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Asymptotic stability in distribution of highly nonlinear stochastic differential equations with G-Brownian motion

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Abstract

Following the analysis on the stability in distribution of stochastic differential equations discussed in Fei, Fei & Mao (2023) [11], this article further investigates the stability in distribution of highly nonlinear stochastic differential equations driven by G-Brownian motion (G-HNSDEs). To this end, by employing the theory on sublinear expectations, the stability in distribution of G-HNSDEs is analysed. Moreover, a sufficient criterion of the stability in distribution of G-HNSDEs is provided for convenient use.

Keywords: G-HNSDEs; sublinear expectation; stability in distribution; Chebyshev inequality; G-Itô formula.

1 Introduction

For the past decades, stochastic differential equations (SDEs) and their stabilities have become an active area of stochastic analysis. The convergence of SDEs

includes four types of convergence in probability theory, namely, stability in distribution, stability in probability as well as moment stability and almost sure stability. And almost sure stability and moment stability have received enormous attention. Recently, in references [3, 4, 24], they investigated the pth moment exponential stability, h-stability in pth moment, and partial asymptotic stability of neutral pantograph stochastic differential equations with Markovian switching. The stability of highly nonlinear hybrid stochastic delay equations is also discussed, see, e.g., [15, 17], and references therein. The main tools used to show the results are the Lyapunov function method, Razumikhin or comparison techniques. As far as we know, there are some studies on the stability in distribution of SDEs. Among them, [35, 36] provided a notable contribution to the stability in distribution. In [8], the authors discussed a new sufficient condition for stability in distribution of stochastic differential delay equations with Markovian switching. [6] improved the results on stability in distribution of nonlinear SDEs in [35]. In [1] and [28], the stability in distribution of neutral delay SDEs is investigated. [2] researched the stability in distribution of Markovian delay SDEs with reflection. In [18], the stability in distribution of neutral stochastic functional differential equations with Markovian switching is discussed. [37] studied the stability in distribution of stochastic delay recurrent neural networks with Markovian switching. [22] analyzed the stability in distribution for SDEs with memory driven by positive semigroups and Lévy processes. [29] explored the asymptotic stability in distribution of stochastic systems with semi-Markovian switching. [30] further investigated the stability in distribution of stochastic functional differential equations. Recently, [20] considered the stabilization in distribution of hybrid stochastic differential equations by feedback control based on discrete-time state observations while the stabilisation in distribution by delay feedback control for hybrid SDEs is studied in [34].

On the other hand, much work of SDEs driven by G-Brownian motion is studied by many researchers, e.g., [7, 9, 10, 12, 14, 16, 19, 21, 26, 27, 32, 38], and the references therein.

Recently, [11] explored the stability in distribution of SDEs driven by G-Brownian motion, where the coefficients of SDEs are linear growth or bounded by linear functions. To the best of our knowledge, the stability in distribution of highly nonlinear SDEs driven by G-Brownian motion (G-HNSDEs) is not investigated yet. In this article, we try to discuss the stability of the following G-HNSDE

$$dX(t) = f(X(t))dt + g(X(t))dB(t) + h(X(t))d < B > (t) \tag{1}$$

on a sublinear expectation space $(\Omega, \mathscr{H}, \hat{\mathbb{E}})$.

The main contributions of this paper are as follows.

• We first investigate the stability in distribution of G-HNSDEs by using theory of sublinear expectations.

• The related criteria of the stability in distribution of G-HNSDEs are given.

• Some mathematical techniques under sublinear expectation are employed.

The remaining part of this paper can briefly be stated as follows: Section 2 provides some definitions and assumptions for the derivation of main results. In Section 3, we prove the main results of the paper. In Section 4, an example is presented to illustrate the obtained results. Sect. 5 concludes the paper and points out some future research.

2 Definitions and Assumptions

Let $(\Omega, \mathscr{H}, \hat{\mathbb{E}})$ to be sublinear expectation space. Here, Ω is a given state set and $\mathscr{H}$ a linear space of real valued functions defined on Ω. The related notions and properties of sublinear expectation are referred to [26]. For the notions of stochastic processes, the uncertainty probability family $\mathscr{P}$ and upper capacity $\mathbb{V}$ (or lower capacity $\mathscr{V}$), we see [11, 12].

Definition 2.1. (i). The distribution $\mathfrak{F}_\xi$ generated by d-dimensional random variable ξ in $\mathscr{H}$ is defined by

$$\mathfrak{F}_\xi(A) = \mathbb{V}\{\omega \in \Omega : \xi(\omega) \in A\} = \hat{\mathbb{E}}[\mathbf{1}_{\{\omega\in\Omega:\xi(\omega)\in A\}}],\ \forall A \in \mathscr{B}(\mathbb{R}^d).$$

(ii). For random variables ξ and η, we denote their distributions by $\mathfrak{F}_\xi$ and $\mathfrak{F}_\eta$, respectively. Define the distance of distributions of random variables ξ and η as follows

$$d_{\mathbb{T}}(\mathfrak{F}_\xi, \mathfrak{F}_\eta) = \sup_{\phi\in\mathbb{T}} |\hat{\mathbb{E}}[\phi(\xi)] - \hat{\mathbb{E}}[\phi(\eta)]|,$$

where $\mathbb{T} = \{\phi : \mathbb{R}^d \to \mathbb{R} : |\phi(x) - \phi(y)| \leq |x-y| \text{ and } |\phi(\cdot)| \leq 1\}$.

(iii). For the stochastic process $(x(t))_{t\geq 0}$ on sublinear expectation space $(\Omega, \mathscr{H}, \hat{\mathbb{E}}, \mathbb{V}, (\mathscr{H}_t)_{t\geq 0})$, we denote the distribution of $x(t)$ by $\mathfrak{F}_{x(t)}$ for each $t \in [0,\infty)$. If there is a distribution $\nu(\cdot)$ of the random variable such that

$$d_{\mathbb{T}}(\mathfrak{F}_{x(t)}, \nu) \to 0, \text{ as } t \to \infty,$$

then the stochastic process $(x(t))_{t\geq 0}$ is called the (asymptotic) stability in distribution. We also call the process $(x(t))_{t\geq 0}$ converges weakly to the distribution ν.

