Stochastic stabilization of hybrid neural networks by periodically intermittent control based on discrete-time state observations

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Abstract

This paper is concerned with stablization of hybrid neural networks by intermittent control based on continuous or discrete-time state observations. By means of exponential martingale inequality and the ergodic property of the Markov chain, we establish a sufficient stability criterion on hybrid neural networks by intermittent control based on continuous-time state observations. Meantime, by M-matrix theory and comparison method, we show that hybrid neural networks can be stabilized by intermittent control based on discrete-time state observations. Finally, two examples are presented to illustrate our theory.

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1 Introduction

As a classical neural network, the Hopfield neural network proposed by Hopfield [?] has become an important class of nonlinear dynamic systems. In the past few decades, the Hopfield network has been studied by many scholars and widely used in signal processing, optimization, control and many other fields. It can be described by the following ordinary differential equation

$$C_k\dot{x}_k(t) = -\frac{1}{R_k}\dot{x}_i(t) + \sum_{j=1}^{n} T_{kj}g_j(x_j(t)), \quad 1 \le k \le n, \tag{1.1}$$

on $t \ge 0$, where the variable $x_k(t)$ represents the voltage on the input of the ith neuron, which is characterized by an input capacitance C_i and nonlinear activation function $g_k(x)$. T_{kj} is the connected matrix element and R_k represents the parallel resistance of each neuron input. By defining $b_k = \frac{1}{C_kR_k}$ and $a_{ki} = \frac{T_{kj}}{C_k}$, Eq.(1.1) can be re-written as

$$\dot{x}(t) = -Bx(t) + Ag(x(t)), \quad t \ge 0 \tag{1.2}$$

where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^\top$, $B = \text{diag}(b_1, b_2, \cdots, b_n)$, $A = (a_{kj})_{n\times n}$, $g(x) = (g_1(x_1), g_2(x_2), \cdots, g_n(x_n))^\top$.

Since Hopfield studied its stability by using the energy function, the stability of neural network has become an important research problem (e.g. [?, ?]). However, due to the uncertainty of system parameters and the disturbance of external random factors, the neural network is not always stable. Therefore, it is necessary to stabilize unstable neural network by means of feedback control. One common strategy is to design feedback control $u(y(t))$ in the drift part, so that the deterministically controlled neural network

$$\dot{y}(t) = -By(t) + Ag(y(t)) + u(y(t)) \tag{1.3}$$

becomes stable (e.g. [?]-[?]). It is obviously that $y(t) \neq x(t)$ and the feedback control $u(y(t))$ changes the state of the system (1.2). While another strategy is to design feedback control $u(z(t))dw(t)$ in the diffusion part, so that stochastically controlled neural network

$$dz(t) = [-Bz(t) + Ag(z(t))]dt + u(z(t))dw(t) \tag{1.4}$$

becomes stable. As a matter of fact, this strategy is called stochastic feedback control, and some stability results have been obtained (e.g. [?]-[?]). Compared with deterministic feedback control, stochastic feedback control has the advantage of preserves the original state in average. In this paper, stochastic feedback control with random noise will be used to stabilize the given unstable neural network. The pioneering work was done by Hasminskii [?], who stabilized a system by using two white noise sources. The theory on stabilization by random noise has since then been develpoed by many authors (e.g. [?]-[?]). It is noted that above mentioned papers are concerned with stochastic feedback control which requires the continuous observation of the state $x(t)$ or $x(t-\tau)$ for all time $t \geq 0$. However, such a continuous time feedback control is expensive and sometimes impossible as the observations are often of discrete time. In order to reduce the cost of continuous time observations, Mao [?] introduced the feedback control based on the discrete time observations of the state at times $0, \tau,\ 2\tau, \cdots$ to stabilize an unstable system. From a practical point of view, stabilization by discrete time feedback control is more realistic and costless. Some recent results on stochastic stabilization with discrete time feedback control may be found in [?]-[?].

On the other hand, let us turn to another discontinuous control strategy. Just like the feedback control based on discrete time observations, intermittent control, which involves working time and rest time, can also reduce the control cost efficiently. Therefore, intermittent control has attracted more interest from many people. For example, Li et al. [?] considered the exponential stabilization problem for stochastic memristive neural network under periodically intermittent control. Zhang et al. [?, ?] studied the stabilization of a given nonlinear system by the intermittent Brownian noise perturbation. Liu [?] and Zhu [?] showed that an unstable system can be stabilized by the intermittent stochastic feedback based on discrete time observation. Wang et al. [?] studied the stabilization of hybrid stochastic complex valued coupled delayed systems by means of periodically intermittent control. Liu et al. [?] investigated the stabilization of highly nonlinear stochastic coupled delayed systems via periodically intermittent control. He et al. [?] showed that the underlying neural networks can be stabilized by the discrete-time intermittent noise. Jiang et al.[?] discussed the stabilization of hybrid stochastic systems by intermittent feedback control based on discrete-time state observations. For the other intermittent control results of stochastic system and neural network, refer to [?]-[?] and the references therein.

As is known to all, many neural networks may experience abrupt changes in their structure and parameters caused by phenomena such as component failures or repairs and changing subsystem interconnections, and sudden environmental disturbances. In this situation, the neural network (1.2) becomes hybrid neural networks

$$\dot{x}(t) = -B(r(t))x(t) + A(r(t))g(x(t)), \quad t \geq 0. \tag{1.5}$$

This paper is concerned with the almost surely exponential stabilization of hybrid neural networks (1.5) by the intermittent stochastic feedback control. That is, assume that the hybrid neural networks (1.5) is not almost surely exponentially stable, our aim is to design an intermittent feedback control based on continuous-time state observations so that

$$\dot{x}(t) = -B(r(t))x(t) + A(r(t))g(x(t)) + \sigma(x(t), r(t))I(t)dw(t), \quad t \geq 0 \tag{1.6}$$

is almost surely exponentially stable. By using the exponential martingale inequality and the ergodic property of Markov chain, the almost surely exponential stability of hybrid neural networks with intermittent stochastic noise (1.6) is obtained. Meantime, the periodic intermittent control based on discrete-time state observations $\sigma(x(\lfloor t/\tau \rfloor \tau), r(t))I(t)dw(t)$ is used to stabilized the hybrid neural networks (1.5), and the upper bound of the duration between two consecutive observations τ is obtained. It should be noted that the intermittent stochastic feedback control does not only achieve sample-path stabilisation but also enable the expectation of the state to be, at all times, equal to the state of the original uncontrolled system (1.5). The main contributions of this paper are as follows:

(1) In this paper, we study a class of stochastic neural network with Markov switching, which is a stochastic network system composed of multiple subsystems under different modes, also known as hybrid stochastic neural networks. Therefore, the stability of hybrid stochastic network systems is more complex. Compared with the existing research, the almost surely exponential stability of hybrid neural networks with intermittent random noise (1.6) is discussed for the first time, and a sufficient criterion for stability is derived.

(2) As far as we know, there is little research on the stochastic stabilization of neural networks with intermittent control. The periodic intermittent stochastic feedback control is used to stabilized the unstable neural networks, in which the derived stability criteria depends not only on the transition rate γ_{ij} and the intermittent control parameters θ, but also on the intensity of stochastic noise $\sigma(x(t), r(t))I(t)dw(t)$.

