{1+\sigma+\frac{2}{\eta\sin\vartheta}}{1-\sigma}\cdot A(r), \tag{1.6}$$

$$\big|\operatorname{Re} q_H\big(re^{i\vartheta}\big)\big| \le \frac{1+\sigma+\frac{1}{\eta\sin\vartheta}}{1-\sigma}\cdot A(r), \tag{1.7}$$

$$\frac{\frac{\eta\sin\vartheta}{2}}{1+|\cos\vartheta|}\cdot\frac{1-\sigma}{1+\sigma}\cdot L(r) \le \operatorname{Im} q_H\big(re^{i\vartheta}\big) \le \frac{\sigma+\frac{2}{\eta\sin\vartheta}}{1-\sigma}\cdot A(r). \tag{1.8}$$

Let us add some comments.

1.2 Remark.

(i) Let us note that the inequalities in (1.6) can also be derived from [12, Theorem 3.2], which is based on a completely different proof. However, the most involved part of our proof is the proof of the first inequality in (1.8), which cannot be deduced from [12].

(ii) By the assumption that neither h_1 nor h_2 vanishes a.e., the equation (1.5) indeed has a unique solution for every $r>0$. To see this, set

$$\mathring{a} := \sup\big\{t\in[a,b):\ m_1(t)m_2(t)=0\big\}, \tag{1.9}$$

which is equal to the right endpoint of a maximal interval of the form (a,c) where $h_1=0$ a.e. or $h_2=0$ a.e. if such an interval exists and equal to a otherwise. Then $(m_1m_2)' = h_1m_2+h_2m_1>0$ a.e. on $(\mathring{a},b)$, and

$$\lim_{t\to b}\big(m_1(t)+m_2(t)\big) = \int_a^b \operatorname{tr} H(s)\,\mathrm{d}s = \infty.$$

This implies that m_1m_2 is a strictly increasing bijection from $[\mathring{a},b)$ onto $[0,\infty)$. It follows then from (1.5) that $\mathring{t}$ is a strictly decreasing bijection from $(0,\infty)$ onto $(\mathring{a},b)$. Since $\lim_{r\to\infty}\mathring{t}(r)=\mathring{a}$, the inequalities in (1.6)–(1.8) relate the behaviour of $q_H(re^{i\vartheta})$ as $r\to\infty$ to the behaviour of $H(t)$ when $t\searrow\mathring{a}$. This shows that the theorem is a perfect instance of the above quoted principle at the level of the Weyl coefficient, in particular, in the case when $\mathring{a}=a$. Note that $\mathring{a}>a$ if and only if one of h_1 and h_2 vanishes locally at a; this case is discussed in more detail in §5.2.

(iii) The constants in (1.6)–(1.8) are symmetric about $\frac{\pi}{2}$ and depend monotonically on ϑ on $(0,\frac{\pi}{2}]$ and $[\frac{\pi}{2},\pi)$ respectively. Hence the estimates are valid in a sector around the positive imaginary axis.

(iv) Alternative forms of $A(r)$ and $L(r)$ are given in Section 2.4. Examples to show how Theorem 1.1 can be applied are discussed in Example 5.8 and Section 5.4.

(v) Solving equation (1.5) may not be possible explicitly or computationally difficult. Using monotonicity properties we can show weaker estimates for q_H already from an estimate of the solution. Details are given in Section 5.1.

(vi) Some properties of $A(r)$ and $L(r)$ and hence of $q_H(re^{i\vartheta})$ can be seen directly from properties of the functions m_j without finding $\mathring{t}(r)$, e.g. $\lim_{r\to\infty}A(r)=0$ if and only if $\lim_{t\to\mathring{a}}\frac{m_1(t)}{m_2(t)}=0$; similar characterisations hold for $\liminf_{r\to\infty}A(r)=0$ or A being bounded.

(vii) Some results about the spectral measure μ_H (see (1.3)) are deduced in Theorems 4.10 and 4.17. There we deal with Kac classes, i.e. weighted integrability of μ_H, and boundedness of tails of μ_H relative to a weight function.

(viii) Since the estimates in (1.6)–(1.8) are valid for all $r\in(0,\infty)$, one can also obtain information about the asymptotic behaviour of $q_H(re^{i\vartheta})$ as $r\to 0$. For this one needs information about $m_i(t)$ as $t\to b$. This relates to the considerations in [44, §5].

(ix) The two cases that are excluded in the theorem are trivial: if $h_1 = 0$ a.e., then $q_H \equiv 0$; if $h_2 = 0$ a.e., then $q_H \equiv \infty$.

$\Diamond$

Using Theorem 1.1 we can prove an independence result and a comparison result for the absolute value of q_H. The first one is an immediate corollary, while the second one requires some arguments; the proof is given in Section 3.3.

1.3 Corollary. *Let H be a Hamiltonian defined on some interval $[a, b)$. Then the absolute value $|q_H(ir)|$ is, up to universal multiplicative constants, independent of the off-diagonal entries of H.*

1.4 Proposition. *Let H and $\widetilde{H}$ be two Hamiltonians defined on some interval $[a, b)$ such that none of the respective diagonal entries $h_1, h_2, \tilde{h}_1, \tilde{h}_2$ vanishes a.e., let $\mathring{a}$ and $\mathring{\tilde{a}}$ be as in (1.9) corresponding to H and $\widetilde{H}$ respectively, and define $\widetilde{m}_j$ and $\widetilde{M}$ analogously to (1.4).*

Assume that there exist constants $c_1, c_2, \gamma_1, \gamma_2 > 0$ and a point $a' \in (\max\{\mathring{a}, \mathring{\tilde{a}}\}, b]$ such that, for all $t \in (\max\{\mathring{a}, \mathring{\tilde{a}}\}, a')$,

$$\frac{1}{c_1} \operatorname{tr} M(t) \leq \operatorname{tr} \widetilde{M}(t) \leq c_2 \operatorname{tr} M(t), \tag{1.10}$$

$$m_1(t) \leq \gamma_1 \widetilde{m}_1(t), \qquad \widetilde{m}_2(t) \leq \gamma_2 m_2(t). \tag{1.11}$$

Then there exist $C > 0$ and $r_0 \geq 0$, such that for all $r > r_0$,

$$|q_H(ir)| \leq C |q_{\widetilde{H}}(ir)|.$$

The constant C depends on $c_1, c_2, \gamma_1, \gamma_2$, but not on $a', H, \widetilde{H}$. Moreover, $r_0 = 0$ when $a' = b$.

Let us point out that some assumption on the absolute sizes of M and $\widetilde{M}$ has to be made in order to have a chance for any kind of comparison result because otherwise, one could rescale the independent variable without changing the Weyl coefficient. We use (1.10) since this is sufficiently flexible in applications; note that it is clearly satisfied if both H and $\widetilde{H}$ are trace-normed. Also note that none of the conditions in (1.11) can be removed as simple examples show.

As a corollary of Proposition 1.4 we obtain a stability result.

1.5 Corollary. *Let H and $\widetilde{H}$ be Hamiltonians defined on some interval $[a, b)$ such that none of the respective diagonal entries $h_1, h_2, \tilde{h}_1, \tilde{h}_2$ vanishes a.e. Further, let $a' \in (\max\{\mathring{a}, \mathring{\tilde{a}}\}, b]$ and assume that $m_1 \asymp \widetilde{m}_1$ and $m_2 \asymp \widetilde{m}_2$ on $(\max\{\mathring{a}, \mathring{\tilde{a}}\}, a')$. Then there exists $r_0 \geq 0$, with $r_0 = 0$ when $a' = b$, such that $|q_H(ir)| \asymp |q_{\widetilde{H}}(ir)|$ on (r_0, ∞).*

Notation. In Corollary 1.5 and for the rest of the paper we use the following notation. We write $f \lesssim g$ if there exists $c > 0$ such that $f(r) \leq c g(r)$ for all r, and we write $f \asymp g$ if $f \lesssim g \wedge g \lesssim f$. Moreover, we use the notation $f \ll g$ for $\frac{f}{g} \to 0$. We deliberately do not always specify the range of values of r; one can think of r belonging to a certain portion of the ray $(0, \infty)$, or to some sequence tending to ∞, or similar.

About the method of proof

The proof of Theorem 1.1 is based on the geometric idea to directly estimate Weyl discs. It follows the approach of H. Winkler in [44].

Recall Weyl's nested discs construction: from the fundamental solution of the system (1.1) a family of discs $\Omega_{t,z}$ is built; see §2.1 for details. Here the parameter t ranges over $[a, b)$, and the spectral parameter z lies in the open upper half-plane. For each fixed $z \in \mathbb{C}^+$, the discs $\Omega_{t,z}$ form a nested family, and, due to our assumption that $\int_a^b \operatorname{tr} H(s)\, \mathrm{d}s = \infty$, they shrink to a single point when t approaches b. The function q_H is then defined by $\bigcap_{t \in (a,b)} \Omega_{t,z} = \{q_H(z)\}$.

The radius of the Weyl disc $\Omega_{t,ir}$ built at a point in t and a point ir in the plane decays to 0 not only for each fixed r when t increases to b, but also for each fixed t when r increases to

∞. Hence, when looking at large r one can afford to use t close to a and still have a relatively small Weyl disc[1].

To produce estimates for q_H, we start from the fact that centres and radii of Weyl discs $\Omega_{t,ir}$ can be expressed explicitly in terms of the fundamental solution. Then we

- ▷ prove estimates for the power series coefficients of the fundamental solution;
- ▷ use these estimates to determine the leading terms of centres and radii of Weyl discs;
- ▷ deduce estimates for q_H.

The first step is the technical core of the proof. It can be carried out only when (t, r) lies in the region indicated in the following picture, and this explains the role of equation (1.5): it describes the border of our method to prove coefficient estimates.

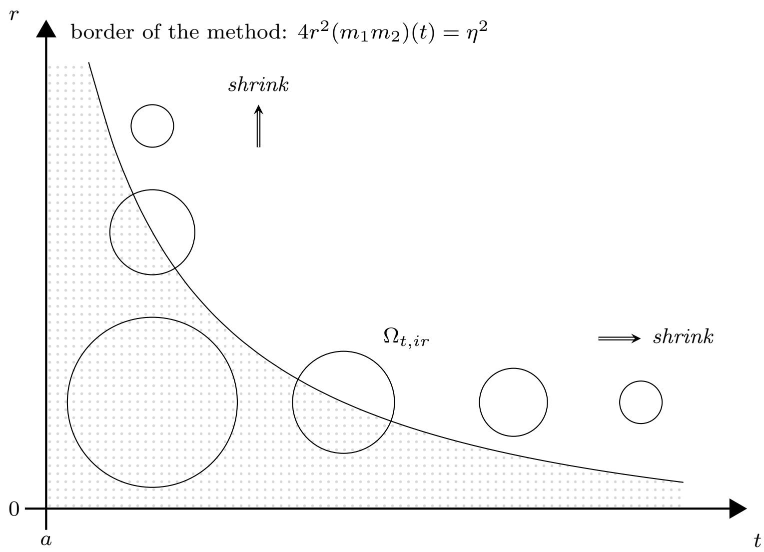

Figure 1: admissible region for coefficient estimates

Since $A(r)$ always correctly describes the absolute value of $q_H(ir)$, and — as we shall see later — the lower bound $L(r)$ is in many situations correct, it seems that this border occurs not "just because of the method", but for intrinsic reasons.

Further perspective

The present work is part of a series of papers where we investigate the high-energy behaviour of the Weyl coefficient. The other parts which are already — or will very soon be — available, are [32, 26, 28]. In [32] limit points of $q_H(ir)$ for $r \to \infty$ are investigated. The contents of the other two papers is discussed below.

Let us first make three observations about Theorem 1.1.

① The absolute value of q_H is fully determined by (1.6) up to some universal constants, while (1.8) gives only estimates for the imaginary part of the Weyl coefficient:

$$L(r) \lesssim \operatorname{Im} q_H(ir) \leq |q_H(ir)| \asymp A(r).$$

② The lower bound $L(r)$ becomes smaller when the relative size of the off-diagonal entries of H becomes larger.

[1] This can be seen as a mathematical reason behind the principle that high-energy behaviour of q_H corresponds to behaviour of H locally at the left endpoint, which we quoted at the very beginning. Another, maybe even more striking, reason is established by a certain group action on the set of Hamiltonians, namely, by the group of rescaling operators. This "rescaling trick" goes probably back to Y. Kasahara in [24].

③ The bounds given in Theorem 1.1 involve universal constants which depend on the parameter η. Consider, for example, the constant on the right-hand side of (1.6). Its value can be minimised by making an appropriate choice of η, but this minimum is larger than 1.

Each of these observations gives rise to a natural question.

Question concerning observation ①.
If $L(r) \asymp A(r)$, then clearly $\operatorname{Im} q_H(ir) \asymp |q_H(ir)|$. What happens if $L(r) \ll A(r)$?

Investigating the relation between the gaps $\frac{L(r)}{A(r)}$ and $\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|}$ more closely, requires very different methods from the presently developed ones, namely, a refined variant of Kasahara's rescaling trick. We settle the above question on a qualitative level in [26], where we prove that the lower bound $L(r)$ is sharp in the sense that $L(r) \ll A(r)$ if and only if $\operatorname{Im} q_H(ir) \ll |q_H(ir)|$. Moreover, we characterise this situation explicitly in terms of H and see that for "most" Hamiltonians $L(r) \asymp A(r)$ holds. This confirms the intuition familiar from complex analysis that "usually" $\operatorname{Im} q_H(ir) \asymp |q_H(ir)|$.

Question concerning observation ②.
Does $\operatorname{Im} q_H(ir)$ show the same behaviour as $L(r)$, namely that it becomes smaller when the relative size of the off-diagonal entries of H becomes larger?

There are indications that this is the case; see, for example, the main theorem in the forthcoming [28]. However, we have no general result confirming this, and it seems that the question is very much related to what is stated as an open problem below.

Question concerning observation ③.
When does q_H have an asymptotic expansion at $i\infty$?

With the present Theorem 1.1 we cannot possibly exclude that $|q_H(ir)|$ oscillates between the bounds given by (1.6); cf. Example 5.11. Dealing with asymptotic expansions requires different methods. We answer this question in [28], where we characterise the presence of a regularly varying asymptotic explicitly in terms of H. The idea will again be to apply a variant of Kasahara's trick already mentioned above.

An open problem. Is it possible to quantitatively compare the gaps $\frac{L(r)}{A(r)}$ and $\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|}$? In particular, we do not know whether there exist a Hamiltonian with $L(r) \asymp \operatorname{Im} q_H(ir) \ll |q_H(ir)|$. It seems that none of the presently available methods is suitable to attack this problem.

Let us give an overview of the contents of the paper. In Section 2 we present some preliminary material that is used in the proof of the main theorem, in particular a discussion about Weyl discs, a power series expansion of the fundamental solution of (1.1) and some matrix algebra that is useful for estimates of the coefficients of the power series. Section 3 contains the proofs of Theorem 1.1 and Proposition 1.4. In Section 4 our estimates for the Weyl coefficient are related to estimates of the spectral measure. In particular, we prove characterisations for μ_H belonging to some Kac classes and for the distribution function of μ_H satisfying certain growth estimates. Section 5 contains further results, which complement Theorem 1.1. In particular, we prove a monotonicity property; we discuss the situation when h_1 or h_2 vanishes on neighbourhood of a; we study a rotation transformation that improves the bounds in (1.6)–(1.8) in certain situations; we discuss some examples; and we express $A(r)$ in terms of the mass function of a related Krein string. Finally, in Section 6 we apply our theorems to Krein strings and Sturm–Liouville equations, and relate our results to previous work in the literature. Two appendices collect some facts about regularly varying functions and generalised inverses.

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2 Weyl discs and matrix algebra

In this section we collect some preliminary information.

2.1 Centre and radius of a Weyl disc

We recall the definition of the Weyl discs $\Omega_{t,z}$ and a basic formula for their centres and radii. Let $W(t,z)$ be the (transpose of) the fundamental solution of the system (1.1), i.e. the unique 2×2-matrix-valued solution of the initial value problem

$$\begin{cases} \dfrac{\partial}{\partial t} W(t,z)J = zW(t,z)H(t), \qquad & t\in[a,b), \\ W(a,z) = I. \end{cases}$$

Equivalently, $W(t,z)$ is the solution of the integral equation

$$W(t,z)J - J = z\int_a^t W(s,z)H(s)\,\mathrm{d}s, \qquad t\in[a,b). \tag{2.1}$$

Note that the transposes of the rows of W are solutions of (1.1). Writing $W(t,z) = \left(\begin{smallmatrix} w_{11}(t,z) & w_{12}(t,z) \\ w_{21}(t,z) & w_{22}(t,z)\end{smallmatrix}\right)$, the Weyl disc $\Omega_{t,z}$ is defined as the image of the closed upper half-plane under the fractional linear transformation

$$\zeta \mapsto \frac{w_{11}(t,z)\zeta + w_{12}(t,z)}{w_{21}(t,z)\zeta + w_{22}(t,z)}\,.$$

The Weyl discs are nested: $\Omega_{t_1,z} \supseteq \Omega_{t_2,z}$ when $a\le t_1\le t_2 < b$. Since we assume that H is in the limit point case at b, i.e. (1.2) holds, the Weyl discs $\Omega_{t,z}$ shrink to a point as $t\to b$ for $z\in\mathbb{C}\setminus\mathbb{R}$. This point is denoted by $q_H(z)$, i.e.

$$q_H(z) := \lim_{t\to b} \frac{w_{11}(t,z)\zeta + w_{12}(t,z)}{w_{21}(t,z)\zeta + w_{22}(t,z)}\,, \qquad z\in\mathbb{C}\setminus\mathbb{R}, \tag{2.2}$$

independent of ζ, and q_H is called the *Weyl coefficient* associated with the Hamiltonian H. The centre and the radius of $\Omega_{t,z}$ can be expressed in a neat way using the function

$$\nabla(t,z) := \int_a^t W(s,z)H(s)W(s,z)^*\,\mathrm{d}s = \frac{W(t,z)JW(t,z)^* - J}{2i(\operatorname{Im} z)}\,; \tag{2.3}$$

the second equality in (2.3) follows from, e.g. [13, (2.5)].

2.1 Lemma. *Let $z\in\mathbb{C}^+$ and let $t\in(a,b)$ be such that $h_2|_{(a,t)}\neq 0$. Then the Weyl disc $\Omega_{t,z}$ is the closed disc with*

$$\textit{centre: } \frac{b(t,z)}{a(t,z)} + i\frac{1}{2(\operatorname{Im} z)a(t,z)} \qquad \Big| \qquad \textit{radius: } \frac{1}{2(\operatorname{Im} z)a(t,z)} \tag{2.4}$$

where $a(t,z)$ and $b(t,z)$ respectively denote the $(2,2)$ and the $(1,2)$ entries of $\nabla(t,z)$.

This is folklore and can be found, for instance, implicitly in [44, §3]. For the convenience of the reader, we recall the argument.

Proof of Lemma 2.1. Let $\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{smallmatrix}\right)\in\mathbb{C}^{2\times 2}$ be an invertible matrix such that $\operatorname{Im}(a_{22}\overline{a_{21}})>0$, and consider the fractional linear transformation $w\mapsto \frac{a_{11}w+a_{12}}{a_{21}w+a_{22}}$. The image of the closed upper half-plane under this map is the disc with

$$\text{centre: } \frac{a_{12}\overline{a_{21}} - a_{11}\overline{a_{22}}}{a_{22}\overline{a_{21}} - a_{21}\overline{a_{22}}} \qquad \Big| \qquad \text{radius: } \left|\frac{a_{11}a_{22} - a_{12}a_{21}}{a_{22}\overline{a_{21}} - a_{21}\overline{a_{22}}}\right|. \tag{2.5}$$

Note that the condition $\operatorname{Im}(a_{22}\overline{a_{21}})>0$ is equivalent to the fact that the image of the closed upper half-plane is a disc.

For each $t \in (a,b)$ and $z \in \mathbb{C}^+$, the Weyl disc $\Omega_{t,z}$ is the image of the closed upper half-plane (considered on the Riemann sphere) under the fractional linear transformation $\zeta \mapsto \frac{w_{11}(t,z)\zeta + w_{12}(t,z)}{w_{21}(t,z)\zeta + w_{22}(t,z)}$. From (2.3) we obtain

$$w_{22}(t,z)\overline{w_{21}(t,z)} - w_{21}(t,z)\overline{w_{22}(t,z)} = \begin{pmatrix}0\\1\end{pmatrix}^* \Big[W(t,z)JW(t,z)^* - J\Big]\begin{pmatrix}0\\1\end{pmatrix}$$
$$= 2i(\operatorname{Im} z)\begin{pmatrix}0\\1\end{pmatrix}^* \nabla(t,z)\begin{pmatrix}0\\1\end{pmatrix} = 2i(\operatorname{Im} z)a(t,z) \tag{2.6}$$

and

$$w_{12}(t,z)\overline{w_{21}(t,z)} - w_{11}(t,z)\overline{w_{22}(t,z)} + 1 = \begin{pmatrix}1\\0\end{pmatrix}^* \Big[W(t,z)JW(t,z)^* - J\Big]\begin{pmatrix}0\\1\end{pmatrix}$$
$$= 2i(\operatorname{Im} z)\begin{pmatrix}1\\0\end{pmatrix}^* \nabla(t,z)\begin{pmatrix}0\\1\end{pmatrix} = 2i(\operatorname{Im} z)b(t,z). \tag{2.7}$$

With $v(t,z) = \binom{w_{21}(t,z)}{w_{22}(t,z)}$, which satisfies (1.1), we have

$$a(t,z) = \int_a^t v(s,z)^* H(s) v(s,z)\,\mathrm{d}s \ge 0.$$

Assume that $a(t,z) = 0$. Then $H(s)v(s,z) = 0$ a.e. and hence v is constant. by (1.1). The initial condition $W(a,z) = I$ yields $v(t,z) \equiv \binom{0}{1}$. This, in turn, implies that $h_2(t) = 0$ a.e. on (a,t), which is a contradiction to the assumption. Therefore $\operatorname{Im}(w_{22}(t,z)\overline{w_{21}(t,z)}) = (\operatorname{Im} z)a(t,z) > 0$ for $z \in \mathbb{C}^+$. Hence we can apply (2.5), which, together with (2.6) and (2.7), yields that the centre of the disc is

$$\frac{w_{12}(t,z)\overline{w_{21}(t,z)} - w_{11}(t,z)\overline{w_{22}(t,z)}}{w_{22}(t,z)\overline{w_{21}(t,z)} - w_{21}(t,z)\overline{w_{22}(t,z)}} = \frac{2i(\operatorname{Im} z)b(t,z) - 1}{2i(\operatorname{Im} z)a(t,z)} = \frac{b(t,z)}{a(t,z)} + i\frac{1}{2(\operatorname{Im} z)a(t,z)}\,,$$

and its radius is

$$\frac{1}{\big|w_{22}(t,z)\overline{w_{21}(t,z)} - w_{21}(t,z)\overline{w_{22}(t,z)}\big|} = \frac{1}{2(\operatorname{Im} z)a(t,z)}\,;$$

this proves (2.4). ❑

2.2 The power series expansion of $\nabla(t,z)$

For $t \in [a,b)$ let $W_n(t)$ be the coefficients in the power series expansion of $W(t,z)$, i.e.

$$W(t,z) = \sum_{n=0}^{\infty} W_n(t) z^n. \tag{2.8}$$

This series converges uniformly on every compact subset of $[a,b) \times \mathbb{C}$. The integral equation (2.1) for $W(t,z)$ shows that the coefficient sequence $(W_n(t))_{n=0}^{\infty}$ is given by the recurrence relation

$$\begin{cases} W_0(t) = I, \\ W_{n+1}(t) = \displaystyle\int_a^t W_n(s)H(s)\,\mathrm{d}s \cdot (-J), \quad n \ge 0. \end{cases} \tag{2.9}$$

The latter implies that

$$\begin{aligned} &W_n(a) = 0, && n \ge 1; \\ &W_n(t) \in \mathbb{R}^{2\times 2}, && n \ge 0,\ t \in [a,b). \end{aligned} \tag{2.10}$$

Plugging the power series expansion of $W(t,z)$ into the definition of $\nabla(t,z)$ in (2.3) we obtain

$$\nabla(t,z) = \sum_{n,m=0}^{\infty} \bigg(\int_a^t W_n(s)H(s)W_m(s)^*\,\mathrm{d}s\bigg) z^n \overline{z}^m.$$

Setting

$$\alpha_{n,m}(t) := \int_a^t W_n(s)H(s)W_m(s)^*\,\mathrm{d}s$$

we have, for $z = re^{i\vartheta}$,

$$\nabla\big(t, re^{i\vartheta}\big) = \sum_{l=0}^{\infty}\bigg(\sum_{n=0}^{l}\alpha_{n,l-n}(t)e^{i\vartheta(2n-l)}\bigg)r^l. \tag{2.11}$$

Note that the coefficients $\alpha_{n,m}$ satisfy the symmetry property $\alpha_{m,n} = \alpha_{n,m}^*$.

Let us list a couple of properties of the function M defined in (1.4).

▷ The recurrence relation (2.9) implies that $W_1 J = M$, which, together with the symmetry of M shows that

$$-JW_1^* = W_1 J. \tag{2.12}$$

▷ The symmetry of M also implies that $MJM = (\det M)J$, and consequently,

$$W_1W_1 = -(\det M)I, \qquad W_1JW_1^* = (\det M)J. \tag{2.13}$$

▷ Since M is the primitive of a pointwise positive semi-definite matrix function, M is itself positive semi-definite and $M(t) \le M(s)$ whenever $t \le s$.

2.3 Some matrix algebra

We frequently use certain algebraic manipulations with matrices. In order to present arguments in a clean way, it is practical to introduce the following operations and relations for matrices.

2.2 Definition. For $U = (u_{ij})_{i,j=1}^2 \in \mathbb{C}^{2\times 2}$ set

$$|U| := \big(|u_{ij}|\big)_{i,j=1}^2.$$

For $U = (u_{ij})_{i,j=1}^2, \tilde{U} = (\tilde{u}_{ij})_{i,j=1}^2 \in \mathbb{R}^{2\times 2}$ define

$$U \preceq \tilde{U} \quad :\Leftrightarrow \quad u_{ij} \le \tilde{u}_{ij}, \quad i,j \in \{1,2\}.$$

$\Diamond$

We will use without further notice a couple of simple properties and rules for these operations and relations. A list of these rules can be found in [27, Lemma 2.3].