Denote the family of capacities on $\mathbb{R}^d$ by $\mathscr{C}(\mathscr{B}(\mathbb{R}^d))$. It is easy to know that the metric $d_{\mathbb{T}}$ on $\mathscr{C}(\mathscr{B}(\mathbb{R}^d))$ is a distance, and $(\mathscr{C}(\mathscr{B}(\mathbb{R}^d)), d_{\mathbb{T}})$ is a Polish space.

Now we consider the stochastic differential equation (1) with initial value $X(0) = x$, where $(B(t))_{t\geq 0}$ is the G-Brownian motion in $\mathbb{R}$ on the generalized nonlinear expectation space $(\Omega, \mathscr{H}, \hat{\mathbb{E}}, \mathbb{V}, (\mathscr{H}_t)_{t\geq 0})$, and

$$f : \mathbb{R}^d \to \mathbb{R}^d,\ g : \mathbb{R}^d \to \mathbb{R}^d,\ h : \mathbb{R}^d \to \mathbb{R}^d.$$

For Eq. (1), we provide the following locally Lipschitzian condition. Let $\mathscr{K}$ denote the family of increasing functions $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\kappa(0) = 0$, and $\mathscr{K}_\infty$ the family of functions $\kappa \in \mathscr{K}$ such that $\kappa(s) \to \infty$ as $s \to \infty$.

Assumption 2.2 *The coefficients of SDE (1) satisfy the local Lipschitzian condition, that is, for each $k \in \mathbb{N}$, there is a $b_k > 0$ such that*

$$|f(x) - f(y)|^2 + |g(x) - g(y)|^2 + |h(x) - h(y)|^2 \leq b_k |x - y|^2, \ \forall |x| \vee |y| \leq k.$$

Assumption 2.3 *There are functions $V \in C^2(\mathbb{R}^d; \mathbb{R}_+)$, $\lambda \in \mathscr{K}_\infty$ and a positive number ϖ such that*

$$\lim_{|x| \to \infty} V(x) = \infty \tag{2}$$

$$\mathbb{L}V(x) \equiv V_x(x) f(x) + G(2V_x(x)h(x) + \mathrm{trace}[g^\top(x) V_{xx}(x) g(x)]) \leq -\lambda(x) + \varpi. \tag{3}$$

Assumption 2.4 *There are functions $W \in C^2(\mathbb{R}^d; \mathbb{R}_+)$, $\upsilon \in \mathscr{K}_\infty$ and $\nu^{\mathfrak{R}} \in \mathscr{K}$ for $\mathfrak{R} > 0$ satisfying*

$$\begin{aligned}
&W(0) = 0 \\
&\upsilon(|x|) \leq W(x), \ x \in \mathbb{R}^d \\
&\mathscr{L}W(x,y) = W_x(x-y)[f(x) - f(y)] + G(2W_x(x-y)[h(x) - h(y)] \\
&\quad + \mathrm{trace}([g(x) - g(y)]^\top W_{xx}(x-y)[g(x) - g(y)])) \leq -\nu^{\mathfrak{R}}(|x-y|), \ \forall |x| \vee |y| \leq \mathfrak{R}.
\end{aligned} \tag{4}$$

3 Main Results

In this section, we shall show our main results on asymptotic stability in distribution of SDEs driven by G-Brownian motion.

Proposition 3.1 *Let Assumptions 2.2 and 2.3 hold. Then, for each initial value $x \in \mathbb{R}^d$, there is a unique global solution, denoted by $X^x(t)$, to SDE (1) with initial value $X(0) = x$. Moreover, for $H > 0$, we can find $\bar{K} = \bar{K}(H) > 0$ such that*

$$\frac{1}{t} \int_0^t \hat{\mathbb{E}}[\lambda(X^x(s))] ds \leq \bar{K}, \ \forall |x| \leq H, t \geq 1.$$

Proof. Under Assumption 2.2, by Fei et al. [14, Theorem 3.4], SDE (1) has a maximal solution $X^x(t)$ on $[0, \sigma_\infty)$, where σ_∞ is the explosion time of solution. We now need to prove $\sigma_\infty = \infty$ q.s. If this is not true, then we can find two positive constants ε, T such that

$$\mathscr{V}\{\sigma_\infty \leq T\} > 2\varepsilon.$$

For any integer $k \geq 1$, define the stopping time

$$\sigma_k = \inf\{t \geq 0 : |X(t)| \geq k\}.$$

Since $\sigma_k \to \sigma_\infty$ q.s. as $k \to \infty$, we can find a sufficiently large integer k_0 such that

$$\mathscr{V}\{\sigma_k \le T\} > \varepsilon,\ k \ge k_0. \tag{5}$$

Fix any $k \ge k_0$, and $t \in [0,T]$, from the G-Itô formula from Peng [26, Proposition 3.6.3] and Assumption 2.3, we get

$$\hat{\mathbb{E}}[V(X(t\wedge\sigma_k))] \le V(x) + \hat{\mathbb{E}}\left[\int_0^{t\wedge\sigma_k} \mathbb{L}V(X^x(s))ds\right] \le V(x) + \varpi T,$$

which shows

$$\hat{\mathbb{E}}[\mathbf{1}_{\{\sigma_k \le t\}} V(X(\sigma_k))] \le V(x) + \varpi T. \tag{6}$$

On the other hand, if we set

$$\ell_k = \inf\{V(x) : |x| \ge k\},$$

then $\ell_k \to \infty$ as $k \to \infty$ by (2). From (5) and (6), we have, $k \ge k_0$,

$$\varepsilon\ell_k \le \ell_k \mathscr{V}\{\sigma_k \le T\} \le \ell_k \mathbb{V}\{\sigma_k \le T\} \le V(x) + \varpi T,$$

which shows a contradiction. Thus, we know $\sigma_\infty = \infty$. q.s.