(3) In this paper, the periodic intermittent control based on discrete-time state observations is designed to ensure the almost surely exponential stability of hybrid stochastic

neural networks (2.2). If $I(t) = 1$, then Eq.(2.2) degenerates into the hybrid stochastic neural networks with the classical feedback control based on discrete time observation, namely, the state is observed at discrete times, say $0,\ \tau,\ 2\tau,\ 3\tau,\ 4\tau,\ 5\tau, \cdots$. For the results of discrete time feedback control, see [**?**]-[**?**]. While in Eq.(2.2), if $T = 5\tau$ and $\theta = 0.6$, the periodic intermittent control implies that the state is only observed at discrete times, say $0,\ \tau,\ 2\tau,\ 5\tau,\ 6\tau,\ 7\tau,\ 10\tau,\ 11\tau,\ 12\tau, \cdots$. Because this control strategy only requires discrete time state observations within the working time of intermittent control period, the control cost is further reduced.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and hypotheses concerning Eq.(2.2). In Section 3 and 4, we investigate the stabilization for hybrid stochastic neural networks via intermittent feedback control based on continuous-time state observations and discrete-time state observations, respectively. While in Section 5 we give two examples to illustrate our theory.

2 Preliminaries

Throughout this paper, unless otherwise specified, we use the following notation. Let $|.|$ denote the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by $A^\top$. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^\top A)}$ while its operator norm is defined by $||A|| = \sup\{|Ax| : |x| = 1\}$. If A is a symmetric matrix, denoted by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalues, respectively.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous, $\mathcal{F}_0$ contains all P-null sets). Let $w(t) = (w_1(t), \cdots, w_m(t))^\top$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t), t \geq 0$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, P)$ taking values in a finite state space $S = \{1, 2 \dots N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by:

$$P(r(t+\Delta) = j | r(t) = i) = \begin{cases} \gamma_{ij}\Delta + \circ(\Delta), & if \quad i \neq j, \\ 1 + \gamma_{ii}\Delta + \circ(\Delta), & if \quad i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j, $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume that the Markov chain $r(t)$ is independent of the Brownian motion $w(t)$ and irreducible. Under this condition, the Markov chain has a unique stationary distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_N) \in R^{1\times N}$ which can be determined by solving the following linear equation $\pi\Gamma = 0$ subject to $\sum_{j=1}^{N} \pi_j = 1$ and $\pi_j > 0$ for all $j \in S$.

Consider the following hybrid stochastic neural networks by intermittent feedback control based on discrete-time state observations

$$\begin{aligned} dx_k(t) &= [-b_k(r(t))x_k(t) + \sum_{j=1}^{n} a_{kj}(r(t))g_j(x_j(t))]dt \\ &+ \sum_{j=1}^{n} \sigma_{kj}(x_k(\lfloor t/\tau \rfloor \tau), r(t))I(t)dw_j(t), \qquad k = 1, 2, \cdots, n. \end{aligned} \tag{2.1}$$

or equivalently

$$dx(t) = [-B(r(t))x(t) + A(r(t))g(x(t))]dt + \sigma(x(\lfloor t/\tau \rfloor \tau), r(t))I(t)dw(t), \tag{2.2}$$

with the initial data $x(0) = x_0 \in R^n$, $r(0) = r_0 \in S$, where $x(t)$ is the state vector associated with the n neurons and for each $i \in S$, $B(i) = \mathrm{diag}(B_1(i), \cdots, B_n(i))$ is a positive diagonal matrix, $A(i) = (a_{kj}(i))_{n\times n}$ is connection weight matrix, $g(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), \cdots, g_n(x_n(t)))^\top$ is a vector valued activation function, $\sigma : R^n \times S \to R^{n\times m}$, $\sigma = (\sigma_{kj})_{n\times m}$ is the diffusion coefficient matrix, τ is the duration between two consecutive observations and $\lfloor t/\tau \rfloor$ is the integer part of t/τ. Here $I : [0, \infty) \to \{0, 1\}$ is defined by

$$I(t) = \sum_{n=0}^{\infty} I_{[nT, nT+\theta T)}(t), t \geq 0,$$

where $T > 0$ denotes the control period and $\theta T > 0$ is the working width satisfying $\theta \in (0, 1)$.

Remark 2.1 *In fact, Eq.(2.2) can also be expressed as follows:*

$$\begin{cases} dx(t) = [-B(r(t))x(t) + A(r(t))g(x(t))]dt + \sigma(x(\lfloor t/\tau \rfloor \tau), r(t))dw(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [nT, nT + \theta T), \\ dx(t) = [-B(r(t))x(t) + A(r(t))g(x(t))]dt, \quad t \in [nT + \theta T, (n + 1)T). \end{cases} \tag{2.3}$$

Note that $I_{[nT,nT+\theta T)}(t)$ is the indicator function of $[nT, nT + \theta T)$ which means that it takes 1 when $t \in [nT, nT + \theta T)$ and 0 otherwise.

In this paper, we impose the following conditions on the functions g and σ.

Assumption 2.2 *Assume that there exists a constant matrix $G \in R^{n\times n}$ such that*

$$|g(x) - g(y)| \le |G(x - y)| \tag{2.4}$$

for all $x, y \in R^n$. Moreover, we assume that $g(0) = 0$.

Assumption 2.3 *Assume that there are nonnegative constants k_i such that*

$$|\sigma(x, i) - \sigma(y, i)| \le k_i|x - y| \tag{2.5}$$

for all $x, y \in R^n$ and $i \in S$. Moreover, we assume that $\sigma(0, i) = 0$ for all $i \in S$.

Remark 2.4 *In fact, the nonlinear activation function $g(.)$ in Assumption 2.2 satisfies the Lipschitz condition. In many literatures, the following Lipschitz condition are imposed on the nonlinear activation function $g(.)$: Assume that the activation functions $g_i(i = 1, 2, \cdots, n)$ are globally Lipschitz continuous, that is, there exists a constant $l_i > 0$ such that*

$$|g_i(x) - g_i(y)| \le l_i|(x - y)|, \quad \forall\ x, y \in R^n. \tag{2.6}$$

Obviously, (2.4) and (2.6) are only different in expression, but they both indicate that $g(.)$ is Lipschitz continuous. For example, by (2.6), we have that

$$\begin{aligned}
|g(x) - g(y)| &= \sqrt{(g_1(x_1) - g_1(y_1))^2 + (g_2(x_2) - g_2(y_2))^2 + \cdots + (g_n(x_n) - g_n(y_n))^2} \\
&\le \sqrt{l_1^2(x_1 - y_1)^2 + l_2^2(x_2 - y_2)^2 + \cdots + l_n^2(x_n - y_n)^2} \\
&= |G(x - y)|,
\end{aligned}$$

where $G = \mathrm{diag}(l_1, l_2, \cdots, l_n)$ and $(x - y) = [x_1 - y_1, x_2 - y_2, \cdots, x_n - y_n]^\top$.

Remark 2.5 *Under Assumptions 2.2 and 2.3, it is easy to conclude that Eq.(2.2) has a unique global solution $x(t)$ on $t \ge 0$ (see, [**?**]).*

3 Intermittent continuous-time stochastic stabilization

3.1 Almost sure stochastic stabilization

Consider a nonlinear unstable neural networks

$$dy(t)/dt = -B(r(t))y(t) + A(r(t))g(y(t)) \tag{3.1}$$

and its corresponding stochastically controlled neural networks

$$dy(t) = [-B(r(t))y(t) + A(r(t))g(y(t))]dt + \sigma(y(t), r(t))I(t)dw(t), \quad t \geq 0 \tag{3.2}$$

with the initial value $y(0) = y_0$ and $r(0) = r_0$.

Similarly, under Assumptions 2.2 and 2.3, Eq.(3.2) has a unique solution (see [?, ?]). Denote the unique solution by $y(t; 0, y_0, r_0)$ on $t \geq 0$.

Assumption 3.1 *Assume that there are nonnegative constants β_i such that*

$$|x^\top \sigma(x, i)| \geq \beta_i |x|^2$$

for all $x \in R^n$ and $i \in S$.