2.4 Alternative forms for the bounds $A(r)$ and $L(r)$

In this short subsection we consider some reformulations of $\mathring{t}$, $A(r)$ and $L(r)$, which are used in the proofs of Theorem 1.1 and other results in later sections. Let us set

$$\mathring{r}(t) := \frac{\eta}{2}\big(m_1(t)m_2(t)\big)^{-\frac{1}{2}}, \qquad t \in (\mathring{a}, b), \tag{2.14}$$

which is the solution for r of the equation (1.5). It follows that $\mathring{t}$ and $\mathring{r}$ are inverses of each other. In particular, $r = \frac{\eta}{2}(m_1m_2)^{-\frac{1}{2}}(\mathring{t}(r))$, which implies the representations

$$A(r) = \frac{2r}{\eta}\cdot m_1(\mathring{t}(r)) = \frac{\eta}{2r}\cdot\frac{1}{m_2(\mathring{t}(r))}, \tag{2.15}$$

$$L(r) = \frac{2}{\eta}\cdot\frac{r\det M(\mathring{t}(r))}{m_2(\mathring{t}(r))} = \frac{4}{\eta^2}\cdot r^2\det M(\mathring{t}(r))\cdot A(r). \tag{2.16}$$

Let us also note that the function $\frac{\det M}{m_2}$ is non-decreasing since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\frac{\det M(t)}{m_2(t)}\bigg) &= \frac{1}{m_2(t)^2}\Big[m_2(t)^2h_1(t) + m_3(t)^2h_2(t) - 2m_2(t)m_3(t)h_3(t)\Big] \\ &= \frac{1}{m_2(t)^2}\begin{pmatrix} m_2(t) \\ -m_3(t)\end{pmatrix}^*\begin{pmatrix} h_1(t) & h_3(t) \\ h_3(t) & h_2(t)\end{pmatrix}\begin{pmatrix} m_2(t) \\ -m_3(t)\end{pmatrix} \ge 0. \end{aligned} \tag{2.17}$$

This is also used in later sections.

3 Proof of Weyl coefficient estimates

In this section we prove Theorem 1.1 and Proposition 1.4. This is done by establishing bounds for the coefficients $\alpha_{n,m}(t)$ of the expansion (2.11) and some related quantities in terms of

$$M^+(t) := \begin{pmatrix} m_1(t) & \sqrt{m_1(t)m_2(t)} \\ \sqrt{m_1(t)m_2(t)} & m_2(t) \end{pmatrix},$$

and

$$m^+(t) := 2 \cdot \sqrt{m_1(t)m_2(t)}.$$

Observe the following properties of $M^+(t)$ and $m^+(t)$.

3.1 Lemma. *With the notation introduced in Definition* 2.2 *we have*

(i) $|M(t)| \preceq \int_a^t |H(s)|\,\mathrm{d}s \preceq M^+(t)$;

(ii) $\big(M^+(t)|J|\big)^n M^+(t) = m^+(t)^n M^+(t), \quad n \geq 0$;

(iii) $\forall s \leq t.\ \ M^+(s) \preceq M^+(t) \ \wedge\ m^+(s) \leq m^+(t)$.

Proof. The first inequality in (i) is clear. Consider the second inequality. The diagonal entries of the left-hand and right-hand sides actually coincide. Since $H(t) \geq 0$, also $|H(t)| \geq 0$, and hence $\int_a^t |H(s)|\,\mathrm{d}s \geq 0$. Thus $m_1(t)m_2(t) - \big(\int_a^t |h_3(s)|\,\mathrm{d}s\big)^2 \geq 0$.

Item (ii) follows from a general observation. Consider a matrix $U = (u_{ij})_{i,j=1}^2 \in \mathbb{R}^{2\times 2}$ that is of the form $U = \alpha\xi_\phi\xi_\phi^*$ with some $\alpha, \phi \in \mathbb{R}$ where $\xi_\phi := (\cos\phi, \sin\phi)^*$. Then

$$\begin{aligned}(U|J|)U &= \alpha\xi_\phi\xi_\phi^*|J| \cdot \alpha\xi_\phi\xi_\phi^* = \alpha\xi_\phi \cdot \alpha\xi_\phi^*|J|\xi_\phi \cdot \xi_\phi^* \\ &= \alpha\xi_\phi \cdot \alpha 2\cos\phi\sin\phi \cdot \xi_\phi^* = 2u_{12} \cdot \alpha\xi_\phi\xi_\phi^* = 2u_{12}U.\end{aligned}$$

Thus, inductively,

$$(U|J|)^n U = (2u_{12})^n U, \quad n \geq 0.$$

The matrix $M^+(t)$ is symmetric with determinant zero, and hence of the considered form.

Item (iii) is clear since m_1 and m_2 are non-decreasing. ❑

3.1 Bounds involving $A(r)$

In order to obtain the bounds (1.6), (1.7), and the estimate from above in (1.8), it is enough to have the following crude estimate.

3.2 Lemma. *We have*

$$|W_n(t)| \preceq m^+(t)^{n-1} M^+(t)|J|, \quad n \geq 1, \tag{3.1}$$

and

$$|\alpha_{n,m}(t)| \preceq m^+(t)^{n+m} M^+(t), \quad n, m \geq 0.$$

This statement has been extracted from the personal communication [43] by H. Winkler[2]. Since this communication remained unpublished, we provide a complete proof.

Proof. To see (3.1), we use induction on n. First, observe that

$$|W_1(t)| = \Big|\int_a^t H(s)\,\mathrm{d}s \cdot (-J)\Big| \preceq \int_a^t |H(s)|\,\mathrm{d}s \cdot |J| \preceq M^+(t)|J|$$

[2]The proofs of the main results of [44] use the estimate [44, Lemma 3.2] of power series coefficients whose proof contains a mistake (and whose assertion is most probably wrong). However, the theorems stated in the paper are still correct. They can be proved using a weaker form of [44, Lemma 3.2], which does hold. This is shown in [43], and Lemma 3.2 is nothing but a clean formulation of the relevant argument.

by Lemma 3.1(i). Second, with Lemma 3.1(ii) we obtain

$$\begin{aligned}|W_{n+1}(t)| &= \Big|\int_a^t W_n(s)H(s)\,\mathrm{d}s\cdot(-J)\Big| \preceq \int_a^t |W_n(s)||H(s)|\,\mathrm{d}s\cdot|J| \\ &\preceq \int_a^t m^+(s)^{n-1}M^+(s)|J||H(s)|\,\mathrm{d}s\cdot|J| \preceq m^+(t)^{n-1}M^+(t)|J|\int_a^t |H(s)|\,\mathrm{d}s\cdot|J| \\ &\preceq m^+(t)^{n-1}\cdot M^+(t)|J|M^+(t)\cdot|J| = m^+(t)^n M^+(t)|J|,\end{aligned}$$

which proves (3.1).

Now we use (3.1) to estimate $\alpha_{n,m}(t)$. For $n,m\geq 1$ we have

$$\begin{aligned}|\alpha_{n,m}(t)| &\preceq \int_a^t |W_n(s)|\cdot|H(s)|\cdot|W_m(s)|^*\,\mathrm{d}s \\ &\preceq \int_a^t m^+(s)^{n+m-2}M^+(s)|J|\cdot|H(s)|\cdot|J|M^+(s)\,\mathrm{d}s \\ &\preceq m^+(t)^{n+m-2}M^+(t)|J|\cdot\int_a^t |H(s)|\,\mathrm{d}s\cdot|J|M^+(t) \\ &\preceq m^+(t)^{n+m-2}M^+(t)|J|\cdot M^+(t)\cdot|J|M^+(t) \preceq m^+(t)^{n+m}M^+(t).\end{aligned}$$

For the case when n or m is equal to zero, the asserted estimate is obtained in an analogous way. The necessary computation can be found in [27]. ❑

Proof of Theorem 1.1 (*upper bounds*). Let $H,\eta,\sigma,\mathring{t}(r)$ and $\vartheta\in(0,\pi)$ be as in the formulation of the theorem. Note that (1.5) is equivalent to $rm^+(\mathring{t}(r))=\eta$. Using (2.11) and Lemma 3.2 we obtain

$$\begin{aligned}&\big|\nabla\big(\mathring{t}(r),re^{i\vartheta}\big)-M(\mathring{t}(r))\big| = \Big|\sum_{l=1}^{\infty}\Big(\sum_{n=0}^{l}\alpha_{n,l-n}(\mathring{t}(r))e^{i\vartheta(2n-l)}\Big)r^l\Big| \preceq \sum_{l=1}^{\infty}\Big(\sum_{n=0}^{l}|\alpha_{n,l-n}(\mathring{t}(r))|\Big)r^l \\ &\preceq \Big(\sum_{l=1}^{\infty}(l+1)m^+(\mathring{t}(r))^l r^l\Big)M^+(\mathring{t}(r)) = \Big(\sum_{l=1}^{\infty}(l+1)\eta^l\Big)M^+(\mathring{t}(r)).\end{aligned}$$

We have

$$\sum_{l=1}^{\infty}(l+1)\eta^l = \sum_{k=1}^{\infty}k\eta^{k-1}-1 = \frac{1}{(1-\eta)^2}-1=\sigma, \tag{3.2}$$

and this implies that

$$\begin{aligned}&\big|a(\mathring{t}(r),re^{i\vartheta})-m_2(\mathring{t}(r))\big|\leq\sigma m_2(\mathring{t}(r)),\\ &\big|b(\mathring{t}(r),re^{i\vartheta})-m_3(\mathring{t}(r))\big|\leq\sigma\sqrt{(m_1m_2)(\mathring{t}(r))}=\frac{\sigma\eta}{2r}.\end{aligned}$$

Remembering that $m_3(t)^2\leq(m_1m_2)(t)$, we therefore obtain

$$\begin{aligned}&(1-\sigma)m_2(\mathring{t}(r))\leq a\big(\mathring{t}(r),re^{i\vartheta}\big)\leq(1+\sigma)m_2(\mathring{t}(r)), \\ &\big|b\big(\mathring{t}(r),re^{i\vartheta}\big)\big|\leq\frac{(1+\sigma)\eta}{2r},\qquad \operatorname{Im} b\big(\mathring{t}(r),re^{i\vartheta}\big)\leq\frac{\sigma\eta}{2r}.\end{aligned}\tag{3.3}$$

Note that the assumption $\eta\in(0,1-\frac{1}{\sqrt{2}})$ implies that $\sigma\in(0,1)$. Now it follows from Lemma 2.1 that (recall from (2.15) that $A(r)=\frac{\eta}{2}\frac{1}{rm_2(\mathring{t}(r))}$)

$$\begin{aligned}\big|q_H(re^{i\vartheta})\big| &\leq \frac{|b(\mathring{t}(r),re^{i\vartheta})|}{a(\mathring{t}(r),re^{i\vartheta})}+2\cdot\frac{1}{2(r\sin\vartheta)a(\mathring{t}(r),re^{i\vartheta})} \\ &\leq \frac{\frac{(1+\sigma)\eta}{2r}}{(1-\sigma)m_2(\mathring{t}(r))}+\frac{1}{(r\sin\vartheta)(1-\sigma)m_2(\mathring{t}(r))} \\ &= \frac{(1+\sigma)\eta+\frac{2}{\sin\vartheta}}{2(1-\sigma)}\cdot\frac{1}{rm_2(\mathring{t}(r))} = \frac{1+\sigma+\frac{2}{\eta\cdot\sin\vartheta}}{1-\sigma}\cdot A(r),\end{aligned}$$

$$
\begin{aligned}
\big|\operatorname{Re}\eta_H(re^{i\vartheta})\big| &\leq \frac{\operatorname{Re} b(\mathring{t}(r), re^{i\vartheta})}{a(\mathring{t}(r), re^{i\vartheta})} + \frac{1}{2(r\sin\vartheta)a(\mathring{t}(r), re^{i\vartheta})} \\
&\leq \frac{\frac{(1+\sigma)\eta}{2r}}{(1-\sigma)m_2(\mathring{t}(r))} + \frac{1}{2(r\sin\vartheta)(1-\sigma)m_2(\mathring{t}(r))} \\
&= \frac{(1+\sigma)\eta + \frac{1}{\sin\vartheta}}{2(1-\sigma)} \cdot \frac{1}{rm_2(\mathring{t}(r))} = \frac{1+\sigma+\frac{1}{\eta\cdot\sin\vartheta}}{1-\sigma} \cdot A(r),
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Im} q_H(re^{i\vartheta}) &\leq \frac{\operatorname{Im} b(\mathring{t}(r), re^{i\vartheta})}{a(\mathring{t}(r), re^{i\vartheta})} + 2\cdot\frac{1}{2(r\sin\vartheta)a(\mathring{t}(r), re^{i\vartheta})} \\
&\leq \frac{\frac{\sigma\eta}{2r}}{(1-\sigma)m_2(\mathring{t}(r))} + \frac{1}{(r\sin\vartheta)(1-\sigma)m_2(\mathring{t}(r))} \\
&= \frac{\sigma\eta + \frac{2}{\sin\vartheta}}{2(1-\sigma)} \cdot \frac{1}{rm_2(\mathring{t}(r))} = \frac{\sigma+\frac{2}{\eta\cdot\sin\vartheta}}{1-\sigma} \cdot A(r).
\end{aligned}
$$ ❑

Proof of Theorem 1.1 (*lower bound for the absolute value*). Let the data be given as in the formulation of the theorem. We use the already established upper bound for the Hamiltonian

$$
\widetilde{H} := -JHJ = \begin{pmatrix} h_2 & -h_3 \\ -h_3 & h_1 \end{pmatrix}.
$$

With the obvious notation we have

$$
q_{\widetilde{H}} = -\frac{1}{q_H}, \qquad \widetilde{m}_1 = m_2, \quad \widetilde{m}_2 = m_1, \qquad \tilde{\mathring{t}}(r) = \mathring{t}(r).
$$

Hence

$$
\widetilde{A}(r) = \sqrt{\frac{\widetilde{m}_1\big(\tilde{\mathring{t}}(r)\big)}{\widetilde{m}_2\big(\tilde{\mathring{t}}(r)\big)}} = \sqrt{\frac{m_2\big(\mathring{t}(r)\big)}{m_1\big(\mathring{t}(r)\big)}} = \frac{1}{A(r)},
$$

and the upper bound from (1.6) gives

$$
\big|q_H(re^{i\vartheta})\big| = \frac{1}{|q_{\widetilde{H}}(re^{i\vartheta})|} \geq \Big(\frac{1+\sigma+\frac{2}{\eta\sin\vartheta}}{1-\sigma}\Big)^{-1} A(r).
$$ ❑

3.2 The lower bound for $\operatorname{Im} q_H$

The bound for $\alpha_{n,m}(t)$ given in Lemma 3.2 puts absolute values everywhere and does not take care of possible cancellations. Proving the lower bound of $\operatorname{Im} q_H$ asserted in (1.8) requires more delicate coefficient estimates.

We start with some preliminary computations.

3.3 Lemma. *We have*

$$
\begin{aligned}
\int_a^t W_n(s)H(s)W_m(s)^*\,\mathrm{d}s &= W_n(t)W_1(t)JW_m(t)^* + \int_a^t W_n(s)W_1(s)H(s)W_{m-1}(s)^*\,\mathrm{d}s \\
&\quad + \int_a^t W_{n-1}(s)H(s)W_1(s)^*W_m(s)^*\,\mathrm{d}s, \qquad n,m\geq 1,
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
W_n(t) = W_{n-1}(t)W_1(t) + \big(\det M(t)\big)W_{n-2}(t) &+ \int_a^t \big(\det M(s)\big)W_{n-3}(s)H(s)J\,\mathrm{d}s \\
- \int_a^t W_{n-2}(s)JW_1(s)^*\cdot JH(s)J\,\mathrm{d}s, &\qquad n\geq 3.
\end{aligned} \tag{3.5}
$$

Proof. The equality (3.4) follows from the definition of W_n by straightforward differentiation which is omitted; the necessary computation can be found in [27].

To show (3.5), we use again integration by parts, (3.4) with (n,m) replaced by $(n-2,1)$, and also (2.13) and (2.12):

$$\begin{aligned}
W_n(t) &= \int_a^t W_{n-1}(s)\big(-H(s)J\big)\,\mathrm{d}s \\
&= W_{n-1}(t)W_1(t) - \int_a^t \big(-W_{n-2}(s)H(s)J\big)W_1(s)\,\mathrm{d}s \\
&= W_{n-1}(t)W_1(t) - \int_a^t W_{n-2}(s)H(s)W_1^*(s)J\,\mathrm{d}s \\
&= W_{n-1}(t)W_1(t) - W_{n-2}(t)W_1(t)JW_1^*(t)J \\
&\quad - \int_a^t W_{n-2}(s)W_1(s)H(s)J\,\mathrm{d}s - \int_a^t W_{n-3}(s)H(s)\big(W_1(s)W_1(s)\big)^*J\,\mathrm{d}s \\
&= W_{n-1}(t)W_1(t) - W_{n-2}(t)\cdot(\det M(t))J\cdot J \\
&\quad - \int_a^t W_{n-2}(s)JW_1^*(s)\cdot JH(s)J\,\mathrm{d}s - \int_a^t W_{n-3}(s)H(s)\big(-\det M(s)\big)I\cdot J\,\mathrm{d}s.
\end{aligned}$$ ❑

Set

$$\beta_{n,m}(t) := W_n(t)JW_m(t)^*, \qquad n,m\geq 0,$$

and note that the symmetry relation $\beta_{m,n}(t) = -\beta_{n,m}(t)^*$ holds.

3.4 Proposition.[3] *For $k,l\geq 0$, $(k,l)\neq(0,0)$ we have*

$$|\beta_{2k+1,2l+1}(t)| \preceq (\det M(t))\cdot\big(1+3(k+l)\big)\cdot m^+(t)^{2(k+l)-1}M^+(t). \tag{3.6}$$

Proof. We divide the proof into three steps.

① Since the matrix on the right-hand side of (3.6) is symmetric, the stated assertion is symmetric in k and l. This means, it holds for a pair (k,l) if and only if it holds for (l,k).

② We proceed by induction on $k+l$. First, assume that $k+l=1$, i.e. $(k,l)\in\{(1,0),(0,1)\}$. By symmetry, it is enough to consider the case $(k,l)=(1,0)$. Using (3.5), (2.13), (3.1) and Lemma 3.1 we compute

$$\begin{aligned}
|\beta_{3,1}(t)| &= |W_3(t)\cdot JW_1(t)^*| \\
&= \Big|\Big(W_2(t)W_1(t) + (\det M(t))W_1(t) + \int_a^t (\det M(s))H(s)J\,\mathrm{d}s \\
&\qquad - \int_a^t W_1(s)JW_1(s)^*\cdot JH(s)J\,\mathrm{d}s\Big)JW_1(t)^*\Big| \\
&\preceq |W_2(t)|\cdot|\underbrace{W_1(t)JW_1(t)^*}_{=(\det M(t))J}| + (\det M(t))|W_1(t)|\cdot|J|\cdot|W_1(t)|^* \\
&\qquad + \int_a^t (\det M(s))|H(s)|\,\mathrm{d}s\cdot|W_1(t)|^* \\
&\qquad + \int_a^t |\underbrace{W_1(s)JW_1(s)^*}_{=(\det M(s))J}|\cdot|J|\cdot|H(s)|\,\mathrm{d}s\cdot|W_1(t)|^* \\
&\preceq (\det M(t))|W_2(t)||J| + (\det M(t))|W_1(t)||J||W_1(t)|^* \\
&\qquad + (\det M(t))\cdot\int_a^t |H(s)|\,\mathrm{d}s\cdot|W_1(t)|^* \\
&\qquad + (\det M(t))\cdot\int_a^t |H(s)|\,\mathrm{d}s\cdot|W_1(t)|^*
\end{aligned}$$

[3] Seeking a finer estimate of the form (3.6) was already suggested in [43] as a potential way to progress.

$$
\begin{aligned}
&\preceq (\det M(t))\Big(m^+(t)M^+(t)|J|\cdot|J| + M^+(t)|J|\cdot|J|\cdot|J|M^+(t) \\
&\quad + 2M^+(t)\cdot|J|\cdot M^+(t)\Big) \\
&= 4(\det M(t))m^+(t)M^+(t),
\end{aligned}
$$

which is (3.6) for $(k,l)=(1,0)$.

③ Let (k,l) with $k+l\geq 2$ be given, and assume that (3.6) holds for all (k',l') with $k'+l'<k+l$. By symmetry, we can assume, w.l.o.g., that $k\geq l$. Then, certainly, $k\geq 1$. It follows from (3.5) that

$$
\begin{aligned}
\beta_{2k+1,2l+1}(t) &= W_{2k+1}(t)JW_{2l+1}^*(t) \\
&= W_{2k}(t)\cdot W_1(t)JW_{2l+1}^*(t) + (\det M(t))W_{2k-1}(t)JW_{2l+1}^*(t) \\
&\quad - \int_a^t (\det M(s))W_{2k-2}(s)H(s)\,\mathrm{d}s\cdot W_{2l+1}^*(t) \\
&\quad + \int_a^t W_{2k-1}(s)JW_1^*(s)\cdot JH(s)\,\mathrm{d}s\cdot W_{2l+1}^*(t).
\end{aligned}
$$

Let us estimate each term on the right-hand side separately. Using (3.1), the induction hypothesis, (2.13) and Lemma 3.1(ii) we obtain

$$
\begin{aligned}
&|W_{2k}(t)\cdot W_1(t)JW_{2l+1}(t)^*| \preceq |W_{2k}(t)|\cdot|W_1(t)JW_{2l+1}(t)^*| \\
&\preceq m^+(t)^{2k-1}M^+(t)|J|\cdot\begin{cases}(\det M(t))(1+3l)m^+(t)^{2l-1}M^+(t), & l>0\\ (\det M(t))|J|, & l=0\end{cases} \\
&= (\det M(t))(1+3l)\cdot\begin{cases}m^+(t)^{2(k+l)-2}M^+(t)|J|M^+(t), & l>0\\ m^+(t)^{2k-1}M^+(t)|J||J|, & l=0\end{cases} \\
&= (\det M(t))(1+3l)m^+(t)^{2(k+l)-1}M^+(t).
\end{aligned}
$$

Moreover, (3.1) and Lemma 3.1 imply that

$$
\begin{aligned}
&\big|(\det M(t))W_{2k-1}(t)JW_{2l+1}(t)^*\big| \\
&\quad\preceq (\det M(t))m^+(t)^{2k-2}M^+(t)|J|\cdot|J|\cdot m^+(t)^{2l}|J|M^+(t) \\
&\quad= (\det M(t))m^+(t)^{2(k+l)-2}M^+(t)|J|M^+(t) \\
&\quad= (\det M(t))m^+(t)^{2(k+l)-1}M^+(t)
\end{aligned}
$$

and

$$
\begin{aligned}
&\bigg|\int_a^t (\det M(s))W_{2k-2}(s)H(s)\,\mathrm{d}s\cdot W_{2l+1}(t)^*\bigg| \\
&\quad\preceq \int_a^t (\det M(s))|W_{2k-2}(s)|\cdot|H(s)|\,\mathrm{d}s\cdot|W_{2l+1}(t)|^* \\
&\quad\preceq \int_a^t (\det M(s))\cdot\begin{Bmatrix}m^+(s)^{2k-3}M^+(s)|J|, & k>1\\ I, & k=1\end{Bmatrix}\cdot|H(s)|\,\mathrm{d}s \\
&\quad\quad\cdot m^+(t)^{2l}|J|M^+(t) \\
&\quad\preceq (\det M(t))\cdot\begin{Bmatrix}m^+(t)^{2k-3}M^+(t)|J|, & k>1\\ I, & k=1\end{Bmatrix}\cdot\underbrace{\int_a^t |H(s)|\,\mathrm{d}s}_{\preceq M^+(t)} \\
&\quad\quad\cdot m^+(t)^{2l}|J|M^+(t)
\end{aligned}
$$

$$\preceq (\det M(t)) \cdot \begin{cases} m^+(t)^{2(k+l)-3}(M^+(t)|J|)^2 M^+(t), & k > 1 \\ m^+(t)^{2l} M^+(t)|J|M^+(t), & k = 1 \end{cases}$$

$$= (\det M(t)) m^+(t)^{2(k+l)-1} M^+(t).$$

Finally, we use again the induction hypothesis to show

$$\Big| \int_a^t W_{2k-1}(s) J W_1(s)^* \cdot JH(s)\,\mathrm{d}s \cdot W_{2l+1}(t)^* \Big|$$

$$\preceq \int_a^t |W_{2k-1}(s) J W_1(s)^*| \cdot |J| \cdot |H(s)|\,\mathrm{d}s \cdot |W_{2l+1}(t)|^*$$

$$\preceq \int_a^t \left\{ \begin{matrix} (\det M(s))(1+3(k-1)) m^+(s)^{2(k-1)-1} M^+(s), & k > 1 \\ (\det M(s))|J|, & k = 1 \end{matrix} \right\}$$

$$\cdot |J| \cdot |H(s)|\,\mathrm{d}s \cdot m^+(t)^{2l}|J|M^+(t)$$

$$\preceq (\det M(t)) \cdot \left\{ \begin{matrix} (1+3(k-1)) m^+(t)^{2(k-1)-1} M^+(t), & k > 1 \\ |J|, & k = 1 \end{matrix} \right\}$$

$$\cdot |J| \cdot \underbrace{\int_a^t |H(s)|\,\mathrm{d}s}_{\preceq M^+(t)} \cdot m^+(t)^{2l}|J|M^+(t)$$

$$\preceq (\det M(t))(1+3(k-1))$$

$$\cdot \begin{cases} m^+(t)^{2(k-1+l)-1}(M^+(t)|J|)^2 M^+(t), & k > 1 \\ m^+(t)^{2l} M^+(t)|J|M^+(t), & k = 1 \end{cases}$$

$$= (\det M(t))(1+3(k-1)) m^+(t)^{2(k+l)-1} M^+(t).$$

Combining all estimates we obtain

$$|\beta_{2k+1,2l+1}(t)| = |W_{2k+1}(t) J W_{2l+1}(t)^*|$$

$$\preceq (\det M(t)) m^+(t)^{2(k+l)-1} M^+(t) \cdot \big[\underbrace{(1+3l) + 1 + 1 + (1+3(k-1))}_{=1+3(k+l)} \big],$$

which finishes the proof. ❑

We also need the following formulae that relate $\alpha_{n,m}$ to $\beta_{k,l}$.