Define stopping time

$$\upsilon_k = \inf\{t \ge 0 : |X^x(t)| \vee |X^y(t)| > k\}, k \in \mathbb{N}.$$

Therefore, we have $\upsilon_k \to \infty$ q.s. as $k \to \infty$. From the G-Itô formula, we have, for each $P \in \mathscr{P}$,

$$E_P[V(X^x(t\wedge\upsilon_k))] \le V(x) + E_P\left[\int_0^{t\wedge\upsilon_k} (\varpi - \lambda(X^x(s)))ds\right]$$

which shows

$$\hat{\mathbb{E}}\left[\int_0^{t\wedge\upsilon_k} \lambda(X^x(s))ds\right] \le V(x) + \varpi t.$$

Letting $k \to \infty$, by Fatou's lemma, we get

$$\frac{1}{t}\hat{\mathbb{E}}\left[\int_0^t \lambda(X^x(s))ds\right] \le \varpi + \frac{1}{t}V(x) \le \bar{K}(H),\ \forall |x| \le H, t \ge 1.$$

Thus, we complete the proof. □

Proposition 3.2 *Let Assumptions 2.2, 2.3 and 2.4 hold. SDE (1) has the property that for positive constants H, ε, δ, there is a positive constant $T = T(H, \varepsilon, \delta)$ such that*

$$\mathscr{V}\{|X^x(t) - X^y(t)| \le \delta,\ \forall t \ge T\} \ge 1 - \varepsilon, \tag{7}$$

for all $|x| \vee |y| \le H$.

Proof. Take arbitrarily $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq H$. Let arbitrarily $\delta > 0$ and $\varepsilon_1 > 0$. First, we shall verify, for bounded stopping times $\tau_1 \leq \tau_2$,

$$\hat{\mathbb{E}}[W(X^x(\tau_2) - X^y(\tau_2))] \leq \hat{\mathbb{E}}[W(X^x(\tau_1) - X^y(\tau_1))] \leq W(x-y) \tag{8}$$

and

$$0 \leq W(x-y) + \hat{\mathbb{E}}\left[\int_0^{\tau_1} \mathscr{L}W(X^x(s), X^y(s))ds\right]. \tag{9}$$

In fact, for the stopping time υ_k defined as above, by using G-Itô formula and (4), we get

$$\begin{aligned} 0 &\leq \hat{\mathbb{E}}[W(X^x(\tau_1 \wedge \upsilon_k) - X^y(\tau_1 \wedge \upsilon_k))] \\ &\leq W(x-y) + \hat{\mathbb{E}}\left[\int_0^{\tau_1 \wedge \upsilon_k} \mathscr{L}W(X^x(s), X^y(s))ds\right] \leq W(x-y). \end{aligned}$$

Consequently, letting $k \to \infty$, by Fatou's lemma, we have $\hat{\mathbb{E}}W(X^x(\tau_1) - X^y(\tau_1)) \leq W(x-y)$. Similarly, we get

$$\hat{\mathbb{E}}[W(X^x(\tau_2) - X^y(\tau_2))] \leq \hat{\mathbb{E}}[W(X^x(\tau_1) - X^y(\tau_1))].$$

Then (8) holds. Note

$$-\hat{\mathbb{E}}\left[\int_0^{\tau_1 \wedge \upsilon_k} \mathscr{L}W(X^x(s), X^y(s))ds\right] \leq W(x-y).$$

Thus, letting $k \to \infty$, we get (9).

Since $\lim_{\zeta \to \infty} \lambda(\zeta) = \infty$, for any $\varepsilon_1 > 0$, there is a sufficient large H such that $\inf_{\zeta \geq H} \lambda(\zeta) \geq 2\bar{K}/\varepsilon_1^2$ with $\bar{K}$ defined by Proposition 3.1. Therefore, by Chebyshev inequality (see [5, Proposition 2.1]) we get

$$\begin{aligned} \mathbb{V}\{|X^x(t)| \vee |X^y(t)| > H\} &\leq \mathbb{V}\{|X^x(t)| > H\} + \mathbb{V}\{|X^y(t)| > H\} \\ &\leq \frac{\varepsilon_1^2}{2\bar{K}}(\hat{\mathbb{E}}[\lambda(|X^x(t)|)] + \hat{\mathbb{E}}[\lambda(|X^y(t)|)]). \end{aligned} \tag{10}$$

Due to $W(0) = 0$, there is a $\rho \in (0, \delta)$ such that

$$\sup_{|u| \leq \rho} W(u)/\upsilon(\delta) \leq \varepsilon_1.$$

Define the stopping time

$$\tau_\rho = \inf\{t \geq 0 : |X^x(t) - X^y(t)| \leq \rho \text{ and } |X^x(t)| \vee |X^y(t)| \leq H\}.$$

Moreover, by (4), we have, for each $P \in \mathscr{P}$,

$$\begin{aligned}
&E_P\left[\int_0^{\tau_\rho \wedge t} \mathscr{L}W(X^x(s), X^y(s))ds\right] \\
&= E_P\left[\int_0^t \mathbf{1}_{\{0\le s\le \tau_\rho\}} \mathscr{L}W(X^x(s), X^y(s))ds\right] \\
&\le E_P\left[\int_0^t \mathbf{1}_{\{|X^x(s)|\vee|X^y(s)|\le H\}} \mathbf{1}_{\{0\le s\le \tau_\rho\}} \mathscr{L}W(X^x(s), X^y(s))ds\right] \\
&\le -\nu^H(\rho) E_P\left[\int_0^t \mathbf{1}_{\{|X^x(s)|\vee|X^y(s)|\le H\}} \mathbf{1}_{\{0\le s\le \tau_\rho\}} ds\right]. \qquad (11)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&E_P\left[\int_0^t \mathbf{1}_{\{0\le s\le \tau_\rho\}} ds\right] - E_P\left[\int_0^t \mathbf{1}_{\{|X^s(s)|\vee|X^y(s)|\le H\}} \mathbf{1}_{\{0\le s\le \tau_\rho\}} ds\right] \\
&\quad = E_P\left[\int_0^t \mathbf{1}_{\{|X^s(s)|\vee|X^y(s)|> H\}} \mathbf{1}_{\{0\le s\le \tau_\rho\}} ds\right].
\end{aligned}$$