We denote by $C^2(R^n \times S; R_+)$ the family of all continuous non-negative functions $V(x, i)$ defined on $R^n \times S$ such that for each $i \in S$, they are continuously twice differentiable in x. For $V(x, i) \in C^2(R^n \times S; R_+)$, we define the function $\mathcal{L}V : R^n \times S \times R_+ \to R$ by

$$\begin{aligned} \mathcal{L}V(x, i, t) &= V_x(x, i)[-B(i)x + A(i)g(x)] \\ &+ \frac{1}{2}[\sigma^\top(x, i)V_{xx}(x, i)\sigma(x, i)I(t)] + \sum_{j=1}^{N} \gamma_{ij} V(x, j) \end{aligned} \tag{3.3}$$

and

$$\mathcal{H}V(x, i, t) = V_x(x, i)\sigma(x, i)I(t) \tag{3.4}$$

where $V_x(x, i) = \left(\frac{\partial V(x,i)}{\partial x_1}, \cdots, \frac{\partial V(x,i)}{\partial x_n}\right), \quad V_{xx}(x, i) = \left(\frac{\partial^2 V(x,i)}{\partial x_i \partial x_j}\right)_{n\times n}$.

Theorem 3.2 *Under Assumptions 2.2, 2.3 and 3.1, the trivial solution $y(t)$ of Eq.(3.2) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log(|y(t)|) \leq \sum_{i\in S} \pi_i[\eta_i + \theta(0.5k_i^2 - \beta_i^2)] \quad a.s., \tag{3.5}$$

where $\eta_i = -\lambda_{\min}(B(i)) + |G|\|A(i)\|$. In particular, if $\sum_{i\in S} \pi_i[\eta_i + \theta(0.5k_i^2 - \beta_i^2)] < 0$, then the trivial solution of Eq.(3.2) is almost surely exponentially stable.

To prove Theorem 3.2, we present two useful lemmas.

Lemma 3.3 *For any $t \geq 0$, $h > 0$ and $i \in S$, then*

$$P(r(s) \neq i \text{ for some } s \in [t, t+h] | r(t) = i) \leq 1 - e^{-\tilde{\gamma} h},$$

where $\tilde{\gamma} = \max_{i \in S}(-\gamma_{ii})$.

The proof of this lemma can refer to [**?**], which is omitted here.

Lemma 3.4 *For any $i \in S$, then*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t [k^2_{r(s)} - 2\beta^2_{r(s)}] I(s) ds \leq \theta \sum_{i\in S} \pi_i (k_i^2 - 2\beta_i^2) \quad a.s. \tag{3.6}$$

Proof. It is obvious that

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t}\int_0^t [k^2_{r(s)} - 2\beta^2_{r(s)}] I(s) ds \\ = &\limsup_{n\to\infty} \frac{1}{(n+1)T} \sum_{j=0}^{n} \int_{jT}^{jT+\theta T} [k^2_{r(s)} - 2\beta^2_{r(s)}] ds \ a.s. \end{aligned} \tag{3.7}$$

Let $\delta \in (0,1)$ be arbitrary and $\varpi = \theta T/m$ for a sufficiently large integer m so that $\varpi < \delta$. Then it follows from (3.7) that

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{(n+1)T} \sum_{j=0}^{n} \int_{jT}^{jT+\theta T} [k^2_{r(s)} - 2\beta^2_{r(s)}] ds \\ \leq &\limsup_{n\to\infty} \frac{1}{(n+1)T} \sum_{v=0}^{m-1}\sum_{j=0}^{n} \int_{jT+v\varpi}^{jT+(v+1)\varpi} [k^2_{r(s)} - 2\beta^2_{r(s)}] ds = \sum_{v=0}^{m-1} Q_v, \end{aligned} \tag{3.8}$$

where

$$Q_v = \limsup_{n\to\infty} \frac{1}{(n+1)T} \sum_{j=0}^{n} \int_{jT+v\varpi}^{jT+(v+1)\varpi} [k^2_{r(s)} - 2\beta^2_{r(s)}] ds.$$

For each $i \in S$, define

$$\tau_0^i = \inf\{j \geq 0 : \ r(jT) = i\} \ \text{ and } \ \tau_n^i = \inf\{j \geq \tau_{n-1}^i : \ r(jT) = i\} \ \text{ for } \ n \geq 1.$$

Then τ_n^i are all finite stopping times such that $0 \leq \tau_0^i < \cdots \tau_n^i \to \infty$ a.s. Set $S_n^i = \inf\{j \geq 0 : \ \tau_j^i \leq n\}$ and T_n^i denote the number of the set S_n^i contains. By the ergodic property of the Markov chain, we have $\lim_{n\to\infty}\frac{T_n^i}{n+1}=\pi_i$. Similar to the proof of Li et al. [?], we can derive that

$$\begin{aligned}Q_0 &= \limsup_{n\to\infty}\frac{1}{(n+1)T}\sum_{i\in S}\sum_{j\in S_n^i}\int_{jT}^{jT+\varpi}[k_{r(s)}^2-2\beta_{r(s)}^2]ds\\ &\le \sum_{i\in S}\frac{\pi_i}{T}\limsup_{n\to\infty}\frac{1}{T_n^i}\sum_{j\in S_n^i}\int_{jT}^{jT+\varpi}[k_{r(s)}^2-2\beta_{r(s)}^2]ds\\ &\le \sum_{i\in S}\frac{\pi_i}{T}\limsup_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}\int_{\tau_j^iT}^{\tau_j^iT+\varpi}[k_{r(s)}^2-2\beta_{r(s)}^2]ds.\end{aligned}\tag{3.9}$$

By the strong Markov property, $\{r(\tau_j^iT+t)\}_{t\ge0}$ forms a Markov chain with the same generator Γ which starts from i. Letting $\zeta_j^i=\int_{\tau_j^iT}^{\tau_j^iT+\varpi}[k_{r(s)}^2-2\beta_{r(s)}^2]ds$, we have that $\{\zeta_j^i\}_{j\ge0}$ are independent identically distributed with mean value $E\zeta_0^i$. By the large number theory,

$$\limsup_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}\int_{\tau_j^iT}^{\tau_j^iT+\varpi}[k_{r(s)}^2-2\beta_{r(s)}^2]ds=E\zeta_0^i.$$

On the other hand, by Lemma 3.3, we can obtain that $E\zeta_0^i\le(k_i^2-2\beta_i^2)\varpi+(\hat{k}^2-2\check{\beta}^2)\sqrt{\tilde{\gamma}\delta}\varpi$, where $\hat{k}=\max_{i\in S}k_i$ and $\check{\beta}=\min_{i\in S}\beta_i$. Inserting this into (3.9), we get

$$Q_0\le\sum_{i\in S}\frac{\pi_i}{T}[(k_i^2-2\beta_i^2)\varpi+(\hat{k}^2-2\check{\beta}^2)\sqrt{\tilde{\gamma}\delta}\varpi].$$

Similarly, we can show

$$Q_v\le\sum_{i\in S}\frac{\pi_i}{T}[(k_i^2-2\beta_i^2)\varpi+(\hat{k}^2-2\check{\beta}^2)\sqrt{\tilde{\gamma}\delta}\varpi],$$

for $v=0,1,2,\cdots,m-1$. Combing these and (3.7) together, we have

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\int_0^t[k_{r(s)}^2-2\beta_{r(s)}^2]I(s)ds &\le m\sum_{i\in S}\frac{\pi_i}{T}[(k_i^2-2\beta_i^2)\varpi+(\hat{k}^2-2\check{\beta}^2)\sqrt{\tilde{\gamma}\delta}\varpi]\\ &\le \theta\sum_{i\in S}\pi_i[(k_i^2-2\beta_i^2)+\sqrt{\tilde{\gamma}\delta}\theta(\hat{k}^2-2\check{\beta}^2)].\end{aligned}\tag{3.10}$$

