Shock models based on renewal processes with matrix Mittag-Leffler distributed inter-arrival times

Dheeraj Goyal[a], Nil Kamal Hazra[a,b], and Maxim Finkelstein*[c,d]

[a]Department of Mathematics, Indian Institute of Technology Jodhpur, 342037, Karwar, Rajasthan, India

[b]School of AI & DS, Indian Institute of Technology Jodhpur, 342037, Karwar, Rajasthan, India

[c]Department of Mathematical Statistics and Actuarial Science, University of the Free State, 339 Bloemfontein 9300, South Africa

[d]Department of Management Science, University of Strathclyde, Glasgow, UK.

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Abstract

Some general shock models are considered under the assumption that shocks occur according to a renewal process with the matrix Mittag-Leffler distributed inter-arrival times. As the class of matrix Mittag-Leffler distributions is wide and well-suited for modeling the heavy tail phenomena, these shock models can be very useful for analysis of lifetimes of systems subject to random shocks with inter-arrival times having heavier tails. Some relevant stochastic properties of the introduced models are described. Finally, two applications, namely, the optimal replacement policy and the optimal mission duration are discussed.

Keywords: Fractional homogeneous Poisson process; matrix Mittag-Leffler distribution; phase-type distribution; shock models; reliability.

1 Introduction

Most of the real-life engineering systems operate in random environments and hence, often continuously are subject to internal or external random impulses (shocks). Therefore, relevant shock models play a significant role in stochastic description of lifetimes of these systems. Existing shock models are usually classified into four broad categories: the extreme shock model, the cumulative shock model, the run shock model and the δ-shock model. In the extreme shock

*Corresponding author, email: FinkelM@ufs.ac.za

model, a system fails if the magnitude of a single shock exceeds some threshold value (Gut and Hüsler [23, 24], Shanthikumar and Sumita [43, 44], Cha and Finkelstein [10], Finkelstein [15], to name a few). In the cumulative shock model, a failure occurs when the cumulative damage due to shocks exceeds the predetermined threshold value (A-Hameed and Proschan [1], Esary et al. [14] and Gut [22], Gong et al. [18], Ranjkesh et al. [40], among others). In the run shock model, a failure of a system takes place when k consecutive shocks with critical magnitude occur (see, e.g., Mallor and Omey [31], Ozkut and Eryilmaz [37], Gong et al. [17]). Lastly, in the classical δ-shock model, a system fails if the time lag between two successive shocks is less than the threshold value δ, i.e., the recovery time of a system from occurrence of a shock is δ (see Li et al. [29], Li and Kong [30], Goyal et al. [21]). Furthermore, various mixed shock models, as a combination of two or more basic shock models, were introduced in the literature. For instance, the extreme shock model with the cumulative shock model (Cha and Finkelstein [9]), the extreme shock model with the run shock model (see Eryimaz and Tekin [13]), the extreme shock model with the δ-shock model (Parvardeh and Balakrishnan [38], Wang and Zhang [46], Goyal et al. [20]), the cumulative shock model with the run shock model (see Mallor et al. [32]), the cumulative shock model with the δ-shock model (Parvardeh and Balakrishnan [38]), the run shock model with the δ-shock model (see, e.g., Eryilmaz [11]).

Shocks arrivals are usually modeled by relevant point processes. The renewal process is often the first candidate for that in practice. However, for arbitrary distributed inter-arrival times, it is not possible to obtain practically useful results in a closed form. Therefore, various specific cases are considered in the literature. The simplest is the homogeneous Poisson process (HPP) as the process with exponentially distributed inter-arrival times. However, the exponential distribution, with its memoryless property, restricts the usage of the HPP in different applications. To overcome this limitation, the Phase-type (PH) distribution was considered in the shocks-related literature. For instance, Eryilmaz [12] have discussed the renewal process of shocks with the PH distributed inter-arrival times. This distribution is mathematically tractable as its probability density function (pdf) and the cumulative distribution function (cdf) can be written in matrix forms that are convenient in computation using different software packages. Moreover, this distribution has the 'denseness property' (i.e., any distribution with a non-negative support can be approximated by a PH distribution). Furthermore, this distribution allows to include multiple failures to the arrivals of shocks in stochastic models (see Montoro-Cazorla and Pérez-Ocón [33, 34] for reference). However, and despite their denseness, the classical PH distributions are always light-tailed (see Example 2.4 in Asmussen and Albrecher [5]), and hence, cannot be used in applications with heavy-tailed shock inter-arrival times. In fact, as many shock arrival processes exhibit the heavy-tailed behavior of inter-arrival times, using the light-tailed distributions may be misleading and incorrect (see Li and Luo [27] and the references therein).

In view of the above, we suggest to employ the matrix Mittag-Leffler (MML) distributions for modeling the inter-arrival times of shock renewal processes for obtaining reliability characteristics of systems operating in a shock environment. Some relevant stochastic orders are also to be investigated. To the best of our knowledge, this was not considered in the literature before.

The class of MML distributions, defined by Albrecher et al. [2], is a wide class of *heavy-tailed* distributions with attractive mathematical properties. Like in case of the PH distributions, it is also dense in the class of all lifetime distributions. Consequently, this distribution may be useful in modeling various real life scenarios where the inter-arrival times of shocks have heavier tails. Furthermore, the class of MML distributions is a fractional generalization of the PH distribution (note that, Albrecher et al. [3] named it as fractional PH distribution). This distribution contains many popular distributions as special cases, namely, the exponential distribution, the Erlang distribution, the PH distribution, the Mittag-Leffler (ML) distribution, the fractional Erlang distribution (a fractional generalization of the Erlang distribution). Thus, a renewal process with the MML distributed inter-arrival times may be very useful in modeling arrivals of shocks with inter-arrival times having heavier tails. From now onward, we will call this process: the renewal process of matrix Mittag-Leffler type (RPMML). One of the important special cases of the RPMML is the fractional homogeneous Poisson process (FHPP) which is a counting process with independent and identically distributed inter-arrival times following the ML distribution. As far the applications of this process are concerned, it has not yet been used in shock models. However, the RPMML (and, in particular, the FHPP) may be very useful in modeling extreme events (shocks), such as, e.g., earthquakes, storms, etc. (see Benson et al. [6], Biard and Saussereau [7], Musson et al. [35] for reference).

To summarize: From a brief literature review, we can conclude that counting processes based on heavy-tailed distributions (namely, the FHPP and the RPMML) have not yet been considered in the literature for modeling the occurrences of shocks. Therefore, the main goal of this paper is to develop the corresponding methodology for considering shock models governed by these point processes. It turns out that FHPP and the RPMML are very useful in modeling extreme events in many real-life applications where the inter-arrival times are heavy-tailed.

The rest of the paper is organized as follows. In Section 2, we give some definitions and supplementary results. In Section 3, we derive the distribution of the lifetime of a system for the generalized extreme shock model and the generalized run shock model. Further, we study some stochastic comparison results for systems operating under random environments. In Section 4, we discuss two applications, namely, the optimal replacement and the optimal mission duration. Finally, concluding remarks are given in Section 5.

2 Definitions and supplementary results

In this section, we provide necessary definitions and properties for stochastic objects considered in the paper. Moreover, some new results are proved also that are intensively used in the forthcoming sections, e.g., for stochastic ordering with respect to the considered lifetime models.

For any random variable U, we denote the cdf by $F_U(\cdot)$, the survival function by $\bar{F}_U(\cdot)$, the pdf by $f_U(\cdot)$, the failure rate function by $r_U(\cdot)$ and the Laplace transform by $L_U(\cdot)$; here $\bar{F}_U(\cdot) = 1 - F_U(\cdot)$, $r_U(\cdot) = f_U(\cdot)/\bar{F}_U(\cdot)$ and $L_U(x) = E(\exp\{-xU\})$, for $x \in (-\infty, \infty)$. We write a matrix A as $A = [A_{ij}]$, where A_{ij} represents the ij-th element of A. For any two matrices $A = [A_{ij}]$ and $B = [B_{ij}]$, $A \otimes B$ is defined as $[A_{ij}B]$, where "$\otimes$" stands for the Kronecker product. By denoting $A = diag(d_1, d_2, \ldots, d_m)$, we mean that A is a diagonal matrix with i-th diagonal entry equal to d_i, $i = 1, 2, \ldots, m$. For any real x, $|x|$ denotes the absolute value of x. By writing $\mathbb{R}$ and $\mathbb{N}$, we mean the set of real numbers and the set of natural numbers, respectively. Below we give a set of acronyms that are used in this paper.

HPP: homogeneous Poisson process
PH: phase-type
RPMML: renewal process of matrix Mittag-Leffler type
FHPP: fractional homogeneous Poisson process
ML: Mittag-Leffler
MML: matrix Mittag-Leffler
pdf: probability density function
cdf: cumulative distribution function

2.1 Stochastic orders

Stochastic ordering is a very effective tool for comparing two or more random variables. In illustrations and reliability applications of the obtained in this paper results, we consider mostly ordering in the sense of the usual stochastic order, as the one that compares relevant survival/reliability functions. However, from the mathematical point of view, it is also interesting to look at other relevant stochastic orders as they were not considered in the literature before with respect to random variables of interest. Moreover, some of them have the well-pronounced practical meaning as well (see below).

Below, for convenience, we provide definitions of some stochastic orders which are used in subsequent sections (Shaked and Shanthikumar [42], Li et al. [28]).