In terms of the Hölder inequality, (10) and Proposition 3.1, we have

$$\begin{aligned}
&\left(\hat{\mathbb{E}}\left[\int_0^t \mathbf{1}_{\{|X^x(s)|\vee|X^y(s)|> H\}} \mathbf{1}_{\{0\le s\le \tau_\rho\}} ds\right]\right)^2 \\
&\le \int_0^t \hat{\mathbb{E}}[\mathbf{1}_{\{|X^x(s)|\vee|X^y(s)|> H\}}]ds \int_0^t \hat{\mathbb{E}}[\mathbf{1}_{\{0\le s\le \tau_\rho\}}]ds \\
&\le t\frac{\varepsilon_1^2}{2\bar{K}} \int_0^t (\hat{\mathbb{E}}[\lambda(|X^x(s)|)] + \hat{\mathbb{E}}[\lambda(|X^y(s)|)])ds \le (\varepsilon_1 t)^2,\ t \ge 1,
\end{aligned}$$

which implies, for each $P \in \mathscr{P}$,

$$E_P\left[\int_0^t \mathbf{1}_{\{|X^x(s)|\vee|X^y(s)|> H\}} \mathbf{1}_{\{0\le s\le \tau_\rho\}} ds\right] \le \varepsilon_1 t,\ t \ge 1.$$

Thus, we get

$$E_P\left[\int_0^t \mathbf{1}_{\{|X^x(s)|\vee|X^y(s)|\le H\}} \mathbf{1}_{\{0\le s\le \tau_\rho\}} ds\right] \ge E_P[\tau_\rho \wedge t] - \varepsilon_1 t,$$

which shows, by (11),

$$\begin{aligned}
&E_P\left[\int_0^{t\wedge \tau_\rho} \mathscr{L}W(X^x(s), X^y(s))ds\right] \\
&\le -\nu^H(\rho) E_P\left[\int_0^t \mathbf{1}_{\{|X^x(s)|\vee|X^y(s)|\le H\}} \mathbf{1}_{\{0\le s\le \tau_\rho\}} ds\right]
\end{aligned}$$

$$\leq -\nu^H(\rho)(E_P[\tau_\rho \wedge t] - \varepsilon_1 t).$$

From the Itô formula, we get, for each $P \in \mathscr{P}$,

$$0 \leq W(x-y) + E_P\left[\int_0^{\tau_\rho} \mathscr{L}W(X^x(s), X^y(s))ds\right].$$

Thus, we have

$$0 \leq W(x-y) + \nu^H(\rho)\varepsilon_1 t - \nu^H(\rho)E_P[\tau_\rho \wedge t]$$

which implies $E_P[\tau_\rho \wedge t] \leq \frac{W(x-y)}{\nu^H(\rho)} + \varepsilon_1 t$. Hence, due to arbitrariness of $P \in \mathscr{P}$, we get

$$t\mathbb{V}\{\tau_\rho \geq t\} \leq \hat{\mathbb{E}}[\tau_\rho \wedge t] \leq \frac{W(x-y)}{\nu^H(\rho)} + \varepsilon_1 t,\ t \geq 1,$$

which shows that there exists a $T > 1$ such that

$$\mathbb{V}\{\tau_\rho \geq T\} \leq 2\varepsilon_1. \tag{12}$$

Now define the stopping time $\rho = \inf\{t \geq \tau_\rho : |X^x(t) - X^y(t)| \geq \delta\}$. Thus, define

$$\sigma_\rho = \begin{cases} \rho, & \text{if } \tau_\rho \leq T, \\ \tau_\rho, & \text{otherwise.} \end{cases}$$

Since $\sigma_\rho \geq \tau_\rho$, we get, from (8),

$$\hat{\mathbb{E}}[W(X^x(\sigma_\rho \wedge t) - X^y(\sigma_\rho \wedge t))] \leq \hat{\mathbb{E}}[W(X^x(\tau_\rho \wedge t) - X^y(\tau_\rho \wedge t))].$$

Furthermore, we have

$$\begin{aligned} &\hat{\mathbb{E}}[\mathbf{1}_{\{\tau_\rho \leq T\}} W(X^x(\sigma_\rho \wedge t) - X^y(\sigma_\rho \wedge t))] \\ &\leq \hat{\mathbb{E}}[\mathbf{1}_{\{\tau_\rho \leq T\}} W(X^x(\tau_\rho \wedge t) - X^y(\tau_\rho \wedge t))]. \end{aligned} \tag{13}$$

Then, by (13) we deduce, for $t > T$,

$$\begin{aligned} &\mathbb{V}(\{\tau_\rho \leq T\} \cap \{\rho < t\})\upsilon(\delta) \\ &\leq \hat{\mathbb{E}}[\mathbf{1}_{\{\tau_\rho \leq T\} \cap \{\rho < t\}} W(X^x(\rho \wedge t) - X^y(\rho \wedge t))] \\ &\leq \hat{\mathbb{E}}[\mathbf{1}_{\{\tau_\rho \leq T\}} W(X^x(\rho \wedge t) - X^y(\rho \wedge t))] \\ &= \hat{\mathbb{E}}[\mathbf{1}_{\{\tau_\rho \leq T\}} W(X^x(\sigma_\rho \wedge t) - X^y(\sigma_\rho \wedge t))] \\ &\leq \hat{\mathbb{E}}[\mathbf{1}_{\{\tau_\rho \leq T\}} W(X^x(\tau_\rho \wedge t) - X^y(\tau_\rho \wedge t))] \\ &\leq \mathbb{V}\{\tau_\rho \leq T\} \sup_{|u| \leq \rho} W(u) \leq \sup_{|u| \leq \rho} W(u). \end{aligned}$$