3.5 Lemma. *We have*

$$\alpha_{n,m+1}(t) - \alpha_{n+1,m}(t) = \beta_{n+1,m+1}(t), \qquad n, m \geq 0,$$

$$\alpha_{n,0}(t) = \beta_{n+1,0}(t), \quad \alpha_{0,n}(t) = -\beta_{0,n+1}(t), \qquad n \geq 0.$$

Proof. The first line follows since

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(W_{n+1}(t) J W_{m+1}(t)^*\big) = W_n(t) H(t) W_{m+1}(t)^* - W_{n+1}(t) H(t) W_m(t)^*$$

and $W_n(a) = 0$, $n \geq 1$. The second line is just the recurrence relation (2.9). ❑

Proof of Theorem 1.1 (*lower bound for the imaginary part*). Let the data be given as in the formulation of the theorem. We first estimate $\operatorname{Im} q_H(z)$ along the imaginary axis. Let $t \in [a, b)$ and $r > 0$ and consider (2.11), which, for $\vartheta = \pi/2$, becomes

$$\nabla(t, ir) = \sum_{l=0}^{\infty} \Big(\sum_{n=0}^{l} \alpha_{n,l-n}(t)(-1)^n \Big)(-i)^l r^l.$$

The coefficient of r^1 is

$$\alpha_{0,1}(t)i^{-1} + \alpha_{1,0}(t)i^1 = -i\beta_{1,1}(t) = -iW_1(t)JW_1(t)^* = -i(\det M(t))J$$

by (2.13), and hence

$$\nabla(t, ir) - M(t) + i(\det M(t))J \cdot r = \sum_{\substack{l=2\\ l \text{ even}}}^{\infty} \Big(\sum_{n=0}^{l} \alpha_{n,l-n}(t)(-1)^n\Big)(-1)^{\frac{l}{2}} r^l$$
$$+ i \sum_{\substack{l=3\\ l \text{ odd}}}^{\infty} \Big(\sum_{n=0}^{l} \alpha_{n,l-n}(t)(-1)^n\Big)(-1)^{\frac{l+1}{2}} r^l.$$

The imaginary part of the right-hand side can be estimated with the help of Proposition 3.4; namely, using Lemma 3.5 to join consecutive summands we obtain

$$\begin{aligned}
&\Big| \sum_{\substack{l=3\\ l \text{ odd}}}^{\infty} \Big(\sum_{n=0}^{l} \alpha_{n,l-n}(t)(-1)^n\Big)(-1)^{\frac{l+1}{2}} r^l \Big| \\
&= \Big| \sum_{\substack{l=3\\ l \text{ odd}}}^{\infty} \Big(\sum_{k=0}^{\frac{l-1}{2}} \big[\alpha_{2k,l-2k}(t) - \alpha_{2k+1,l-2k-1}(t)\big]\Big)(-1)^{\frac{l+1}{2}} r^l \Big| \\
&\preceq \sum_{\substack{l=3\\ l \text{ odd}}}^{\infty} \Big(\sum_{k=0}^{\frac{l-1}{2}} \big|\beta_{2k+1,l-2k}(t)\big|\Big) r^l = \sum_{\substack{l=3\\ l \text{ odd}}}^{\infty} \Big(\sum_{k=0}^{\frac{l-1}{2}} \big|\beta_{2k+1,\,2(\frac{l-1}{2}-k)+1}(t)\big|\Big) r^l \\
&\preceq \sum_{\substack{l=3\\ l \text{ odd}}}^{\infty} \Big(\sum_{k=0}^{\frac{l-1}{2}} (\det M(t))\Big[1 + 3\Big(k + \frac{l-1}{2} - k\Big)\Big] \cdot m^+(t)^{2(k+\frac{l-1}{2}-k)-1} M^+(t) r^l \\
&= \det M(t) \cdot \Big(\sum_{\substack{l=3\\ l \text{ odd}}}^{\infty} \frac{l+1}{2} \cdot \frac{3l-1}{2} m^+(t)^{l-2} r^l\Big) M^+(t).
\end{aligned}$$

Using the definition of $a(t,z)$ and $b(t,z)$ in Lemma 2.1 we obtain

$$\begin{aligned}
&\big| \operatorname{Im} b(t, ir) - (\det M(t))r \big| = \Big| \operatorname{Im}\Big(\begin{pmatrix}1\\0\end{pmatrix}^* \nabla(t, ir) \begin{pmatrix}0\\1\end{pmatrix}\Big) - (\det M(t))r \Big| \\
&= \Big| \operatorname{Im} \begin{pmatrix}1\\0\end{pmatrix}^* \Big(\nabla(t, ir) - M(t) + i(\det M(t))J \cdot r\Big) \begin{pmatrix}0\\1\end{pmatrix} \Big| \\
&\le \begin{pmatrix}1\\0\end{pmatrix}^* \det M(t) \cdot \Big(\sum_{\substack{l=3\\ l \text{ odd}}}^{\infty} \frac{l+1}{2} \cdot \frac{3l-1}{2} m^+(t)^{l-2} r^l\Big) M^+(t) \begin{pmatrix}0\\1\end{pmatrix} \\
&\le (\det M(t))r \cdot \sum_{\substack{l=3\\ l \text{ odd}}}^{\infty} \frac{(l+1)(3l-1)}{8} \big(m^+(t)r\big)^{l-1} \\
&\le (\det M(t))r \cdot \sum_{k=1}^{\infty} \frac{(2k+2)\big(3(2k+1)-1\big)}{8} \big(m^+(t)r\big)^{2k} \\
&= (\det M(t))r \cdot \sum_{k=1}^{\infty} (k+1) \frac{3k+1}{2} \big(m^+(t)r\big)^{2k}.
\end{aligned}$$

Recall that $m^+(\mathring{t}(r))r = \eta$. Moreover, it is easy to check that $\eta \le \frac{1}{2}$ implies

$$\frac{3k+1}{2}\eta^k \le 1, \qquad k \in \mathbb{N}. \tag{3.7}$$

Continuing the above chain of inequalities with $t = \mathring{t}(r)$ and using (3.7) and (3.2) we obtain

$$\begin{aligned}
\big|\operatorname{Im} b(\mathring{t}(r), ir) - (\det M(\mathring{t}(r)))r\big| &\le (\det M(\mathring{t}(r)))r \cdot \sum_{k=1}^{\infty}(k+1)\frac{3k+1}{2}\eta^{2k} \\
&\le (\det M(\mathring{t}(r)))r \cdot \sum_{k=1}^{\infty}(k+1)\eta^k = \sigma(\det M(\mathring{t}(r)))r,
\end{aligned}$$

and hence $\operatorname{Im} b(\mathring{t}(r), ir) \ge (1-\sigma)(\det M(\mathring{t}(r)))r$. This, together with (3.3) and Lemma 2.1, implies that

$$\operatorname{Im} q_H(ir) \ge \frac{\operatorname{Im} b(\mathring{t}(r), ir)}{a(\mathring{t}(r), ir)} \ge \frac{(1-\sigma)\det M(\mathring{t}(r))r}{(1+\sigma)m_2(\mathring{t}(r))} = \frac{1-\sigma}{1+\sigma}\cdot\frac{\eta}{2}L(r),$$

which is the lower bound in (1.8) for $\vartheta = \pi/2$. Having an estimate along the imaginary axis, (1.8) for general ϑ follows using a standard property of the Poisson kernel. Using the Herglotz integral representation (1.3) of q_H we obtain, for each $\vartheta \in (0,\pi)$, that

$$\begin{aligned}
\operatorname{Im} q_H(re^{i\vartheta}) &= \beta_H r\sin\vartheta + \int_{\mathbb{R}} \frac{r\sin\vartheta}{|t-re^{i\vartheta}|^2}\,\mathrm{d}\mu_H(t) \\
&\ge \beta_H r\sin\vartheta + \int_{\mathbb{R}} \frac{r\sin\vartheta}{(t^2+r^2)(1+|\cos\vartheta|)}\,\mathrm{d}\mu_H(t) \\
&\ge \frac{\sin\vartheta}{1+|\cos\vartheta|}\underbrace{\bigg(\beta_H r + \int_{\mathbb{R}}\frac{r}{t^2+r^2}\,\mathrm{d}\mu_H(t)\bigg)}_{=\operatorname{Im} q_H(ir)},
\end{aligned}$$

which finishes the proof of Theorem 1.1. ❑

3.3 The comparison result

Let H and $\widetilde{H}$ be Hamiltonians on $[a,b)$ such that none of their diagonal entries vanishes a.e. We use the freedom in the choice of the parameter η in Theorem 1.1, and therefore also make a notational distinction: set

$$\mathring{r}_{(\eta)}(t) := \frac{\eta}{2}\big(m_1(t)m_2(t)\big)^{-\frac{1}{2}}, \qquad \mathring{t}_{(\eta)}(r) := r_{(\eta)}^{-1}(r), \qquad A_{(\eta)}(r) := \sqrt{\frac{m_1(\mathring{t}_{(\eta)}(r))}{m_2(\mathring{t}_{(\eta)}(r))}}\,.$$

for $t \in (\mathring{a}, b)$ and $r > 0$, where $\mathring{a}$ is defined in (1.9). Moreover, recall that $A_{(\eta)}(r)$ can be written as

$$A_{(\eta)}(r) = \frac{2r}{\eta}\cdot m_1\big(\mathring{t}_{(\eta)}(r)\big) = \frac{\eta}{2r}\cdot\frac{1}{m_2(\mathring{t}_{(\eta)}(r))}. \tag{3.8}$$

Analogous notation applies to $\widetilde{H}$.

3.6 Lemma. *Let $c, \gamma > 0$ and let $t \in (\mathring{a}, b)$. Assume that $t \le \mathring{\tilde{a}}$ or that*

$$\operatorname{tr}\widetilde{M}(t) \le c\operatorname{tr} M(t), \qquad \widetilde{m}_2(t) \le \gamma m_2(t), \qquad m_1(t) \ge \frac{1}{2}\operatorname{tr} M(t). \tag{3.9}$$

Further, let $\eta \in (0, \frac{1}{\sqrt{2c\gamma}}[1-\frac{1}{\sqrt{2}}])$, and set

$$\tilde{\eta} := \eta\cdot\sqrt{2c\gamma}, \quad \tilde{\sigma} := \frac{1}{(1-\tilde{\eta})^2} - 1, \quad \tilde{\delta} := \bigg(\frac{1+\tilde{\sigma}+\frac{2}{\tilde{\eta}}}{1-\tilde{\sigma}}\bigg)^{-1}\frac{2}{\tilde{\eta}}.$$

Then

$$\frac{1}{\mathring{r}_{(\eta)}(t)}\big|q_{\widetilde{H}}\big(i\mathring{r}_{(\eta)}(t)\big)\big| \ge \tilde{\delta}\cdot\widetilde{m}_1(t).$$

Proof. First assume that $t > \mathring{\tilde{a}}$. Then, by assumption, (3.9) holds, from which we obtain

$$\widetilde{m}_1(t) \leq \operatorname{tr} \widetilde{M}(t) \leq c \cdot \operatorname{tr} M(t) \leq 2c \cdot m_1(t),$$
$$\widetilde{m}_1(t)\widetilde{m}_2(t) \leq 2c\gamma \cdot m_1(t)m_2(t),$$

and hence

$$\mathring{r}_{(\eta)}(t) = \frac{\eta}{2}\big(m_1(t)m_2(t)\big)^{-\frac{1}{2}} \leq \frac{\eta\sqrt{2c\gamma}}{2}\big(\widetilde{m}_1(t)\widetilde{m}_2(t)\big)^{-\frac{1}{2}} = \mathring{\tilde{r}}_{(\tilde{\eta})}(t).$$

Since $\mathring{\tilde{t}}_{(\tilde{\eta})}$ is decreasing, it follows that

$$\mathring{\tilde{t}}_{(\tilde{\eta})}\big(\mathring{r}_{(\eta)}(t)\big) \geq \mathring{\tilde{t}}_{(\tilde{\eta})}\big(\mathring{\tilde{r}}_{(\eta)}(t)\big) = t.$$

In the case when $t \leq \mathring{\tilde{a}}$ we have

$$\mathring{\tilde{t}}_{(\tilde{\eta})}\big(\mathring{r}_{(\eta)}(t)\big) > \mathring{\tilde{a}} \geq t.$$

In both cases we can apply Theorem 1.1 to estimate

$$\begin{aligned}\frac{1}{\mathring{r}_{(\eta)}(t)}\big|q_{\widetilde{H}}\big(i\mathring{r}_{(\eta)}(t)\big)\big| &\geq \left(\frac{1+\tilde{\sigma}+\frac{2}{\tilde{\eta}}}{1-\tilde{\sigma}}\right)^{-1}\frac{1}{\mathring{r}_{(\eta)}(t)}\tilde{A}_{(\tilde{\eta})}\big(\mathring{r}_{(\eta)}(t)\big)\\ &= \left(\frac{1+\tilde{\sigma}+\frac{2}{\tilde{\eta}}}{1-\tilde{\sigma}}\right)^{-1}\frac{2}{\tilde{\eta}}\cdot\widetilde{m}_1\Big(\mathring{\tilde{t}}_{(\tilde{\eta})}\big(\mathring{r}_{(\eta)}(t)\big)\Big) \geq \left(\frac{1+\tilde{\sigma}+\frac{2}{\tilde{\eta}}}{1-\tilde{\sigma}}\right)^{-1}\frac{2}{\tilde{\eta}}\cdot\widetilde{m}_1(t).\end{aligned}$$

❑

Proof of Proposition 1.4. Let the data be given as in the statement of the proposition and assume first that $a' < b$. Choose $\eta > 0$ such that

$$\max\big\{\eta,\, \eta\sqrt{2c_1\gamma_1},\, \eta\sqrt{2c_2\gamma_2}\big\} < 1 - \frac{1}{\sqrt{2}},$$

and set

$$r_0 := \max\big\{\mathring{r}_{(\eta)}(a'), \mathring{\tilde{r}}_{(\eta)}(a')\big\}. \tag{3.10}$$

Further, let $r > r_0$. We distinguish three cases.

① Assume that $A_{(\eta)}(r) \geq 1$.

Set $t := \mathring{t}_{(\eta)}(r)$; then $t \in (\mathring{a}, a')$. Since $A_{(\eta)}(r) \geq 1$, we have $m_1(t) \geq m_2(t)$, and hence $m_1(t) \geq \frac{1}{2}\operatorname{tr} M(t)$. Set

$$\eta_2 := \eta \cdot \sqrt{2c_2\gamma_2}, \qquad \sigma_2 := \frac{1}{(1-\eta_2)^2} - 1, \qquad \delta_2 := \left(\frac{1+\sigma_2+\frac{2}{\eta_2}}{1-\sigma_2}\right)^{-1}\frac{2}{\eta_2}.$$

The assumptions of Lemma 3.6 are satisfied with $c = c_2$, $\gamma = \gamma_2$, and hence

$$\begin{aligned}\frac{1}{r}|q_{\widetilde{H}}(ir)| &\geq \delta_2 \cdot \widetilde{m}_1(t) \geq \frac{\delta_2}{\gamma_1}\cdot m_1(t) = \frac{\delta_2}{\gamma_1}\cdot\frac{\eta}{2r}A_{(\eta)}(r)\\ &\geq \frac{\delta_2}{\gamma_1}\cdot\frac{\eta}{2r}\cdot\left(\frac{1+\sigma+\frac{2}{\eta}}{1-\sigma}\right)^{-1}\cdot|q_H(ir)|.\end{aligned} \tag{3.11}$$

② Assume that $\widetilde{A}_{(\eta)}(r) \leq 1$.

We proceed similarly. Set $t := \mathring{\tilde{t}}_{(\eta)}(r)$; then $t \in (\mathring{\tilde{a}}, a')$. Moreover, we have $\widetilde{m}_1(t) \leq \widetilde{m}_2(t)$, and hence $\widetilde{m}_2(t) \geq \frac{1}{2}\operatorname{tr}\widetilde{M}(t)$. We apply Lemma 3.6 with $-J\widetilde{H}J$ in place of H and $-JHJ$ in place of $\widetilde{H}$. The assumptions of the lemma are now satisfied with $c = c_1$, $\gamma = \gamma_1$. Set

$$\eta_1 := \eta \cdot \sqrt{2c_1\gamma_1}, \qquad \sigma_1 := \frac{1}{(1-\eta_1)^2} - 1, \qquad \delta_1 := \left(\frac{1+\sigma_1+\frac{2}{\eta_1}}{1-\sigma_1}\right)^{-1}\frac{2}{\eta_1};$$

then

$$\frac{1}{r}\Big|\frac{-1}{q_H(ir)}\Big| = \frac{1}{r}\big|q_{-JHJ}(ir)\big| \geq \delta_1 \cdot m_2(t) \geq \frac{\delta_1}{\gamma_2}\cdot \widetilde{m}_2(t) = \frac{\delta_1}{\gamma_2}\cdot\frac{\eta}{2r}\big[\widetilde{A}_{(\eta)}(r)\big]^{-1}$$
$$\geq \frac{\delta_1}{\gamma_2}\cdot\frac{\eta}{2r}\cdot\left[\Big(\frac{1+\sigma+\frac{2}{\eta}}{1-\sigma}\Big)\cdot|q_{\widetilde{H}}(ir)|\right]^{-1}. \tag{3.12}$$

③ Assume that $A_{(\eta)}(r) < 1$ and $\widetilde{A}_{(\eta)}(r) > 1$.
Then

$$|q_H(ir)| < \frac{1+\sigma+\frac{2}{\eta}}{1-\sigma}, \qquad |q_{\widetilde{H}}(ir)| > \Big(\frac{1+\sigma+\frac{2}{\eta}}{1-\sigma}\Big)^{-1},$$

and we see that

$$|q_H(ir)| \leq \Big(\frac{1+\sigma+\frac{2}{\eta}}{1-\sigma}\Big)^2\cdot|q_{\widetilde{H}}(ir)|. \tag{3.13}$$

The above cases together cover the whole ray (r_0,∞). From (3.11), (3.12) and (3.13) we obtain that, for all r on this ray, $|q_H(ir)| \leq C|q_{\widetilde{H}}(ir)|$ where C is the maximum of the three terms

$$\frac{\gamma_1}{\delta_2}\cdot\frac{2}{\eta}\Big(\frac{1+\sigma+\frac{2}{\eta}}{1-\sigma}\Big), \qquad \frac{\gamma_2}{\delta_1}\cdot\frac{2}{\eta}\Big(\frac{1+\sigma+\frac{2}{\eta}}{1-\sigma}\Big), \qquad \Big(\frac{1+\sigma+\frac{2}{\eta}}{1-\sigma}\Big)^2.$$

Observe that C is independent of $H, \widetilde{H}$ and a'. Finally, the statement that we can choose $r_0 = 0$ when $a' = b$ follows from the fact that C is independent of a' and from (3.10), which shows that $r_0 \to 0$ as $a' \to b$. ❑

4 Asymptotic behaviour of the spectral measure

Having available the estimates of the Weyl coefficient q_H from Theorem 1.1, we employ Abelian–Tauberian-type theorems to translate knowledge about the growth of $\operatorname{Im} q_H(ir)$ for $r \to \infty$ to knowledge about the growth of the distribution function of the spectral measure μ_H towards infinity. This enables us to give conditions for weighted integrability and boundedness of tails of μ_H relative to suitable comparison functions. Thereby we use functions of regular variation (see Appendix A) to compare with. This class of regularly varying functions provides a much finer scale than just the class of power functions; see, e.g. Examples 4.13 and 4.21.

In this section we assume that neither h_1 nor h_2 vanishes in a neighbourhood of the left endpoint a, i.e. we assume that $\mathring{a} = a$, where $\mathring{a}$ is defined in (1.9). The reason is the following. If $h_1 = 0$ in a neighbourhood of a, then the measure μ_H is finite and therefore the growth of the distribution function of μ_H trivial. If $h_2 = 0$ in a neighbourhood of a, then the linear term $\beta_H z$ in (1.3) dominates the integral when $z \to \infty$ non-tangentially. Hence the growth of $\operatorname{Im} q_H(ir)$ does not determine the growth of the distribution function of the spectral measure μ_H. These two special cases are considered in more detail in §5.2.

Throughout this section we use the following notation.

4.1 Definition. If μ is a positive Borel measure on the real line, then we denote by $\tilde{\mu}$ the push-forward measure of μ under the map $t \mapsto |t|$, and by $\overset{\leftrightarrow}{\mu}$ the distribution function of $\tilde{\mu}$. Explicitly, this means

$$\tilde{\mu}([0,r)) = \mu((-r,r)) = \overset{\leftrightarrow}{\mu}(r), \;\; r \geq 0, \qquad \tilde{\mu}((-\infty,0)) = 0. \tag{4.1}$$

◊

4.1 Membership of Kac classes

Recall that a Nevanlinna function q is said to belong to the *Kac class* with index $\alpha \in (0,2)$ if the measure μ in its Herglotz integral representation (cf. (1.3)) satisfies

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu(r)}{1+|r|^\alpha} < \infty$$

and no linear term is present. These classes have been investigated for a long time because of their role in the spectral theory of the string; see [21, 11.6°], [22, 18]. They are also known to play a role in a broader operator-theoretic context [45, 14]. More general classes occur in [20] where weighted integrability conditions for the spectral measure of a Krein string are characterised in terms of the mass distribution function of the string. Kac's result is formulated in a way to allow arbitrary non-decreasing comparison functions, while we prefer to give more explicit conditions on the cost of restricting the class of comparison functions to regularly varying functions. See Appendix A for the definition and some properties of regularly varying functions.

4.2 Definition. Let g be a continuous, regularly varying function with index $\alpha \leq 2$ and $\lim_{r\to\infty} g(r) = \infty$. We denote by $\mathcal{M}_g$ the set of all positive Borel measures μ on $\mathbb{R}$ such that

$$\int_{[1,\infty)} \frac{\mathrm{d}\tilde{\mu}(r)}{g(r)} < \infty.$$

$\lozenge$

4.3 Remark. The assumption that $\alpha \leq 2$ is natural. Namely, our aim is to relate the spectral measure of a canonical system with the Hamiltonian of the system. Such measures are always Poisson integrable, i.e. $\int_{[1,\infty)} r^{-2}\,\mathrm{d}\tilde{\mu}(r) < \infty$.

Contrasting this, the assumption that $\lim_{r\to\infty} g(r) = \infty$ is a restriction. The necessity to impose this assumption comes from the fact that finite measures μ correspond to Hamiltonians which start with an interval where $h_1 = 0$. Some cases of finite measures μ can be reduced to the case of an infinite measure. Explicit formulae that relate the Hamiltonians with spectral measures μ and $t^2\,\mathrm{d}\mu(t)$ are known; see [41, Rule 6]. Iterating these formulae and combining them with our results below one can obtain corresponding results for a class of finite measures; however, these formulae will be very lengthy (and presumably hard to apply in practice). Hence, we do not go into further details in this respect. $\lozenge$

Theorem 4.10 below is a generalisation of [20, Theorem] to the case of a non-diagonal Hamiltonian; the connection is worked out in detail in Section 6.1. In the formulation of the theorem we use the following notation: for a regularly varying function g with index $\alpha \leq 2$ set

$$g_\star(r) := \int_1^r \frac{t}{g(t)}\,\mathrm{d}t, \quad r \geq 1. \tag{4.2}$$

By Karamata's theorem (see Theorem A.4 (i)) the function $g_\star$ is regularly varying with index $2-\alpha$, and

$$g_\star(r) \begin{cases} \asymp \frac{r^2}{g(r)}, & \alpha < 2, \\ \gg \frac{r^2}{g(r)}, & \alpha = 2. \end{cases} \tag{4.3}$$

For the following, it turns out to be more convenient to use a slight modification of the class $\mathcal{M}_g$.

4.4 Definition. Let g be a regularly varying function with index $\alpha \leq 2$ and $\lim_{r\to\infty} g(r) = \infty$. We denote by $\widehat{\mathcal{M}}_g$ the set of all positive Borel measures μ on $\mathbb{R}$ such that

$$\int_1^\infty \overset{\leftrightarrow}{\mu}(r)\frac{g_\star(r)}{r^3}\,\mathrm{d}r < \infty,$$

where $\overset{\leftrightarrow}{\mu}$ and $g_\star$ are defined in (4.1) and (4.2) respectively. $\lozenge$

For $\alpha \in (0,2)$ the classes $\mathcal{M}_g$ and $\widehat{\mathcal{M}}_g$ coincide as the following proposition shows.

4.5 Proposition. *Let g be a regularly varying function with index $\alpha \leq 2$ and $\lim_{r\to\infty} g(r) = \infty$.*

(i) *If $\alpha \in (0,2)$, then $\widehat{\mathcal{M}}_g = \mathcal{M}_g$.*

(ii) *If $\alpha = 0$ or $\alpha = 2$, then $\widehat{\mathcal{M}}_g \subseteq \mathcal{M}_g$.*

(iii) *If* $\alpha \in [0,2)$, *then*

$$\mu \in \widehat{\mathcal{M}}_{\mathfrak{g}} \quad \Leftrightarrow \quad \int_1^\infty \overleftrightarrow{\mu}(r) \frac{\mathrm{d}r}{r\mathfrak{g}(r)} < \infty.$$

(iv) *If* $\alpha \in (0,2]$, *then*

$$\mu \in \mathcal{M}_{\mathfrak{g}} \quad \Leftrightarrow \quad \int_1^\infty \overleftrightarrow{\mu}(r) \frac{\mathrm{d}r}{r\mathfrak{g}(r)} < \infty.$$

Before we prove Proposition 4.5 we formulate a lemma about integration by parts in a measure-theoretic form; see, e.g. [19, Lemma 2]. An explicit proof can be found in [27].

4.6 Lemma. *Let* $-\infty < a < b \leq \infty$ *and let* μ *and* ν *be positive Borel measures on* $[a,b)$. *Then*

$$\int_{[a,b)} \mu([a,t))\,\mathrm{d}\nu(t) = \int_{[a,b)} \nu((t,b))\,\mathrm{d}\mu(t). \tag{4.4}$$

If these integrals are finite, then $\lim\limits_{t\to b} \mu([a,t))\nu([t,b)) = 0$.

Proof of Proposition 4.5. If $\int_1^\infty \frac{\mathrm{d}t}{t\mathfrak{g}(t)} < \infty$, then

$$\int_r^\infty \frac{\mathrm{d}t}{t\mathfrak{g}(t)} \begin{cases} \asymp \frac{1}{\mathfrak{g}(r)}, & \alpha > 0, \\ \gg \frac{1}{\mathfrak{g}(r)}, & \alpha = 0, \end{cases} \tag{4.5}$$

by Theorem *A*.4 (ii). Using (4.3), Lemma 4.6 and (4.5) we then obtain the following implications:

$$\begin{aligned} \mu \in \widehat{\mathcal{M}}_{\mathfrak{g}} \quad &\Leftrightarrow \quad \int_1^\infty \frac{\overleftrightarrow{\mu}(r)}{r^3}\mathfrak{g}_\star(r)\,\mathrm{d}r < \infty \\ \left\{\begin{matrix} \overset{\text{if } \alpha<2}{\Longleftrightarrow} \\ \underset{\text{if } \alpha=2}{\Longrightarrow} \end{matrix}\right\} \quad &\int_1^\infty \frac{\overleftrightarrow{\mu}(r)}{r\mathfrak{g}(r)}\,\mathrm{d}r < \infty \quad \Leftrightarrow \quad \int_{[1,\infty)} \left(\int_r^\infty \frac{1}{t\mathfrak{g}(t)}\,\mathrm{d}t\right) \mathrm{d}\tilde{\mu}(r) < \infty \\ \left\{\begin{matrix} \overset{\text{if } \alpha>0}{\Longleftrightarrow} \\ \underset{\text{if } \alpha=0}{\Longrightarrow} \end{matrix}\right\} \quad &\int_{[1,\infty)} \frac{\mathrm{d}\tilde{\mu}(r)}{\mathfrak{g}(r)} < \infty \quad \Leftrightarrow \quad \mu \in \mathcal{M}_{\mathfrak{g}}, \end{aligned}$$

which shows all assertions. ❑

The following proposition contains the core of the argument in the proof of Theorem 4.10 below. However, it is more flexible and is also used in Section 6.1.