Definition 2.1 *Let Y_1 and Y_2 be two absolutely continuous non-negative random variables. Then Y_1 is said to be greater than Y_2 in the*

- *(i) usual stochastic order, denoted by $Y_1 \geq_{st} Y_2$, if $\bar{F}_{Y_1}(x) \geq \bar{F}_{Y_2}(x)$ for all $x > 0$;*
- *(ii) Laplace transform order, denoted by $Y_1 \geq_{lt} Y_2$, if $L_{Y_2}(x) \geq L_{Y_1}(x)$ for all $x > 0$;*
- *(iii) Laplace transform ratio order, denoted by $Y_1 \geq_{lt-r} Y_2$, if $L_{Y_1}(x)/L_{Y_2}(x)$ is decreasing in $x > 0$;*
- *(iv) reversed Laplace transform ratio order, denoted by $Y_1 \geq_{r-lt-r} Y_2$, if $[1 - L_{Y_1}(x)]/[1 - L_{Y_2}(x)]$ is decreasing in $x > 0$;*

(*v*) *differentiated Laplace transform ratio order, denoted by* $Y_1 \geq_{d-lt-r} Y_2$, *if* $L'_{Y_1}(x)/L'_{Y_2}(x)$ *is decreasing in* $x > 0$.

Stochastic orders based on Laplace transforms have also many applications in reliability, insurance and actuarial science. For instance, suppose that an item, with the survival function $\bar{F}$ 'produces' one unit of output per hour when it is operating. Then the *present value* of one unit produced at time t is given by $1.\exp\{-st\}$, where s is the discount rate. Further, the expected present value of the total output produced during the lifetime of the item is $\int_0^\infty \exp\{-st\}\bar{F}(t)dt$. Consequently, $Y_1 \geq_{lt} Y_2$ implies that an item with a lifetime described by the survival function $\bar{F}_{Y_2}(\cdot)$ produces the smaller expected total present value as compared with an item with the survival function $\bar{F}_{Y_1}(\cdot)$ (see Alzaid et al. [4]). Similarly, $Y_1 \geq_{r-lt-r} Y_2$ implies that the expected present value of an item with the survival function $\bar{F}_{Y_1}(\cdot)$, relative to the expected present value of an item with the survival function $\bar{F}_{Y_2}(\cdot)$, increases as s gets smaller (see Shaked and Wong [41]). Other applications of these orders may be found in Alzaid et al. [4], Li et al. [28] and the references therein.

Proposition 2.1 *The following results hold true.*

(*i*) *If* $Y_1 \geq_{st} Y_2$ *then* $Y_1 \geq_{lt} Y_2$ *(Theorem 5.A.16 in Shaked and Shanthikumar [42]);*

(*ii*) *If* $Y_1 \geq_{lt-r} Y_2$ *or* $Y_1 \geq_{r-lt-r} Y_2$, *then* $Y_1 \geq_{lt} Y_2$ *(Theorem 5.B.10 in Shaked and Shanthikumar [42]);*

(*iii*) *If* $Y_1 \geq_{d-lt-r} Y_2$, *then* $Y_1 \geq_{lt-r} Y_2$ *and* $Y_1 \geq_{r-lt-r} Y_2$ *(Theorem 3.2 of Li et al. [28]).*

2.2 ML distribution

Below we give the definition of the ML distribution which is a generalization of the exponential distribution (Pillai [39], Kataria and Vellaisamy [26]).

Definition 2.2 *A random variable* X *is said to have the ML distribution with parameters* α *and* λ, *if its pdf is given by*

$$f_X(x) = \lambda x^{\alpha-1} E_{\alpha,\alpha}(-\lambda x^\alpha), \quad x \geq 0,$$

where $E_{\alpha,\beta}(x)$, $\alpha > 0, \beta > 0$, *is the two parameter* ML *function defined as*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta + k\alpha)}, \quad x \in \mathbb{R}.$$

We write $X \sim ML(\alpha, \lambda)$ to indicate that X follows ML distribution with parameters α and λ. Note that, when $\alpha = 1$, this distribution reduces to the well known exponential distribution with parameter $\lambda > 0$.

Some useful for our further discussion properties of the ML distribution are given in the following proposition (Pillai [39], Kataria and Vellaisamy [26]).

Proposition 2.2 *Let $X \sim ML(\alpha, \lambda), \alpha > 0, \lambda > 0$. Then*

(i) the survival function of X is given by $\bar{F}_X(x) = E_{\alpha,1}(-\lambda x^\alpha)$;

(ii) the Laplace transform of X is given by $L_X(s) = \frac{\lambda}{\lambda + s^\alpha}, s > 0$.

Next we provide the definition of the completely monotonic function (Gorenflo et al. [19]).

Definition 2.3 *A function $f : (0, \infty) \to \mathbb{R}$ is said to be completely monotonic if it possesses all order derivatives $f^{(n)}(x)$, $n = 0, 1, 2, \ldots$, and the derivatives are alternating in sign, i.e.,*

$$(-1)^n f^{(n)}(x) \geq 0, \quad \textit{for all } x \in (0, \infty).$$

The following proposition is borrowed from Gorenflo et al. [19].

Proposition 2.3 *The ML function of negative argument, $E_{\alpha,1}(-x)$, is completely monotonic for all $0 \leq \alpha \leq 1$.*

In the next theorem we compare two ML distributions with respect to different stochastic orders. These results will be used in the subsequent sections.

Theorem 2.1 *Let X_1 and X_2 be two random variables with $X_1 \sim ML(\alpha, \lambda_1)$ and $X_2 \sim ML(\alpha, \lambda_2)$, $0 < \alpha \leq 1, \lambda_1 > 0, \lambda_2 > 0$. If $\lambda_1 \geq \lambda_2$ then $X_1 \leq_{st} X_2$ and $X_1 \leq_{d-lt-r} X_2$.*

Proof: From Proposition 2.3, we have that $E_{\alpha,1}(-x)$ is completely monotonic and hence, $E^{(1)}_{\alpha,1}(-x) \leq 0$, for all $x \in (0, \infty)$. This means that $E_{\alpha,1}(-x)$ is decreasing in $x > 0$, or equivalently, $E_{\alpha,1}(-\lambda_1 x^\alpha) \leq E_{\alpha,1}(-\lambda_2 x^\alpha)$, for all $x \in (0, \infty)$. Using Proposition 2.2(i), the previous inequality can equivalently be written as $\bar{F}_{X_1}(x) \leq \bar{F}_{X_2}(x)$, for all $x \in (0, \infty)$ and hence, $X_1 \leq_{st} X_2$. Again, from Proposition 2.2(ii), we have $L_{X_1}(s) = \frac{\lambda_1}{\lambda_1 + s^\alpha}$ and $L_{X_2}(s) = \frac{\lambda_2}{\lambda_2 + s^\alpha}$. Consequently, $L'_{X_1}(s) = \frac{-\lambda_1 \alpha s^{\alpha-1}}{(\lambda_1 + s^\alpha)^2}$, $L'_{X_2}(s) = \frac{-\lambda_2 \alpha s^{\alpha-1}}{(\lambda_2 + s^\alpha)^2}$ and

$$\left(\frac{L'_{X_2}(s)}{L'_{X_1}(s)} \right)' = \frac{2\lambda_2 \alpha s^{\alpha-1}(\lambda_1 + s^\alpha)(\lambda_2 - \lambda_1)}{\lambda_1(\lambda_2 + s^\alpha)^3},$$

which is non-positive because $\lambda_1 \geq \lambda_2$. Hence, $X_1 \leq_{d-lt-r} X_2$. □

The following corollary follows from Proposition 2.1 and Theorem 2.1.

Corollary 2.1 *If $\lambda_1 \geq \lambda_2$ then $X_1 \leq_{lt} X_2$, $X_1 \leq_{lt-r} X_2$ and $X_1 \leq_{r-lt-r} X_2$.* □

The next proposition follows from Theorem 2.1 in view of Theorem 1.A.4 of Shaked and Shanthikumar [42], and Theorem 3.5 of Li et al. [28].

Proposition 2.4 *Let $\{X_i : i = 1, 2, \ldots\}$ and $\{Y_i : i = 1, 2, \ldots\}$ be two sequences of independent random variables with $X_i \sim ML(\alpha, \mu_i)$ and $Y_i \sim ML(\alpha, \nu_i)$, $i = 1, 2, \ldots,$. Further, let N be a nonnegative integer-valued random variable independent of the X_i's and the Y_i's. Let $L_1 = \sum_{i=1}^N X_i$ and $L_2 = \sum_{i=1}^N Y_i$. If $\mu_i \geq \nu_i$, for all $i = 1, 2, \ldots$, then $L_1 \leq_{st} L_2$, $L_1 \leq_{lt} L_2$, $L_1 \leq_{lt-r} L_2$, $L_1 \leq_{r-lt-r} L_2$ and $L_1 \leq_{d-lt-r} L_2$.*

In the next proposition, we compare two series systems with components having independent ML distribution. The proof of the corollary follows from Proposition 2.3, Theorem 2.1, and Theorem 5.A.19 of Shaked and Shanthikumar [42].