Due to $\sup_{|u|\le\rho} W(u) \le \varepsilon_1 \upsilon(\delta)$, it follows that

$$\mathbb{V}(\{\tau_\rho \le T\} \cap \{\rho < t\}) < \varepsilon_1,\ \forall t > T,$$

which shows, letting $t \to \infty$,

$$\mathbb{V}(\{\tau_\rho \le T\} \cap \{\rho < \infty\}) \le \varepsilon_1.$$

Moreover, through (12), we have

$$\begin{aligned}&\mathscr{V}(\{\tau_\rho \le T\} \cap \{\rho = \infty\})\\ &\ge 1 - \mathbb{V}(\tau_\rho > T) - \mathbb{V}(\{\tau_\rho \le T\} \cap \{\rho < \infty\}) \ge 1 - 3\varepsilon_1.\end{aligned}$$

We know that if $\omega \in \{\tau_\rho \le T\} \cap \{\rho = \infty\}$, then $|X^x(t) - X^y(t)| < \delta,\ \forall t \ge T$. Thus, for positive constants $H, \delta, \varepsilon = 3\varepsilon_1$, there is a $T = T(H, \delta, \varepsilon)$ such that

$$\mathscr{V}\Big\{\sup_{t\ge T} |X^x(t) - X^y(t)| \le \delta\Big\} > 1 - \varepsilon,\ \forall |x| \vee |y| \le H.$$

Thus, the proof is complete. □

Lemma 3.3 *Let Assumptions 2.2, 2.3 and 2.4 hold. Then, for any compact subset K of $\mathbb{R}^d$, we have*

$$\lim_{t\to\infty} d_{\mathbb{T}}[\mathfrak{F}_{X^x(t)}, \mathfrak{F}_{X^y(t)}] = 0 \tag{14}$$

uniformly in $x, y \in K$.

Proof. From Proposition 3.1, it is easy to show that for positive numbers $\mathfrak{R}, \delta, \varepsilon$, we can find a positive constant $H = H(\mathfrak{R}, \delta, \varepsilon)$ such that

$$\mathscr{V}\Big\{\sup_{0\le t\le T} |X^x(t)| \le H\Big\} > 1 - \varepsilon,\ |x| \le \mathfrak{R}.$$

Together with Proposition 3.2, we can deduce the desired claim. Thus, the proof is complete. □

We now prove our main result below.

Theorem 3.4 *Let Assumptions 2.2, 2.3 and 2.4 hold. Then, there exists an invariant measure (upper capacity) $\nu \in \mathscr{C}(\mathscr{B}(\mathbb{R}^d))$ such that*

$$\lim_{t\to\infty} d_{\mathbb{T}}(\mathfrak{F}_{X^x(t)}, \nu) = 0. \tag{15}$$

That is, the transition measures $\mathfrak{F}_{X^x(t)}$ converge weakly to the invariant measure ν for all $x \in \mathbb{R}^d$.

Proof. Fix probability measure $P^{(\sigma.)} \in \mathscr{P}$ arbitrarily. Let the invariant measure of random variable ξ_0

$$\nu = \sup_{P^{(\sigma.)} \in \mathscr{P}} \nu^{P^{(\sigma.)}}$$

while we set

$$\underline{\nu} = \inf_{P^{(\sigma.)} \in \mathscr{P}} \nu^{P^{(\sigma.)}}.$$

Furthermore, there exists an invariant probability measure $\nu^{(\sigma.)} = \nu^{P^{(\sigma.)}}$ from [25, Theorem 4.5]. Denote the classical transition probability measure by $p^{(\sigma.)}(t,z,du)$ under probability measure $P^{(\sigma.)}$. We easily know that

$$\begin{aligned}
E_{P^{(\sigma.)}}[\phi(\xi_0)] &= \int_{\mathbb{R}^d} \phi(u)\nu^{(\sigma.)}(du) \\
&= \int_{\mathbb{R}^d} \phi(u)\Big(\int_{\mathbb{R}^d} p^{(\sigma.)}(t,\zeta,du)\nu^{(\sigma.)}(d\zeta)\Big) \\
&= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \phi(u)p^{(\sigma.)}(t,\zeta,du)\Big)\nu^{(\sigma.)}(d\zeta) \\
&= \int_{\mathbb{R}^d} E_{P^{(\sigma.)}}[\phi(X^{\zeta}(t))]\nu^{(\sigma.)}(d\zeta). \qquad (16)
\end{aligned}$$

For any $z \in \mathbb{R}^d$, from (16), we derive

$$\begin{aligned}
&|E_{P^{(\sigma.)}}[\phi(X^z(t))] - E_{P^{(\sigma.)}}[\phi(\xi_0)]| \\
&= \Big|\int_{\mathbb{R}^d} \Big(E_{P^{(\sigma.)}}[\phi(X^z(t))] - E_{P^{(\sigma.)}}[\phi(X^{\zeta}(t))]\Big)\nu^{(\sigma.)}(d\zeta)\Big| \\
&\le \int_{\mathbb{B}_m} \Big|E_{P^{(\sigma.)}}[\phi(X^z(t))] - E_{P^{(\sigma.)}}[\phi(X^{\zeta}(t))]\Big|\nu^{(\sigma.)}(d\zeta) \\
&\quad + \int_{\mathbb{B}_m^c} \Big|E_{P^{(\sigma.)}}[\phi(X^z(t))] - E_{P^{(\sigma.)}}[\phi(X^{\zeta}(t))]\Big|\nu^{(\sigma.)}(d\zeta) \\
&\le \sup_{\zeta \in \mathbb{B}_m} |E_{P^{(\sigma.)}}[\phi(X^z(t))] - E_{P^{(\sigma.)}}[\phi(X^{\zeta}(t))]|\nu(\mathbb{B}_m) + 2\nu(\mathbb{B}_m^c), \qquad (17)
\end{aligned}$$

where $\mathbb{B}_m = \{x \in \mathbb{R}^d : |x| \le m\}$ and $\mathbb{B}_m^c = (\mathbb{R}^d \setminus \mathbb{B}_m)$ and $m \in \mathbb{N}$ is selected to be sufficiently large such that $\underline{\nu}(\mathbb{B}_m) \ge 1 - \frac{\varepsilon}{4}$, which shows $\nu(\mathbb{B}_m^c) \le \varepsilon/4$. By the classical property of stability in distribution (see, e.g. Theorem 2.3 in Dang [6]), there exists a $T = T(\mathbb{B}_m, \varepsilon) > 0$ such that

$$\sup_{\zeta \in \mathbb{B}_m} |E_{P^{(\sigma.)}}[\phi(X^z(t))] - E_{P^{(\sigma.)}}[\phi(X^{\zeta}(t))] < \frac{\varepsilon}{2}.$$