4.7 Proposition. *Let* H *be a Hamiltonian defined on some interval* $[a,b)$, *and assume that neither* h_1 *nor* h_2 *vanishes on a neighbourhood of the left endpoint* a. *Let* $\mathfrak{f}$ *be a continuous, non-decreasing, regularly varying function, and denote by* μ_H *the spectral measure of* H.

Then the statements

(i) $\exists a' \in (a,b)$ *such that*

$$\int_a^{a'} h_1(x) \cdot \mathfrak{f}\big((m_1 m_2)(t)^{-\frac{1}{2}}\big)\,\mathrm{d}t < \infty;$$

(ii)

$$\int_1^\infty \overleftrightarrow{\mu}_H(r) \frac{\mathfrak{f}(r)}{r^3}\,\mathrm{d}r < \infty; \tag{4.6}$$

(iii) $\exists a' \in (a,b)$ *such that*

$$\int_a^{a'} \frac{1}{m_2(t)^2} \begin{pmatrix} m_2(t) \\ -m_3(t) \end{pmatrix}^* H(t) \begin{pmatrix} m_2(t) \\ -m_3(t) \end{pmatrix} \cdot \mathfrak{f}\big((m_1 m_2)(t)^{-\frac{1}{2}}\big)\,\mathrm{d}t < \infty.$$

satisfy (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

If, in addition, $\mathfrak{f}$ *is differentiable and* $\mathfrak{f}'$ *is regularly varying, then*

$$\text{(i)} \quad \Leftrightarrow \quad \exists a' \in (a,b).\ \int_a^{a'} m_1(t)\mathfrak{f}'\big((m_1m_2)(t)^{-\frac{1}{2}}\big)\frac{(m_1m_2)'(t)}{(m_1m_2)(t)^{\frac{3}{2}}}\,\mathrm{d}t < \infty,$$

$$\text{(iii)} \quad \Leftrightarrow \quad \exists a' \in (a,b).\ \int_a^{a'} \frac{\det M(t)}{m_2(t)}\mathfrak{f}'\big((m_1m_2)(t)^{-\frac{1}{2}}\big)\frac{(m_1m_2)'(t)}{(m_1m_2)(t)^{\frac{3}{2}}}\,\mathrm{d}t < \infty.$$

4.8 Remark. It can be seen from the proof below that, in the first part, instead of assuming that $\mathfrak{f}$ is regularly varying, it is sufficient to assume that $\mathfrak{f} \in OR$, i.e. for every $\lambda > 0$ there exist $c_1, c_2 > 0$ such that $c_1 \leq \frac{\mathfrak{f}(\lambda r)}{\mathfrak{f}(r)} \leq c_2$, $r \in [1,\infty)$; for the latter definition see, e.g. [4, §2.0.2]. $\Diamond$

For the proof of Proposition 4.7 we use a simple Abelian–Tauberian-type theorem for the Poisson integral of a positive measure. This is folklore; an explicit proof can be found in, e.g. [20, Lemma 4][4].

4.9 Lemma. *Let* μ *be a positive Borel measure on* $\mathbb{R}$ *with* $\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty$*, let* $r_0 > 0$*, and let* ξ *be a positive Borel measure on* $[r_0,\infty)$*. Define* $\overset{\leftrightarrow}{\mu}$ *as in* (4.1)*, set*

$$\vec{\xi}(r) := \xi([r_0,r)), \quad r \geq r_0,$$

and let $\mathcal{P}[\mu](z)$ *be the Poisson integral*

$$\mathcal{P}[\mu](z) := \int_{\mathbb{R}} \operatorname{Im}\frac{1}{t-z}\,\mathrm{d}\mu(t), \quad z \in \mathbb{C}^+, \tag{4.7}$$

of μ*. Then*

$$\int_{[r_0,\infty)} \frac{1}{r}\mathcal{P}[\mu](ir)\,\mathrm{d}\xi(r) < \infty \quad \Leftrightarrow \quad \int_{[r_0,\infty)} \frac{\overset{\leftrightarrow}{\mu}(r)\vec{\xi}(r)}{r^3}\,\mathrm{d}r < \infty.$$

Proof of Proposition 4.7. First note that finiteness of the integrals in the proposition clearly does not depend on $a' \in (a,b)$.

Let ξ be the measure on $[1,\infty)$ such that $\mathfrak{f}(r) = \xi([1,r))$, $r \geq 1$. It follows from Lemma 4.9 that

$$\int_{[1,\infty)} \frac{\mathcal{P}[\mu_H](ir)}{r}\,\mathrm{d}\xi(r) < \infty \quad \Leftrightarrow \quad \int_1^\infty \frac{\overset{\leftrightarrow}{\mu}_H(r)\mathfrak{f}(r)}{r^3}\,\mathrm{d}r < \infty.$$

Fix $\eta \in (0, 1-\frac{1}{\sqrt{2}})$ and let $\mathring{r}$ be as in (2.14). By Theorem 1.1 with $A(r)$ and $L(r)$ in the form of (2.15) and (2.16), we have

$$\frac{\det M(\mathring{t}(r))}{m_2(\mathring{t}(r))} \lesssim \frac{\mathcal{P}[\mu_H](ir)}{r} \lesssim m_1(\mathring{t}(r)).$$

Hence

$$\int_{[1,\infty)} m_1\big(\mathring{t}(r)\big)\,\mathrm{d}\xi(r) < \infty \;\Rightarrow\; (4.6) \;\Rightarrow\; \int_1^\infty \frac{\det M\big(\mathring{t}(r)\big)}{m_2\big(\mathring{t}(r)\big)}\,\mathrm{d}\xi(r) < \infty. \tag{4.8}$$

Let ν be the measure on $(0,\infty)$ such that $\nu((r,\infty)) = m_1(\mathring{t}(r))$, $r > 0$, and let $\mathring{t}_*\nu$ be the push-forward measure of ν under the mapping $\mathring{t}$. For $t \in (a,b)$ we have $\mathring{t}_*\nu((a,t)) = \nu((\mathring{t}^{-1}(t),\infty)) = m_1(t)$. Moreover, recall that $\mathring{r}$, defined in (2.14), is the inverse function of $\mathring{t}$. Hence, with Lemma 4.6 we can rewrite the first integral in (4.8) as follows:

$$\begin{aligned}\int_{[1,\infty)} m_1\big(\mathring{t}(r)\big)\,\mathrm{d}\xi(r) &= \int_{[1,\infty)} \nu\big((r,\infty)\big)\,\mathrm{d}\xi(r) = \int_{[1,\infty)} \vec{\xi}(r)\,\mathrm{d}\nu(r) = \int_{[1,\infty)} \mathfrak{f}(r)\,\mathrm{d}\nu(r)\\ &= \int_{(a,\mathring{t}(1)]} \mathfrak{f}\big(\mathring{r}(t)\big)\,\mathrm{d}(\mathring{t}_*\nu)(t) = \int_{(a,\mathring{t}(1)]} \mathfrak{f}\big(\mathring{r}(t)\big)\,\mathrm{d}m_1(t) = \int_a^{\mathring{t}(1)} \mathfrak{f}\big(\mathring{r}(t)\big)h_1(t)\,\mathrm{d}t.\end{aligned}$$

[4]For the case when $\mathfrak{g}(r) = r^\alpha$ it goes back at least to [17, Theorema 1].

Since $\mathpzc{f}$ is regularly varying, the last integral is finite if and only the integral in (i) is finite.

In a similar way one can rewrite the last integral in (4.8) by using a measure ν such that $\nu((r,\infty)) = \frac{\det M(\mathring{t}(r))}{m_2(\mathring{t}(r))}$, $r>0$, which is possible since $t \mapsto \frac{\det M(t)}{m_2(t)}$ is non-decreasing by (2.17).

For the last part let us assume that $\mathpzc{f}$ is differentiable and that $\mathpzc{f}'$ is regularly varying. Using a substitution we can rewrite the first integral in (4.8) differently:

$$\begin{aligned}\int_{[1,\infty)} m_1\big(\mathring{t}(r)\big)\,\mathrm{d}\xi(r) &= \int_1^\infty m_1\big(\mathring{t}(r)\big)\mathpzc{f}'(r)\,\mathrm{d}r = \int_{\mathring{t}(1)}^a m_1(t)\mathpzc{f}'\big(\mathring{r}(t)\big)\mathring{r}'(t)\,\mathrm{d}t\\ &= \frac{\eta}{4}\int_a^{\mathring{t}(1)} m_1(t)\mathpzc{f}'\big(\mathring{r}(t)\big)\frac{(m_1m_2)'(t)}{(m_1m_2)(t)^{\frac{3}{2}}}\,\mathrm{d}t.\end{aligned}$$

Since $\mathpzc{f}'$ is assumed to be regularly varying, the last integral is finite if and only if it is finite with $\mathpzc{f}'(\mathring{r}(t))$ replaced by $\mathpzc{f}'((m_1m_2)(t)^{-\frac{1}{2}})$. In exactly the same way one can rewrite the last integral in (4.8). ❑

The following theorem is the main result of this subsection. It provides, in particular, information when the spectral measure μ_H belongs to the class $\widehat{\mathcal{M}}_{\mathpzc{g}}$.

4.10 Theorem. *Let H be a Hamiltonian defined on some interval $[a,b)$, and assume that neither h_1 nor h_2 vanishes on a neighbourhood of the left endpoint a. Let $\mathpzc{g}$ be a continuous, regularly varying function with index $\alpha \leq 2$ and $\lim_{r\to\infty}\mathpzc{g}(r)=\infty$, let $\mathpzc{g}_\star$ be as in (4.2), and denote by μ_H the spectral measure of H as in (1.3). For every $a' \in (a,b)$ the following statements*

(i) $$\int_a^{a'} h_1(t)\cdot \mathpzc{g}_\star\big((m_1m_2)(t)^{-\frac{1}{2}}\big)\,\mathrm{d}t < \infty,$$

(i)′ $$\int_a^{a'} m_1(t)\cdot \frac{(m_1m_2)'(t)}{(m_1m_2)(t)^2\mathpzc{g}\big((m_1m_2)(t)^{-\frac{1}{2}}\big)}\,\mathrm{d}t < \infty,$$

(i)″ $$\int_a^{a'} h_1(t)\cdot \frac{\mathrm{d}t}{(m_1m_2)(t)\mathpzc{g}\big((m_1m_2)(t)^{-\frac{1}{2}}\big)} < \infty,$$

(ii) $\mu_H \in \widehat{\mathcal{M}}_{\mathpzc{g}}$,

(ii)′ $\mu_H \in \mathcal{M}_{\mathpzc{g}}$,

(iii) $$\int_a^{a'} \frac{1}{m_2(t)^2}\binom{m_2(t)}{-m_3(t)}^{*} H(t)\binom{m_2(t)}{-m_3(t)}\cdot \mathpzc{g}_\star\big((m_1m_2)(t)^{-\frac{1}{2}}\big)\,\mathrm{d}t < \infty,$$

(iii)′ $$\int_a^{a'} \frac{\det M(t)}{m_2(t)}\cdot \frac{(m_1m_2)'(t)}{(m_1m_2)(t)^2\mathpzc{g}\big((m_1m_2)(t)^{-\frac{1}{2}}\big)}\,\mathrm{d}t < \infty,$$

(iii)″ $$\int_a^{a'} \frac{1}{m_2(t)^2}\binom{m_2(t)}{-m_3(t)}^{*} H(t)\binom{m_2(t)}{-m_3(t)}\frac{\mathrm{d}t}{(m_1m_2)(t)\mathpzc{g}\big((m_1m_2)(t)^{-\frac{1}{2}}\big)} < \infty$$

satisfy the relations:

$$\begin{array}{ccccc}
\text{(i)} & \Longleftrightarrow & \text{(i)}' & \underset{\textit{if } \alpha=2}{\overset{\textit{if } \alpha<2}{\Longleftrightarrow \atop \Longrightarrow}} & \text{(i)}''\\
\Downarrow & & & & \\
\text{(ii)} & \underset{\textit{if } \alpha\in\{0,2\}}{\overset{\textit{if } \alpha\in(0,2)}{\Longleftrightarrow \atop \Longrightarrow}} & \text{(ii)}' & & \\
\Downarrow & & & & \\
\text{(iii)} & \Longleftrightarrow & \text{(iii)}' & \underset{\textit{if } \alpha=2}{\overset{\textit{if } \alpha<2}{\Longleftrightarrow \atop \Longrightarrow}} & \text{(iii)}''.
\end{array}$$

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and the equivalences (i) $\Leftrightarrow$ (i)$'$ and (iii) $\Leftrightarrow$ (iii)$'$ follow directly from Proposition 4.7 with $\mathfrak{f} = \mathfrak{g}_\star$. The relations between (i) and (i)$''$ and between (iii) and (iii)$''$ follow from (4.3). Finally, Proposition 4.5 implies the relations between (ii) and (ii)$'$. ❑

4.11 Remark. Let us consider the case of the regularly varying function $\mathfrak{g}(r) = r^\alpha(\log r)^\beta$ with $\alpha \in [0,2]$ and $\beta \in \mathbb{R}$. It is easy to check that (i)$''$ is equivalent to

$$\int_a^{a'} \frac{h_1(t)\,\mathrm{d}t}{\big[(m_1m_2)(t)\big]^{1-\frac{\alpha}{2}}\big|\log\big((m_1m_2)(t)\big)\big|^\beta} < \infty.$$

◊

The following corollary shows that for diagonally-dominant Hamiltonians we obtain a characterisation when μ_H belongs to $\widehat{\mathcal{M}}_{\mathfrak{g}}$.

4.12 Corollary. *Consider the situation from Theorem* 4.10*, and assume, in addition, that* $\limsup_{t\to a} \frac{m_3(t)^2}{(m_1m_2)(t)} < 1$ (*this holds in particular if H is diagonal*). *Then, for every $a' \in (a,b)$, also* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof. The additional hypothesis just means that $\frac{\det M(t)}{m_2(t)} \asymp m_1(t)$, which implies that (i)$'$ $\Leftrightarrow$ (iii)$'$. ❑

4.13 Example. Let $\rho_1, \rho_2 > 0$ and let H be a Hamiltonian on $[0,b)$ with $0 < b \leq \infty$ such that $h_1(t) \asymp t^{\rho_1-1}$ and $m_2(t) \asymp t^{\rho_2}$ as $t \to 0$ and $\limsup_{t\to 0} \frac{m_3(t)^2}{(m_1m_2)(t)} < 1$. Let us consider the regularly function $\mathfrak{g}(r) = r^\alpha(\log r)^\beta$ with $\alpha \in (0,2)$ and $\beta \in \mathbb{R}$. It follows from Theorem 4.10 and Remark 4.11 that $\mu_H \in \mathcal{M}_{\mathfrak{g}}$ if and only if

$$\int_0^{a'} \frac{t^{\rho_1-1}}{t^{(\rho_1+\rho_2)(1-\frac{\alpha}{2})}|\log t|^\beta}\,\mathrm{d}t < \infty,$$

which, in turn, is equivalent to

$$\alpha > \frac{2\rho_2}{\rho_1+\rho_2} \qquad \text{or} \qquad \Big(\alpha = \frac{2\rho_2}{\rho_1+\rho_2} \quad \text{and} \quad \beta > 1\Big).$$

◊

4.2 Limit superior conditions

In this section we investigate lim sup-conditions for the quotient $\frac{\overleftrightarrow{\mu}_H(r)}{\mathfrak{g}(r)}$ instead of integrability conditions. Let us introduce the corresponding classes of measures.

4.14 Definition. Let $\mathfrak{g}(r)$ be a regularly varying function with index $\alpha \leq 2$ and $\lim_{r\to\infty} \mathfrak{g}(r) = \infty$. Then we set

$$\mathcal{F}_{\mathfrak{g}} := \big\{\mu\colon\ \overleftrightarrow{\mu}(r) = \mathrm{O}(\mathfrak{g}(r))\big\}, \qquad \mathcal{F}^0_{\mathfrak{g}} := \big\{\mu\colon\ \overleftrightarrow{\mu}(r) = \mathrm{o}(\mathfrak{g}(r))\big\},$$

where again $\overleftrightarrow{\mu}(r) := \mu((-r,r))$. ◊

Clearly, we have $\mathcal{F}^0_{\mathfrak{g}} \subseteq \mathcal{F}_{\mathfrak{g}}$. In the next proposition relations among $\mathcal{F}_{\mathfrak{g}}$, $\mathcal{F}^0_{\mathfrak{g}}$ and the classes $\mathcal{M}_{\mathfrak{g}}$ and $\widehat{\mathcal{M}}_{\mathfrak{g}}$ from Section 4.1 are discussed.

4.15 Proposition. *Let $\mathfrak{g}$, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be continuous, regularly varying functions with indices $\alpha, \alpha_1, \alpha_2 \leq 2$ respectively, such that $\mathfrak{g}(r), \mathfrak{g}_1(r), \mathfrak{g}_2(r) \to \infty$ as $r \to \infty$.*

(i) *If $\mathfrak{g}$ is non-decreasing, then $\mathcal{M}_{\mathfrak{g}} \subseteq \mathcal{F}^0_{\mathfrak{g}}$.*

(ii) *Assume that*

$$\alpha_2 < 2 \qquad \text{and} \qquad \int_1^\infty \frac{g_1(r)}{r g_2(r)} \, \mathrm{d}r < \infty \tag{4.9}$$

or that $\alpha_1 < \alpha_2$. *Then* $\mathcal{F}_{g_1} \subseteq \widehat{\mathcal{M}}_{g_2}$.

Proof.

(i) Let $\mu \in \mathcal{M}_g$, i.e. $\int_1^\infty \frac{1}{g(t)} \, \mathrm{d}\tilde{\mu}(t) < \infty$, where $\tilde{\mu}$ is as in (4.1). Lemma 4.6 (with a measure ν such that $\nu((t,\infty)) = 1/g(t)$) yields

$$\lim_{r\to\infty} \frac{\overset{\leftrightarrow}{\mu}(r)}{g(r)} = \lim_{r\to\infty} \nu\big((r,\infty)\big)\tilde{\mu}\big([0,r)\big) = 0.$$

(ii) Let $\mu \in \mathcal{F}_{g_1}$. Then

$$\overset{\leftrightarrow}{\mu}(r)\frac{(g_2)_\star(r)}{r^3} \lesssim g_1(r)\frac{(g_2)_\star(r)}{r^3}. \tag{4.10}$$

First, we consider the case when (4.9) is satisfied. It follows from (4.3) that

$$g_1(r)\frac{(g_2)_\star(r)}{r^3} \asymp g_1(r)\frac{r^2}{g_2(r)} \cdot \frac{1}{r^3}.$$

This, together with (4.10) and the second relation in (4.9), implies that $\int_1^\infty \overset{\leftrightarrow}{\mu}(r)\frac{(g_2)_\star(r)}{r^3} \, \mathrm{d}r < \infty$ and hence $\mu \in \widehat{\mathcal{M}}_{g_2}$. Let us now assume that $\alpha_1 < \alpha_2$. It follows from the sentence around (4.3) that $(g_2)_\star$ is regularly varying with index $2 - \alpha_2$. Hence the right-hand side of (4.10) is regularly varying with index $\alpha_1 + 2 - \alpha_2 - 3 < -1$ and therefore integrable by Theorem *A*.2. It follows again that $\mu \in \widehat{\mathcal{M}}_{g_2}$. ❑

4.16 Remark.

(i) If $\alpha > 0$ in Proposition 4.15 (i), then the assumption that g is non-decreasing is not necessary because, by Theorem *A*.3, there exists a non-decreasing, regularly varying function $\bar{g}$ such that $\bar{g}(r) \sim g(r)$ as $r \to \infty$.

(ii) The inclusion in Proposition 4.15 (i) is the analogue on the level of measures to the fact that an entire function of convergence class is necessarily of minimal type.

(iii) Let g_1, g_2 be non-decreasing functions as in Proposition 4.15 (ii). We can combine items (i) and (ii) of Proposition 4.15 with Proposition 4.5 to obtain the following chain of inclusions:

$$\mathcal{M}_{g_1} \subseteq \mathcal{F}^0_{g_1} \subseteq \mathcal{F}_{g_1} \subseteq \widehat{\mathcal{M}}_{g_2} \subseteq \mathcal{M}_{g_2}. \tag{4.11}$$

This is satisfied, in particular, when $g_i(r) = r^{\alpha_i}$, $i = 1, 2$, with $\alpha_1 < \alpha_2$, or when $g_i(r) = r^\alpha (\log r)^{\beta_i}$ with $\beta_2 > \beta_1 + 1$.

◊

The next theorem can be viewed as a generalisation of [24, Theorem 4] to the case of a non-diagonal Hamiltonian; the connection is worked out in detail in Section 6.2.

4.17 Theorem. *Let H be a Hamiltonian defined on some interval $[a,b)$, and assume that neither h_1 nor h_2 vanishes on a neighbourhood of the left endpoint a. Let $g(r)$ be a regularly varying function with index $\alpha \leq 2$ and $\lim_{r\to\infty} g(r) = \infty$, and denote by μ_H the spectral measure of H. Then the following implications hold:*

(i) $\displaystyle \limsup_{t\to a} \frac{m_1(t)}{(m_1m_2)(t) g\big((m_1m_2)(t)^{-\frac{1}{2}}\big)} < \infty \;\Rightarrow\; \mu_H \in \mathcal{F}_g$;

(ii) $\displaystyle \lim_{t\to a} \frac{m_1(t)}{(m_1m_2)(t) g\big((m_1m_2)(t)^{-\frac{1}{2}}\big)} = 0 \;\Rightarrow\; \mu_H \in \mathcal{F}^0_g$.

If, in addition, $\alpha < 2$, then

(iii) $\displaystyle \mu_H \in \mathcal{F}_g \;\Rightarrow\; \limsup_{t\to a} \frac{\det M(t)/m_2(t)}{(m_1m_2)(t) g\big((m_1m_2)(t)^{-\frac{1}{2}}\big)} < \infty$;

(iv) $\mu_H \in \mathcal{F}^0_{\mathscr{g}} \quad\Rightarrow\quad \lim_{t\to a} \dfrac{\det M(t)/m_2(t)}{(m_1m_2)(t)\mathscr{g}\big((m_1m_2)(t)^{-\frac{1}{2}}\big)} = 0.$

The proof of Theorem 4.17 relies on the following Abelian–Tauberian-type result. Most probably this fact is folklore, but we do not know an explicit reference and therefore provide the proof.

4.18 Lemma. *Let μ be a positive Borel measure on $\mathbb{R}$ with $\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty$, define $\overleftrightarrow{\mu}(r)$ as in (4.1), denote by $\mathcal{P}[\mu](z)$ the Poisson integral of μ as in (4.7), and let $\mathscr{g}(r)$ be regularly varying with index $\alpha \in [0,2]$.*

(i) *We have*

$$\limsup_{r\to\infty}\Big(\frac{r}{\mathscr{g}(r)}\mathcal{P}[\mu](ir)\Big) \geq \Big(1-\frac{\alpha}{2}\Big)^{1-\frac{\alpha}{2}}\Big(\frac{\alpha}{2}\Big)^{\frac{\alpha}{2}} \cdot \limsup_{r\to\infty}\frac{\overleftrightarrow{\mu}(r)}{\mathscr{g}(r)}, \tag{4.12}$$

where the constant in front of $\limsup$ *on the right-hand side is interpreted as* 1 *if* α *equals* 0 *or* 2.

(ii) *If $\alpha < 2$ and $\lim_{r\to\infty}\mathscr{g}(r) = \infty$, then*

$$\limsup_{r\to\infty}\Big(\frac{r}{\mathscr{g}(r)}\mathcal{P}[\mu](ir)\Big) \leq \mathrm{B}\Big(1+\frac{\alpha}{2}, 1-\frac{\alpha}{2}\Big) \cdot \limsup_{r\to\infty}\frac{\overleftrightarrow{\mu}(r)}{\mathscr{g}(r)}, \tag{4.13}$$

where B *denotes Euler's beta function.*

Proof. For the proof of (i) observe that, for every $x > 0$ and $r > 0$,

$$\begin{aligned}
\frac{r}{\mathscr{g}(r)}\mathcal{P}[\mu](ir) &= \frac{r}{\mathscr{g}(r)}\int_{\mathbb{R}}\frac{r}{t^2+r^2}\,\mathrm{d}\mu(t) \geq \frac{r}{\mathscr{g}(r)}\int_{(-xr,xr)}\frac{r}{t^2+r^2}\,\mathrm{d}\mu(t) \\
&\geq \frac{r}{\mathscr{g}(r)}\int_{(-xr,xr)}\frac{r}{(1+x^2)r^2}\,\mathrm{d}\mu(t) = \frac{1}{1+x^2}\cdot\underbrace{\frac{\mathscr{g}(xr)}{\mathscr{g}(r)}}_{\to x^\alpha}\cdot\frac{\mu\big((-xr,xr)\big)}{\mathscr{g}(xr)}.
\end{aligned}$$

If $\alpha \in (0,2)$, then the function $x \mapsto \frac{x^\alpha}{1+x^2}$ attains the maximum at $x_0 = \sqrt{\alpha/(2-\alpha)}$; with this x_0 the above inequality yields (4.12). For $\alpha = 0$ we use arbitrarily small x, and for $\alpha = 2$ we use x that are arbitrarily close to 2.