Since $\delta>0$ is arbitrary, we must have

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t[k_{r(s)}^2-2\beta_{r(s)}^2]I(s)ds \le \theta\sum_{i\in S}\pi_i(k_i^2-2\beta_i^2).\tag{3.11}$$

The proof is therefore complete. $\square$

Proof of Theorem 3.2. Fix any initial value $y_0 \neq 0$ and $r_0 \in S$, write $y(t; 0, y_0, r_0) = y(t)$. By Mao and Yuan [**?**], we have that $y(t) \neq 0$ for all $t \geq 0$ almost surely. Define the Lyapunov function $V(y, i) = |y|^2$ for $(y, i) \in (R^n - \{0\}) \times S$. We can therefore apply the Itô formula to $\log V(y(t), r(t))$ to obtain that

$$\begin{aligned} d[\log(V(y(t), r(t)))] &= \frac{1}{V(y(t), r(t))}\Big[\mathcal{L}V(y(t), r(t), t)dt + \mathcal{H}V(y(t), r(t), t)dw(t)\Big] \\ &- \frac{1}{2V^2(y(t), r(t))}|\mathcal{H}V(y(t), r(t), t)|^2 dt. \end{aligned} \tag{3.12}$$

Inserting (3.3) and (3.4) into (3.12), we have

$$\begin{aligned} \log|y(t)|^2 &= \log|y_0|^2 + \int_0^t \frac{1}{|y(t)|^2}\Big(2y^\top(s)[-B(r(s))y(s) + A(r(s))g(y(s))] \\ &+ |\sigma(y(s), r(s))|^2 I(s)\Big)ds - \int_0^t \frac{2}{|y(t)|^4}|y^\top(s)\sigma(x(s), r(s))|^2 I(s)ds \\ &+ \int_0^t \frac{2}{|y(t)|^2} y^\top(s)\sigma(y(s), r(s))I(s)dw(t). \end{aligned}$$

By Assumptions 2.2 and 2.3, we obtain that

$$\begin{aligned} \log|y(t)|^2 &\leq \log|y_0|^2 + 2\int_0^t \eta_{r(s)}ds + \int_0^t k_{r(s)}^2 I(s)ds \\ &- \int_0^t \frac{2}{|y(t)|^4}|y^\top(s)\sigma(x(s), r(s))|^2 I(s)ds + M(t), \end{aligned} \tag{3.13}$$

where $\eta_{r(s)} = -\lambda_{\min}(B(r(s)) + |G|\|A(r(s))\|$ and $M(t) = \int_0^t \frac{2}{|y(t)|^2} y^\top(s)\sigma(y(s), r(s))I(s)dw(s)$ is a continuous martingale vanishing at $t = 0$. The quadratic variation of this martingale is given by

$$\langle M(t), M(t)\rangle = 4\int_0^t \frac{|y^\top(s)\sigma(y(s), r(s))|^2 I(s)}{|y(s)|^4} ds.$$

Assign $\varepsilon_2 \in (0, 1)$ arbitrarily and let $k = 1, 2, \cdots$. By the exponential martingale inequality, we have

$$P\Big\{\sup_{0\leq t\leq n}\Big[M(t) - \frac{\varepsilon}{2}\langle M(t), M(t)\rangle\Big] > \frac{2}{\varepsilon}\log n\Big\} \leq \frac{1}{n^2}.$$

Applying the Borel-Cantelli lemma we see that for almost all $\omega \in \Omega$, there is an integer $n_0 = n_0(\omega)$ such that if $n \geq n_0$,

$$M(t) \leq \frac{2}{\varepsilon}\log n + \frac{\varepsilon}{2}\langle M(t), M(t)\rangle$$

holds for all $0 \leq t \leq n$. Substituting this into (3.13) and then using Assumption (3.1), we obtain that

$$\begin{aligned} \log |y(t)|^2 &\leq \log |y_0|^2 + \frac{2}{\varepsilon} \log n + 2\int_0^t \eta_{r(s)} ds \\ &+ \int_0^t [k_{r(s)}^2 - 2(1-\varepsilon)\beta_{r(s)}^2] I(s) ds \end{aligned} \tag{3.14}$$

for all $0 \leq t \leq n$, $n \geq n_0$ almost surely. So for almost all $\omega \in \Omega$, if $n-1 \leq t \leq n$ and $n \geq n_0$, then

$$\begin{aligned} \frac{1}{t} \log |y(t)|^2 &\leq \frac{1}{n-1}\Big(\log |y_0|^2 + \frac{2}{\varepsilon} \log n\Big) + 2\frac{1}{t}\int_0^t \eta_{r(s)} ds \\ &+ \frac{1}{t}\int_0^t [k_{r(s)}^2 - 2(1-\varepsilon)\beta_{r(s)}^2] I(s) ds. \end{aligned}$$

This implies

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{t} \log(|y(t)|^2) &\leq 2\limsup_{t\to\infty} \frac{1}{t}\int_0^t \eta_{r(s)} ds \\ &+ \limsup_{t\to\infty} \frac{1}{t}\int_0^t [k_{r(s)}^2 - 2(1-\varepsilon)\beta_{r(s)}^2] I(s) ds. \end{aligned} \tag{3.15}$$

By Lemma 3.4 and the ergodic property of the Markov chain $r(t)$, we have

$$\limsup_{t\to\infty} \frac{1}{t} \log(|y(t)|^2) \leq \sum_{i\in S} 2\pi_i\eta_i + \theta \sum_{i\in S} \pi_i [k_i^2 - 2(1-\varepsilon)\beta_i^2]. \tag{3.16}$$

Letting $\varepsilon \to 0$ yields the desired assertion (3.5). The proof is therefore complete. □

Remark 3.5 *Note that the deterministic neural networks (3.1) is unstable, then we have*

$$\sum_{i\in S} \pi_i\eta_i > 0, \tag{3.17}$$

otherwise it is already almost surely exponentially stable. In other words, since the deterministic neural networks (3.1) is stable, it is not necessary to add a feedback control $\sigma(y(t), r(t))I(t)dw(t)$ *to stabilize (3.1). Therefore, combing (**??**) and* $\sum_{i\in S} \pi_i[\eta_i + \theta(0.5k_i^2 - \beta_i^2)] < 0$ *together, we have* $\sum_{i\in S} \pi_i\theta(\beta_i^2 - 0.5k_i^2) > 0$. *Note that* $\theta > 0$, *we obtain* $\sum_{i\in S} \pi_i(\beta_i^2 - 0.5k_i^2) > 0$ *and* $\frac{\sum_{i\in S} \pi_i\eta_i}{\sum_{i\in S} \pi_i(\beta_i^2 - 0.5k_i^2)} > 0$.

Remark 3.6 *From Theorem 3.2, we can conclude that the trivial solution of (3.2) is almost surely exponentially stable if and only if* $\theta \in (\frac{\sum_{i\in S} \pi_i\eta_i}{\sum_{i\in S} \pi_i(\beta_i^2 - 0.5k_i^2)}, 1)$. *Moreover, it is showed that the speed at which the solution of (3.2) converges to the equilibrium not only depends on* k_i *and* β_i, *but also on the parameter* θ.

Remark 3.7 *The periodic intermittent control in this paper requires that each subsystem has the same working time or rest time. In practice, the working time or rest time of each subsystem may not be the same. Therefore, we generalize the definition of* $I(t)$ *to* $I(t, r(t))$*. That is,* $I(t, r(t))$ *can be defined as follows*

$$I(t, r(t)) = \sum_{n=0}^{\infty} I_{[nT, nT+\theta_{r(t)}T)}(t), \quad t \geq 0. \tag{3.18}$$

Similar to the discussion of Theorem 3.3 ([**?**]), we can get the following corollary.