Proposition 2.5 *Let $X_i \sim ML(\alpha, \mu_i)$ and $Y_i \sim ML(\alpha, \nu_i)$, $i = 1, 2, \ldots, n$. Assume that X_i's, Y_i's are independent. If $\mu_i \geq \nu_i$, for all $i = 1, 2, \ldots, n$, then*

$$min\{X_1, X_2, \ldots, X_n\} \leq_{lt} min\{Y_1, Y_2, \ldots, Y_n\}.$$

2.3 PH distribution

Neuts [36] have defined the set of PH distributions as a generalization of the exponential distribution. Below, for convenience, we provide the formal definition of the PH distribution (He [25]).

Definition 2.4 *A non-negative random variable X is said to have a PH distribution if*

$$F_X(x) = 1 - \boldsymbol{\pi} \exp\{Tx\}\boldsymbol{e} = 1 - \boldsymbol{\pi} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} T^n \right) \boldsymbol{e}, \quad x \geq 0, \tag{2.1}$$

where

(i) $\boldsymbol{e}$ is the column vector with all elements being one;

(ii) $\boldsymbol{\pi}$ is a substochastic vector of order m, i.e., $\boldsymbol{\pi}$ is a row vector, all elements of $\boldsymbol{\pi}$ are nonnegative, and $\boldsymbol{\pi e} \leq 1$, where m is a positive integer; and

(iii) T is a subgenerator of order m, i.e., T is an $m \times m$ matrix such that: (a) all diagonal elements are negative; (b) all off-diagonal elements are nonnegative; (c) all row sums are non-positive; and (d) T is invertible. □

We call T and the pair $(\boldsymbol{\pi}, T)$ as the PH generator and the PH representation of order m, respectively. We write $X \sim PH(\boldsymbol{\pi}, T)$ to indicate that X follows the PH distribution with the PH representation $(\boldsymbol{\pi}, T)$. Further, the corresponding pdf is given by

$$f_X(x) = \boldsymbol{\pi} \exp\{Tx\}\boldsymbol{T^0}, \ \ x \geq 0, \tag{2.2}$$

where $\boldsymbol{T^0} = -T\boldsymbol{e}$.

Remark 2.1 *The following observations can be made (He [25]).*

(i) A PH distribution with the PH representation $(\boldsymbol{\pi}, T)$, where $\boldsymbol{\pi} = 1$ and $T = -\theta$, is the exponential distribution with parameter θ;

(ii) A PH distribution with the PH representation $(\boldsymbol{\pi}, T)$, where

$$\boldsymbol{\pi} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{1\times m} \text{ and } T = \begin{pmatrix} -\theta & 0 & \cdots & 0 & 0 \\ \theta & -\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \theta & -\theta \end{pmatrix}_{m\times m},$$

is the Erlang distribution with the set of parameters $\{m, \theta\}$. □

We define now the class of discrete PH distributions (Eryilmaz [12], Bozbulut and Eryilmaz [8]). Note that, a discrete PH distribution can be seen as the distribution of the time to absorption in an absorbing Markov chain. They can be effectively used for describing the criterion of failure for systems operating under shocks (see later).

Definition 2.5 *A discrete random variable N is said to have the discrete PH distribution if its probability mass function is given by*

$$P(N = n) = \boldsymbol{a}Q^{n-1}\boldsymbol{u}, \text{ for } n = 1, 2, 3, \ldots,$$

where Q is a matrix of order m such that $I - Q$ is non-singular, and $\boldsymbol{u} = (I - Q)\boldsymbol{e}$, $\boldsymbol{a}\boldsymbol{e} = 1$, and I is the identity matrix.

We write $N \sim DPH(\boldsymbol{a}, Q)$ to represent that N has a discrete PH distribution with parameter set $\{\boldsymbol{a}, Q\}$. The survival function of N is given by $\bar{F}_N(n) = \boldsymbol{a}Q^n\boldsymbol{e}$.

2.4 MML distribution

The class of MML distributions was defined and studied by Albrecher et al. [2]. These distributions have heavier tails.

Definition 2.6 *Let $(\boldsymbol{\pi}, T)$ be a PH reperesentation and $0 < \alpha \leq 1$. A random variable X is said to have a MML distribution, denoted by $X \sim MML(\alpha, \boldsymbol{\pi}, T)$, if its Laplace transform is given by*

$$L_X(u) = \boldsymbol{\pi}(u^\alpha I - T)^{-1}\boldsymbol{T^0},$$

where $\boldsymbol{T^0} = -T\boldsymbol{e}$.

The following lemma is borrowed from Albrecher et al. [2].

Lemma 2.1 *Let $X \sim MML(\alpha, \boldsymbol{\pi}, T)$. Then*

(i) the pdf of X is given by

$$f_X(x) = x^{\alpha-1}\boldsymbol{\pi}E_{\alpha,\alpha}(Tx^\alpha)\boldsymbol{T^0};$$

(ii) *the cdf of* X *is given by*

$$F_X(x) = 1 - \boldsymbol{\pi} E_{\alpha,1}(Tx^{\alpha})\boldsymbol{e};$$

(iii) *the failure rate function of* X *is given by*

$$r_X(x) = \frac{x^{\alpha-1}\boldsymbol{\pi} E_{\alpha,\alpha}(Tx^{\alpha})\boldsymbol{T^0}}{\boldsymbol{\pi} E_{\alpha,1}(Tx^{\alpha})\boldsymbol{e}},$$

where $\boldsymbol{T^0} = -T\,\boldsymbol{e}$.

Remark 2.2 *Let* $X \sim MML(\alpha, \boldsymbol{\pi}, T)$. *Then the following observations can be made.*

(i) *If* $\alpha = 1$ *then* $X \sim PH(\boldsymbol{\pi}, T)$;

(ii) *If* $\boldsymbol{\pi} = 1$ *and* $T = -\lambda$ *then* $X \sim ML(\alpha, \lambda)$, $\lambda > 0$;

(iii) *If* $\alpha = 1$, $\boldsymbol{\pi} = 1$ *and* $T = -\lambda$ *then* X *follows the well known exponential distribution with parameter* $\lambda > 0$;

(iv) *If* $(\boldsymbol{\pi}, T)$ *is a PH representation of the Erlang distribution then* X *follows fractional Erlang distribution.*

We now prove some stochastic comparison results for the MML distribution. Before stating the results, we provide the following lemma.

Lemma 2.2 *Let* $\boldsymbol{\gamma}$ *and* $\boldsymbol{v}$ *be a row and a column vectors with real entries, respectively. Further, let* V *be a real square matrix of size* n *such that* $V = PDP^{-1}$, *where* P *is a non-singular matrix and* D *is a diagonal matrix with non-negative entries. Let* $\boldsymbol{C_i^P}$ *and* $\boldsymbol{R_i^{P^{-1}}}$ *be the* i*-th column and the* i*-th row of* P *and* P^{-1}, *respectively. If* $\boldsymbol{\gamma} \boldsymbol{C_i^P} \boldsymbol{R_i^{P^{-1}}} \boldsymbol{v} \geq 0$, *for all* $i = 1, 2, \ldots, n$, *then* $\boldsymbol{\gamma} V \boldsymbol{v} \geq 0$.

Proof: Clearly, we can write

$$P = \begin{pmatrix} \boldsymbol{C_1^P} & \boldsymbol{C_2^P} & \ldots & \boldsymbol{C_n^P} \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} \boldsymbol{R_1^{P^{-1}}} \\ \boldsymbol{R_2^{P^{-1}}} \\ \vdots \\ \boldsymbol{R_n^{P^{-1}}} \end{pmatrix}.$$

Consider

$$\begin{aligned}
\boldsymbol{\gamma} V \boldsymbol{v} &= \boldsymbol{\gamma} P D P^{-1} \boldsymbol{v} \\
&= \gamma \begin{pmatrix} \boldsymbol{C_1^P} & \boldsymbol{C_2^P} & \dots & \boldsymbol{C_n^P} \end{pmatrix} D \begin{pmatrix} \boldsymbol{R_1^{P^{-1}}} \\ \boldsymbol{R_2^{P^{-1}}} \\ \vdots \\ \boldsymbol{R_n^{P^{-1}}} \end{pmatrix} \boldsymbol{v} \\
&= \begin{pmatrix} \boldsymbol{\gamma C_1^P} & \boldsymbol{\gamma C_2^P} & \dots & \boldsymbol{\gamma C_n^P} \end{pmatrix} diag(d_1, d_2, \dots, d_n) \begin{pmatrix} \boldsymbol{R_1^{P^{-1}}} \boldsymbol{v} \\ \boldsymbol{R_2^{P^{-1}}} \boldsymbol{v} \\ \vdots \\ \boldsymbol{R_n^{P^{-1}}} \boldsymbol{v} \end{pmatrix} \\
&= \sum_{i=1}^{n} d_i (\boldsymbol{\gamma C_i^P R_i^{P^{-1}} v}),
\end{aligned}$$

where d_i, $i = 1, 2, \dots n$, are non-negative diagonal entries of D. Thus, $\boldsymbol{\gamma} V \boldsymbol{v} \geq 0$ follows from the assumption $\boldsymbol{\gamma C_i^P R_i^{P^{-1}} v} \geq 0$ for all i.