We come to the proof of (ii). Let $\tilde{\mu}$ be the measure defined in (4.1). For every $r_0 > 0$ we estimate as follows (where we use Lemma 4.6):

$$\begin{aligned}
\mathcal{P}[\mu](ir) &= \int_{\mathbb{R}}\frac{r}{t^2+r^2}\,\mathrm{d}\mu(t) = \int_{[0,\infty)}\frac{r}{t^2+r^2}\,\mathrm{d}\tilde{\mu}(t) \\
&= r\int_0^\infty \underbrace{\tilde{\mu}\big([0,t)\big)}_{=\overleftrightarrow{\mu}(t)}\cdot\frac{2t}{(t^2+r^2)^2}\,\mathrm{d}t \\
&= r\int_0^{r_0}\overleftrightarrow{\mu}(t)\cdot\frac{2t}{(t^2+r^2)^2}\,\mathrm{d}t + r\int_{r_0}^\infty\frac{\overleftrightarrow{\mu}(t)}{\mathscr{g}(t)}\cdot\frac{2t\mathscr{g}(t)}{(t^2+r^2)^2}\,\mathrm{d}t \\
&\leq \frac{2r_0}{r^3}\int_0^{r_0}\overleftrightarrow{\mu}(t)\,\mathrm{d}t + 2r\Big(\sup_{t\geq r_0}\frac{\overleftrightarrow{\mu}(t)}{\mathscr{g}(t)}\Big)\cdot\int_{r_0}^\infty\frac{t\mathscr{g}(t)}{(t^2+r^2)^2}\,\mathrm{d}t.
\end{aligned}$$

The first summand tends to 0 when multiplied by $\frac{r}{\mathscr{g}(r)}$ and hence does not contribute to the limit superior on the left-hand side of (4.13). The integral in the second summand is estimated by

$$\int_{r_0}^\infty\frac{t\mathscr{g}(t)}{(t^2+r^2)^2}\,\mathrm{d}t \leq \int_0^\infty\frac{t\mathscr{g}(t)}{(t^2+r^2)^2}\,\mathrm{d}t = r^{\alpha-2}\int_0^\infty\frac{x^{1+\alpha}}{(x^2+1)^2}\cdot\frac{\mathscr{g}(rx)}{(rx)^\alpha}\,\mathrm{d}x.$$

Since $\alpha < 2$, we can apply [39, Theorems 2.6, 2.7] and obtain

$$\lim_{r\to\infty}\bigg(\int_0^\infty\frac{x^{1+\alpha}}{(x^2+1)^2}\cdot\frac{\mathscr{g}(rx)}{(rx)^\alpha}\,\mathrm{d}x\Big/\frac{\mathscr{g}(r)}{r^\alpha}\bigg) = \int_0^\infty\frac{x^{1+\alpha}}{(x^2+1)^2}\,\mathrm{d}x = \frac{1}{2}\,\mathrm{B}\Big(1+\frac{\alpha}{2}, 1-\frac{\alpha}{2}\Big).$$

Putting these estimates together we obtain

$$\begin{aligned}
\limsup_{r\to\infty}\Big(\frac{r}{\mathfrak{g}(r)}\mathcal{P}[\mu](ir)\Big) &\le \Big(\sup_{t\ge r_0}\frac{\overleftrightarrow{\mu}(t)}{\mathfrak{g}(t)}\Big)\limsup_{r\to\infty}\Big[\frac{2r^2}{\mathfrak{g}(r)}\int_{r_0}^{\infty}\frac{t\mathfrak{g}(t)}{(t^2+r^2)^2}\,\mathrm{d}t\Big]\\
&\le \Big(\sup_{t\ge r_0}\frac{\overleftrightarrow{\mu}(t)}{\mathfrak{g}(t)}\Big)\limsup_{r\to\infty}\Big[\frac{2r^\alpha}{\mathfrak{g}(r)}\int_{0}^{\infty}\frac{x^{1+\alpha}}{(x^2+1)^2}\cdot\frac{\mathfrak{g}(rx)}{(rx)^\alpha}\,\mathrm{d}x\Big]\\
&= \Big(\sup_{t\ge r_0}\frac{\overleftrightarrow{\mu}(t)}{\mathfrak{g}(t)}\Big)\,\mathrm{B}\Big(1+\frac{\alpha}{2},1-\frac{\alpha}{2}\Big);
\end{aligned}$$

since r_0 can be chosen arbitrarily large, (4.13) follows. ❑

Note that for $\alpha > 2$ both $\limsup$ appearing in Lemma 4.18 are equal to 0, and for $\alpha < 0$ both are equal to $+\infty$ (unless $\mu = 0$).

Proof of Theorem 4.17. Theorem 1.1 with $A(r)$ in the form (2.15), together with (4.12), implies that there exist $c_1, c_2 > 0$ such that

$$\limsup_{r\to\infty}\frac{\overleftrightarrow{\mu}_H(r)}{\mathfrak{g}(r)} \le c_1\limsup_{r\to\infty}\Big(\frac{r}{\mathfrak{g}(r)}\operatorname{Im} q_H(ir)\Big) \le c_2\limsup_{r\to\infty}\Big(\frac{r^2}{\mathfrak{g}(r)}m_1\big(\mathring{t}(r)\big)\Big).$$

With the substitution $t = \mathring{t}(r)$ the last $\limsup$ can be rewritten as

$$\begin{aligned}
\limsup_{r\to\infty}\Big(\frac{r^2}{\mathfrak{g}(r)}m_1\big(\mathring{t}(r)\big)\Big) &= \limsup_{t\to a}\frac{\big(\frac{\eta}{2}(m_1m_2)(t)^{-\frac12}\big)^2}{\mathfrak{g}\big(\frac{\eta}{2}(m_1m_2)(t)^{-\frac12}\big)}m_1(t)\\
&= \frac{(\frac{\eta}{2})^2}{(\frac{\eta}{2})^\alpha}\limsup_{t\to a}\frac{m_1(t)}{(m_1m_2)(t)\mathfrak{g}\big((m_1m_2)(t)^{-\frac12}\big)}.
\end{aligned}$$

Items (iii) and (iv) are shown in a similar way, when Theorem 1.1 with $L(r)$ in the form (2.16) and (4.13) are used. ❑

Analogously to Corollary 4.12, we obtain the obvious corollary.

4.19 Corollary. *Consider the situation from Theorem* 4.17*, and assume in addition that* $\limsup_{t\to a}\frac{m_3(t)^2}{(m_1m_2)(t)} < 1$ *(this holds in particular if H is diagonal). Moreover, let $\alpha\in[0,2)$. Then*

$$\mu_H\in\mathcal{F}_{\mathfrak{g}} \;\Leftrightarrow\; \limsup_{t\to a}\frac{m_1(t)}{(m_1m_2)(t)\mathfrak{g}\big((m_1m_2)(t)^{-\frac12}\big)}<\infty,$$

$$\mu_H\in\mathcal{F}^0_{\mathfrak{g}} \;\Leftrightarrow\; \lim_{t\to a}\frac{m_1(t)}{(m_1m_2)(t)\mathfrak{g}\big((m_1m_2)(t)^{-\frac12}\big)}=0.$$

We finish this section with two examples, the first of which is also used in Section 6.4.

4.20 Example. Let $\rho_1,\rho_2>0$ and let H be a Hamiltonian on $[0,b)$ with $0<b\le\infty$ such that $m_j(t)\asymp t^{\rho_j}$, $j=1,2$, as $t\to 0$ and $\limsup_{t\to 0}\frac{m_3(t)^2}{(m_1m_2)(t)}<1$. Let us consider $\mathfrak{g}_\alpha(r):=r^\alpha$ with $\alpha\in(0,2)$. Then

$$\frac{m_1(t)}{(m_1m_2)(t)\mathfrak{g}_\alpha\big((m_1m_2)(t)^{-\frac12}\big)} = m_1(t)^{\frac{\alpha}{2}}m_2(t)^{\frac{\alpha}{2}-1}\asymp t^{\rho_1\frac{\alpha}{2}+\rho_2(\frac{\alpha}{2}-1)}.$$

It follows from Corollary 4.19 that

$$\begin{aligned}
\mu_H\in\mathcal{F}_{\mathfrak{g}_\alpha}\setminus\mathcal{F}^0_{\mathfrak{g}_\alpha} \quad&\Leftrightarrow\quad 0<\limsup_{t\to 0}\frac{m_1(t)}{(m_1m_2)(t)\mathfrak{g}_\alpha\big((m_1m_2)(t)^{-\frac12}\big)}<\infty\\
&\Leftrightarrow\quad \rho_1\frac{\alpha}{2}+\rho_2\Big(\frac{\alpha}{2}-1\Big)=0 \quad\Leftrightarrow\quad \alpha=\frac{2\rho_2}{\rho_1+\rho_2}.
\end{aligned}$$

Now (4.11) implies that

$$\mu_H \in \mathcal{M}_{\mathfrak{g}_\alpha} \quad \Leftrightarrow \quad \alpha > \frac{2\rho_2}{\rho_1+\rho_2}\,.$$

Moreover, we have $\mu_H \in \mathcal{F}^0_{\mathfrak{g}_\alpha}$ if $\alpha > \frac{2\rho_2}{\rho_1+\rho_2}$, and $\mu_H \notin \mathcal{F}_{\mathfrak{g}_\alpha}$ if $\alpha < \frac{2\rho_2}{\rho_1+\rho_2}$. $\Diamond$

4.21 Example. Let $\rho > 0$ and let H be a Hamiltonian on $[0,b)$ with $0 < b \leq \infty$ such that $m_1(t) \asymp e^{-\frac{1}{t}}$ and $m_2(t) \asymp t^\rho$ as $t \to 0$ and $\limsup_{t\to 0} \frac{m_3(t)^2}{(m_1m_2)(t)} < 1$. We choose the slowly varying function $\mathfrak{g}(r) = (\log r)^\beta$ with $\beta \in \mathbb{R}$. Since

$$\frac{m_1(t)}{(m_1m_2)(t)\mathfrak{g}\big((m_1m_2)(t)^{-\frac{1}{2}}\big)} \asymp \frac{e^{-\frac{1}{t}}}{e^{-\frac{1}{t}}t^\rho\big|\log\big(e^{-\frac{1}{t}}t^\rho\big)\big|^\beta} \sim t^{\beta-\rho}$$

as $t \to 0$, it follows from Corollary 4.19 that $\mu_H \in \mathcal{F}_{\mathfrak{g}} \setminus \mathcal{F}^0_{\mathfrak{g}}$ if and only if $\beta = \rho$. $\Diamond$

5 Further discussion of the main theorem

5.1 A monotonicity property

In many situations one cannot determine the solution $\mathring{t}(r)$ of (1.5) exactly. Having a lower bound can often be sufficient to obtain estimates for $A(r)$ and $L(r)$. With the following lemma we can easily prove Corollary 5.2 below.

5.1 Lemma. *Let H be as in Theorem 1.1, and let $r > 0$. If $\hat{t} \in (\mathring{a}, \mathring{t}(r))$, then*

$$\frac{2}{\eta}\cdot rm_1(\hat{t}) \leq A(r) \leq \frac{\eta}{2}\cdot\frac{1}{rm_2(\hat{t})}, \qquad \frac{2}{\eta}\cdot\frac{r\det M(\hat{t})}{m_2(\hat{t})} \leq L(r).$$

If $\check{t} \in (\mathring{t}(r), b)$, then

$$\frac{\eta}{2}\cdot\frac{1}{rm_2(\check{t})} \leq A(r) \leq \frac{2}{\eta}\cdot rm_1(\check{t}), \qquad L(r) \leq \frac{2}{\eta}\cdot\frac{r\det M(\check{t})}{m_2(\check{t})}.$$

Proof. Since m_1 and m_2 are non-decreasing functions, the assertions involving $A(r)$ are obvious from (2.15). To show the assertion involving $L(r)$, we use the middle term in (2.16) and the monotonicity of $\frac{\det M}{m_2}$, shown in (2.17). ❑

The following statement is now obvious.

5.2 Corollary. *Let H be a Hamiltonian defined on some interval $[a,b)$, and assume that neither $h_1 = 0$ a.e. nor $h_2 = 0$ a.e.*

Let $\hat{t}\colon (0,\infty) \to (\mathring{a}, b)$ be a function such that for all $r > 0$

$$\frac{\eta}{2}(m_1m_2)^{-\frac{1}{2}}\big(\hat{t}(r)\big) \geq r. \tag{5.1}$$

Then, for each $\vartheta \in (0,\pi)$,

$$rm_1(\hat{t}(r)) \lesssim \big|q_H\big(re^{i\vartheta}\big)\big| \lesssim \frac{1}{rm_2(\hat{t}(r))},$$
$$\big|\operatorname{Re} q_H\big(re^{i\vartheta}\big)\big| \lesssim \frac{1}{rm_2(\hat{t}(r))},$$
$$\frac{r\det M(\hat{t}(r))}{m_2(\hat{t}(r))} \lesssim \operatorname{Im} q_H\big(re^{i\vartheta}\big) \lesssim \frac{1}{rm_2(\hat{t}(r))}.$$

The various constants in "$\lesssim$" depend on ϑ but not on H.

Note that the validity of (5.1) just means that the points $(r, \hat{t}(r))$ belong to the dotted region in Figure 1.

5.2 Hamiltonians starting with an indivisible interval of type 0 or $\pi/2$

For $\phi \in \mathbb{R}$ we set

$$\xi_\phi := \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}.$$

5.3 Definition. Let H be a Hamiltonian defined on some interval $[a,b)$. An interval $(c,d) \subseteq (a,b)$ is called *H-indivisible* if there exists $\phi \in \mathbb{R}$ such that

$$H(t) = \operatorname{tr} H(t) \cdot \xi_\phi \xi_\phi^*, \qquad t \in (c,d) \text{ a.e.}$$

Equivalently, one may say that $J\xi_\phi = (-\sin\phi, \cos\phi)^* \in \ker H(t)$ for $t \in (c,d)$ a.e.

The number ϕ is uniquely determined up to integer multiples of π; it will always be understood modulo π, and is called the *type* of the indivisible interval (c,d). $\Diamond$

The property that $\mathring{a} > a$ can now be formulated as: H does not start with an indivisible interval of type 0 or $\frac{\pi}{2}$.

Although we can apply Theorem 1.1 in the situation when $\mathring{a} > a$, we can also split off an interval of type 0 or $\frac{\pi}{2}$ adjacent to a to get an asymptotic expansion with a leading-order term, which is a power of z with exponent 1 or -1, and an estimate for the remainder term.

5.4 Remark. Assume that H starts with an indivisible interval of type 0, i.e. let $(a,\mathring{a})$ with $\mathring{a} > a$ be the maximal interval such that the Hamiltonian H is of the form

$$H(t) = \begin{pmatrix} h_1(t) & 0 \\ 0 & 0 \end{pmatrix}, \qquad t \in (a,\mathring{a}) \text{ a.e.}$$

Then

$$\lim_{r\to\infty} \frac{1}{ir} q_H(ir) = m_1(\mathring{a}) = \int_a^{\mathring{a}} h_1(t)\,\mathrm{d}t = \beta_H, \tag{5.2}$$

where β_H is as in (1.3); see, e.g. [21, 42]. Nevertheless, Theorem 1.1 provides additional information as it yields explicit bounds. Clearly, (5.2) implies that $A(r) \asymp 1$, $r \to \infty$, which can also be seen directly from the behaviour of the functions m_i. Further, we also obtain a good lower bound for the imaginary part, which is seen as follows. For $t > \mathring{a}$ we have

$$\begin{aligned} m_3(t)^2 &= \left| \int_{\mathring{a}}^t h_3(s)\,\mathrm{d}s \right|^2 \le \left[\int_{\mathring{a}}^t |h_3(s)|\,\mathrm{d}s \right]^2 \le \left[\int_{\mathring{a}}^t \sqrt{h_1(s)h_2(s)}\,\mathrm{d}s \right]^2 \\ &\le \int_{\mathring{a}}^t h_1(s)\,\mathrm{d}s \cdot \int_{\mathring{a}}^t h_2(s)\,\mathrm{d}s = \big(m_1(t) - m_1(\mathring{a})\big) m_2(t), \end{aligned}$$

which implies

$$\begin{aligned} 1 &\ge \limsup_{t\searrow\mathring{a}} \frac{\det M(t)}{m_1(t)m_2(t)} \ge \liminf_{t\searrow\mathring{a}} \frac{\det M(t)}{m_1(t)m_2(t)} \\ &\ge \liminf_{t\searrow\mathring{a}} \frac{m_1(t)m_2(t) - \big(m_1(t) - m_1(\mathring{a})\big)m_2(t)}{m_1(t)m_2(t)} = \liminf_{t\searrow\mathring{a}} \frac{m_1(\mathring{a})}{m_1(t)} = 1. \end{aligned}$$

From this we obtain $L(r) \sim A(r)$, $r \to \infty$.

On the other hand, we can also split off the indivisible interval; namely, one can show that

$$q_H(z) = m_1(\mathring{a}) z + q_{\widehat{H}}(z)$$

where $\widehat{H} = H|_{(\mathring{a},b)}$, and then apply Theorem 1.1 to obtain an estimate for the remainder term.

Note that the case when $h_2 = 0$ a.e. on $(a,\mathring{a})$ with $\mathring{a} > a$ corresponds to μ_H including a "point mass at infinity". $\Diamond$

5.5 Remark. Assume now that H starts with an indivisible interval of type $\frac{\pi}{2}$, i.e. let $(a,\mathring{a})$ with $\mathring{a} > a$ be the maximal interval such that the Hamiltonian H is of the form

$$H(t) = \begin{pmatrix} 0 & 0 \\ 0 & h_2(t) \end{pmatrix}, \qquad t \in (a,\mathring{a}) \text{ a.e.}$$

Then the representation in (1.3) can be rewritten as $q_H(z) = \int_{\mathbb{R}} \frac{1}{t-z}\,\mathrm{d}\mu_H(t)$ with the finite measure μ_H, and

$$\lim_{r\to\infty}\big(-irq_H(ir)\big) = \frac{1}{m_2(\mathring{a})} = \mu_H(\mathbb{R}).$$

Again Theorem 1.1 can be applied to obtain explicit bounds, and as in Remark 5.4, we have $L(r) \sim A(r)$. One can also split off the indivisible interval, namely, q_H can be rewritten as

$$q_H(z) = -\frac{1}{m_2(\mathring{a})z - \frac{1}{q_{\check{H}}(z)}},$$

where $\check{H} = H|_{(\mathring{a},b)}$, and one can apply Theorem 1.1 to $\check{H}$ in order to obtain an estimate for the remainder term. ◊

Assume we are given a Hamiltonian H which starts with a finite number of consecutive indivisible intervals whose types alternate between 0 and $\frac{\pi}{2}$. Iterating the splitting-off procedure from Remarks 5.4 and 5.5 we obtain a representation of q_H as a continued fraction of finite length and a certain remainder term to which Theorem 1.1 can be applied. Thus, we obtain information about the size of the remainder.

5.3 A rotation transformation

Sometimes it is possible to improve the bounds (1.8) by applying a transformation to H. In this subsection we consider the situation when q_H has a real, non-zero limit. More specifically, let us assume that

$$\lim_{r\to\infty} q_H(ir) = \cot\phi \tag{5.3}$$

with some $\phi \in (0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi)$. Obviously we have

$$A(r) \asymp |q_H(ir)| \asymp 1, \qquad L(r) \lesssim \operatorname{Im} q_H(ir) \ll 1,$$

and hence (1.8) is certainly not strong enough to fully (i.e. up to universal constants) determine $\operatorname{Im} q_H(ir)$. The following lemma shows that in this situation a transformation, where the dependent variable in (1.1) is rotated in the two-dimensional space $\mathbb{C}^2$, strictly improves the upper bound for the imaginary part of q_H and that the lower bound does not get worse. In Proposition 5.7 below the transformation is made more explicit in terms of the Hamiltonian H.

5.6 Lemma. *Let H be a Hamiltonian such that (5.3) with $\phi \in (0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi)$ is satisfied. Set*

$$Q := \begin{pmatrix} \sin\phi & -\cos\phi \\ \cos\phi & \sin\phi \end{pmatrix}, \qquad \widetilde{H} := QHQ^T. \tag{5.4}$$

Then

$$\operatorname{Im} q_{\widetilde{H}}(ir) \asymp \operatorname{Im} q_H(ir), \qquad \widetilde{A}(r) \ll A(r), \qquad \widetilde{L}(r) \gtrsim L(r) \tag{5.5}$$

as $r \to \infty$.

Proof. The Weyl coefficients of H and $\widetilde{H}$ are related via

$$q_{\widetilde{H}}(z) = \frac{\sin\phi\cdot q_H(z) - \cos\phi}{\cos\phi\cdot q_H(z) + \sin\phi}, \qquad \operatorname{Im} q_{\widetilde{H}}(z) = \frac{\operatorname{Im} q_H(z)}{|\cos\phi\cdot q_H(z) + \sin\phi|^2};$$

see, e.g. [11, (3.20)]. Thus we have

$$\lim_{r\to\infty} q_{\widetilde{H}}(ir) = 0, \qquad \lim_{r\to\infty} \frac{\operatorname{Im} q_{\widetilde{H}}(ir)}{\operatorname{Im} q_H(ir)} = \sin^2\phi > 0,$$

and therefore, in particular,

$$\operatorname{Im} q_{\widetilde{H}}(ir) \asymp \operatorname{Im} q_H(ir), \qquad \widetilde{A}(r) \asymp |q_{\widetilde{H}}(ir)| \ll |q_H(ir)| \asymp A(r),$$

which proves the first two relations in (5.5).

The proof of the third relation in (5.5) requires slightly more effort. Let us first consider the case when H starts with an indivisible interval, say (a,c), which must be of type ϕ. Since $\phi \notin \{0,\pi/2\}$, we have $\mathring{a} = a$, where $\mathring{a}$ is as in (1.9). Clearly, $\det M(t) = 0$ for $t \in [a,c]$, which shows that $L(r) = 0$ for large enough r; so the third relation in (5.5) is trivially satisfied in this case.

Now let us assume that H does not start with an indivisible interval. Then $\widetilde{H}$ does not start with an indivisible interval either, and hence $\mathring{\tilde{a}} = a$, where $\mathring{\tilde{a}}$ is as in (1.9) for H replaced by $\widetilde{H}$. First, note that

$$\det \widetilde{M}(t) = \det M(t), \qquad \operatorname{tr} \widetilde{M}(t) = \operatorname{tr} M(t),$$

and, since $A(r) \asymp 1$ and $\widetilde{A}(r) \ll 1$, we have

$$m_1(t) \asymp m_2(t), \qquad \widetilde{m}_1(t) \ll \widetilde{m}_2(t).$$

Together, it follows that

$$m_1(t) \asymp m_2(t) \asymp \operatorname{tr} M(t) \asymp \widetilde{m}_2(t) \gg \widetilde{m}_1(t),$$

and hence

$$\mathring{r}(t) = \frac{\eta}{2}\big(m_1(t)m_2(t)\big)^{-\frac{1}{2}} \ll \frac{\eta}{2}\big(\widetilde{m}_1(t)\widetilde{m}_2(t)\big)^{-\frac{1}{2}} = \mathring{\tilde{r}}(t),$$

where $\mathring{r}$ and $\mathring{\tilde{r}}$ are as in (2.14). In particular, $\mathring{r}(t) \leq \mathring{\tilde{r}}(t)$ for small enough t, and therefore $\mathring{t}(r) \leq \mathring{\tilde{t}}(r)$ for large enough r, say, $r \geq r_0$.

Now we use the monotonicity property in (2.17) and the representation (2.16) of $L(r)$ and $\widetilde{L}(r)$ to obtain

$$L(r) = \frac{2r}{\eta} \cdot \frac{\det M(\mathring{t}(r))}{m_2(\mathring{t}(r))} \leq \frac{2r}{\eta} \cdot \frac{\det M(\mathring{\tilde{t}}(r))}{m_2(\mathring{\tilde{t}}(r))} \asymp \frac{2r}{\eta} \cdot \frac{\det \widetilde{M}(\mathring{\tilde{t}}(r))}{\widetilde{m}_2(\mathring{\tilde{t}}(r))} = \widetilde{L}(r)$$

for $r \geq r_0$. ❑

According to [6, Corollary 3.2], relation (5.3) holds if and only if

$$\lim_{t\to a} \frac{1}{m_1(t)+m_2(t)} \begin{pmatrix} m_1(t) & m_3(t) \\ m_3(t) & m_2(t) \end{pmatrix} = \begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix} \tag{5.6}$$

with $c_1, c_2 > 0$, $c_3 \in \mathbb{R}$ such that $c_1c_2 - c_3^2 = 0$ and $\frac{c_3}{c_2} = \cot\phi$ (note that the Weyl coefficient is defined slightly differently in [6]). We can use this fact to obtain the following proposition, which provides a transformation that is in terms of the asymptotic behaviour of M at a.

5.7 Proposition. *Let H be a Hamiltonian defined on some interval $[a,b)$, let M be as in (1.4), and assume that (5.6) holds with $c_1, c_2 > 0$, $c_3 \in \mathbb{R}$ such that $c_1c_2 - c_3^2 = 0$. Then there exist Q and $\widetilde{H}$ as in (5.4) such that $\widetilde{M}(t) := \int_a^t \widetilde{H}(s)\,\mathrm{d}s$ satisfies*

$$\widetilde{M} = \begin{pmatrix} c_2m_1 + c_1m_2 - 2c_3m_3 & c_3(m_1-m_2) + (c_2-c_1)m_3 \\ c_3(m_1-m_2) + (c_2-c_1)m_3 & c_1m_1 + c_2m_2 + 2c_3m_3 \end{pmatrix}. \tag{5.7}$$

Hence

$$L(r) \lesssim \widetilde{L}(r) \lesssim \operatorname{Im} q_H(ir) \asymp \operatorname{Im} q_{\widetilde{H}}(ir) \lesssim \widetilde{A}(r) \ll A(r). \tag{5.8}$$

Proof. Let $\phi \in (0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi)$ such that $\frac{c_3}{c_2} = \cot\phi$. It follows from (5.6) that $c_1 + c_2 = 1$. Hence

$$c_2^2(1-\sin^2\phi) = c_2^2\cos^2\phi = c_3^2\sin^2\phi = c_2(1-c_2)\sin^2\phi,$$

which yields $c_2 = \sin^2\phi$. Since $\sin\phi > 0$, we arrive at $\sin\phi = \sqrt{c_2}$ and $\cos\phi = \frac{c_3}{\sqrt{c_2}}$. Applying the transformation (5.4) to H we obtain $\widetilde{H}$, whose primitive $\widetilde{M}$ satisfies

$$\widetilde{M} = QMQ^T = \begin{pmatrix} \sqrt{c_2} & -\frac{c_3}{\sqrt{c_2}} \\ \frac{c_3}{\sqrt{c_2}} & \sqrt{c_2} \end{pmatrix} \begin{pmatrix} m_1 & m_3 \\ m_3 & m_2 \end{pmatrix} \begin{pmatrix} \sqrt{c_2} & \frac{c_3}{\sqrt{c_2}} \\ -\frac{c_3}{\sqrt{c_2}} & \sqrt{c_2} \end{pmatrix}$$

$$= \begin{pmatrix} c_2m_1 - 2c_3m_3 + \frac{c_3^2}{c_2}m_2 & c_3m_1 - \frac{c_3^2}{c_2}m_3 + c_2m_3 - c_3m_2 \\ c_3m_1 + c_2m_3 - \frac{c_3^2}{c_2}m_3 - c_3m_2 & \frac{c_3^2}{c_2}m_1 + 2c_3m_3 + c_2m_2 \end{pmatrix}.$$

Using again $c_1c_2 = c_3^2$ we see that this is equal to the right-hand side of (5.7). ❑

Let us consider an example where this transformation trick is beneficial.