Corollary 3.8 *Under Assumptions 2.2, 2.3 and 3.1, the trivial solution* $y(t)$ *of Eq.(3.2) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log(|y(t)|) \quad \leq \quad \sum_{i\in S} \pi_i[\eta_i + \theta_i(0.5k_i^2 - \beta_i^2)] \quad a.s., \tag{3.19}$$

where $\eta_i = -\lambda_{\min}(B(i)) + |G|\|A(i)\|$*. In particular, if*

$$\sum_{i\in S} \pi_i[\eta_i + \theta_i(0.5k_i^2 - \beta_i^2)] < 0, \tag{3.20}$$

then the trivial solution of Eq.(3.2) is almost surely exponentially stable.

Remark 3.9 *By corollary* **??***, we show that the trivial solution of Eq.(3.2) is almost surely exponentially stable as long as condition (***??***) is satisfied. The only disadvantage is that we cannot give the range of each control parameter* θ_i*. While, for subsystems with the same working time, we can get the range of parameter* θ*. In fact, in Theorem 3.2, we have that if* $\theta \in (\frac{\sum_{i\in S}\pi_i\eta_i}{\sum_{i\in S}\pi_i(\beta_i^2 - 0.5k_i^2)}, 1)$*, then the trivial solution of Eq.(3.2) is almost surely exponentially stable. This is why we focus on considering that each subsystem with the same parameters* θ*.*

Remark 3.10 *Obviously,* $I(t, r(t))$ *is more practical than* $I(t)$ *and each subsystem does not need the same working time, so this control strategy can be more flexible. For example, in Example* **??***, we obtain that hybrid neural networks (***??***) can be stabilized by intermittent stochastic perturbation with* $\theta \in (0.6434, 1)$*. Once the range of* θ *is determined with* $\theta \in (0.6434, 1)$*, we know that it is feasible when all* $\theta_i, i = 1, 2$ *are greater than 0.6434. Meantime, we can also select* θ_1 *which is greater than 0.6434, and* θ_2 *which is less than 0.6434. If we choose* $\theta_1 = 0.8$ *and* $\theta_2 = 0.4$ *then we have that* $\sum_{i\in S} \pi_i[\eta_i + \theta_i(0.5k_i^2 - \beta_i^2)] = -0.0276 < 0$*.*

3.2 Design of linear feedback control with intermittent noise

Now, we shall show that the hybrid neural networks

$$dy(t)/dt = -B(r(t))y(t) + A(r(t))g(y(t)). \tag{3.21}$$

can be stabilized by linear feedback control with intermittent noise. We suppose $g(x)$ satisfies Assumption 2.2. Let

$$\sigma(x,i) = (G_{1,i}x, G_{2,i}x, \cdots, G_{m,i}x),$$

where $G_{k,i} \in R^{n\times n}$ are all $n \times n$ matrices. Then the hybrid stochastic neural network (3.2) becomes

$$dy(t) = [-B(r(t))y(t) + A(r(t))g(y(t))]dt + \sum_{k=1}^{m} G_{k,r(t)}y(t)I(t)dw(t). \tag{3.22}$$

For any $(x,i) \in R^n \times S$, we have

$$|\sigma(x,i)|^2 = \sum_{k=1}^{m} |G_{k,i}x|^2 \le \Big(\sum_{k=1}^{m} \|G_{k,i}\|^2\Big)|x|^2$$

and

$$|x^\top \sigma(x,i)|^2 = \sum_{k=1}^{m} |x^\top G_{k,i}x|^2 \ge \Big(\sum_{k=1}^{m} \lambda_{min}^2(G_{k,i})\Big)|x|^4.$$

These imply that k_i and β_i in Assumptions 2.3 and 3.1 have the forms

$$k_i^2 = \sum_{k=1}^{m} \|G_{k,i}\|^2 \quad \text{and} \quad \beta_i^2 = \sum_{k=1}^{m} \lambda_{min}^2(G_{k,i}).$$

In other words, the coefficients of Eq.(**??**) satisfy Assumptions 2.2, 2.3 and 3.1, then by Theorem 3.2, we have the trivial solution of Eq.(**??**) is almost surely exponentially stable if $\sum_{i\in S}\pi_i[\eta_i + \theta(0.5k_i^2 - \beta_i^2)] < 0$.

In fact, there are many choices for the matrices $G_{k,i}$ in order to stabilize the given hybrid neural networks (**??**). Let us now discuss some special cases of hybrid stochastic neural networks (**??**).

Case 1. Let $G_{k,i} = \sigma_{k,i}I$ for $1 \le k \le m$, $i \in S$, where I is the $n \times n$ identity and $\sigma_{k,i}$ are constants. Then the hybrid stochastic neural networks (**??**) becomes

$$dy(t) = [-B(r(t))y(t) + A(r(t))g(y(t))]dt + \sum_{k=1}^{m} \sigma_{k,r(t)}y(t)I(t)dw(t). \tag{3.23}$$

Note that for each $i \in S$,

$$\sum_{k=1}^{m} |G_{k,i}x|^2 = \Big(\sum_{k=1}^{m} \sigma_{k,i}^2\Big)|x|^2 \text{ and } \sum_{k=1}^{m} |x^\top G_{k,i}x|^2 = \Big(\sum_{k=1}^{m} \sigma_{k,i}^2\Big)|x|^4$$

for all $x \in R^n$. By Theorem 3.2, we conclude that the solution of (??) satisfies

$$\limsup_{t\to\infty} \frac{1}{t}\log(|y(t)|) \quad \le \quad \sum_{i\in S}\pi_i\eta_i - 0.5\theta\sum_{i\in S}\pi_i(\sum_{k=1}^{m}\sigma_{k,i}^2). \quad a.s.$$

In particular, if $\theta \in \Big(\frac{\sum_{i\in S}\pi_i\eta_i}{0.5\sum_{i\in S}\pi_i(\sum_{k=1}^{m}\sigma_{k,i}^2)}, 1\Big)$, then the hybrid stochastic neural networks (??) is almost surely exponentially stable. Now if we choose $\sigma_{k,i} = 0$ for all $2 \le k \le m$, then (??) becomes

$$dy(t) = [-B(r(t))y(t) + A(r(t))g(y(t))]dt + \sigma_{1,r(t)}y(t)I(t)dw_1(t).$$

That is we only use a scalar Brownian motion as the source of stochastic perturbation. This stochastic networks is almost surely exponentially stable provided $\theta \in \Big(\frac{\sum_{i\in S}\pi_i\eta_i}{0.5\sum_{i\in S}\pi_i\sigma_{1,i}^2}, 1\Big)$.