Theorem 2.2 *Let X_1 and X_2 be two random variables such that $X_1 \sim MML(\alpha, \boldsymbol{\pi}, \lambda_1 T)$ and $X_2 \sim MML(\alpha, \boldsymbol{\pi}, \lambda_2 T)$, $\lambda_1 > 0, \lambda_2 > 0$. Assume that T is a diagonalizable matrix of order m with real eigenvalues such that $T = PDP^{-1}$. Let $\boldsymbol{C_i^P}$ and $\boldsymbol{R_i^{P^{-1}}}$ be the i-th column and the i-th row of P and P^{-1}, respectively. If $\lambda_1 \geq \lambda_2$ and $\boldsymbol{\pi C_i^P R_i^{P^{-1}} e} \geq 0$, for all $i = 1, 2, \dots, m$, then $X_1 \leq_{st} X_2$.*

Proof: Let $\epsilon_1, \epsilon_2, \dots, \epsilon_m$ be the eigenvalues of T and hence, $D = diag(\epsilon_1, \epsilon_2, \dots, \epsilon_m)$. Further, let $X \sim MML(\alpha, \boldsymbol{\pi}, \lambda T)$. Then, by Lemma 2.1, we have $\bar{F}_X(x) = \boldsymbol{\pi} E_{\alpha,1}(\lambda T x^{\alpha}) \boldsymbol{e}$. Consider

$$\begin{aligned}
E_{\alpha,1}(\lambda T x^{\alpha}) &= \sum_{k=0}^{\infty} \frac{(\lambda T)^k x^{\alpha k}}{\Gamma(1+\alpha k)} \\
&= \sum_{k=0}^{\infty} \frac{(\lambda P D P^{-1})^k x^{\alpha k}}{\Gamma(1+\alpha k)} \\
&= P \left(\sum_{k=0}^{\infty} \frac{(\lambda D)^k x^{\alpha k}}{\Gamma(1+\alpha k)} \right) P^{-1} \\
&= P \; diag \left(\sum_{k=0}^{\infty} \frac{(\lambda \epsilon_1)^k x^{\alpha k}}{\Gamma(1+\alpha k)}, \sum_{k=0}^{\infty} \frac{(\lambda \epsilon_2)^k x^{\alpha k}}{\Gamma(1+\alpha k)}, \dots, \sum_{k=0}^{\infty} \frac{(\lambda \epsilon_m)^k x^{\alpha k}}{\Gamma(1+\alpha k)} \right) P^{-1} \\
&= P \; diag \left(E_{\alpha,1}(\lambda \epsilon_1 x^{\alpha}), E_{\alpha,1}(\lambda \epsilon_2 x^{\alpha}), \dots, E_{\alpha,1}(\lambda \epsilon_m x^{\alpha}) \right) P^{-1}. \qquad (2.3)
\end{aligned}$$

Now,

$$\begin{aligned}
&\bar{F}_{X_2}(x) - \bar{F}_{X_1}(x) \\
&\quad = \boldsymbol{\pi} E_{\alpha,1}(\lambda_2 T x^{\alpha}) \boldsymbol{e} - \boldsymbol{\pi} E_{\alpha,1}(\lambda_1 T x^{\alpha}) \boldsymbol{e} \\
&\quad = \boldsymbol{\pi} P \; diag \left(\left(E_{\alpha,1}(\lambda_2 \epsilon_1 x^{\alpha}) - E_{\alpha,1}(\lambda_1 \epsilon_1 x^{\alpha}) \right), \dots, \left(E_{\alpha,1}(\lambda_2 \epsilon_m x^{\alpha}) - E_{\alpha,1}(\lambda_1 \epsilon_m x^{\alpha}) \right) \right) P^{-1} \boldsymbol{e},
\end{aligned}$$

where the last equality follows from (2.3). Since $\lambda_1 \geq \lambda_2$, we have, from Theorem 2.1 and Remark 2.2(ii), that $E_{\alpha,1}(\lambda_2\epsilon_j x^\alpha) - E_{\alpha,1}(\lambda_1\epsilon_j x^\alpha) \geq 0$, for all $j = 1, 2, \ldots, m$ and for all $x > 0$. Again, from the assumption, we have that $\boldsymbol{\pi} \boldsymbol{C_i^P} \boldsymbol{R_i^{P^{-1}}} \boldsymbol{e} \geq 0$ for all $i = 1, 2, \ldots, m$. Hence, by using Lemma 2.2, we get that $\bar{F}_{X_2}(x) - \bar{F}_{X_1}(x) \geq 0$, for all $x > 0$. Thus, the result follows.

□

The following corollary immediately follows from the above theorem in view of Proposition 2.1 (i).

Corollary 2.2 *If $\lambda_1 \geq \lambda_2$ and $\boldsymbol{\pi} \boldsymbol{C_i^P} \boldsymbol{R_i^{P^{-1}}} \boldsymbol{e} \geq 0$, for all $i = 1, 2, \ldots, m$, then $X_1 \leq_{lt} X_2$.* □

The next example illustrates the result given in Theorem 2.2.

Example 2.1 *Let $\boldsymbol{\pi} = (1, 0)$ and $T = \begin{pmatrix} -x_1 & 0 \\ x_2 & -x_3 \end{pmatrix}$ such that $x_2 \leq x_3$ and $x_1 \neq x_3$, where x_1, x_2 and x_3 are non-negative real numbers. Then T can be written as $T = PDP^{-1}$, where $D = diag(-x_1, -x_3)$, $P = \begin{pmatrix} \frac{x_3 - x_1}{x_2} & 0 \\ 1 & 1 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} \frac{x_2}{x_3 - x_1} & 0 \\ \frac{-x_2}{x_3 - x_1} & 1 \end{pmatrix}$. Clearly, $\boldsymbol{\pi} \boldsymbol{C_1^P} \boldsymbol{R_1^{P^{-1}}} \boldsymbol{e} = 1$ and $\boldsymbol{\pi} \boldsymbol{C_2^P} \boldsymbol{R_2^{P^{-1}}} \boldsymbol{e} = 0$. Thus, $X_1 \leq_{st} X_2$ follows from Theorem 2.2.* □

In the following theorem, we prove the same result as in Theorem 2.2 and Corollary 2.2 under a different set of sufficient conditions. The proof follows in the same line as in Theorem 2.2 and hence, omitted.

Theorem 2.3 *Let T_1 and T_2 be two real diagonal matrices of order m such that $T_1 = diag(\epsilon_1, \epsilon_2, \ldots, \epsilon_m)$ and $T_2 = diag(\sigma_1, \sigma_2, \ldots, \sigma_m)$, where σ_i and ϵ_i are non-positive real numbers, for all $i = 1, 2, \ldots, m$. Further, let X_1 and X_2 be two random variables such that $X_1 \sim MML(\alpha, \boldsymbol{\pi}, T_1)$ and $X_2 \sim MML(\alpha, \boldsymbol{\pi}, T_2)$, $0 < \alpha \leq 1$. If $|\epsilon_i| \geq |\sigma_i|$, for all $i = 1, 2, \ldots, m$, then $X_1 \leq_{st} X_2$ and $X_1 \leq_{lt} X_2$.*

2.5 RPMML

In this subsection, we define the RPMML. Before that we give the definition of the FHPP.

Definition 2.7 *The FHPP with parameter set $\{\alpha, \lambda\}$ is a renewal process with inter-arrival times following the ML distribution with parameters α and λ.*

Definition 2.8 *A renewal process with inter-arrival times following the MML distribution with parameter set $\{\alpha, \boldsymbol{\pi}, T\}$ is called the RPMML with parameter set $\{\alpha, \boldsymbol{\pi}, T\}$.*

Remark 2.3 *The following observations can be made:*

(i) *The RPMML with parameter set $\{1, \boldsymbol{\pi}, T\}$ is the renewal process of phase-type;*

(ii) *The RPMML with parameter set $\{\alpha, 1, -\lambda\}$, $\lambda > 0$ is the FHPP with parameter set $\{\alpha, \lambda\}$.*

3 Shock models based on the RPMML

In this section, we study different shock models based on the RPMML. Let L be the lifetime of a system subject to random shocks that occur according to the RPMML with parameter set $\{\alpha, \boldsymbol{\pi}, T\}$. Further, let X_i be the inter-arrival time between the i-th and the $(i-1)$-th shocks, $i = 1, 2, \ldots$. Let N be a random variable representing the number of a fatal shock, i.e., event "$N = n$", means that the system has failed on the arrival of the n-th shock and, consequently, $L = \sum_{i=1}^{n} X_i$.

As with inter-arrival times, one cannot go further and obtain in full generality the distribution of a lifetime of a system under shocks assuming arbitrary distribution of N. Therefore, we assume the fairly general discrete PH distribution that in combination with the MML distribution for inter-arrival times allows for 'compact' results. Note that, the discrete PH distribution for N was used in the literature, for example, in the extreme shock model and the generalized extreme shock model (studied by Bozbulut and Eryilmaz [8]), the run shock model (studied by Tank and Eryilmaz [45]) and in the generalized run shock model (studied by Gong et al. [17]). This was done for the light-tail, PH distributions for inter-arrival times, whereas we are obtaining our results for the heavy-tailed MML distribution. Therefore, the corresponding new methodology had to be developed.