5.8 Example. Let H be a Hamiltonian on $[0,\infty)$ whose primitive is

$$M(t) = \begin{pmatrix} 4t + t^\gamma & 2t + t^\gamma \\ 2t + t^\gamma & t + t^\gamma \end{pmatrix}$$

with some $\gamma > 1$. Let us first determine the bounds we obtain when we apply Theorem 1.1 directly. Since $(m_1m_2)(t) \sim 4t^2$, we have $\mathring{t}(r) \asymp \frac{1}{r}$. Further,

$$\sqrt{\frac{m_1(t)}{m_2(t)}} \to 2, \qquad \frac{\det M(t)}{(m_1m_2)(t)} = \frac{t^{\gamma+1}}{4t^2 + 5t^{\gamma+1} + t^{2\gamma}} \sim \frac{1}{4}t^{\gamma-1},$$

and hence

$$A(r) \asymp 1, \qquad L(r) \asymp \frac{\det M(\mathring{t}(r))}{(m_1m_2)(\mathring{t}(r))} \asymp r^{-(\gamma-1)}.$$

Let us now apply the transformation in Proposition 5.7. Since $m_1(t) + m_2(t) \sim 5t$, we obtain $c_1 = \frac{4}{5}$, $c_2 = \frac{1}{5}$ and $c_3 = \frac{2}{5}$. According to (5.7) the primitive of the transformed Hamiltonian $\widetilde{H}$ is

$$\begin{aligned} \widetilde{M} &= \begin{pmatrix} \frac{1}{5}(4t + t^\gamma) + \frac{4}{5}(t + t^\gamma) - \frac{4}{5}(2t + t^\gamma) & \frac{2}{5}\times 3t - \frac{3}{5}(2t + t^\gamma) \\ \frac{2}{5}\times 3t - \frac{3}{5}(2t + t^\gamma) & \frac{4}{5}(4t + t^\gamma) + \frac{1}{5}(t + t^\gamma) + \frac{4}{5}(2t + t^\gamma) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{5}t^\gamma & -\frac{3}{5}t^\gamma \\ -\frac{3}{5}t^\gamma & 5t + \frac{9}{5}t^\gamma \end{pmatrix}. \end{aligned}$$

Since $(\widetilde{m}_1\widetilde{m}_2) \sim t^{\gamma+1}$, we have $\mathring{\tilde{t}}(r) \asymp r^{-\frac{2}{\gamma+1}}$. Further, the relations

$$\sqrt{\frac{\widetilde{m}_1(t)}{\widetilde{m}_2(t)}} \sim \frac{1}{5}t^{\frac{\gamma-1}{2}}, \qquad \frac{\det \widetilde{M}(t)}{(\widetilde{m}_1\widetilde{m}_2)(t)} = \frac{t^{\gamma+1}}{t^{\gamma+1} + \frac{9}{25}t^{2\gamma}} \to 1$$

and (5.8) yield

$$\operatorname{Im} q_H(ir) \asymp \widetilde{A}(r) \asymp \widetilde{L}(r) \asymp r^{-\frac{\gamma-1}{\gamma+1}}.$$

Hence, in this example the actual asymptotic behaviour of $\operatorname{Im} q_H(ir)$ lies strictly between $L(r)$ and $A(r)$:

$$\begin{array}{ccccc} L(r) & \ll & \operatorname{Im} q_H(ir) & \ll & A(r). \\ \asymp & & \asymp & & \asymp \\ r^{-(\gamma-1)} & & r^{-\frac{\gamma-1}{\gamma+1}} & & 1 \end{array}$$

♢

5.9 Remark. Example 5.8 also shows that the lower bound for $\operatorname{Im} q_H(ir)$ from (1.8) may be far too small. However, this particular example is in a sense not proper since it occurred only because the Hamiltonian is "turned in the wrong direction" (thinking of matrices Q as above as rotation matrices). It indicates that the crux for finding out whether or not we may have $L(r) \ll \operatorname{Im} q_H(ir)$ in an intrinsic and essential manner, is to understand the situation when $q_H(ir)$ tends to 0.

So far we have no example of a Hamiltonian H such that $\lim_{r\to\infty} q_H(ir) = 0$ and $L(r) \ll \operatorname{Im} q_H(ir) \ll A(r)$. We expect that there exist large classes of Hamiltonians with these properties. ♢

5.4 Examples

Let us consider more examples, in particular, such where the Hamiltonian oscillates in a neighbourhood of the left endpoint.

5.10 Example. Let $\phi \in (0, \frac{\pi}{2})$, let $(0,\infty) = I_+ \cup I_-$ be a partition into two disjoint measurable sets, and consider on $(0,\infty)$ the Hamiltonian (recall the notation $\xi_\phi := (\cos\phi, \sin\phi)^*$)

$$H(t) := \begin{cases} \xi_\phi \xi_\phi^*, & t \in I_+, \\ \xi_{-\phi}\xi_{-\phi}^*, & t \in I_-. \end{cases} \tag{5.9}$$

Let λ denote the Lebesgue measure, and set

$$l_+(t) := \lambda\big(I_+ \cap (0,t)\big), \qquad l_-(t) := \lambda\big(I_- \cap (0,t)\big)$$

for $t > 0$. Note that $l_+(t) + l_-(t) = t$. It follows easily that

$$m_1(t) = t\cos^2\phi, \quad m_2(t) = t\sin^2\phi, \quad m_3(t) = \big(l_+(t) - l_-(t)\big)\cos\phi\sin\phi.$$

Let $\eta \in (0, 1 - \frac{1}{\sqrt{2}})$. Then

$$A(r) = \sqrt{\frac{m_1(\mathring{t}(r))}{m_2(\mathring{t}(r))}} = \cot\phi \in (0,\infty).$$

Further, we have

$$\mathring{r}(t) = \frac{\eta}{2}(t\cos\phi\sin\phi)^{-1} = \frac{\alpha}{t},$$

with

$$\alpha := \frac{\eta}{2\cos\phi\sin\phi}. \tag{5.10}$$

and hence

$$\mathring{t}(r) = \frac{\alpha}{r}.$$

From this and

$$\begin{aligned} \frac{\det M(t)}{m_1(t)m_2(t)} &= 1 - \frac{m_3(t)^2}{m_1(t)m_2(t)} = 1 - \Big(\frac{l_+(t)}{t} - \frac{l_-(t)}{t}\Big)^2 \\ &= \Big(1 + \frac{l_+(t)}{t} - \frac{l_-(t)}{t}\Big)\Big(1 - \frac{l_+(t)}{t} + \frac{l_-(t)}{t}\Big) = 4\frac{l_+(t)}{t}\cdot\frac{l_-(t)}{t} \end{aligned}$$

we obtain

$$L(r) = 4\cot\phi \cdot \frac{l_+(\mathring{t}(r))}{\mathring{t}(r)} \cdot \frac{l_-(\mathring{t}(r))}{\mathring{t}(r)} = \frac{4\cot\phi}{\alpha^2}\cdot r^2 l_+\Big(\frac{\alpha}{r}\Big) l_-\Big(\frac{\alpha}{r}\Big). \tag{5.11}$$

Hence, if the sets I_+ and I_- both have positive density at 0 in the sense that

$$\liminf_{t\to 0}\frac{l_+(t)}{t} > 0 \qquad \text{and} \qquad \liminf_{t\to 0}\frac{l_-(t)}{t} > 0,$$

then $L(r) \asymp A(r) \asymp 1$ as $r \to \infty$. ◊

Specialising the sets I_+, I_- in Example 5.10 we obtain an example where $|q_H|$ and $\operatorname{Im} q_H$ oscillate between the bounds given by Theorem 1.1.

5.11 Example. Let $\phi \in (0, \frac{\pi}{2})$, set

$$I_+ := \bigcup_{n=-\infty}^{\infty} \big(2^{2n-1}, 2^{2n}\big], \qquad I_- := \bigcup_{n=-\infty}^{\infty} \big(2^{2n}, 2^{2n+1}\big],$$

and let H be as in (5.9). We have $2 \cdot (I_+ \cap (0,t)) = I_- \cap (0,2t)$, and hence $\frac{1}{2t} l_-(2t) = \frac{1}{t} l_+(t)$. Analogously, we find $\frac{1}{2t} l_+(2t) = \frac{1}{t} l_-(t)$. Putting this together we obtain

$$L(2r) = L(r), \quad r > 0.$$

Let $r \in [\alpha, 2\alpha]$ where α is as in (5.10); then $\mathring{t}(r) = \frac{\alpha}{r} \in [\frac{1}{2}, 1]$. Since $l_-(t) = \frac{1}{3}$ for $t \in [\frac{1}{2}, 1]$, it follows from (5.11) that

$$L(r) = 4 \cot\phi \cdot \left(1 - \frac{l_-(\mathring{t}(r))}{\mathring{t}(r)}\right) \cdot \frac{l_-(\mathring{t}(r))}{\mathring{t}(r)} = 4 \cot\phi \cdot \left(1 - \frac{r}{3\alpha}\right) \cdot \frac{r}{3\alpha}, \quad r \in [\alpha, 2\alpha],$$

which is quadratic in r and satisfies $L(\alpha) = L(2\alpha) = \frac{8 \cot\phi}{9}$ and $L(\frac{3\alpha}{2}) = \cot\phi$. The Hamiltonian H satisfies

$$H\Big(\frac{1}{2}x\Big) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} H(x) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since the Weyl coefficient for the Hamiltonian on the left-hand side of the latter equation is $q_H(2z)$ and the Weyl coefficient for the Hamiltonian on the right-hand side is $-q_H(-z)$, we obtain that $q_H(2z) = -q_H(-z)$. From this we see that

$$|q_H(i \cdot 2r)| = |q_H(ir)|, \quad \operatorname{Im} q_H(i \cdot 2r) = \operatorname{Im} q_H(ir), \quad \operatorname{Re} q_H(i \cdot 2r) = -\operatorname{Re} q_H(ir),$$

for $r > 0$. Of course, q_H is not constant since H is not constant. Hence the limit $\lim_{r\to\infty} q_H(ir)$ does not exist.

Let us collect what we have computed. First, the absolute value $|q_H(ir)|$ is a non-constant function which oscillates between the constant bounds (1.6) and is 2-periodic on a logarithmic scale. Second, the imaginary part $\operatorname{Im} q_H(ir)$ is a non-constant “2-periodic” function which lies in between the constant upper bound and the “2-periodic” lower bound from (1.8). $\Diamond$

In the above example it seems that the lower bound $L(r)$ mimics the behaviour of $|q_H|$ and $\operatorname{Im} q_H$ better than the upper bound $A(r)$ in the sense that it oscillates with the same period. For $|q_H|$ this is not always the case as the following example shows in a striking way. For the imaginary part it is not so clear how well $L(r)$ describes its behaviour.

5.12 Example. Choose a sequence $(\phi_n)_{n=1}^\infty$ which oscillates between $+1$ and -1 with decaying step width so that $\{\phi_n : n \in \mathbb{N}\}$ is dense in $[-1, 1]$. For example, let

$$(\phi_n)_{n=1}^\infty := \Big(1, \frac{1}{2}, 0, -\frac{1}{2}, -1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1, \frac{3}{4}, \dots\Big).$$

Further, set $t_n := 2^{1-n^2}$ and consider the Hamiltonian

$$H(t) := \begin{cases} \dfrac{1}{2} \begin{pmatrix} 1 & \sin(\frac{\pi}{2}\phi_n) \\ \sin(\frac{\pi}{2}\phi_n) & 1 \end{pmatrix}, & t \in (t_{n+1}, t_n], \\[2ex] \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & t \in (1, \infty). \end{cases}$$

This definition is made so that [32, Theorem 4.1] is applicable. It follows that

$$\lim_{r\to\infty} |q_H(ir)| = 1$$

$$\forall \alpha \in [0,1]\ \exists r_n > 0.\quad \lim_{n\to\infty} r_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \operatorname{Im} q_H(ir_n) = \alpha.$$

Loosely speaking we may say that $\operatorname{Im} q_H(ir)$ oscillates between 0 and $|q_H(ir)|$. On the other hand, $m_1(t) = m_2(t)$, and hence $A(r) = 1$. Let us consider $L(r)$. For $n \in \mathbb{N}$ we have

$$\begin{aligned} m_3(t_n) &= \frac{1}{2}\Big[t_n \sin\Big(\frac{\pi}{2}\phi_n\Big) - t_{n+1} \sin\Big(\frac{\pi}{2}\phi_n\Big) + \sum_{k=n+1}^\infty \big(t_k - t_{k+1}\big) \sin\Big(\frac{\pi}{2}\phi_k\Big)\Big] \\ &= \frac{1}{2} t_n \sin\Big(\frac{\pi}{2}\phi_n\Big) + \rho_n \end{aligned}$$

where $|\rho_n| \leq \frac{1}{2}(t_{n+1} + t_{n+1}) = t_{n+1}$. Since $(m_1 m_2)(t) = \frac{1}{4}r^2$, the relation $\mathring{t}(r) = \frac{\eta}{r}$ holds with η as in Theorem 1.1. With $r_n := \mathring{r}(t_n) = \frac{\eta}{t_n} \to \infty$ we obtain

$$\begin{aligned}
L(r_n) &= \frac{\det M(t_n)}{(m_1 m_2)(t_n)} = \frac{\frac{1}{4}t_n^2 - \big(\frac{1}{2}t_n \sin(\frac{\pi}{2}\phi_n) + \rho_n\big)^2}{\frac{1}{4}t_n^2} \\
&= \cos^2\Big(\frac{\pi}{2}\phi_n\Big) - 4\frac{\rho_n}{t_n}\sin\Big(\frac{\pi}{2}\phi_n\Big) - 4\Big(\frac{\rho_n}{t_n}\Big)^2 = \cos^2\Big(\frac{\pi}{2}\phi_n\Big) + \mathrm{O}\Big(\frac{t_{n+1}}{t_n}\Big) \\
&= \cos^2\Big(\frac{\pi}{2}\phi_n\Big) + \mathrm{O}\big(4^{-n}\big),
\end{aligned}$$

which oscillates between 0 and 1. ◊

5.5 $A(r)$ in terms of the associated string

A Krein string is a pair $\mathcal{S}[L, \mathfrak{m}]$ where

- ▷ L is a number in $[0, \infty]$;
- ▷ $\mathfrak{m}$ is a non-decreasing, left-continuous, $[0, \infty)$-valued function on $[0, L)$ with $\mathfrak{m}(0) = 0$.

The number L is called the length of the string, and the function $\mathfrak{m}$ its mass distribution function. The string equation can be written as an integro-differential equation,

$$y'_+(x) + z\int_{[0,x]} y(t)\,\mathrm{d}\mathfrak{m}(t) = 0,$$

where $z \in \mathbb{C}$ is the spectral parameter and y'_+ denotes the right-hand derivative of y; see, e.g. [22]. Further, set $\mathfrak{m}(L-) := \lim_{x \nearrow L} \mathfrak{m}(x)$ and $\ell := \sup\{x \in [0, L) : \mathfrak{m}(x) < \mathfrak{m}(L-)\}$; then a string is called *regular* if both ℓ and $\mathfrak{m}(L-)$ are finite.

Given a string, one can construct a function q_{S}, the *principal Titchmarsh–Weyl coefficient* of the string; see [22]. This function belongs to the Stieltjes class, i.e. it is analytic on $\mathbb{C} \setminus [0, \infty)$, has non-negative imaginary part in the upper half-plane, and takes positive values on $(-\infty, 0)$. It is a fundamental theorem proved by M. G. Krein that the assignment

$$\mathcal{S}[L, \mathfrak{m}] \mapsto q_{\mathsf{S}}$$

sets up a bijection between the set of all strings and the Stieltjes class.

Given a Hamiltonian H on some interval $[a, b)$, we can define a string by setting

$$L := \lim_{t \to b} m_1(t), \qquad \mathfrak{m}(x) := (m_2 \circ m_1^-)(x),\ x \in [0, L), \tag{5.12}$$

where m_1^- denotes the generalised inverse of m_1; see Definition B.1. Note that $\mathfrak{m}$ is well defined and satisfies the properties above by Lemma B.2 (i)–(iii). Further, this string does not depend on the off-diagonal entry h_3.

The correspondence between strings and canonical systems is studied in detail in [23]. In our present context, we need two additions, which are given in the following lemma, namely, that the definition of the string associated with H as above does not depend on the parameterisation of H, and a characterisation when the string is regular. Recall from Definition 5.3 that an interval (c, d) is H-indivisible of type 0 (respectively type $\frac{\pi}{2}$) if and only if $h_2(t) = 0$ (respectively $h_1(t) = 0$) for a.e. $t \in (c, d)$. Further, set

$$\mathring{a}_i := \sup\big\{t \in [a, b) : m_i(t) = 0\big\}, \qquad i \in \{1, 2\}. \tag{5.13}$$

Then H starts with an indivisible interval of type 0 (respectively type $\frac{\pi}{2}$) at the left endpoint a if and only if $\mathring{a}_2 > a$ (respectively $\mathring{a}_1 > a$).

5.13 Lemma. *Let H be a Hamiltonian defined on $[a, b)$, let $\mathcal{S}[L, \mathfrak{m}]$ be the string associated with H via* (5.12) *and let $\mathring{a}_i$ be as in* (5.13).

(i) *The following relations hold:*

$$m_1(\mathring{a}_2) = x_0 := \sup\{x \in [0,L) : \boldsymbol{m}(x) = 0\}, \tag{5.14}$$

$$m_2(\mathring{a}_1) = \boldsymbol{m}(0+) := \lim_{x\searrow 0} \boldsymbol{m}(x). \tag{5.15}$$

In particular, the following equivalences are true:

$$\mathring{a}_2 > a \quad \Leftrightarrow \quad x_0 > 0, \tag{5.16}$$

$$\mathring{a}_1 > a \quad \Leftrightarrow \quad \boldsymbol{m}(0+) > 0. \tag{5.17}$$

(ii) *The string associated with H via* (5.12) *is regular if and only if there exists $c \in [a,b)$ such that (c,b) is H-indivisible of type* 0 *or of type $\frac{\pi}{2}$.*

(iii) *Let $\widetilde{H}$ be a second Hamiltonian defined on $[\tilde{a},\tilde{b})$ and assume that H and $\widetilde{H}$ are related via*

$$\widetilde{H} = (H \circ \varphi) \cdot \varphi' \quad a.e.$$

where $\varphi\colon [\tilde{a},\tilde{b}) \to [a,b)$ is an increasing bijection such that φ and φ^{-1} are absolutely continuous. Then the strings associated with H and $\widetilde{H}$ coincide.

Proof. (i) Since m_1 is continuous, it follows from Lemma $B.2$ (vi) that, for $x \in [0,L)$,

$$\boldsymbol{m}(x) = 0 \quad \Leftrightarrow \quad m_1^-(x) \le \mathring{a}_2 \quad \Leftrightarrow \quad x \le m_1(\mathring{a}_2),$$

which proves (5.14) and the equivalence in (5.16).

Next let us prove (5.15). Since m_1 is continuous, Lemma $B.2$ (iv) implies that, for $x \in (0,L)$, we have $m_1(m_1^-(x)) = x > 0$ and hence $m_1^-(x) > \mathring{a}_1$, which shows $\lim_{x\searrow 0} m_1^-(x) \ge \mathring{a}_1$. Further, there exist $t_n \in (\mathring{a}_1, b)$, $n \in \mathbb{N}$, such that $t_n \searrow \mathring{a}_1$ and t_n is not right endpoint of an interval where m_1 is constant. Now Lemma 3.3 (iii) yields $m_1^-(m_1(t_n)) = t_n$. Since $\lim_{t\to\infty} m_1(t_n) = 0$, we obtain $\lim_{x\searrow 0} m_1^-(x) = \mathring{a}_1$. Together with the continuity of m_2, this yields (5.15), and the equivalence in (5.17) follows.

(ii) If $m_1 \equiv 0$, then $L = 0$ and the string is regular. From now on assume that $m_1 \not\equiv 0$ and let ℓ be defined as above. We have $\ell < L$ if and only if m_2 is constant on an interval of the form $[c,b)$ with $c \in [a,b)$, i.e. (c,b) is an indivisible interval of type 0. If $\ell < L$, then $\boldsymbol{m}(L-) = \boldsymbol{m}(x) < \infty$ for $x \in (\ell, L)$ and hence the string is regular.

Now assume that $\ell = L$ and set $t_0 := \sup\{t \in [a,b) : m_1(t) < L\}$. By Lemma $B.2$ (vii) we have $\lim_{x\nearrow L} m_1^-(x) = t_0$. Hence it follows from the continuity of m_1 and Lemma $B.2$ (iv) that

$$L + \boldsymbol{m}(L-) = \lim_{x\nearrow L}\big(x + \boldsymbol{m}(x)\big) = \lim_{x\nearrow L}\big[m_1(m_1^-(x)) + m_2(m_1^-(x))\big]$$

$$= \lim_{x\nearrow L} \int_a^{m_1^-(x)} \operatorname{tr} H(t)\, \mathrm{d}t = \int_a^{t_0} \operatorname{tr} H(t)\, \mathrm{d}t.$$

Since we assumed that H is in the limit point case at b, the right-hand side is finite if and only if $t_0 < b$, which, in turn, is equivalent to the fact that $h_1(t) = 0$ for a.e. $t \in (c,b)$ for some $c \in [a,b)$.

(iii) Let the notation $\widetilde{m}_j$ correspond to $\widetilde{H}$. Then we have

$$\widetilde{m}_j(u) = \int_{a'}^{u} h_j\big(\varphi(v)\big)\varphi'(v)\, \mathrm{d}v = m_j(\varphi(u)).$$

By Lemma $B.2$ (ix) we therefore have

$$\widetilde{m}_2 \circ \widetilde{m}_1^- = (m_2 \circ \varphi) \circ (\varphi^- \circ m_1^-) = m_2 \circ m_1^-. \qquad \square$$

The next lemma contains the relation between the Weyl coefficient q_H of the Hamiltonian and the principal Titchmarsh–Weyl coefficient q_{S} of the string. We also state a relation between the corresponding spectral measures, which is used in §6.1. Since q_{S} belongs to the Stieltjes class, it has the following representation:

$$q_{\mathsf{S}}(z) = \alpha_{\mathsf{S}} + \int_{[0,\infty)} \frac{1}{t-z}\,\mathrm{d}\mu_{\mathsf{S}}(t), \qquad z \in \mathbb{C}\setminus[0,\infty) \tag{5.18}$$

with $\alpha_{\mathsf{S}} \geq 0$ and μ_{S} a measure supported on $[0,\infty)$ satisfying $\int_{[0,\infty)} \frac{\mathrm{d}\mu_{\mathsf{S}}(t)}{1+t} < \infty$. According to M. G. Krein the measure μ_{S} is called *principal spectral measure* of the string $\mathcal{S}[L,\mathfrak{m}]$.

5.14 Lemma. *Let H be a Hamiltonian defined on some interval $[a,b)$, and assume that H is diagonal, i.e. $h_3 = 0$. Moreover, let $\mathcal{S}[L,\mathfrak{m}]$ be its associated string. Let q_H, q_{S}, μ_H and μ_{S} be the Weyl coefficients and spectral measures of H and $\mathcal{S}[L,\mathfrak{m}]$ respectively, and let α_H, β_H and α_{S} be the constants in* (1.3) *and* (5.18) *respectively. Define the mapping $\tau : \mathbb{R} \to \mathbb{R}, t \mapsto t^2$ and let $\tau_*\mu_H$ be the push-forward measure of μ_H. Then*

$$q_H(z) = zq_{\mathsf{S}}(z^2), \qquad \mu_{\mathsf{S}} = \tau_*\mu_H, \qquad \alpha_{\mathsf{S}} = \beta_H \qquad \textit{and} \qquad \alpha_H = 0. \tag{5.19}$$

Proof. The first relation in (5.19) is shown in [23, Theorem 4.2] for trace-normed Hamiltonians (i.e. $\operatorname{tr} H = 1$ a.e.). Every Hamiltonian can be reparameterised to a trace-normed Hamiltonian, and this changes neither its Weyl coefficient nor the associated string by Lemma 5.13. The second relation in (5.19) can either be deduced from [23, Theorem 2.1] or follows from the following considerations. Since H is diagonal, q_H is an odd function and hence $\alpha_H = 0$ in the representation (1.3) of q_H, and μ_H is a symmetric measure; see [23, Lemma 2.2]. Hence

$$\begin{aligned} q_H(z) &= \beta_H z + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\,\mathrm{d}\mu_H(t) \\ &= \beta_H z + \frac{1}{2}\bigg[\int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\,\mathrm{d}\mu_H(t) + \int_{\mathbb{R}} \Big(\frac{1}{-t-z} + \frac{t}{1+t^2}\Big)\,\mathrm{d}\mu_H(t)\bigg] \\ &= z\bigg[\beta_H + \int_{\mathbb{R}} \frac{1}{t^2-z^2}\,\mathrm{d}\mu_H(t)\bigg] = z\bigg[\beta_H + \int_{[0,\infty)} \frac{1}{s-z^2}\,\mathrm{d}(\tau_*\mu_H)(s)\bigg]. \end{aligned}$$

Now the first relation in (5.19) and the uniqueness of the integral representation imply that $\mu_{\mathsf{S}} = \tau_*\mu_H$. The relations among the constants α_H, β_H and α_{S} are clear. ❑

We can now give a formula which represents $A(r)$ in term of the associated string.

5.15 Proposition. *Let H be a Hamiltonian defined on some interval $[a,b)$ and assume that neither $h_1 = 0$ a.e. nor $h_2 = 0$ a.e. Let $\mathcal{S}[L,\mathfrak{m}]$ be the string associated with H, and set*

$$\not{\ell}(x) := \begin{cases} x\mathfrak{m}(x) & \text{if } x \in [0,L), \\ L\mathfrak{m}(L-) & \text{if } L < \infty \text{ and } x = L, \\ \infty & \text{if } L < \infty \text{ and } x \in (L,\infty). \end{cases} \tag{5.20}$$

Moreover, let $\eta \in (0, 1 - \frac{1}{\sqrt{2}})$. Then

$$A(r) = \frac{2r}{\eta}\not{\ell}^-\Big(\frac{\eta^2}{4r^2}\Big), \qquad r > 0. \tag{5.21}$$

Proof. We divide the proof into three steps.