Case 2. For $i \in S$ and $1 \le k \le m$, choose a symmetric positive definite matrix $D_{k,i}$ such that $x^\top D_{k,i}x \ge \frac{\sqrt{10}}{4}$. Let σ be a real number and define $G_{k,i} = \sigma D_{k,i}$. Then the hybrid stochastic neural networks (??) becomes

$$dy(t) = [-B(r(t))y(t) + A(r(t))g(y(t))]dt + \sum_{k=1}^{m}\sigma D_{k,r(t)}y(t)I(t)dw(t). \tag{3.24}$$

Note that for each $i \in S$,

$$\sum_{k=1}^{m} |G_{k,i}x|^2 \le \sigma^2\Big(\sum_{k=1}^{m} \|D_{k,i}\|^2\Big)|x|^2 \text{ and } \sum_{k=1}^{m} |x^\top G_{k,i}x|^2 \ge \frac{5}{8}\sigma^2\Big(\sum_{k=1}^{m} \|D_{k,i}\|^2\Big)|x|^4$$

for all $x \in R^n$. By Theorem 3.2, we obtain that the solution of (??) satisfies

$$\limsup_{t\to\infty} \frac{1}{t}\log(|y(t)|) \quad \le \quad \sum_{i\in S}\pi_i\eta_i - 0.125\theta\sigma^2\sum_{i\in S}\pi_i(\sum_{k=1}^{m}\|D_{k,i}\|^2) \quad a.s.$$

So if $\theta \in \Big(\frac{8\sum_{i\in S}\pi_i\eta_i}{\sigma^2\sum_{i\in S}\pi_i(\sum_{k=1}^{m}\|D_{k,i}\|^2)}, 1\Big)$, then the hybrid stochastic neural network (??) is almost surely exponentially stable. $\square$

4 Intermittent discrete-time stochastic stabilization

In this section, we will establish a sufficient stability criterion on hybrid stochastic neural networks by intermittent feedback control based on discrete-time state observations.

Let us form Eq.(2.2) as a hybrid stochastic differential delay equation (HSDDEs). In fact, if we define the variable delay $\delta : [0, \infty) \to [0, \tau]$ by

$$\delta(t) = t - k\tau \text{ for } \quad k\tau \le t < (k+1)\tau, \quad k = 0, 1, 2, \cdots,$$

then Eq.(2.2) can be re-written as the following HSDDEs:

$$dx(t) = [-B(r(t))x(t) + A(r(t))g(x(t))]dt + \sigma(x(t - \delta(t)), r(t))I(t)dw(t), \quad t \ge 0. \tag{4.1}$$

In the previous section, we can show that Eq.(3.2) is almost surely exponentially stable by using the Lyapunov function method. Unfortunately, the solution of (**??**) may reach the origin provided that $x_0 \neq 0$, so we cannot apply the Itô formula to $\log |x(t)|^2$ in this delay case. Therefore, we adopt a comparative method to study the almost sure exponential stability of the solution to Eq.(**??**). Our aim here is to show that if auxiliary hybrid stochastic neural networks by continuous-time intermittent feedback control (3.2) is pth moment exponentially stable, then so is the hybrid stochastic neural networks by discrete-time intermittent feedback control (2.2) provided τ is sufficiently small.

Assumption 4.1 *There exists a $p \in (0, 1)$ such that*

$$\mathcal{A}_p := \text{diag}(\rho_1(p), \cdots, \rho_N(p)) - \Gamma \tag{4.2}$$

is a nonsingular M-matrix, where $\rho_i(p) = 0.5p[(2-p)\beta_i^2 - k_i^2] - p\eta_i$.

Note that $\mathcal{A}_p$ is a nonsingular M-matrix, by Theorem 2.10 of [**?**], it follows that $\mathcal{A}_p^{-1} \ge 0$. Set $(\psi_1, \cdots, \psi_N)^\top := \mathcal{A}_p^{-1}\overrightarrow{1}$, where $\overrightarrow{1} = (1, \cdots, 1)^\top$, we can obtain that $\psi_i > 0, \ i \in S$.

The following lemma shows that the corresponding auxiliary hybrid stochastic neural networks (3.2) is exponentially stable in the pth moment.

Lemma 4.2 *Under Assumptions 2.2, 2.3, 3.1 and* **??**. *If $\theta \in (\frac{\bar{\psi}}{\psi_M^{-1} + \bar{\psi}}, 1)$, then the solution $y(t)$ of Eq.(3.2) satisfies*

$$E|y(t)|^p \le \frac{\psi_M}{\psi_m} E|y_0|^p e^{-[(\psi_M^{-1} + \bar{\psi})\theta - \bar{\psi}]t}, \quad \forall \ t \ge 0, \tag{4.3}$$

where

$$\psi_m = \min_{i \in S} \psi_i, \psi_M = \max_{i \in S} \psi_i, \bar{\psi} = \max_{i \in S} \frac{1}{\psi_i}\Big(p\eta_i\psi_i + \sum_{j=1}^{N} \gamma_{ij}\psi_j\Big).$$

In other words, the trivial solution of Eq.(3.2) is pth moment exponentially stable.

Proof. Let $V(y,i)=\psi_i|y|^p$. Clearly, $\psi_m|y|^p \le V(y,i) \le \psi_M|y|^p$. By (3.3) and Assumption **??**, we compute the operator $\mathcal{L}V$ as follows:

$$\begin{aligned}\mathcal{L}V(y,i,t) &= p\psi_i|y|^{p-2}y^\top[-B(i)y+A(i)g(y)]+0.5p\psi_i|y|^{p-2}|\sigma(y,i)|^2I(t)\\ &- 0.5p(2-p)\psi_i|y|^{p-4}|y^\top\sigma(y,i)|^2I(t)+\sum_{j=1}^N\gamma_{ij}\psi_j|y|^p\\ &\le p\psi_i[-\lambda_{\min}(B(i))+|G|\|A(i)\|]|y|^p+0.5p\psi_ik_i^2|y|^pI(t)\\ &- 0.5p(2-p)\psi_i\beta_i^2|y|^pI(t)+\sum_{j=1}^N\gamma_{ij}\psi_j|y|^p\\ &\le \Big\{\Big(p\eta_i+0.5p[\beta_i^2-(2-p)\sigma_i^2]\Big)\psi_i+\sum_{j=1}^N\gamma_{ij}\psi_j\Big\}I(t)|y|^p\\ &+ \Big(p\eta_i\psi_i+\sum_{j=1}^N\gamma_{ij}\psi_j\Big)(1-I(t))|y|^p \le [\bar\psi-(\psi_M^{-1}+\bar\psi)I(t)]V(y,i). \end{aligned}\tag{4.4}$$

For any $t\ge 0$, the generalized Itô formula shows that

$$\begin{aligned}&E\Big[V(y(t),r(t))e^{-\int_0^t[\bar\psi-(\psi_M^{-1}+\bar\psi)I(s)]ds}\Big]-EV(y_0,r_0)\\ =\;& E\int_0^t e^{-\int_0^s[\bar\psi-(\psi_M^{-1}+\bar\psi)I(u)]du}\Big(LV(y(s),r(s),s)-[\bar\psi-(\psi_M^{-1}+\bar\psi)I(s)]V(y(s),r(s))\Big)ds.\end{aligned}$$

This implies

$$E|y(t)|^p \le \frac{\psi_M}{\psi_m}E|y_0|^pe^{\int_0^t[\bar\psi-(\psi_M^{-1}+\bar\psi)I(s)]ds}. \tag{4.5}$$

By the condition $\theta\in(\frac{\bar\psi}{\psi_M^{-1}+\bar\psi},1)$, we have

$$\int_0^t[\bar\psi-(\psi_M^{-1}+\bar\psi)I(s)]ds \le [\bar\psi-(\psi_M^{-1}+\bar\psi)\theta]t$$

for any $t\in[kT,(k+1)T)$. Hence, we conclude that

$$E|y(t)|^p \le \frac{\psi_M}{\psi_m}E|y_0|^pe^{[\bar\psi-(\psi_M^{-1}+\bar\psi)\theta]t}.$$

The proof is therefore complete.

Remark 4.3 *As $\theta\to 0$, Eq.(3.2) will degenerate into a hybrid neural networks (**??**). Note that we are only interested in the case when $\bar\psi>0$ in this paper; otherwise, the given hybrid neural networks (**??**) is already stable and there is no need to stabilize it using feedback control. As $\theta\to 1$, Eq.(3.2) will become continuous hybrid stochastic neural networks*

$$dy(t)=[-B(r(t))y(t)+A(r(t))g(y(t))]dt+\sigma(y(t),r(t))dw(t). \tag{4.6}$$

From (??), we can obtain that the solution of Eq.(??) is pth moment exponentially stable and the pth moment Lyapunov exponent is no more than $-\psi_M^{-1}$.