Theorem 3.1 *Let $N \sim DPH(\boldsymbol{a}, Q)$ and let $\{X_n : n \in \mathbb{N}\}$ be a sequence of i.i.d. random variables with $X_i \sim MML(\alpha, \boldsymbol{\pi}, T)$, where $\boldsymbol{\pi e} = 1$, for all $i \in \mathbb{N}$. Then*

$$L = \sum_{i=1}^{N} X_i \sim MML(\alpha, \boldsymbol{\pi} \otimes \boldsymbol{a}, T \otimes I + (\boldsymbol{T^0 \pi}) \otimes Q).$$

Proof: The Laplace transform of L is given by

$$\begin{aligned}
E(\exp\{-uL\}) &= E(E(\exp\{-uL\}|N)) \\
&= \sum_{n=1}^{\infty} (E(\exp\{-uX_1\}))^n P(N=n) \\
&= \sum_{n=1}^{\infty} (\boldsymbol{\pi}(u^\alpha I - T)^{-1} \boldsymbol{T^0})^n (\boldsymbol{a} Q^{n-1}(I-Q)\boldsymbol{e}) \\
&= \sum_{n=1}^{\infty} \left[\boldsymbol{\pi} \left((u^\alpha I - T)^{-1} \boldsymbol{T^0 \pi}\right)^{n-1} (u^\alpha I - T)^{-1} \boldsymbol{T^0}\right] (\boldsymbol{a} Q^{n-1}(I-Q)\boldsymbol{e}) \\
&= (\boldsymbol{\pi} \otimes \boldsymbol{a}) \sum_{n=1}^{\infty} \left[\left((u^\alpha I - T)^{-1} \boldsymbol{T^0 \pi}\right)^{n-1} \otimes Q^{n-1}\right] \left((u^\alpha I - T)^{-1} \boldsymbol{T^0} \otimes (I-Q)\boldsymbol{e}\right) \\
&= (\boldsymbol{\pi} \otimes \boldsymbol{a}) \sum_{n=1}^{\infty} \left((u^\alpha I - T)^{-1} \boldsymbol{T^0 \pi} \otimes Q\right)^{n-1} \left((u^\alpha I - T)^{-1} \boldsymbol{T^0} \otimes (I-Q)\boldsymbol{e}\right) \\
&= (\boldsymbol{\pi} \otimes \boldsymbol{a}) \left[I - \left((u^\alpha I - T)^{-1} \boldsymbol{T^0 \pi} \otimes Q\right)\right]^{-1} \left((u^\alpha I - T)^{-1} \boldsymbol{T^0} \otimes (I-Q)\boldsymbol{e}\right) \\
&= (\boldsymbol{\pi} \otimes \boldsymbol{a}) \left[I - \left((u^\alpha I - T)^{-1} \boldsymbol{T^0 \pi} \otimes Q\right)\right]^{-1} ((u^\alpha I - T) \otimes I)^{-1} (\boldsymbol{T^0} \otimes (I-Q)\boldsymbol{e})
\end{aligned}$$

$$
\begin{aligned}
&= (\boldsymbol{\pi} \otimes \boldsymbol{a}) \left[(u^{\alpha} I - T) \otimes I - (\boldsymbol{T^0}\boldsymbol{\pi} \otimes Q) \right]^{-1} (\boldsymbol{T^0} \otimes (I-Q)\boldsymbol{e}) \\
&= (\boldsymbol{\pi} \otimes \boldsymbol{a})(u^{\alpha} I - (T \otimes I + \boldsymbol{T^0}\boldsymbol{\pi} \otimes Q))^{-1} (\boldsymbol{T^0} \otimes (I-Q)\boldsymbol{e}). \qquad (3.1)
\end{aligned}
$$

Now, consider

$$
\begin{aligned}
(T \otimes I + \boldsymbol{T^0}\boldsymbol{\pi} \otimes Q)^0 &= -(T \otimes I + \boldsymbol{T^0}\boldsymbol{\pi} \otimes Q)\boldsymbol{e} \\
&= -(T \otimes I + \boldsymbol{T^0}\boldsymbol{\pi} \otimes Q)(\boldsymbol{e} \otimes \boldsymbol{e}) \\
&= (-T \otimes I)(\boldsymbol{e} \otimes \boldsymbol{e}) - (\boldsymbol{T^0}\boldsymbol{\pi} \otimes Q)(\boldsymbol{e} \otimes \boldsymbol{e}) \\
&= (-T\boldsymbol{e} \otimes \boldsymbol{e}) - (\boldsymbol{T^0}\boldsymbol{\pi}\boldsymbol{e} \otimes Q\boldsymbol{e}) \\
&= (\boldsymbol{T^0} \otimes \boldsymbol{e}) - (\boldsymbol{T^0} \otimes Q\boldsymbol{e}) = \boldsymbol{T^0} \otimes (I-Q)\boldsymbol{e}.
\end{aligned}
$$

By using the above equality in (3.1), we get

$$
E(\exp\{-uL\}) = (\boldsymbol{\pi} \otimes \boldsymbol{a})(u^{\alpha} I - (T \otimes I + \boldsymbol{T^0}\boldsymbol{\pi} \otimes Q))^{-1} (T \otimes I + \boldsymbol{T^0}\boldsymbol{\pi} \otimes Q)^0,
$$

which implies that $L \sim MML(\alpha, \boldsymbol{\pi} \otimes \boldsymbol{a}, T \otimes I + (\boldsymbol{T^0}\boldsymbol{\pi}) \otimes Q)$ and hence, the result is proved. □

Remark 3.1 *Note that, for $\alpha = 1$, the result given in Theorem 3.1 coincides with that of Eryilmaz [12].*

The following corollary immediately follows from Theorem 3.1.

Corollary 3.1 *Let $N \sim DPH(\boldsymbol{a}, Q)$ and let $\{X_n : n \in \mathbb{N}\}$ be a sequence of i.i.d. random variables with $X_i \sim ML(\alpha, \lambda)$, where $\lambda > 0$, for all $i \in \mathbb{N}$, i.e., shocks occur according to the FHPP with parameter set $\{\alpha, \lambda\}$. Then*

$$
L = \sum_{i=1}^{N} X_i \sim MML(\alpha, \boldsymbol{a}, -\lambda(I-Q)).
$$

3.1 Generalized extreme shock model

The main results provided in this and the following subsections are practically important special cases of our general Theorem 3.1. On the other hand, the relevant stochastic comparisons are due to Proposition 2.4 and Theorem 2.3.

In the classical extreme shock model, there is only one source of shocks impacting a system. As a generalization of this model, Bozbulut and Eryilmaz [8] introduced two models with m possible sources, whereas at any instant of time only one source is 'operable'. Let θ_i be the probability that shocks come from source i, and let p_i be the probability that the magnitude of a shock from source i exceeds the critical level d; $i = 1, 2, \ldots, m$ (Model 2 in Bozbulut and Eryilmaz [8]). In this model, a system fails upon occurrence of a shock of size, at least, d. In other words, a shock that comes from the source i is harmless with probability $1 - p_i$, $i = 1, 2, \ldots, m$. Note that, in this model, $N \sim DPH(\boldsymbol{a_1}, Q_1)$, where $\boldsymbol{a_1} = (\theta_1, \theta_2, \ldots, \theta_m)$ and

$Q_1 = diag(1-p_1, 1-p_2, \ldots, 1-p_m)$ with $\theta_i > 0$, $i = 1, 2, \ldots, m$, and $\sum_{i=1}^{m} \theta_i = 1$ (see Bozbulut and Eryilmaz [8]). Different applications of this shock model were discussed in Bozbulut and Eryilmaz [8]. Note that, when $m = 1$, this model reduces to the classical extreme shock model, therefore, our results are meaningful generalizations for this basic case as well.

In the following theorem, we derive the distribution of the lifetime of a system for the described generalized extreme shock model. The proof, obviously, follows from Theorem 3.1.

Theorem 3.2 *Assume that shocks arrive according to the RPMML with parameter set* $\{\alpha, \boldsymbol{\pi}, T\}$. *Then* $L \sim MML(\alpha, \boldsymbol{\pi} \otimes \boldsymbol{a_1}, T \otimes I + (\boldsymbol{T^0}\boldsymbol{\pi}) \otimes Q_1)$.

The following corollary follows from the above theorem.

Corollary 3.2 *Assume that shocks arrive according to the FHPP with parameter set* $\{\alpha, \lambda\}$, *where* $0 < \alpha \leq 1$ *and* $\lambda > 0$. *Then* $L \sim MML(\alpha, \boldsymbol{a_1}, -\lambda S)$, *where* $S = diag(p_1, p_2, \ldots, p_m)$.

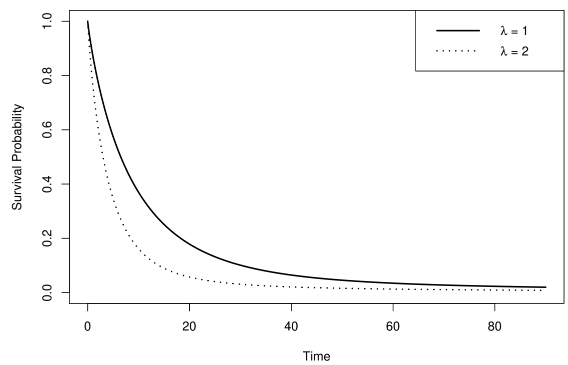
The next results comparing the lifetimes of two systems operating under two different random environments follow from Proposition 2.4 and Theorem 2.3.