Moreover, from Lemma [11, Lemma 2.2], we get

$$\sup_{\zeta \in \mathbb{B}_m} d_{\mathbb{T}}(\mathfrak{F}_{X^z(t)}, \mathfrak{F}_{X^{\zeta}(t)}) < \frac{\varepsilon}{2}, \ \forall t \ge T.$$

Since ϕ is taken arbitrarily, it follows from (16) and (17) that $d_{\mathbb{T}}(\mathfrak{F}_{X^z(t)}, \nu) < \varepsilon$, $\forall t \geq T$. Thus the proof is complete. □

4 Example

In this section, we provide an example for illustrating our conclusions.

Example 4.1. We now consider stability in distribution of the stochastic logistic model

$$dX(t) = X(t)(a - bX(t))dt + cX^2(t)dB(t), \tag{18}$$

where a, b, c are positive constants. Denote the family of uncertain probability measures of G-HNSDE (18) by $\mathscr{P}$ associated to sublinear expectation $\hat{\mathbb{E}}$. For each $P \in \mathscr{P}$, by Luo and Mao [23, Theorem 2.2], we have that if $x > 0$, then $0 < X^x(t) = X(t) < \infty$ P-a.s., which easily shows $0 < X^x(t) = X(t) < \infty$ $\mathscr{P}$-q.s. Let $Y(t) = \ln X(t)$. By G-Itô formula, SDE (18) becomes

$$dY(t) = \Big(a - be^{Y(t)}\Big)dt - \frac{c^2}{2}e^{2Y(t)}d < B > (t) + ce^{Y(t)}dB(s). \tag{19}$$

For equation (19), the local Lipschitzian condition (Assumption 2.2) is obviously fulfilled. Furthermore, we consider the function $V(y) = e^y + e^{-y} > 0$, $\forall y \in \mathbb{R}$, we deduce

$$\mathbb{L}V(y) = a(e^y - e^{-y}) - b(e^{2y} - 1) + \bar{\sigma}^2 c^2 e^y.$$

Note

$$\begin{aligned}\varpi &= \sup_{y\in\mathbb{R}}\{\mathbb{L}V(y) + \vartheta V(y)\}\\ &= \sup_{y\in\mathbb{R}}\Big\{(a + \bar{\sigma}^2c^2 + \vartheta)e^y - (a - \vartheta)e^{-y} - b(e^{2y} - 1)\Big\}.\end{aligned}$$

Then, $\varpi < \infty$ as $0 < \vartheta < a$. Hence, for the case of $0 < \vartheta < a$, we have $\mathbb{L}V(y) \leq \varpi - \vartheta V(y)$, $\forall y \in \mathbb{R}$. Thus, Assumption 2.3 holds for $0 < \vartheta < a$.

On the other hand, put $W(u) = u^2$. We consider two solutions of (19) with initial values being y, z. Thus, we deduce

$$\begin{aligned}\mathscr{L}W(y,z) &= -2(y-z)b(e^y - e^z)\\ &\quad + G\Big(-2c^2(y-z)(e^{2y} - e^{2z}) + 2c^2(e^y - e^z)^2\Big)\\ &= -2b(y-z)(e^y - e^z) + 2c^2G\Big(-(y-z)(e^y - e^z)(e^y + e^z - \frac{e^y - e^z}{y-z})\Big).\end{aligned}$$

Note that $e^y + e^z - \frac{e^y - e^z}{y-z} > 0$, $\forall (y,z) \in \mathbb{R} \times \mathbb{R}$. Moreover, by the definition of $G(\cdot)$, we have

$$G\Big(-(y-z)(e^y - e^z)(e^y + e^z - \frac{e^y - e^z}{y-z})\Big) \leq 0.$$

Thus we have

$$\mathscr{L}W(y,z) \leq -2b(y-z)(e^y - e^z),\ \forall (y,z) \in \mathbb{R} \times \mathbb{R}.$$

Since $\mathscr{L}W(y,z) \to 0$ as $y \to -\infty$ and $z \to -\infty$, there is no $\varphi \in \mathscr{K}$ such that $\mathscr{L}W(y,z) \leq -\varphi(|y-z|)$. It shows that the function $W(u) = u^2$ does not satisfy Assumption 2.4. But, for each $R > 0$, and $|y| \vee |z| \leq R$, we have $\mathscr{L}W(y,z) \leq -2b(y-z)(e^y - e^z) \leq -C_R(y-z)^2$ with $C_R = 2be^{-R}$. Thus, Assumption 2.4 holds for the function $W(\cdot)$. Moreover, SDE (19) is asymptotically stable in distribution. Consequently, by Theorem 3.4, SDE (18) is asymptotically stable in distribution on state space $(0, +\infty)$.

(**Algorithm and simulation**) In order to illustrate stability in distribution of solution of SDE (18), we provide an algorithm as follows. We select $\underline{\sigma} = \sigma_0 < \sigma_1 < \cdots < \sigma_m = \bar{\sigma}$ such that $\sigma_{i+1} - \sigma_i = \sigma_i - \sigma_{i-1}, i = 1, \cdots, m$. Let h be a small positive number. For any $t > 0$, there a positive integer k such that $t \in [(k-1)h, kh)$. The discrete approximation solution of SDE (4.1) with probability measure $P^{(\sigma_i)}$ can be expressed as

$$\begin{aligned} X^j(kh, \sigma_i) =& X^j((k-1)h, \sigma_i) + X^j((k-1)h, \sigma_i)(1 - X^j((k-1)h, \sigma_i))h \\ &+ (X^j((k-1)h, \sigma_i))^2 \sigma_i \Delta w^j, \quad k \geq 1, \quad i \geq 0, \end{aligned}$$

where Δw^j $(j = 1, \cdots, n)$ are random numbers from the normal distribution $\Delta w^j \sim N(0,h)$. Define the empirical distribution function of random variable $X(kh)$ as follows

$$\Upsilon_k^i(x) := \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}_{\{X^j(kh, \sigma_i) \leq x\}},\ \forall x \in \mathbb{R}.$$