Remark 4.4 *Clearly, (??) means that the pth moment of the solution will tend to 0 exponentially fast. It follows from (??) that*

$$\limsup_{t\to\infty}\frac{1}{t}\log(E|y(t)|^p)<0.$$

However, by (??), we can also get this asymptotic property. That is, we have

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log(E|y(t)|^p) &\le \bar{\psi}-(\psi_M^{-1}+\bar{\psi})\Big(\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(s)]ds\Big)\\ &= \bar{\psi}-\theta(\psi_M^{-1}+\bar{\psi})<0\end{aligned}$$

as long as $\theta>\frac{\bar{\psi}}{\psi_M^{-1}+\bar{\psi}}$.

Lemma 4.5 *Let Assumptions 2.2, 2.3 hold and $p\in(0,1)$. Then, for any $t\ge 0$,*

$$E|x(t)|^p \le |x_0|^p e^{(\hat{\eta}+0.5\hat{k}^2)pt}, \tag{4.7}$$

$$E|x(t)-x(\delta_t)|^p \le |x_0|^p e^{(\hat{\eta}+0.5\hat{k}^2)pt}H_1(p,\tau), \tag{4.8}$$

where $\hat{\eta}=\max_{i\in S}\eta_i$, $\hat{k}=\max_{i\in S}k_i$, $H_1(p,\tau)=[2\tau(2\hat{C}+\hat{k}^2)]^{\frac{p}{2}}$, $\hat{C}=\max_{i\in S}(\|B(i)\|^2+|G|^2\|A(i)\|^2)$.

Proof. By the Itô formula, it is easy to show that, for $t\ge 0$,

$$\begin{aligned}|x(t)|^2 &= |x_0|^2+\int_0^t\Big\{2x^\top(s)[-B(r(s))x(s)+A(r(s))g(x(s))]+|\sigma(x(\delta_s),r(s))|^2I(s)\Big\}ds\\ &+ \int_0^t 2x^\top(s)\sigma(x(\delta_s),r(s))I(s)dw(s).\end{aligned}$$

By Assumptions 2.2 and 2.3, we get

$$\begin{aligned}E|x(t)|^2 &\le |x_0|^2+2\eta_i\int_0^t E|x(s)|^2ds+k_i^2\int_{t_0}^t E|x(\delta_s)|^2ds\\ &\le |x_0|^2+(2\max_{i\in S}\eta_i+\max_{i\in S}k_i^2)\int_0^t\sup_{0\le u\le s}E|x(s)|^2ds.\end{aligned}$$

Noting that the right-hand-side term of the above inequality is increasing in t, we hence have

$$\sup_{0\le u\le t}E|x(u)|^2 \le |x_0|^2+(2\hat{\eta}+\hat{k}^2)\int_0^t\sup_{0\le u\le s}E|x(s)|^2ds.$$

Consequently, the Gronwall inequality gives

$$\sup_{0\le u\le t} E|x(u)|^2 \;\le\; |x_0|^2 e^{(2\hat\eta+\hat k^2)t}.$$

By the Hölder inequality, we then have

$$E|x(t)|^p \;\le\; \Big(E|x(t)|^2\Big)^{\frac{p}{2}} \le |x_0|^p e^{(\hat\eta+0.5\hat k^2)pt}.$$

On the other hand, we can show that

$$\begin{aligned} E|x(t)-x(\delta_t)|^2 \;&\le\; 2E|\int_{\delta_t}^t [-B(r(s))x(s)+A(r(s))g(x(s))]ds|^2 \\ &+\; 2E|\int_{\delta_t}^t [\sigma(x(\delta_s),r(s))]ds|^2. \end{aligned}$$

By the Burkholder-Davis-Gundy inequality, Assumptions 2.2 and 2.3, we have

$$\begin{aligned} E|x(t)-x(\delta_t)|^2 \;&\le\; 4\tau\max_{i\in S}(||B(i)||^2+|G|^2||A(i)||^2)\int_{\delta_t}^t E|x(s)|^2 ds \\ &+\; 2\hat k^2\int_{\delta_t}^t E|x(\delta_s)|^2 ds \le 2\tau(2\hat C\tau+\hat k^2)e^{(2\hat\eta+\hat k^2)t}. \end{aligned}$$

Once again, by the Hölder inequality, we have

$$E|x(t)-x(\delta_t)|^p \;\le\; |x_0|^p e^{(\hat\eta+0.5\hat k^2)pt} H_1(p,\tau).$$

The proof is therefore complete. $\square$

Lemma 4.6 *Let Assumptions 2.2, 2.3 hold and $p\in(0,1)$. Then, for all $t\ge 0$,*

$$E|x(t)-y(t)|^p \;\le\; |x_0|^p H_2(p,\tau)[e^{(2\hat\eta+\hat k^2)t}-1]^{\frac{p}{2}} e^{(\hat\eta+\hat k^2)pt},$$

where $H_2(p,\tau)=\left[\frac{2\hat k^2 H_1(\tau,2)}{2\hat\eta+\hat k^2}\right]^{\frac{p}{2}}$.

Proof. By the Itô formula and Assumption 2.2, we can show that

$$\begin{aligned} & E|x(t)-y(t)|^2 \\ =\;& E\int_0^t \Big(2[x(s)-y(s)]^\top[-B(r(s))(x(s)-y(s)) \\ +\;& A(r(s))(g(x(s)-g(y(s)))] + |\sigma(x(\delta_s),r(s))-\sigma(y(s),r(s))|^2 I(s)\Big)ds \\ \le\;& 2\hat\eta\int_0^t E|x(s)-y(s)|^2 ds + E\int_0^t |\sigma(x(\delta_s),r(s))-\sigma(y(s),r(s))|^2 ds. \end{aligned} \tag{4.9}$$

By Assumption 2.3, we obtain that

$$\begin{aligned} & E\int_0^t |\sigma(x(\delta_s),r(s))-\sigma(y(s),r(s))|^2 ds \\ \leq\ & 2E\int_0^t |\sigma(x(\delta_s),r(s))-\sigma(x(s),r(s))|^2 ds + 2E\int_0^t |\sigma(x(s),r(s))-\sigma(y(s),r(s))|^2 ds \\ \leq\ & 2\hat{k}^2 E\int_0^t |x(\delta_s)-x(s)|^2 ds + 2\hat{k}^2 E\int_0^t |x(s)-y(s)|^2 ds. \end{aligned}$$

By Lemma **??**, we get

$$\begin{aligned} & E\int_0^t |\sigma(x(\delta_s),r(s))-\sigma(y(s),r(s))|^2 ds \\ \leq\ & 2\hat{k}^2 H_1(\tau,2)|x_0|^2 E\int_0^t e^{(2\hat{\eta}+\hat{k}^2)s} ds + 2\hat{k}^2 E\int_0^t |x(s)-y(s)|^2 ds \\ \leq\ & \frac{2\hat{k}^2 H_1(\tau,2)|x_0|^2}{2\hat{\eta}+\hat{k}^2}[e^{(2\hat{\eta}+\hat{k}^2)t}-1] + 2\hat{k}^2 E\int_0^t |x(s)-y(s)|^2 ds. \end{aligned} \tag{4.10}$$