Proposition 3.1 *Let* L_1 *and* L_2 *be the lifetimes of two systems subject to different FHPP shock processes with parameter sets* $\{\alpha, \lambda_1\}$ *and* $\{\alpha, \lambda_2\}$, *respectively, where* $0 < \alpha \leq 1$ *and* $\lambda_1, \lambda_2 > 0$. *Assume that the same source of shocks impacts both systems. If* $\lambda_1 \geq \lambda_2$, *then*

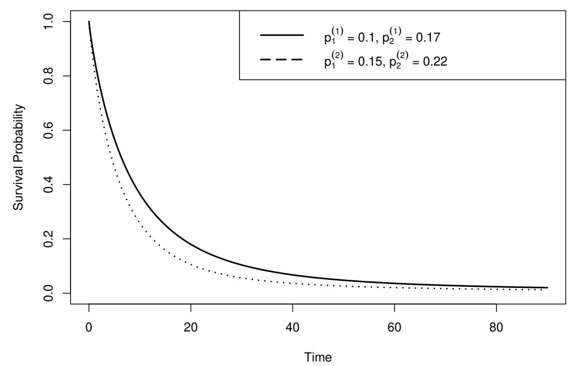
$L_1 \leq_{st} L_2$, $L_1 \leq_{lt} L_2$, $L_1 \leq_{lt-r} L_2$, $L_1 \leq_{r-lt-r} L_2$ *and* $L_1 \leq_{d-lt-r} L_2$.

Proposition 3.2 *Let* L_1 *and* L_2 *be the lifetimes of two systems subject to the same FHPP shock process with parameter set* $\{\alpha, \lambda\}$, *where* $0 < \alpha \leq 1$ *and* $\lambda > 0$. *Assume that shocks impacting both systems come from the same source* i *with probability* θ_i, $i = 1, 2, \ldots, m$. *Further, assume that* $p_i^{(1)}$ *and* $p_i^{(2)}$ *be the probabilities of the event "the magnitude of a shock from source* i *exceeds the critical level* d*" for the first and the second systems, respectively. If* $p_i^{(1)} \geq p_i^{(2)}$, *for all* $i = 1, 2, \ldots, m$, *then* $L_1 \leq_{st} L_2$ *and* $L_1 \leq_{lt} L_2$. □

To illustrate the obtained stochastic comparisons, in Figure 1a, we plot the survival function of the system over $t \in [0, 90]$ for the fixed $\alpha = 0.9, p_1 = 0.11, p_2 = 0.15, \theta_1 = 0.5$ and $\theta_2 = 0.5$. From this figure, we see that an increment in the parameter λ of FHPP decreases the survival function of the system (see Proposition 3.1). Further, in Figure 1b, we plot the survival function of the system over $t \in [0, 90]$ for fixed $\alpha = 0.9, \lambda = 1, \theta_1 = 0.5$ and $\theta_2 = 0.5$. This also shows that the system lifetime decreases as the magnitudes of shocks increase (see Proposition 3.2). In Figure 2, we plot the system's failure rate over $t \in (0, 20]$ by assuming that the shock process is the FHPP with parameter set $\{0.9, 4\}$. Further, we assume $p_1 = 0.5$, $p_2 = 0.3$, $\theta_1 = 0.5$ and $\theta_2 = 0.5$. This figure shows the decreasing shape of the failure rate.



(a) For $\alpha = 0.9$, $p_1 = 0.11$ and $p_2 = 0.15$, $\theta_1 = 0.5$ and $\theta_2 = 0.5$.



(b) For $\alpha = 0.9$, $\lambda = 1$, $\theta_1 = 0.5$ and $\theta_2 = 0.5$.

Figure 1: Plot of system's survival function over $t \in [0, 90]$.

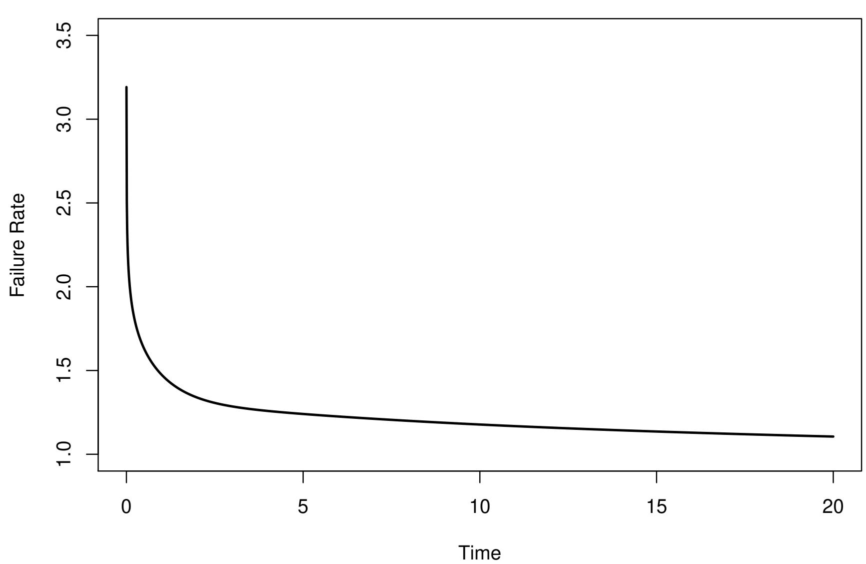


Figure 2: Plot of system's failure rate over $t \in (0, 20]$.

3.2 Generalized run shock model

In this subsection, we discuss the generalized run shock model introduced by Gong et al. [17]. In the classical run shock model (Mallor and Omey [31]), a system fails when a sequence of shocks with magnitudes above a threshold arrives in succession, whereas two different thresholds are considered in the generalized shock model. Below we give a description of this model.

Define two critical levels d_1 and d_2 ($d_1 < d_2$). For given two positive integers k_1 and k_2, a system is considered to be failed if at least k_1 consecutive shocks with magnitudes above d_1 or at least k_2 consecutive shocks with magnitudes above d_2 occur. As $d_1 < d_2$, it is obvious that $k_1 > k_2$. Let Y_i be the magnitude of the i-th shock, $i = 1, 2, \ldots$, and let $p_1 = P(Y_i \le d_1)$, $p_2 = P(d_1 < Y_i < d_2)$ and $p_3 = P(Y_i \ge d_2)$. Then $N \sim DPH(\boldsymbol{a_2}, Q_2)$ (see Gong et al. [17]), where $\boldsymbol{a_2} = (1, 0, \ldots, 0)$ and

$$Q_2 = \begin{pmatrix} E_{k_1} + S_{k_1} & T_{k_1} & & & \\ E_{k_1-1} & S_{k_1-1} & T_{k_1-1} & & \\ E_{k_1-2} & & S_{k_1-2} & \ddots & \\ \vdots & & & \ddots & T_{k_1-k_2+2} \\ E_{k_1-k_2+1} & & & & S_{k_1-k_2+1} \end{pmatrix},$$

$$E_i = \begin{pmatrix} p_1 & 0 & \ldots & 0 \\ p_1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_1 & 0 & \ldots & 0 \end{pmatrix}_{i \times k_1}, \quad S_i = \begin{pmatrix} 0 & p_2 & \ldots & 0 \\ 0 & \vdots & \ddots & 0 \\ \vdots & 0 & \ldots & p_2 \\ 0 & 0 & \ldots & 0 \end{pmatrix}_{i \times i}, \quad T_i = \begin{pmatrix} p_3 & \ldots & 0 \\ \vdots & \ddots & 0 \\ 0 & \ldots & p_3 \\ 0 & \ldots & 0 \end{pmatrix}_{i \times i-1}.$$

Note that, when $d_2 = \infty$, this model reduces to the classical run shock model (see Mallor and Omey [31], Tank and Eryilmaz [45]). Further, when $k_2 = 1$, this model reduces to a mixed shock model studied by Eryilmaz and Tekin [13].

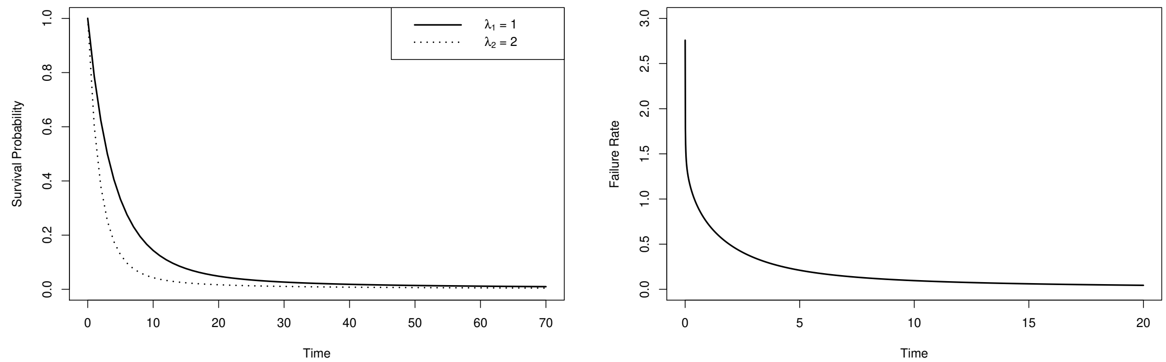
In the following theorem, we derive the lifetime of a system for the generalized run shock model. The proof follows from Theorem 3.1.

Theorem 3.3 *Let shocks arrive according to the RPMML with parameter set* $\{\alpha, \boldsymbol{\pi}, T\}$. *Then* $L \sim MML(\alpha, \boldsymbol{\pi} \otimes \boldsymbol{a_2}, T \otimes I + (\boldsymbol{T^0}\boldsymbol{\pi}) \otimes Q_2)$. □

The next corollary is an immediate consequence of the above theorem.