① Let $\mathring{a}$ and $\mathring{a}_i$ be as in (1.9) and (5.13) respectively. It follows from Lemma 5.13 (i) that, for $x \in [0,L)$,

$$\mathfrak{m}(x) = 0 \quad \Leftrightarrow \quad \not{\ell}(x) = 0 \quad \Leftrightarrow \quad x \leq m_1(\mathring{a}_2).$$

Hence $\not{\ell}$ is strictly increasing on $[m_1(\mathring{a}), L)$.

Let us consider the continuity of $\not{\ell}^-$. Let $y \in (0, L\mathfrak{m}(L-))$; then

$$\not{\ell}^-(y) = \inf\big\{x \in [0,L) : \not{\ell}(x) \geq y\big\} \geq m_1(\mathring{a}_2), \tag{5.22}$$

and it follows from Lemma $B.2$ (v) that ℓ^- is continuous at y. The definition of ℓ^- shows that

$$\ell^-(y) = L, \qquad y \in \big[Lm(L-), \infty\big). \tag{5.23}$$

Since ℓ^- is left-continuous at $Lm(L-)$ by Lemma $B.2$ (ii), ℓ^- is continuous on $(0, \infty)$.

② Our aim is to show

$$m_1(t) = \ell^-\big(m_1(t)m_2(t)\big) \tag{5.24}$$

for $t \in (\mathring{a}, b)$. Set $t_0 := \sup\{t \in [a, b) : m_1(t) < L\}$, which is the left endpoint of the maximal interval of the form (c, b) on which h_1 vanishes if such an interval exists, and is equal to b otherwise.

First, let $t \in (\mathring{a}, t_0] \cap (\mathring{a}, b)$ be such that t is not right endpoint of an interval where m_1 is constant. Then Lemma $B.2$ (iii) implies that

$$\ell(m_1(t)) = m_1(t)m_2\big(m_1^-(m_1(t))\big) = m_1(t)m_2(t). \tag{5.25}$$

Since $t > \mathring{a}$ and t is not right endpoint of an interval where m_1 is constant, we have $m_1(t) > m_1(\mathring{a}) \geq m_1(\mathring{a}_2)$, and hence $m_1(t)$ is not right endpoint of an interval where ℓ is constant. Now Lemma $B.2$ (iii), together with (5.25), implies that

$$m_1(t) = \ell^-\big(\ell(m_1(t))\big) = \ell^-\big(m_1(t)m_2(t)\big).$$

Next, let $[c, d] \subseteq [\mathring{a}, t_0)$ with $c < d$ be a maximal interval where m_1 is constant. There exist $d_n \in (d, t_0)$, $n \in \mathbb{N}$, such that $d_n \searrow d$ and d_n is not right endpoint of an interval where m_1 is constant. Hence (5.24) holds for $t = d_n$. Since m_1 and m_2 are continuous at d and ℓ^- is continuous at $m_1(d)m_2(d) > 0$, we can take the limit as $n \to \infty$, and therefore (5.24) holds also for $t = d$. Now let $s \in (c, d]$. If $c > \mathring{a}$, then (5.24) holds also for $t = c$, which yields

$$m_1(s) = m_1(c) = \ell^-\big(m_1(c)m_2(c)\big) \leq \ell^-\big(m_1(s)m_2(s)\big).$$

If $c = \mathring{a}$, then we must have $\mathring{a} = \mathring{a}_2 > a$, and (5.22) implies

$$m_1(s) = m_1(c) = m_1(\mathring{a}_2) \leq \ell^-\big(m_1(s)m_2(s)\big).$$

In both cases we obtain

$$m_1(s) \leq \ell^-\big(m_1(s)m_2(s)\big) \leq \ell^-\big(m_1(d)m_2(d)\big) = m_1(d) = m_1(s).$$

Since we must have equality everywhere, it follows that (5.24) holds for $s \in [c, d] \cap (\mathring{a}, t_0)$.

Finally, let $t \in [t_0, b)$. It follows from (5.25) that

$$m_1(t)m_2(t) \geq m_1(t_0)m_2(t_0) = \ell\big(m_1(t_0)\big) = \ell(L) = Lm(L-).$$

From (5.23) we obtain $m_1(t) = L = \ell^-(m_1(t)m_2(t))$. This finishes the proof of (5.24) for all $t \in (\mathring{a}, b)$.

③ Now (2.15) and (5.24) yield

$$A(r) = \frac{2r}{\eta} m_1\big(\mathring{t}(r)\big) = \frac{2r}{\eta} \ell^-\Big(m_1\big(\mathring{t}(r)\big)m_2\big(\mathring{t}(r)\big)\Big) = \frac{2r}{\eta} \ell^-\bigg(\frac{\eta^2}{4r^2}\bigg)$$

for $r > 0$.

❑

6 Relation to previous work

6.1 Strings and I. S. Kac's theorem

Let $\mathcal{S}[L, m]$ be a Krein string where the length L and the mass distribution function m are as at the beginning of §5.5. The aim of the current section is to establish a relation between the asymptotic behaviour of the spectral measure of the string at infinity and the asymptotic behaviour of the mass distribution function at 0. To this end, let H be a diagonal Hamiltonian that is related to the string as in [23, §4], and that, in particular, (5.12) is satisfied. For a given string, such a Hamiltonian can be constructed as follows. Set

$$\widetilde{m}(x) := \begin{cases} x + m(x) & \text{if } x \in [0, L), \\ L + m(L-) & \text{if } L < \infty \text{ and } x = L, \\ \infty & \text{if } L < \infty \text{ and } x \in (L, \infty). \end{cases}$$

The generalised inverse $\widetilde{m}^-$ of $\widetilde{m}$ is defined on $[0, \infty)$ (it is even defined on $[0, \infty]$ if $L < \infty$) and absolutely continuous; see Appendix B for the definition of the generalised inverse. A trace-normed Hamiltonian H on $[0, \infty)$ that satisfies (5.12) is then given by

$$H(t) := \begin{pmatrix} \frac{\mathrm{d}\widetilde{m}^-(t)}{\mathrm{d}t} & 0 \\ 0 & 1 - \frac{\mathrm{d}\widetilde{m}^-(t)}{\mathrm{d}t} \end{pmatrix}, \qquad t \in [0, \infty); \tag{6.1}$$

see [23, §4] and [6, §5].

Using this diagonal Hamiltonian we can now prove the following characterisation of the asymptotic behaviour of the spectral measure in the sense of an integrability condition. This essentially reproves the main theorem in [20] by I. S. Kac for the case when no potential is present. The theorem in [20] is stated only for regular strings although it is stated that it can be carried over to singular strings. In order to apply the results from §4, we have to assume that the corresponding Hamiltonian H does not start with an indivisible interval of type 0 or $\frac{\pi}{2}$, which, by Lemma 5.13 (i), is equivalent to the fact that $m(0+) = 0$ and $m(x) > 0$ for $x \in (0, L)$.

6.1 Corollary. *Let $\mathcal{S}[L, m]$ be a Krein string such that $m(0+) = 0$ and $m(x) > 0$ for every $x \in (0, L)$. Further, let μ_{S} the principal spectral measure of the string as in (5.18), set $\vec{\mu_{\mathsf{S}}}(r) := \mu_{\mathsf{S}}([0, r))$, $r > 0$, and let g be a continuous, non-decreasing, regularly varying function. Then*

$$\int_1^\infty \vec{\mu_{\mathsf{S}}}(s) \frac{g(s)}{s^2}\, \mathrm{d}s < \infty \quad \Leftrightarrow \quad \exists x_0 \in (0, L).\ \int_0^{x_0} g\left(\frac{1}{\int_0^x m(t)\, \mathrm{d}t}\right) \mathrm{d}x < \infty. \tag{6.2}$$

Before we prove this corollary, we need the following lemma.

6.2 Lemma. *Let $F : (0, \infty) \to (0, \infty)$ be a non-increasing function and let $c \in (0, b)$. Then*

$$\begin{aligned} \frac{1}{2} \int_0^{2m_1(c)} F\Big(\int_0^x m(\xi)\, \mathrm{d}\xi\Big)\, \mathrm{d}x &\leq \int_0^c F\big((m_1 m_2)(t)\big) h_1(t)\, \mathrm{d}t \\ &\leq \int_0^{m_1(c)} F\Big(\int_0^x m(\xi)\, \mathrm{d}\xi\Big)\, \mathrm{d}x. \end{aligned}$$

Proof. To show the second inequality, we apply [9, Proposition 1] to obtain

$$\begin{aligned} \int_0^c F\big((m_1 m_2)(t)\big) h_1(t)\, \mathrm{d}t &= \int_0^c F\big(m_1(t) m_2(t)\big)\, \mathrm{d}m_1(t) \\ &= \int_0^{m_1(c)} F\Big[m_2\big(m_1^-(x)\big) \underbrace{m_1\big(m_1^-(x)\big)}_{=x}\Big]\, \mathrm{d}x \\ &= \int_0^{m_1(x)} F\Big[m(x) \int_0^x \mathrm{d}\xi\Big]\, \mathrm{d}x \leq \int_0^{m_1(c)} F\Big[\int_0^x m(\xi)\, \mathrm{d}\xi\Big]\, \mathrm{d}x, \end{aligned}$$

where in the last step the monotonicity of $\mathpzc{m}$ and F was used.

For the first inequality we use [9, Proposition 1] again

$$
\begin{aligned}
&\int_0^c F\big((m_1m_2)(t)\big)h_1(t)\,\mathrm{d}t = \frac{1}{2}\int_0^c F\big(m_1(t)m_2(t)\big)\,\mathrm{d}(2m_1)(t) \\
&= \frac{1}{2}\int_0^{2m_1(c)} F\Bigg[m_2\Big(m_1^-\Big(\frac{x}{2}\Big)\Big)\underbrace{m_1\Big(m_1^-\Big(\frac{x}{2}\Big)\Big)}_{=\frac{x}{2}}\Bigg]\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{2m_1(c)} F\Bigg[\mathpzc{m}\Big(\frac{x}{2}\Big)\int_{x/2}^x \mathrm{d}\xi\Bigg]\,\mathrm{d}x \geq \frac{1}{2}\int_0^{2m_1(c)} F\Bigg[\int_{x/2}^x \mathpzc{m}(\xi)\,\mathrm{d}\xi\Bigg]\,\mathrm{d}x \\
&\geq \frac{1}{2}\int_0^{2m_1(c)} F\Bigg[\int_0^x \mathpzc{m}(\xi)\,\mathrm{d}\xi\Bigg]\,\mathrm{d}x,
\end{aligned}
$$

where in the penultimate step the monotonicity of $\mathpzc{m}$ and F was used. ❑

Proof of Corollary 6.1. Let H be the Hamiltonian in (6.1), and let τ and $\tau_*\mu_H$ be as in Lemma 5.14. From Lemma 5.14 we obtain that

$$
\vec{\mu}_{\mathsf{S}}(r^2) = \mu_{\mathsf{S}}\big([0,r^2)\big) = \tau_*\mu_H\big([0,r^2)\big) = \mu_H\big((-r,r)\big) = \overleftrightarrow{\mu}_H(r).
$$

Define $\mathpzc{f}(r) := \mathpzc{g}(r^2)$, $r > 0$, which is also a continuous, non-decreasing, regularly varying function. A substitution yields

$$
\int_1^\infty \vec{\mu}_{\mathsf{S}}(s)\frac{\mathpzc{g}(s)}{s^2}\,\mathrm{d}s = 2\int_1^\infty \vec{\mu}_{\mathsf{S}}(r^2)\frac{\mathpzc{g}(r^2)}{r^3}\,\mathrm{d}r = 2\int_1^\infty \overleftrightarrow{\mu}_H(r)\frac{\mathpzc{f}(r)}{r^3}\,\mathrm{d}r.
$$

The assumptions on $\mathpzc{m}$ imply that neither h_1 nor h_2 vanish in a neighbourhood of 0; see Lemma 5.13 (i). Hence we can apply Proposition 4.7, which, for the diagonal Hamiltonian H, gives the second equivalence in the following chain:

$$
\begin{aligned}
\int_1^\infty \vec{\mu}_{\mathsf{S}}(s)\frac{\mathpzc{g}(s)}{s^2}\,\mathrm{d}s < \infty \quad &\Leftrightarrow \quad \int_1^\infty \overleftrightarrow{\mu}_H(r)\frac{\mathpzc{f}(r)}{r^3}\,\mathrm{d}r < \infty \\
&\Leftrightarrow \quad \exists c > 0.\quad \int_0^c h_1(t)\mathpzc{f}\big((m_1m_2)(t)^{-\frac{1}{2}}\big)\,\mathrm{d}t < \infty \\
&\Leftrightarrow \quad \exists c > 0.\quad \int_0^c h_1(t)\mathpzc{g}\Big(\frac{1}{(m_1m_2)(t)}\Big)\,\mathrm{d}t < \infty.
\end{aligned}
$$

It follows from Lemma 6.2 with $F(u) = \mathpzc{g}(\frac{1}{u})$ that the last condition in the above chain is equivalent to the condition on the right-hand side of (6.2). ❑

6.2 The work of Y. Kasahara

Y. Kasahara's paper [24] deals with Krein strings as discussed in §5.5. It was a milestone in the study of high-energy asymptotics of the principal Titchmarsh–Weyl coefficient[5]. Its recent successor [25] extends the results to so-called Kotani strings. In these papers also some theorems about pointwise relations between the mass distribution function of the string and its principal Titchmarsh–Weyl coefficient are proved; see [24, Theorem 4], [25, Section 3].

Since Krein strings can be considered as diagonal canonical systems, we can use our present theorems to obtain quite precise information about the principal Titchmarsh–Weyl coefficient; see Proposition 6.3 below. Concerning its content, our proposition is a variant of Kasahara's Theorem 4. It is weaker than [24, Theorem 4] in the sense that we have worse universal constants in the estimates, but stronger in the sense that we do not restrict ourselves to invertible comparison functions[6]. This allows us to give a formulation which is similar to [25, Theorem 3.4].

[5] Kasahara works with the function $q_{\mathsf{S}}(-z)$ and calls it the "characteristic function of the string".

[6] This assumption is not explicitly stated in [24, Theorem 4]. However, it is needed in the proof.

That result, however, is not comparable with ours since on the one hand it deals with a more general type of strings (for which it is not clear how we could treat them) but on the other hand restricts to the particular situation when q_{S} oscillates around a regularly varying function.

6.3 Proposition. *Let $\mathcal{S}[L,\mathfrak{m}]$ be a string with $L>0$ and $\mathfrak{m}\not\equiv 0$. Denote by q_{S} its principal Titchmarsh–Weyl coefficient, and define $\mathfrak{f}$ as in* (5.20)*. Then*

$$q_{\mathsf{S}}(-y)\asymp \mathfrak{f}^{-}\Big(\frac{1}{y}\Big),\qquad y\in(0,\infty).$$

The constants in "$\asymp$" are independent of the string.

Proof. Let H be a diagonal Hamiltonian associated with the string $\mathcal{S}[L,\mathfrak{m}]$, e.g. as in (6.1). By assumption, neither $h_1=0$ a.e. nor $h_2=0$ a.e. Moreover, fix $\eta\in(0,1-\frac{1}{\sqrt{2}})$. By Theorem 1.1 and Proposition 5.15 we have

$$|q_H(ir)|\asymp A(r)=\frac{2r}{\eta}\mathfrak{f}^{-}\Big(\frac{\eta^2}{4r^2}\Big).$$

The values $q_H(ir)$ are purely imaginary since H is diagonal, and Lemma 5.14 gives $q_H(ir)=(ir)q_{\mathsf{S}}(-r^2)$. Setting $r=\sqrt{y}$ we obtain

$$q_{\mathsf{S}}(-y)=\frac{|q_H(i\sqrt{y})|}{\sqrt{y}}\asymp \mathfrak{f}^{-}\Big(\frac{\eta^2}{4y}\Big)\asymp \mathfrak{f}^{-}\Big(\frac{1}{y}\Big),$$

where in the last step we used Lemma B.3. ❑

6.3 Sturm–Liouville equations

In this subsection we consider Sturm–Liouville equations of the form

$$-(py')'+qy=\lambda wy \tag{6.3}$$

on an interval (a,b), where $p(x),w(x)>0$, $q(x)\in\mathbb{R}$ for a.e. $x\in(a,b)$, $\frac{1}{p},q,w\in L^1_{\mathrm{loc}}([a,b))$, and λ is the spectral parameter. Further, we assume that the equation is in the limit-point case at the right endpoint b. The Titchmarsh–Weyl coefficient corresponding to a Dirichlet boundary condition at a is defined as follows. Let $\theta(\cdot;\lambda),\phi(\cdot;\lambda)$ be solutions of (6.3) that satisfy the initial conditions

$$\begin{aligned}\theta(a;\lambda)&=1, &\quad (p\theta')(a;\lambda)&=0,\\ \phi(a;\lambda)&=0, &\quad (p\phi')(a;\lambda)&=1.\end{aligned}$$

Then $q_{\mathsf{D}}(\lambda)$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, is the unique number such that $\theta(\cdot\,;\lambda)+q_{\mathsf{D}}(\lambda)\phi(\cdot\,;\lambda)\in L^2_w(a,b)$ where $L^2_w(a,b)$ denotes the weighted L^2-space with inner product $(f,g)=\int_a^b f\overline{g}w$. Denote by A_{D} and A_{N} the operators $y\mapsto\frac{1}{w}\big(-(py')'+qy\big)$ in $L^2_w(a,b)$ associated with (6.3) and Dirichlet ($y(a)=0$) or Neumann boundary condition ($(py')(a)=0$) at a respectively. The Titchmarsh–Weyl coefficient can be analytically continued to $\mathbb{C}\setminus[\min\sigma(A_{\mathsf{D}}),\infty)$.

Let us first consider the case when $q\equiv 0$. In this situation we can apply our main result, Theorem 1.1, directly. Note that q_{D} is defined at least on $\mathbb{C}\setminus[0,\infty)$ in this case.

6.4 Corollary. *Let $\kappa\in(0,\frac12-\frac{1}{2\sqrt{2}})$, set $\sigma:=\frac{1}{(1-2\kappa)^2}-1$, and, for $r>0$, let $\hat{x}(r)\in(a,b)$ be the unique solution of the equation*

$$\int_a^{\hat{x}(r)} w(t)\,\mathrm{d}t\cdot\int_a^{\hat{x}(r)}\frac{1}{p(t)}\,\mathrm{d}t=\frac{\kappa^2}{r}\,. \tag{6.4}$$

Then the Titchmarsh–Weyl coefficient q_{D} for (6.3) *with $q\equiv 0$ satisfies*

$$C_{1,\vartheta}B(r)\le|q_{\mathsf{D}}(re^{i\vartheta})|\le C_{2,\vartheta}B(r),\qquad r>0,\ \vartheta\in(0,2\pi), \tag{6.5}$$

where

$$B(r)=\frac{r}{\kappa}\int_a^{\hat{x}(r)} w(t)\,\mathrm{d}t=\frac{\kappa}{\int_a^{\hat{x}(r)}\frac{1}{p(t)}\,\mathrm{d}t}\tag{6.6}$$

and

$$C_{2,\vartheta}=\frac{1+\sigma+\frac{1}{\kappa\sin(\vartheta/2)}}{1-\sigma},\qquad C_{1,\vartheta}=\frac{1}{C_{2,\vartheta}}.$$

Proof. With the given p,w we consider the Hamiltonian

$$H=\begin{pmatrix} w & 0\\ 0 & \frac{1}{p}\end{pmatrix}.$$

Comparing (6.4) and (1.5) (with $\eta=2\kappa$) we see that $\hat{x}(r)=\mathring{t}(r^{\frac{1}{2}})$. It follows from [29, (3.1)] that $q_{\mathsf{D}}(z^2)=zq_H(z)$ for $z\in\mathbb{C}\setminus\mathbb{R}$. Now (1.6) and (2.15) imply

$$\begin{aligned}|q_{\mathsf{D}}(re^{i\vartheta})|&=r^{\frac{1}{2}}\big|q_H\big(r^{\frac{1}{2}}e^{i\frac{\vartheta}{2}}\big)\big|\le r^{\frac{1}{2}}\frac{1+\sigma+\frac{1}{\kappa\sin(\vartheta/2)}}{1-\sigma}A(r^{\frac{1}{2}})\\ &=C_{2,\vartheta}r^{\frac{1}{2}}\frac{r^{\frac{1}{2}}}{\kappa}m_1\big(\mathring{t}(r^{\frac{1}{2}})\big)=C_{2,\vartheta}\frac{r}{\kappa}\int_a^{\hat{x}(r)}w(t)\,\mathrm{d}t=C_{2,\vartheta}\frac{\kappa}{\int_a^{\hat{x}(r)}\frac{1}{p(t)}\,\mathrm{d}t},\end{aligned}$$

which is the second inequality in (6.5); the lower bound follows similarly. ❑

Let us now consider the case where also the potential q is present. We use a transformation to reduce this situation to the previous case with no potential. Asymptotic estimates have been proved in, e.g. [15, Theorem 3], [2, Theorems 1 and 2] and [3, Theorem 3.3]. The following corollary is similar to the latter two references although in those theorems also sign-changing p is allowed. On the other hand, in the following corollary the bounds and the range of validity depend — apart from an a priori lower bound of the corresponding Neumann operator — explicitly and uniformly on certain integrals over the coefficients in a neighbourhood of the left endpoint a, in contrast to the results in the literature we know of. Note that in some of the papers the Neumann Titchmarsh–Weyl coefficient $q_{\mathsf{N}}=-\frac{1}{q_{\mathsf{D}}}$ is considered.

6.5 Corollary. *Let p,q,w be as at the beginning of this subsection and let κ, σ, $C_{1,\vartheta}$, $C_{2,\vartheta}$, $\hat{x}(r)$ and B be as in Corollary 6.4. Further, assume that the Neumann operator A_{N} is bounded below, let $\lambda_0<\min\sigma(A_{\mathsf{N}})$, choose $x_0\in(a,b)$ such that*

$$\int_a^{x_0}\frac{1}{p(t)}\,\mathrm{d}t\le\frac{1}{3},\qquad\int_a^{x_0}\big|q(t)-\lambda_0w(t)\big|\,\mathrm{d}t\le\frac{1}{3}\tag{6.7}$$

and set

$$r_0:=9\kappa^2\bigg[\int_a^{x_0}w(t)\,\mathrm{d}t\cdot\int_a^{x_0}\frac{1}{p(t)}\,\mathrm{d}t\bigg]^{-1}.$$

Then

$$\frac{C_{1,\vartheta}}{36}B(9r)\le\big|q_{\mathsf{D}}\big(\lambda_0+re^{i\vartheta}\big)\big|\le\frac{9C_{2,\vartheta}}{4}B(9r),\qquad r\ge r_0,\ \vartheta\in(0,2\pi).\tag{6.8}$$

Proof. Let v be the solution of the initial value problem

$$-(pv')'+qv=\lambda_0wv,\qquad v(a)=1,\quad (pv')(a)=0.\tag{6.9}$$

It follows in exactly the same way as in [31, Lemma 2.3] that $v(x)>0$ for all $x\in[a,b)$.

In order to obtain an explicit estimate for v, we rewrite the initial value problem (6.9) in a standard way: with $u=\binom{v}{pv'}$, (6.9) is equivalent to

$$u(x)=u_0+\int_a^x\begin{pmatrix}0 & \frac{1}{p(t)}\\ q(t)-\lambda_0w(t) & 0\end{pmatrix}u(t)\,\mathrm{d}t$$

where $u_0 = \binom{1}{0}$. Let T be the operator in the space $C([a,x_0])^2$ (with norm $\left\|\binom{f_1}{f_2}\right\| = \|f_1\|_\infty + \|f_2\|_\infty$) that maps u onto the integral on the right-hand side. It follows from (6.7) that $\|T\| \le \frac{1}{3}$. Hence

$$\|u - u_0\| = \left\|(I-T)^{-1}u_0 - u_0\right\| \le \sum_{n=1}^{\infty} \|T\|^n \|u_0\| \le \frac{1}{2},$$

and, in particular,

$$\frac{1}{2} \le v(x) \le \frac{3}{2}, \qquad x \in [a, x_0]. \tag{6.10}$$

We use the following transformation: set $P := v^2 p$, $W := v^2 w$ and define the unitary mapping

$$U : L^2_w(a,b) \to L^2_W(a,b), \quad y \mapsto \frac{y}{v}.$$

Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and let ψ be a non-trivial solution of $-(p\psi')' + q\psi - \lambda_0 w\psi = \lambda w\psi$ which is in $L^2_w(a,b)$. Then $q_{\mathsf{D}}(\lambda_0 + \lambda) = \frac{(p\psi')(a)}{\psi(a)}$. Set $\widetilde{\psi} := U\psi$, which belongs to $L^2_W(a,b)$. It follows from [31, Lemma 3.2] that $-(P\widetilde{\psi}')' = \lambda W\widetilde{\psi}$. Let $\widetilde{q}_{\mathsf{D}}$ be the Titchmarsh–Weyl coefficient corresponding to the equation $-(Py')' = \lambda W y$; then $\widetilde{q}_{\mathsf{D}}(\lambda) = \frac{(P\widetilde{\psi}')(a)}{\widetilde{\psi}(a)}$. The two Titchmarsh–Weyl coefficients are related as follows:

$$\begin{aligned} q_{\mathsf{D}}(\lambda_0 + \lambda) &= \frac{(p\psi')(a)}{\psi(a)} = \frac{\big(p(v\widetilde{\psi})'\big)(a)}{(v\widetilde{\psi})(a)} = \frac{v(a)\lim_{x\to a}\big(p(x)\widetilde{\psi}'(x)\big) + (pv')(a)\widetilde{\psi}(a)}{v(a)\widetilde{\psi}(a)} \\ &= \frac{1}{v(a)^2} \cdot \frac{(P\widetilde{\psi}')(a)}{\widetilde{\psi}(a)} + \frac{(pv')(a)}{v(a)} = \widetilde{q}_{\mathsf{D}}(\lambda) \end{aligned}$$

since v satisfies the initial conditions in (6.9).