Define the error of two empirical distributions of $\Upsilon_{k_1}^i(x)$ and $\Upsilon_{k_2}^i(x)$ by

$$e(t_{k_1}, t_{k_2};\, x) = \frac{1}{m+1} \sum_{i=0}^{m} |\Upsilon_{k_1}^i(x) - \Upsilon_{k_2}^i(x)|,\ \forall x \in \mathbb{R}.$$

If the empirical error $e(t_k, t_k + \ell;\, x)$ converges to zero uniformly in $x \in \mathbb{R}$ as $k, \ell \to \infty$, then we claim stability in distribution of solution of SDE (4.1). A simulation finds $e(t_k, t_k + \ell) \to 0$ as $k \to \infty$ (see Figure 1) which verifies our theoretical assertion. In real simulation, the empirical distribution at $t = 10$ is regarded as a true one. We observe that other empirical distributions at time $t \in [0,5]$ approximate the degree of the true distribution. The differences between the empirical distributions and the true

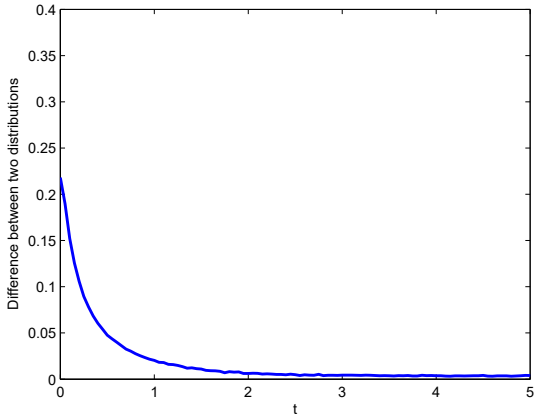

Fig. 1: The computer simulation of the empirical errors $e(t_k, t_k + \ell)$ with $a = b = c = 1, \underline{\sigma} = 0.8, \bar{\sigma} = 1, x_0 = 1, h = 0.005, n = 2000, m = 10$.

distribution along the time line are plotted in Figure 1. It can be shown that as time advances the difference tends to zero, which verifies that the solution of SDE (18) is the stability in distribution.

5 Conclusions

Presently, the analysis on the stability in distribution of stochastic differential equations is mainly based on linear or nonlinear SDEs which coefficients are bounded linear functions. However, our paper further studies the stability in distribution of highly nonlinear stochastic differential equations under sublinear expectation framework. For our aim, by using the stochastic analysis on G-Brownian motion, we investigate the stability in distribution of highly nonlinear stochastic differential equations disturbed by G-Brownian motion. The sufficient criterion of the stability in distribution of G-HNSDEs is given explicitly.

Recently, with classical probability space, [13] discussed the stabilization of highly nonlinear hybrid stochastic differential equations under feedback control based on discrete-time state observations. In [31], authors investigated the discrete-state-feedback stabilization of highly nonlinear hybrid stochastic differential equations by Razumikhin method. The stabilization in distribution of hybrid stochastic differential equations by feedback control based on discrete-time state observations is discussed in [20]. On nonlinear expectation space, [33] explored stabilization of stochastic differential equations driven by G-Brownian motion with feedback control based on discrete-time observations. Thus, we shall investigate the stabilization in distribution of highly nonlinear stochastic differential equations driven by G-Brownian motion by using feedback control based on discrete-time observations.

Declarations

Conflict of Interests

The authors have no conflicts of interest to declare.

Authors' contributions

Chen Fei: Conceptualization, Methodology, Original draft preparation;
Weiyin Fei: Writing-Review & Editing, Validation;
Shounian Deng: Validation, Numerical simulation;
Xuerong Mao: Formal analysis, Validation.

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Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] Bao, J., Hou, Z., & Yuan, C. (2009). Stability in distribution of neutral stochastic differential delay equations with Markovian switching, Statistics and Probability Letters. *Statistics and Probability Letters*, *79*, 1663-1673.

[2] Bo, L., & Yuan, C. (2016). Stability in distribution of Markov-modulated stochastic differential delay equations with reflection. *Stochastic Models*, *32*(3), 392-413.

[3] Caraballo, T., Belfeki, M., Mchiri, L. & Rhaima, M. (2021). h-stability in pth moment of neutral pantograph stochastic differential equations with Markovian switching driven by Lévy noise. *Chaos, Solitons and Fractals*, *151*, 111249.

[4] Caraballo, T., Mchiri, L., Mohsen, B., & Rhaima, M. (2021). pth moment exponential stability of neutral stochastic pantograph differential equations with Markovian switching. *Commun. Nonlinear Sci. Numer. Simul.*, *102*, 105916.

[5] Chen, Z., Wu, P., & Li, B. (2013). A strong law of large numbers for non-additive probabilities. *International Journal of Approximate Reasoning*, *54*, 365-377.

[6] Dang, N. (2014). A note on sufficient conditions for asymptotic stability in distribution of stochastic differential equations with Markovian switching. *Nonlinear Analysis: TMA*, *95*, 625-631.

[7] Deng, S., Fei, C., Fei, W., & Mao, X. (2019). Stability equivalence between the stochastic differential delay equations driven by *G*-Brownian motion and the Euler-Maruyama method. *Applied Mathematics Letters*, *96*, 138-146.

[8] Du, N., Dang, N., & Dieu, N. (2014). On stability in distribution of stochastic differential delay equations with Markovian switching. *Systems & Control Letters*, *65*, 43-49.