Corollary 3.3 *Let shocks arrive according to the FHPP with parameter set* $\{\alpha, \lambda\}$, *where* $0 < \alpha \le 1$ *and* $\lambda > 0$. *Then* $L \sim MML(\alpha, \boldsymbol{a_2}, -\lambda(I - Q_2))$. □

We will now compare the lifetimes of two systems operating under two different random environments. The proof follows from Proposition 2.4 and hence, omitted.



(a) System's survival function over $t \in [0, 70]$.

(b) System's failure rate over $t \in (0, 20]$.

Figure 3: Plot of system's survival function and failure rate.

Proposition 3.3 *Let L_1 and L_2 be the lifetimes of two systems subject to two FHPP shock processes with parameter sets $\{\alpha, \lambda_1\}$ and $\{\alpha, \lambda_2\}$, respectively, where $0 < \alpha \leq 1$ and $\lambda_1, \lambda_2 > 0$. Assume that shocks impacting both systems come from the same FHPP. If $\lambda_1 \geq \lambda_2$ then $L_1 \leq_{st} L_2$, $L_1 \leq_{lt} L_2$, $L_1 \leq_{lt-r} L_2$, $L_1 \leq_{r-lt-r} L_2$ and $L_1 \leq_{d-lt-r} L_2$.* □

Below we give an illustration of the above proposition. Let $k_1 = 2$, $k_2 = 1$ and

$$Q_2 = \begin{pmatrix} p_1 & p_2 \\ p_1 & 0 \end{pmatrix}.$$

In Figure 3a, we plot the survival function of the system, for fixed $\alpha = 0.90$, $p_1 = 0.5$ and $p_2 = 0.3$. The figure shows that an increment in the parameter λ of the shock process decreases the lifetime of the system. In Figure 3b, we plot the failure rate of the system by assuming that $p_1 = 0.5$, $p_2 = 0.3$, and the shock process is the FHPP with parameter set $\{0.8, 4\}$. This figure shows that the system's failure rate is decreasing over time.

4 Applications

In this section, we discuss two applications of the proposed model for systems that can fail due to impact of external shocks that occur according to the RPMML, whereas the criterion of failure is defined in accordance with the generalized run shock model. In the same line, these applications can also be discussed for the generalized extreme shock model.

4.1 Optimal replacement time

In this subsection, we consider the classical age replacement policy when a system is replaced by a new one either upon failure or on reaching the predetermined age, whichever comes first.

Thus, the optimal replacement time t^* should be obtained that minimizes the total long-run average cost per unit time.

Assumptions:

(i) Let a new system with a lifetime described by the generalized run shock model be incepted into operation at $t = 0$.

(ii) Assume that shocks constitute the only cause of failure and occur according to the RP-MML with parameter set $\{\alpha, \boldsymbol{\pi}, T\}$.

(iii) The system is replaced by a new one either upon failure or after reaching the predetermined age, whichever occurs first.

(iv) Let c_1 and c_2 be the costs of replacing a non-failed and failed systems, respectively. As any failure incurs an additional penalty, we assume that $c_1 < c_2$.

Based on the above assumptions, we derive the optimal replacement time. Let $C(t)$ be the mean cost rate per unit time. Then $C(t)$ should be a function of the replacement age t. Consequently,

$$\begin{aligned} C(t) &= \frac{c_1 P(L > t) + c_2 P(L \leq t)}{E(min(L, t))} \\ &= \frac{c_2 + (c_1 - c_2) P(L > t)}{E(min(L, t))}. \end{aligned}$$

As the denominator in this equation tends to 0 as $t \to \infty$, we cannot consider the problem on the infinite horizon. In real life, due to specifications and internal degradation processes the wearing items should be replaced in any case (not as a result of an optimal decision) at some large time t^u. In this way, our setting can be regularized by updating Assumption (iii).

Updated Assumption (iii): The system is replaced by a new one either upon its failure or after reaching its age to a predetermined threshold value t not exceeding t^u, whichever occurs first.

Thus, we should obtain optimal t^* $(\leq t^u)$ such that $C(t^*) < C(t^u)$ and $C(t^*) = \min_{t \in (0, t^u]} C(t)$. To evaluate $C(t)$, we first derive $P(L > t)$ and $E(min(L, t))$. From Theorem 3.1, we have that $L = \sum_{i=1}^{N} X_i \sim MML(\alpha, \boldsymbol{\pi} \otimes \boldsymbol{a}, T \otimes I + (T^0 \boldsymbol{\pi}) \otimes Q)$ and hence,

$$P(L > t) = (\boldsymbol{\pi} \otimes \boldsymbol{a}) E_{\alpha,1}((T \otimes I + (T^0 \boldsymbol{\pi}) \otimes Q) t^{\alpha}) \boldsymbol{e}$$

and

$$\begin{aligned} E(min(L, t)) &= \int_0^{\infty} P(min(L, t) > x) dx \\ &= \int_0^t P(L > x) dx = \int_0^t (\boldsymbol{\pi} \otimes \boldsymbol{a}) E_{\alpha,1}((T \otimes I + (T^0 \boldsymbol{\pi}) \otimes Q) x^{\alpha}) \boldsymbol{e} dx \end{aligned}$$

Table 1: Optimal values of the replacement time and the mean cost function

(p_1, p_2)	α	t^*	$C(t^*)$	$C(t^u)$
$(0.2, 0.4)$	1	1.432	2.055016	2.571429
	0.99	1.472	2.060097	2.386248
	0.98	1.520	2.064255	2.214772
$(0.3, 0.5)$	1	1.864	1.406771	1.833333
	0.99	1.896	1.407717	1.701328
	0.98	1.944	1.408333	1.579092

$$= \quad t(\boldsymbol{\pi} \otimes \boldsymbol{a})E_{\alpha,2}((T \otimes I + (T^0\boldsymbol{\pi}) \otimes Q)t^\alpha)\boldsymbol{e}.$$

Consequently,

$$C(t) = \frac{c_2 + (c_1 - c_2)(\boldsymbol{\pi} \otimes \boldsymbol{a})E_{\alpha,1}((T \otimes I + (T^0\boldsymbol{\pi}) \otimes Q)t^\alpha)\boldsymbol{e}}{t(\boldsymbol{\pi} \otimes \boldsymbol{a})E_{\alpha,2}((T \otimes I + (T^0\boldsymbol{\pi}) \otimes Q)t^\alpha)\boldsymbol{e}}, \quad t > 0.$$

In Table 1, we compute the optimal values of the replacement time and the mean cost function for different values of parameters p_1, p_2 and α. We assume other model parameters as $c_1 = 1$, $c_2 = 10$, $\lambda = 1$,

$$\boldsymbol{\pi} = (0, 1),\ T = \begin{pmatrix} -\lambda & 0 \\ \lambda & -\lambda \end{pmatrix},\ \boldsymbol{a} = (1, 0),\ Q = \begin{pmatrix} p_1 & p_2 \\ p_1 & 0 \end{pmatrix}$$

and $t^u = 1000$. From Table 1, we observe that, if the value of α decreases, for fixed (p_1, p_2), then the optimal replacement time increases.

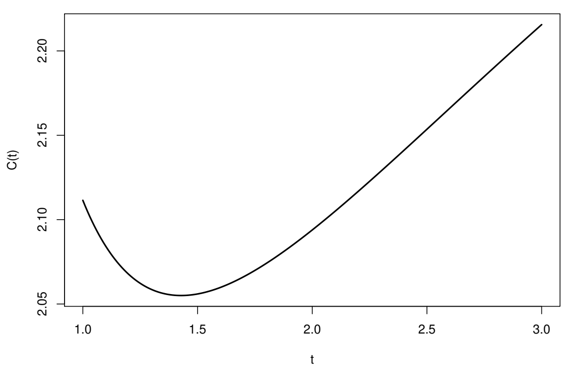
In Figure 4, we plot $C(t)$ with respect to time t, for $(\alpha, p_1, p_2) = (1, 0.2, 0.4), (0.9, 0.2, 0.4), (1, 0.3, 0.5)$ and $(0.9, 0.3, 0.5)$. Here, we also assume that $\lambda = 1$, $c_1 = 1$ and $c_2 = 10$,

$$\boldsymbol{\pi} = (0, 1),\ T = \begin{pmatrix} -\lambda & 0 \\ \lambda & -\lambda \end{pmatrix},\ \boldsymbol{a} = (1, 0),\ Q = \begin{pmatrix} p_1 & p_2 \\ p_1 & 0 \end{pmatrix}.$$

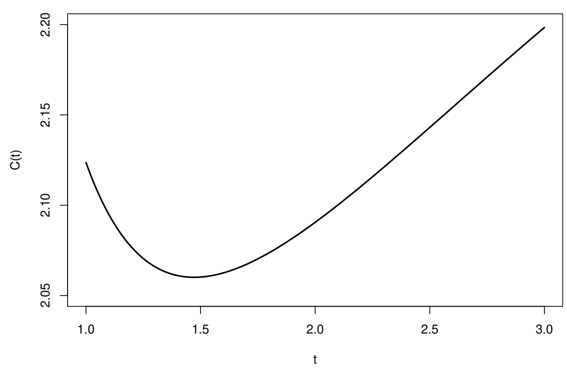
From Figure 4, we observe that the cost function $C(t)$ has U-shape in initial time period.