We want to apply Corollary 6.4 with P and W instead of p and w respectively. Let $\tilde{x}(r)$ be the unique solution of

$$\int_a^{\tilde{x}(r)} W(t)\,\mathrm{d}t \cdot \int_a^{\tilde{x}(r)} \frac{1}{P(t)}\,\mathrm{d}t = \frac{\kappa^2}{r}$$

for $r > 0$. It follows from (6.10) that, for r such that $\tilde{x}(r) \le x_0$, we have

$$\begin{aligned} &\int_a^{\tilde{x}(r)} w(t)\,\mathrm{d}t \cdot \int_a^{\tilde{x}(r)} \frac{1}{p(t)}\,\mathrm{d}t = \int_a^{\tilde{x}(r)} \frac{W(t)}{v(t)^2}\,\mathrm{d}t \cdot \int_a^{\tilde{x}(r)} \frac{v(t)^2}{P(t)}\,\mathrm{d}t \\ &\ge \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{3}{2}\right)^2} \int_a^{\tilde{x}(r)} W(t)\,\mathrm{d}t \cdot \int_a^{\tilde{x}(r)} \frac{1}{P(t)}\,\mathrm{d}t = \frac{1}{9} \cdot \frac{\kappa^2}{r} = \int_a^{\hat{x}(9r)} w(t)\,\mathrm{d}t \cdot \int_a^{\hat{x}(9r)} \frac{1}{p(t)}\,\mathrm{d}t \end{aligned}$$

and hence $\tilde{x}(r) \ge \hat{x}(9r)$. In a similar way one proves $\tilde{x}(r) \le \hat{x}(\frac{r}{9})$. In particular, $\tilde{x}(r) \le x_0$ is satisfied if $\hat{x}(\frac{r}{9}) \le x_0$, which, in turn, is equivalent to $r \ge r_0$.

Now Corollary 6.4 applied to $-(Py')' = \lambda W y$ yields

$$\begin{aligned} |q_{\mathsf{D}}(\lambda_0 + re^{i\vartheta})| &= |\widetilde{q}_{\mathsf{D}}(re^{i\vartheta})| \le C_{2,\vartheta}\,\kappa \bigg[\int_a^{\tilde{x}(r)} \frac{1}{P(t)}\,\mathrm{d}t\bigg]^{-1} \le C_{2,\vartheta}\,\kappa \Big(\frac{3}{2}\Big)^2 \bigg[\int_a^{\tilde{x}(r)} \frac{1}{p(t)}\,\mathrm{d}t\bigg]^{-1} \\ &\le \frac{9C_{2,\vartheta}}{4} \cdot \kappa \bigg[\int_a^{\hat{x}(9r)} \frac{1}{p(t)}\,\mathrm{d}t\bigg]^{-1} = \frac{9C_{2,\vartheta}}{4} B(9r) \end{aligned}$$

and

$$\begin{aligned} |q_{\mathsf{D}}(\lambda_0 + re^{i\vartheta})| &= |\widetilde{q}_{\mathsf{D}}(re^{i\vartheta})| \ge C_{1,\vartheta} \frac{r}{\kappa} \int_a^{\tilde{x}(r)} W(t)\,\mathrm{d}t \ge C_{1,\vartheta} \frac{r}{\kappa} \cdot \Big(\frac{1}{2}\Big)^2 \int_a^{\tilde{x}(r)} w(t)\,\mathrm{d}t \\ &\ge \frac{C_{1,\vartheta}}{4} \cdot \frac{r}{\kappa} \int_a^{\hat{x}(9r)} w(t)\,\mathrm{d}t = \frac{C_{1,\vartheta}}{36} \cdot \frac{9r}{\kappa} \int_a^{\hat{x}(9r)} w(t)\,\mathrm{d}t = \frac{C_{1,\vartheta}}{36} B(9r) \end{aligned}$$

for $r \ge r_0$, which proves (6.8). ❑

6.6 Remark. It follows from Corollary 6.5 and the second form of $B(r)$ in (6.6) that $|q_{\mathsf{D}}(\lambda_0 + re^{i\vartheta})|$ is bounded above and below by constants times the function B, which is monotonic increasing and tends to ∞ as $r \to \infty$. ♢

6.4 The work of H. Winkler

Let us now discuss [44, 43]. These papers were of utmost importance for us since the method to estimate Weyl discs is taken from there. H. Winkler proves three theorems about membership of Kac classes; in our notation, these are the classes $\mathcal{M}_{g_\alpha}$ with $g_\alpha(r) := r^\alpha$, $\alpha \in (0,2)$, where the classes $\mathcal{M}_g$ are defined in Definition 4.2.

We start with a slightly more general situation that covers Theorems 4.2 and 4.4 in [44]. Let $\rho_1, \rho_2 > 0$, $c_1, c_2 > 0$ and assume that the primitive M of a Hamiltonian H, defined on $[0,\infty)$, satisfies

$$m_i(t) = c_i t^{\rho_i} + \mathrm{o}(t^{\rho_i}), \qquad t \to 0, \ \ i = 1,2.$$

Further, assume that one of the following conditions is satisfied:

$$\begin{aligned} &\text{(i)} \ \ \rho_1 \neq \rho_2; \\ &\text{(ii)} \ \ \rho_1 = \rho_2, \quad m_3(t) = c_3 t^{\rho_1} + \mathrm{o}(t^{\rho_1}), \quad c_3^2 < c_1 c_2. \end{aligned} \tag{6.11}$$

It follows from [44, Lemma 4.3] in the case when H is trace normed (or from the much more general Theorem 6.1 in [26]) that if (i) is satisfied, then

$$\limsup_{t\to 0} \frac{m_3(t)^2}{m_1(t)m_2(t)} \leq \left(\frac{\sqrt{\rho_1\rho_2}}{\frac{1}{2}(\rho_1+\rho_2)} \right)^2 < 1.$$

In the case when (ii) is satisfied, we obtain $\limsup_{t\to 0} \frac{m_3(t)^2}{m_1(t)m_2(t)} = \frac{c_3^2}{c_1c_2} < 1$. It now follows from Example 4.20 that

$$\mu_H \in \mathcal{F}_{g_{\alpha_0}} \setminus \mathcal{F}^0_{g_{\alpha_0}} \qquad \text{where} \ \ \alpha_0 := \frac{2\rho_2}{\rho_1+\rho_2} \tag{6.12}$$

and hence

$$\mu_H \in \mathcal{M}_{g_\alpha} \quad \Leftrightarrow \quad \alpha > \alpha_0. \tag{6.13}$$

The two special cases considered in [44] are the following. In that paper the Hamiltonian is always trace normed, which implies $\min\{\rho_1, \rho_2\} = 1$.

① Theorem 4.2 in [44].

In that theorem the situation (ii) with $\rho_1 = \rho_2 = 1$ is considered, i.e. $m_i(t) = c_i t + \mathrm{o}(t)$, $i = 1,2,3$, with $c_3^2 < c_1c_2$. In this case we have $\alpha_0 = 1$. One should note that this situation is actually more specific: by [6, Theorem 3.1] one has

$$\lim_{r\to\infty} q_H(ir) = \zeta \qquad \text{with some } \zeta \in \mathbb{C}^+;$$

in particular, $\operatorname{Im} q_H(ir) \asymp 1$, and this also implies the result.

② Theorem 4.4 in [44].

In that theorem the situation (i) is considered, i.e. either $\rho_1 = 1, \rho_2 > 1$ or $\rho_1 > 1, \rho_2 = 1$, which leads to $\alpha_0 = \frac{2\rho_2}{\rho_2+1}$ and $\alpha_0 = \frac{2}{\rho_1+1}$ respectively. We should note that in [44] only the classes $\mathcal{M}_g$ are studied and hence only (6.13) is proved but not the statement in (6.12).

Finally, let us consider a situation that is slightly more complicated.

③ Theorem 4.5 in [44].

Consider a Hamiltonian H defined on $[0,\infty)$ such that M satisfies

$$\begin{aligned} m_i(t) &= c_i t + d_i t^\gamma + \mathrm{o}(t^\gamma), \qquad i = 1,2, \\ m_3(t) &= c_3 t + d_3 t^\delta + \mathrm{o}(t^\delta), \end{aligned} \tag{6.14}$$

as $t \to 0$, where

$$c_1, c_2 > 0, \quad c_3^2 = c_1c_2, \quad \gamma, \delta > 1, \quad d_i \in \mathbb{R}.$$

We cannot argue in the same way as in ① and ② since $\frac{\det M(t)}{(m_1m_2)(t)} \to 0$ and hence $L(r) \ll A(r)$. However, we can apply the transformation from §5.3. In particular, (5.7) yields the primitive $\widetilde{M}$ of a new Hamiltonian $\widetilde{H}$ such that

$$\begin{aligned}
\widetilde{m}_1(t) &= c_2m_1(t) + c_1m_2(t) - 2c_3m_2(t) = (c_2d_1 + c_1d_2)t^\gamma - 2c_2d_3t^\delta + \mathrm{o}(t^\kappa),\\
\widetilde{m}_2(t) &= c_1m_1(t) + c_2m_2(t) + 2c_3m_3(t) = (c_1^2 + c_2^2 + 2c_3^2)t + \mathrm{o}(t),\\
\widetilde{m}_3(t) &= c_3\big(m_1(t) - m_2(t)\big) + (c_2 - c_1)m_3(t)\\
&\qquad = c_3(d_1 - d_2)t^\gamma + (c_2 - c_1)d_3t^\delta + \mathrm{o}(t^\kappa),
\end{aligned}$$

where $\kappa := \min\{\gamma, \delta\}$. Let us assume that $\widetilde{m}_1(t) \sim ct^\kappa$ with some $c > 0$, i.e. that one of the following three conditions is satisfied:

- $\gamma < \delta, \quad c_2d_1 + c_1d_2 > 0$;
- $\delta < \gamma, \quad d_3 < 0$;
- $\gamma = \delta, \quad c_2d_1 + c_1d_2 - 2c_2d_3 > 0$.

(If $\widetilde{m}_1(t) = \mathrm{o}(t^\kappa)$, then one needs more information on the small o terms in the representations of m_i.) The Hamiltonian $\widetilde{M}$ satisfies (i) in (6.11) with $\rho_1 = \kappa$, $\rho_2 = 1$. Hence (6.12) holds with H replaced by $\widetilde{H}$ and $\alpha_0 = \frac{2}{\kappa+1}$. By Proposition 5.7 we have $\operatorname{Im} q_H(ir) \asymp \operatorname{Im} q_{\widetilde{H}}(ir)$ and therefore also $\overleftrightarrow{\mu}_H(r) \asymp \overleftrightarrow{\mu}_{\widetilde{H}}(r)$; see Lemma 4.18. This implies that (6.12) holds also for H and that

$$\mu_H \in \mathcal{M}_{g_\alpha} \quad \Leftrightarrow \quad \alpha > \frac{2}{\kappa + 1}.$$

This covers most cases in [44, Theorem 4.5]; some cases where $\widetilde{m}_1(t) = \mathrm{o}(t^\kappa)$ and more information on m_i is known are also treated there.

Let us note that, as in ①, the situation in (6.14) is more specific. By the main theorem of the forthcoming paper [28], $q_H(ir)$ has a power asymptotic for $r \to \infty$. However, contrasting ①, this does not lead to a proof of the assertion concerning Kac classes since the leading term of this asymptotic expansion is real.

Appendix A. Regularly varying functions

To quantify speed of growth, we use comparison functions which behave roughly like a power in Karamata's sense of regular variation. In this appendix we recall the definition and some facts about such functions. A very good source for the theory of regular variation is [4]; and this is our standard reference.

A.1 Definition. A function $g\colon (0, \infty) \to (0, \infty)$ is called *regularly varying with index* $\alpha \in \mathbb{R}$ if it is measurable and

$$\forall \lambda > 0. \quad \lim_{r\to\infty} \frac{g(\lambda r)}{g(r)} = \lambda^\alpha. \tag{A.1}$$

♢

Examples of regularly varying functions include functions g behaving, for large r, like

$$r^\alpha \cdot \big(\log r\big)^{\beta_1} \cdot \big(\log\log r\big)^{\beta_2} \cdot \ldots \cdot \big(\underbrace{\log \cdots \log}_{m^{\text{th}} \text{ iterate}} r\big)^{\beta_m},$$

where $\alpha, \beta_1, \ldots, \beta_m \in \mathbb{R}$, which were studied already in the context of entire functions. Other examples are $g(r) = r^\alpha e^{(\log r)^\beta}$ with $\beta \in (0, 1)$, or $g(r) = r^\alpha e^{\frac{\log r}{\log\log r}}$; see [4, §1.3].

The following theorem shows that regularly varying functions with index α are asymptotically strictly between powers with exponents strictly larger than α and powers with exponents strictly smaller than α. It follows, e.g. from the Potter bounds; see [4, Theorem 1.5.6 (iii)].

A.2 Theorem. *Let $g : (0, \infty) \to (0, \infty)$ be a regularly varying function with index $\alpha \in \mathbb{R}$. For every $\rho \in \mathbb{R}$, $\rho \neq \alpha$, there exist $r_0 > 0$ and $C > 0$ such that*

$$\begin{aligned} g(r) &\leq C r^\rho \qquad \text{if } \rho > \alpha, \\ g(r) &\geq C r^\rho \qquad \text{if } \rho < \alpha, \end{aligned}$$

for $r \geq r_0$.

A regularly varying function with a strictly positive index is asymptotically equivalent to a monotonic increasing, regularly varying function with the same index, as the following theorem shows; see [4, Theorems 1.5.3 and 1.3.1].

A.3 Theorem. *Let $g : (0, \infty) \to (0, \infty)$ be a locally bounded, regularly varying function with index $\alpha > 0$. Then*

$$\bar{g}(r) := \sup\{g(t) : 0 \leq t \leq r\} \sim g(r)$$

as $r \to \infty$, and $\bar{g}$ is regularly varying with index α.

Another fundamental result, due to J. Karamata, determines what happens when a regularly varying function is integrated against a power. We recall this theorem in a comprehensive formulation collecting what is proved in [4, Section 1.5.6].

A.4 Theorem (Karamata)**.** *Let g be regularly varying at ∞ with index $\alpha \in \mathbb{R}$ and assume that g is locally bounded.*

(i) *Let $\delta \in \mathbb{R}$ and assume that $\delta + \alpha + 1 \geq 0$. Then the function $x \mapsto \int_1^x t^\delta g(t)\,\mathrm{d}t$ is regularly varying with index $\delta + \alpha + 1$, and*

$$\lim_{x \to \infty} \left(x^{\delta+1} g(x) \Big/ \int_1^x t^\delta g(t)\,\mathrm{d}t \right) = \delta + \alpha + 1.$$

(ii) *Let $\delta \in \mathbb{R}$ and assume that $\int_1^\infty t^\delta g(t)\,\mathrm{d}t < \infty$. Then $\delta + \alpha + 1 \leq 0$, the function $x \mapsto \int_x^\infty t^\delta g(t)\,\mathrm{d}t$ is regularly varying with index $\delta + \alpha + 1$, and*

$$\lim_{x \to \infty} \left(x^{\delta+1} g(x) \Big/ \int_x^\infty t^\delta g(t)\,\mathrm{d}t \right) = -(\delta + \alpha + 1).$$

Appendix B. The generalised inverse of a non-decreasing function

Let us recall the notion of a generalised inverse; see, e.g. [7, Definition 2.1]. We slightly adapt the definition and, in particular, allow that the given function may attain the value $+\infty$; this is convenient when we apply our results to Krein strings. On the set $(-\infty, \infty]$ we use the obvious order structure.

B.1 Definition. Let $-\infty < a < b \leq \infty$, let $f : [a, b) \to (-\infty, \infty]$ be a non-decreasing function and set $\mathcal{R}_f := \operatorname{conv}(\operatorname{ran} f)$, the convex hull of the range of f. The function f^-, defined by

$$f^-(y) := \inf\bigl\{x \in [a, b) \colon f(x) \geq y\bigr\}, \qquad y \in \mathcal{R}_f, \tag{B.1}$$

is called the *generalised inverse* of f. ◊

Note that in this definition the function f is neither assumed to be strictly increasing nor to be continuous. Further, note that, with $b' := \lim_{x \to b} f(x)$, we have $\mathcal{R}_f = [f(a), b']$ if $b' \in \operatorname{ran} f$ and $\mathcal{R}_f = [f(a), b')$ otherwise.

In the next two lemmata we state some facts about generalised inverses which we use in the present paper. They are folklore; some can be found in [7], some in [23], and most probably in many other references. For the sake of completeness and because the setting is slightly different, we give proofs.

B.2 Lemma. *Let f be as in Definition B.1 and set $b' := \lim_{x\to b} f(x)$. Then the following statements hold.*

(i) $f^-(y) \in [a,b)$ *for all* $y \in \mathcal{R}_f$.

(ii) f^- *is non-decreasing and left-continuous.*

(iii) *Let* $x \in [a,b)$. *Then*

$$f^-\big(f(x)\big) = \inf\{\xi \in [a,x] : f(\xi) = f(x)\} \le x$$

with equality if and only if x is not right endpoint of an interval where f is constant. In particular, $f^-(f(a)) = a$.

(iv) *Let* $y \in \mathcal{R}_f$. *Then*

$$f\big(f^-(y)\big) \le y \qquad \text{if } f \text{ is left-continuous at } f^-(y),$$

$$f\big(f^-(y)\big) \ge y \qquad \text{if } f \text{ is right-continuous at } f^-(y).$$

In particular, if f is continuous on $[a,b)$, then $f(f^-(y)) = y$.

(v) *Let $y_0 \in \mathcal{R}_f$ with $y_0 < b'$ and assume that f is strictly increasing on the interval $[f^-(y_0), f^-(y_0)+\varepsilon]$ for some $\varepsilon > 0$. Then f^- is continuous at y_0.*

(vi) *Assume hat f is continuous and let $x \in [a,b)$ and $y \in \mathcal{R}_f$. Then*

$$f^-(y) \le x \quad \Leftrightarrow \quad y \le f(x).$$

(vii) *Set $x_0 := \sup\{\xi \in [a,b) : f(\xi) < b'\}$. Then $\lim\limits_{y \nearrow b'} f^-(y) = x_0$.*

(viii) *Let g be another function as in Definition B.1 defined on the same interval $[a,b)$. If $f(x) \le g(x)$ for all $x \in [a,b)$, then*

$$f^-(y) \ge g^-(y), \qquad y \in \mathcal{R}_f \cap \mathcal{R}_g.$$

(ix) *Let $-\infty < c < d \le \infty$, let $I' \subseteq (-\infty,\infty]$ be an interval, and let $\varphi\colon [c,d) \to [a,b)$ and $\psi : \mathcal{R}_f \to I'$ be increasing bijections. Then*

$$\big(\psi \circ f \circ \varphi\big)^-(v) = \big(\varphi^{-1} \circ f^- \circ \psi^{-1}\big)(v), \qquad v \in I'.$$

(x) *Let $\tilde f$ be another function as in Definition B.1 defined on $[a,\tilde b)$ such that $\tilde b \ge b$ and $f(x) = \tilde f(x)$ for $x \in [a,b)$. Then $\tilde f^-(y) = f^-(y)$ for $y \in \mathcal{R}_f$.*

Proof.

(i) For $y \in \mathcal{R}_f$ the set in (B.1), of which the infimum is taken, is non-empty and contained in $[a,b)$.

(ii) To show that f^- is non-decreasing, let $y_1 \le y_2$. Then

$$\big\{x \in [a,b)\colon\ f(x) \ge y_1\big\} \supseteq \big\{x \in [a,b)\colon\ f(x) \ge y_2\big\},$$

and hence $f^-(y_1) \le f^-(y_2)$. In order to prove the left-continuity of f^-, we first show the following implications

$$\begin{aligned} x < f^-(y) \quad &\Rightarrow \quad f(x) < y, \\ x > f^-(y) \quad &\Rightarrow \quad f(x) \ge y. \end{aligned} \tag{B.2}$$

Indeed, if $x < f^-(y)$, then $x \notin \{\xi \in [a,b) : f(\xi) \ge y\}$ and hence $f(x) < y$; if $x > f^-(y)$, then there exists $\xi \in [a,x)$ such that $f(\xi) \ge y$ and therefore $f(x) \ge f(\xi) \ge y$.

Now let $y_0 \in \mathcal{R}_f$ with $y_0 > f(a)$ and assume that

$$x_- := \lim_{y \nearrow y_0} f^-(y) < f^-(y_0).$$

Choose an arbitrary number $x' \in (x_-, f^-(x_0))$. For every $y \in (f(a), y_0)$ we have $f^-(y) \le x_- < x' < f^-(y_0)$. Hence, two applications of (B.2) yield $y \le f(x') < y_0$, which is a contradiction to the arbitrariness of y in $(f(a), y_0)$.

(iii) For $x \in [a, b)$ we have

$$\begin{aligned} f^-\bigl(f(x)\bigr) &= \inf\{\xi \in [a,b) : f(\xi) \ge f(x)\} = \inf\{\xi \in [a,x] : f(\xi) \ge f(x)\} \\ &= \inf\{\xi \in [a,x] : f(\xi) = f(x)\}. \end{aligned}$$

The remaining assertions are now clear.

(iv) The implications follow from (B.2), e.g. if f is left-continuous at $f^-(y)$, then $f(f^-(y)) = \lim_{x \nearrow f^-(y)} f(x) \le y$. The last statement clearly follows from this when $f^-(y) \in (a, b)$; when $f^-(y) = a$, then $f(f^-(y)) = f(a) \le y$ since $y \in \mathcal{R}_f$, and the other inequality follows since f is right-continuous at $f^-(y)$.

(v) In light of (ii) it is sufficient to show right-continuity. Suppose that

$$f^-(y_0) < \lim_{y \searrow y_0} f^-(y)$$

and set

$$x_+ := \min\Bigl\{\lim_{y \searrow y_0} f^-(y), f^-(y_0) + \varepsilon\Bigr\}.$$

Further, choose $x_1, x_2 \in (f^-(y_0), x_+)$ with $x_1 < x_2$. For every $y \in (y_0, b')$ we have $f^-(y_0) < x_1 < x_2 < x_+ \le f^-(y)$. Hence, applying (B.2) twice and using the strict monotonicity of f on $[f^-(y_0), x_+]$ we obtain

$$y_0 \le f(x_1) < f(x_2) < y,$$

which is a contradiction to the arbitrariness of y in (y_0, b').

(vi) Assume that $f^-(y) \le x$. Since f is continuous, it follows from (iv) that $y = f(f^-(y)) \le f(x)$. Now assume that $f^-(y) > x$; then $y > f(x)$ by (B.2).

(vii) We distinguish two cases. When $x_0 < b$, then $b' \in \mathcal{R}_f$ and $f^-(b') = x_0$. Hence the left-continuity of f^-, shown in (ii), implies the assertion.

Now assume that $x_0 = b$. It is clear from (i) that $f^-(y) < b$ for every $y < b'$; therefore $\lim_{y \nearrow b'} f^-(y) \le x_0$. To show equality in the latter relation, let $x \in [a, x_0)$ be arbitrary. By the definition of x_0 we have $f(x) < b'$. Hence there exists $y \in (f(x), b')$, which, by (B.2), implies that $f^-(y) \ge x$.

(viii) For $y \in \mathcal{R}_f \cap \mathcal{R}_g$ we have

$$\bigl\{x \in [a,b)\colon\ f(x) \ge y\bigr\} \subseteq \bigl\{x \in [a,b)\colon\ g(x) \ge y\bigr\}$$

and hence $f^-(y) \ge g^-(y)$.

(ix) Let $v \in I'$. Then

$$\begin{aligned} \bigl\{t \in [c,d)\colon\ (\psi \circ f \circ \varphi)(t) \ge v\bigr\} &= \bigl\{t \in [c,d)\colon\ f(\varphi(t)) \ge \psi^{-1}(v)\bigr\} \\ &= \varphi^{-1}\bigl(\bigl\{x \in [a,b)\colon\ f(x) \ge \psi^{-1}(v)\bigr\}\bigr), \end{aligned}$$

from which the desired relation follows.

(x) For $y \in \mathcal{R}_f$ we have

$$\tilde{f}^-(y) = \inf\{x \in [a, \tilde{b}) : \tilde{f}(x) \ge y\} = \inf\{x \in [a,b) : \tilde{f}(x) \ge y\},$$

which implies the assertion. ❑

The following lemma is used in §6.2. To avoid distinction of cases, we use the relation $\frac{\infty}{x} = \infty$ for $x \in (0, \infty)$.

B.3 Lemma. *Let $b \in (0, \infty]$, and let $f\colon [0, b) \to [0, \infty]$ be a non-decreasing function such that $f(0) = 0$, $\lim_{x \to b} f(x) = \infty$ and $x \mapsto \frac{f(x)}{x^\rho}$ is non-decreasing on $(0, b)$ for some $\rho > 0$. Then for all $c > 0$ we have $f^-(cy) \asymp f^-(y)$, $y \in [0, \infty)$.*

Proof. First note that we have either $\mathcal{R}_f = [0,\infty)$ or $\mathcal{R}_f = [0,\infty]$.

It is sufficient to consider the case when $c > 1$. From the monotonicity of f^- it follows that

$$f^-(y) \le f^-(cy), \qquad y \in [0,\infty). \tag{B.3}$$

Our aim is to show a reverse inequality with a multiplicative constant. If $b < \infty$, we extend f as follows:

$$\tilde{f}(x) := \begin{cases} f(x), & x \in [0,b), \\ +\infty, & x \in [b,\infty). \end{cases}$$

In the case when $b = \infty$, we set $\tilde{f} := f$. It follows from Lemma $B.2$ (x) that $\tilde{f}^-(y) = f^-(y)$ for $y \in \mathcal{R}_f$. For $x \in (0, c^{-\frac{1}{\rho}} b)$ we have

$$f(x) = x^\rho \frac{f(x)}{x^\rho} \le x^\rho \frac{f\big(c^{\frac{1}{\rho}} x\big)}{\big(c^{\frac{1}{\rho}} x\big)^\rho} = \frac{f\big(c^{\frac{1}{\rho}} x\big)}{c}.$$

We can extend this inequality to

$$\tilde{f}(x) \le \frac{1}{c} \tilde{f}\big(c^{\frac{1}{\rho}} x\big), \qquad x \in [0,\infty). \tag{B.4}$$

Define the functions $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(x) = c^{\frac{1}{\rho}} x$ and $\psi : \mathcal{R}_{\tilde{f}} \to \mathcal{R}_{\tilde{f}}$, $\psi(y) = \frac{1}{c} y$. Then (B.4) can be written as $\tilde{f}(x) \le (\psi \circ \tilde{f} \circ \varphi)(x)$, $x \in [0,\infty)$. Hence Lemma $B.2$ (viii), (ix) imply that, for $y \in \mathcal{R}_f$,

$$\begin{aligned} f^-(y) = \tilde{f}^-(y) &\ge \big(\psi \circ \tilde{f} \circ \varphi\big)^-(y) = \big(\varphi^{-1} \circ \tilde{f}^- \circ \psi^{-1}\big)(y) \\ &= c^{-\frac{1}{\rho}} \tilde{f}^-(cy) = c^{-\frac{1}{\rho}} f^-(cy). \end{aligned}$$

In particular, this inequality is true for $y \in [0,\infty)$, which, together with (B.3), proves the assertion. ❑

Note that Lemma $B.3$ says that under the given assumptions the function f^- belongs to the class OR defined in [4, §2.0.2]; see also Remark 4.8.

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