[9] Fei, C. (2021). Optimal stochastic control and optimal consumption and portfolio with *G*-Brownian motion. *Acta Mathematicae Applicata Sinica*, *44*(3), 355-381. (in Chinese)

[10] Fei, C., & Fei, W. (2019). Consistency of least squares estimation to the parameter for stochastic differential equations under distribution uncertainty. *Acta Mathematica Scientia*, *39A*(6), 1499-1513 . (in Chinese)

[11] Fei, C., Fei, W., & Mao, X. (2023). A note on sufficient conditions of asymptotic stability in distribution of stochastic differential equations with G-Brownian motion,. *Applied Mathematics Letters*, *136*, 108448.

[12] Fei, C., Fei, W., Mao, X., & Yan, L. (2022). Delay-dependent asymptotic stability of highly nonlinear stochastic differential delay equations driven by *G*-Brownian motion. *Journal of the Franklin Institute*, *359*(9), 4366-4392.

[13] Fei, C., Fei, W., Mao, X., Xia, D., & Yan, L. (2020). Stabilisation of highly nonlinear hybrid systems by feedback control based on discrete-time state observations. *IEEE Transaction Automatic Control*, *65*(7), 2899-2912.

[14] Fei, C., Fei, W., & Yan, L. (2019). Existence-uniqueness and stability of solutions to highly nonlinear stochastic differential delay equations driven by *G*-Brownian motions. *Applied Mathematics-A Journal of Chinese Universities*, *34*(2), 184-204.

[15] Fei, C., Shen, M., Fei, W., Mao, X., & Yan, L. (2019). Stability of highly nonlinear hybrid stochastic integro-differential delay equations. *Nonlinear Analysis: Hybrid Systems*, *31*, 180-199.

[16] Gao, F. (2009). Pathwise properties and homeomorphic flows for stochastic differential equations driven by *G*-Brownian motion. *Stochastic Processes and their Applications*, *119*(10), 3356-3382.

[17] Hu, J., Mao, W., & Mao, X. (2023). Advances in nonlinear hybrid stochastic differential delay equations: Existence, boundedness and stability. *Automatica*,

147, 110682.

[18] Hu, G., & Wang, K. (2012). Stability in distribution of neutral stochastic functional differential equations with Markovian switching. *Journal of Mathematical Analysis and Applications*, *385*, 757-769.

[19] Hu, M., Ji, X., & Liu, G. (2021). On the strong Markov property for stochastic differential equations driven by *G*-Brownian motion. *Stochastic Processes and their Applications*, *131*, 417-453.

[20] Li, X., Liu, W., Luo, Q., & Mao, X. (2022). Stabilization in distribution of hybrid stochastic differential equations by feedback control based on discrete-time state observations. *Automatica*, *140*, 110210.

[21] Li, Y., Fei, W., & Deng, S. (2022). Delay feedback stabilization of stochastic differential equations driven by *G*-Brownian motion. *Internatinal Journal of Control*, *95*(9), 2499-2509.

[22] Liu, K. (2019). Stability in distribution for stochastic differential equations with memory driven by positive semigroups and Lévy processes. *Applied Mathematics and Computation*, *362*, 124580.

[23] Luo, Q., & Mao, X. (2007). Stochastic population dynamics under regime switching. *Journal of Mathematical Analysis and Applications*, *334*, 69-84.

[24] Mchiri, L., Caraballo, T., & Rhaima, M. (2022). Partial asymptotic stability of neutral pantograph stochastic differential equations with Markovian switching. *Adv. Contin. Discret. Model.*, *2022*, 18.

[25] Meyn, S.,& Tweedie, R. (1993). Stability of Markovian processes III: Foster-Lyapunov criteria for continuous-time processes. *Advances in Applied Probability*, *25*, 518-548..

[26] Peng, S. (2019). *Nonlinear Expectations and Stochastic Calculus under Uncertainty*. Springer, Berlin.

[27] Ren,Y., Yin, W., & Sakthivel, R. (2018). Stabilization of stochastic differential equations driven by *G*-Brownian motion with feedback control based on discrete-time state observation. *Automatica*, *95*, 146-151.

[28] Tan, L., Jin, W., & Suo, Y. (2015) . Stability in distribution of neutral stochastic functional differential equations. *Statistics and Probability Letters*, *107*, 27-36.

[29] Wang, B., & Zhu, Q. (2019) . Asymptotic stability in distribution of stochastic systems with semi-Markovian switching. *International Journal of Control*, *92*(6), 1314-1324.

[30] Wang, Y., Wu, F., & Mao, X. (2019). Stability in distribution of stochastic functional differential equations. *Systems & Control Letters*, *132*, 104513.

[31] Xu, H., & Mao, X. Advances in discrete-state-feedback stabilization of highly nonlinear hybrid systems by Razumikhin technique. *IEEE Transactions on Automatic Control*, DOI 10.1109/TAC.2022.3232989

[32] Yao, Z., & Zong, X. (2020). Delay-dependent stability of a class of stochastic delay systems driven by *G*-Brownian motion. *IET Control Theory and Applications*, *14*(6), 834-842.

[33] Yin, W., Cao, J., Ren, Y., & Zheng, G. Improved results on stabilization of?*G*-SDEs by feedback control based on discrete-time observations. *SIAM J. Control Optim.*, *59*, 1927-1950.

[34] You, S., Hu, L., Lu, J., & Mao, X. (2021). Stabilisation in distribution by delay feedback control for hybrid stochastic differential equations. *IEEE Transactions on Automatic Control*, DOI: 10.1109/TAC.2021.3075177.

[35] Yuan, C., & Mao, X. (2003). Asymptotic stability in distribution of stochastic differential equations with Markovian switching. *Stochastic Processes and Applications*, *79*, 45-67.

[36] Yuan, C., Zou, J., & Mao, X. (2003). Stability in distribution of stochastic differential delay equations with Markovian switching. *Systems & Control Letters*, *50*, 195-207.

[37] Zhu, E., Yin, G., & Yuan, Q. (2016). Stability in distribution of stochastic delay recurrent neural networks with Markovian switching. *Neural Computing and Applications*, *27*, 2141-2151.

[38] Zhang F., Fei, C., & Fei, W. (2023). Stability of stochastic Hopfield neural networks driven by *G*-Brownian motion with time-varying and distributed delays. *Neurocomputing*, *520*, 320-330.