4.2 Optimal mission duration

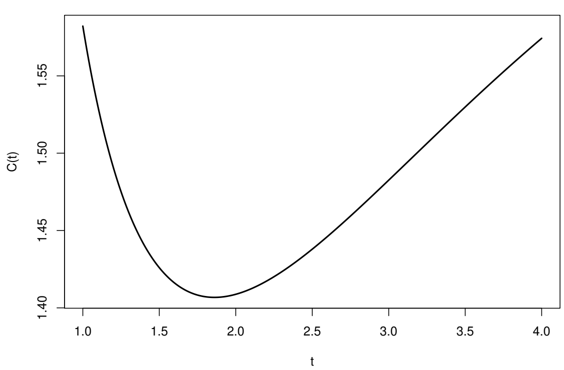
In this subsection, for illustration of the obtained in this paper results to the case of shocks arriving in accordance with RPMML (the run shock model), we generalize the approach introduced and studied by Finkelstein and Levitin [16], where the HPP of shocks was considered. As a failure of a system during a mission can result in substantial losses/penalties, sometimes it is reasonable to abort the mission before its completion. In other words, the completion of a mission may not always be beneficial in terms of cost for a degrading system. A mission abort decision usually results in a reward that depends on the system's operation time and a



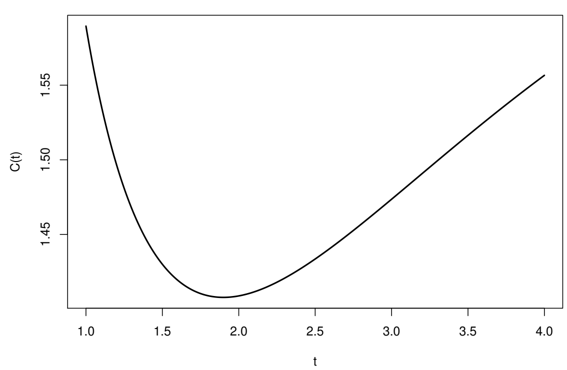
(a) For fixed $\alpha = 1$ and $(p_1, p_2) = (0.2, 0.4)$



(b) For fixed $\alpha = 0.9$ and $(p_1, p_2) = (0.2, 0.4)$



(c) For fixed $\alpha = 1$ and $(p_1, p_2) = (0.3, 0.5)$



(d) For fixed $\alpha = 0.9$ and $(p_1, p_2) = (0.3, 0.5)$

Figure 4: Plot of the mean cost function against time.

penalty. On the other hand, the mission completion results in an additional reward. Moreover, the failure of a system during the mission also results in a penalty because it incurs additional cost due to failure of the mission. The decision about the mission termination, at a given time τ, should be made if the profit for the mission termination exceeds the expected profit in the case of its continuation with respect to risk associated with the system failure. The problem is to find the optimal mission time that minimizes the difference of the expected profit in the case of its continuation with respect to risk associated with the system failure and the profit in the case of the termination. Below we give a list of assumptions to be used for further analysis.

Assumptions:

(i) Let a new system be incepted into operation at time $\tau = 0$, and let its lifetime be defined via RPMML of arriving shocks and the generalized run shock model. Further, let L and t (> 0) be the lifetime of the system and the mission duration, respectively. The mission can be aborted at time $\tau \in (0, t]$.

(ii) Assume that shocks constitute the only failure mode of a system and arrive according to the RPMML with parameter set $\{\alpha, \boldsymbol{\pi}, T\}$.

(iii) The profit $C(t)$ is obtained when the mission is completed (i.e., the system does not fail during the mission or the mission is not aborted in $[0, t]$). The per time unit reward, when the system is operating, is c_p and the per time unit operational cost is c_0, where $c_0 < c_p$.

(iv) A penalty C_f is imposed if the system fails during the mission. In case of the premature termination, the fixed penalty C_{pt} $(C_{pt} < C_f)$ is administrated. Further, C_r is an additional reward for the mission completion.

(v) Reward after the failure is discarded.

Based on aforementioned assumptions, the profit $C(t)$ upon mission completion can be expressed as

$$C(t) = (c_p - c_0)t + C_r.$$

Note that the mission is aborted at time τ if the total profit at termination exceeds the expected profit in case of mission continuation. The profit at termination at time τ is equal to $(c_p - c_0)\tau - C_{pt}$. On the other hand, the expected profit in the case of mission continuation is given by

$$\frac{\bar{F}_L(t)}{\bar{F}_L(\tau)}\left((c_p - c_0)\, t + C_r\right) - \left(1 - \frac{\bar{F}_L(t)}{\bar{F}_L(\tau)}\right) C_f,$$

where $\bar{F}_L(t)/\bar{F}_L(\tau)$ is the probability that a system does not fail in the remaining mission time given that it is operable at time τ; here $P(L > t) = (\boldsymbol{\pi} \otimes \boldsymbol{a_2}) E_{\alpha,1}((T \otimes I + (T^0 \boldsymbol{\pi}) \otimes Q_2) t^{\alpha}) \boldsymbol{e}$ (from Theorem 3.3). Thus, for some τ, if the expression

$$A(\tau) \overset{\text{def.}}{=} \frac{\bar{F}_L(t)}{\bar{F}_L(\tau)}\left((c_p - c_0)\, t + C_r\right) - \left(1 - \frac{\bar{F}_L(t)}{\bar{F}_L(\tau)}\right) C_f - ((c_p - c_0)\tau - C_{pt})$$

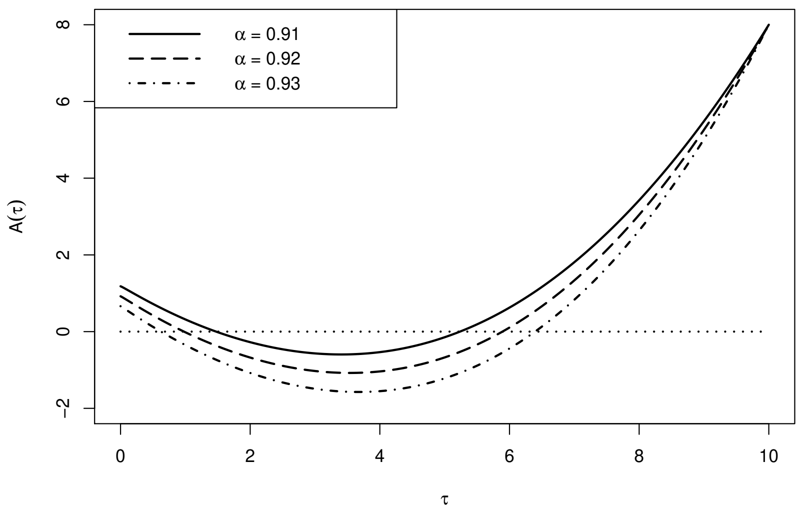


Figure 5: Plot of $A(\tau)$ against $\tau \in [0, 10]$

is non-negative, then the mission should not be terminated at time τ. Clearly, $A(0) \geq 0$, as there is no need to terminate the mission that had just started. Since the expression of $A(\tau)$ is complicated, it is not analytically possible to find out the values of τ for which $A(\tau) \geq 0$. Thus, we consider the following numerical example.

Let us assume $t = 10, c_p = 2.5, c_0 = 0.5, C_r = 3, C_f = 8, C_{pt} = 5$,

$$\boldsymbol{\pi} = (0, 1),\ T = \begin{pmatrix} -\lambda & 0 \\ \lambda & -\lambda \end{pmatrix},\ \boldsymbol{a_2} = (1, 0),\ Q_2 = \begin{pmatrix} p_1 & p_2 \\ p_1 & 0 \end{pmatrix},$$

$\lambda = 1$, $p_1 = 0.5$ and $p_2 = 0.3$. Based on these parameter values, we plot the profit comparison function $A(\tau)$ against $\tau \in [0, 10]$, for $\alpha = 0.91,\ 0.92$ and 0.93. Figure 5 shows that $A(\tau)$ is in U-shaped. Further, note that, for $\alpha = 0.91$, it takes negative values in the interval $\tau \in [1.48, 5.23]$. This implies that the mission should not be terminated in the interval $[0, 1.48)$ and $(5.23, 10]$, whereas it should be aborted just at $\tau = 3.41$ as it is the optimal solution. In case the mission is not terminated at time $\tau = 3.41$, it may be terminated at any time in the interval $[1.48, 5.23]$. Further, if this is not done, then the mission should not be terminated at all because its termination in the interval $(5.23, 10]$ is not beneficial. Similar conclusions can be made for other values of α as well.

5 Concluding remarks

In this paper, we have introduced a renewal process with inter-arrival times following the matrix Mittag-Leffler distribution, i.e., the renewal process of the matrix Mittag-Leffler type (RP-MML). The class of matrix Mittag-Leffler distributions are heavy-tailed and dense in the class of all lifetime distributions. Therefore, the corresponding models are fairly general. Furthermore, this distribution is also mathematically tractable. Due to these properties, the RPMML can be very useful in modeling the lifetimes of systems subject to random shocks with inter-arrival times having the heavy-tailed behavior.

Based on the described properties of the RPMML, we have studied two shock models, namely, the generalized extreme shock model and the generalized run shock model. For this, the relevant mathematical results have been obtained and the corresponding methodology discussed. Finally, two real-life applications illustrate our findings.

Apart from the shock models discussed in this paper, the study of other shock models based on the RPMML may be considered as a potential problem yet to be explored. Specifically, relevant generalizations of the δ-shock model can be of interest in applications.

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