Inserting (**??**) into (**??**), we have

$$\begin{aligned} E|x(t)-y(t)|^2 \ \leq\ & \frac{2\hat{k}^2 H_1(\tau,2)|x_0|^2}{2\hat{\eta}+\hat{k}^2}[e^{(2\hat{\eta}+\hat{k}^2)t}-1] \\ +\ & 2(\hat{\eta}+\hat{k}^2)E\int_0^t |x(s)-y(s)|^2 ds. \end{aligned}$$

Then, the Gronwall inequality implies that

$$E|x(t)-y(t)|^2 \ \leq\ \frac{2\hat{k}^2 H_1(\tau,2)|x_0|^2}{2\hat{\eta}+\hat{k}^2}[e^{(2\hat{\eta}+\hat{k}^2)t}-1]e^{2(\hat{\eta}+\hat{k}^2)t}.$$

By the Hölder inequality, we have

$$E|x(t)-y(t)|^p \ \leq\ |x_0|^p\Big[\frac{2\hat{k}^2 H_1(\tau,2)}{2\hat{\eta}+\hat{k}^2}\Big]^{\frac{p}{2}}[e^{(2\hat{\eta}+\hat{k}^2)t}-1]^{\frac{p}{2}}e^{(\hat{\eta}+\hat{k}^2)pt},$$

which is the required assertion. The proof is therefore complete. □

Lemma 4.7 *Let Assumptions 2.2, 2.3, 3.1 and* **??** *hold. Choose a free parameter $\varepsilon\in(0,1)$. Let $\tau^*\geq 0$ be the unique root to the equation*

$$H_2(p,\tau)[e^{(2\hat{\eta}+\hat{k}^2)(\tau+\frac{\log(\frac{\psi_M}{\psi_m\varepsilon})}{\gamma})}-1]^{\frac{p}{2}}e^{(\hat{\eta}+\hat{k}^2)p(\tau+\frac{\log(\frac{\psi_M}{\psi_m\varepsilon})}{\gamma})}=1-\varepsilon, \tag{4.11}$$

where $\gamma=(\psi_M^{-1}+\bar{\psi})\theta-\bar{\psi}$ and $H_2(p,\tau)$ has been given in lemma **??**. *If $\tau<\tau^*$, then there is a pair of positive integer $\tilde{k}$ and constant λ such that the solution of Eq.(2.2) satisfies*

$$E|x(i\tilde{k}\tau)|^p\leq|x_0|^p e^{-i\lambda\tilde{k}\tau},\quad \forall\ i=1,2,3,\cdots. \tag{4.12}$$

Proof. Let $\tilde{k}$ be a positive integer that is no less than

$$\frac{\log(\frac{\psi_M}{\psi_m \varepsilon})}{\gamma\tau} \le \tilde{k} < 1 + \frac{\log(\frac{\psi_M}{\psi_m \varepsilon})}{\gamma\tau}. \tag{4.13}$$

This implies $\frac{\psi_M}{\psi_m} e^{-\gamma\tilde{k}\tau} \le \varepsilon$. Write $y(\tilde{k}\tau; x_0, r_0) = y_{\tilde{k}}$. By Lemma **??**,

$$E|y_{\tilde{k}}|^p \le \frac{\psi_M}{\psi_m}|x_0|^p e^{-\gamma\tilde{k}\tau} \le \varepsilon|x_0|^p. \tag{4.14}$$

By the basic inequality $(a+b)^p \le a^p + b^p$ for any $a, b \ge 0$ and $0 < p < 1$, we can obtain that

$$E|x_{\tilde{k}}|^p \le E|y_{\tilde{k}}|^p + E|x_{\tilde{k}} - y_{\tilde{k}}|^p.$$

Using (**??**) and lemma **??**, we get

$$E|x_{\tilde{k}}|^p \le |x_0|^p \Big(\varepsilon + H_2(p,\tau)[e^{(2\hat{\eta}+\hat{k}^2)\tilde{k}\tau} - 1]^{\frac{p}{2}} e^{(\hat{\eta}+\hat{k}^2)p\tilde{k}\tau}\Big). \tag{4.15}$$

By (**??**), we have

$$\begin{aligned}
&\varepsilon + H_2(p,\tau)[e^{(2\hat{\eta}+\hat{k}^2)\tilde{k}\tau} - 1]^{\frac{p}{2}} e^{(\hat{\eta}+\hat{k}^2)p\tilde{k}\tau} \\
&< \varepsilon + H_2(p,\tau)[e^{(2\hat{\eta}+\hat{k}^2)(\tau+\frac{\log(\frac{\psi_M}{\psi_m\varepsilon})}{\gamma})} - 1]^{\frac{p}{2}} e^{(\hat{\eta}+\hat{k}^2)p(\tau+\frac{\log(\frac{\psi_M}{\psi_m\varepsilon})}{\gamma})} \le 1.
\end{aligned}$$

Thus, we may choose $\lambda > 0$ such that

$$\varepsilon + H_2(p,\tau)[e^{(2\hat{\eta}+\hat{k}^2)\tilde{k}\tau} - 1]^{\frac{p}{2}} e^{(\hat{\eta}+\hat{k}^2)p\tilde{k}\tau} = e^{-\lambda\tilde{k}\tau}.$$

It then follows from (**??**) that

$$E|x_{\tilde{k}}|^p \le |x_0|^p e^{-\lambda\tilde{k}\tau}.$$

Let us now consider the solution $x(t)$ of Eq.(2.2) on $t \ge \tilde{k}\tau$. Due to the time-homogeneous property of Eq.(2.2), we can get

$$E(|x_{2\tilde{k}}|^p|\mathcal{F}_{\tilde{k}\tau}) \le |x_{\tilde{k}}|^p e^{-\lambda\tilde{k}\tau}.$$

This implies $E|x_{2\tilde{k}}|^p \le E|x_{\tilde{k}}|^p e^{-\lambda\tilde{k}\tau} \le |x_0|^p e^{-2\lambda\tilde{k}\tau}$. Repeating this procedure, we have

$$E|x_{i\tilde{k}}|^p \le E|x_{(i-1)\tilde{k}}|^p e^{-\lambda\tilde{k}\tau} \le |x_0|^p e^{-i\lambda\tilde{k}\tau}, \quad i = 1, 2, \cdots.$$

The proof is therefore complete. $\square$

Theorem 4.8 *Let Assumptions 2.2, 2.3, 3.1 and* **??** *hold. Then, there is a positive number τ^* such that Eq.(2.2) is almost surely exponentially stable provided $\tau \le \tau^*$.*

Proof. Fix $\tau \in (0, \tau^*)$ and the initial data $x_0 \in R^n, r_0 \in S$. For simplicity, we write $x(t; 0, x_0, r_0) = x(t)$, $r(t; 0, r_0) = r(t)$. For any $t \ge 0$, we can find a unique integer i such that $t \in [i\tilde{k}\tau, (i+1)\tilde{k}\tau)$. By the time-homogeneous property of Eq.(2.2), we see from Lemma **??** that

$$\begin{aligned} E(|x(t)|^p|\mathcal{F}_{\tilde{k}\tau}) &\le |x_{i\tilde{k}}|^p e^{(\hat{\eta}+0.5\hat{k}^2)p(t-i\tilde{k}\tau)} \\ &\le |x_{i\tilde{k}}|^p e^{(\hat{\eta}+0.5\hat{k}^2)p\tilde{k}\tau}. \end{aligned} \tag{4.16}$$

This, together with Lemma **??**, implies

$$\begin{aligned} E|x(t)|^p| &\le E|x_{i\tilde{k}}|^p e^{(\hat{\eta}+0.5\hat{k}^2)p\tilde{k}\tau} \le |x_0|^p e^{-i\lambda\tilde{k}\tau} e^{(\hat{\eta}+0.5\hat{k}^2)p\tilde{k}\tau} \\ &\le C(p,\tau)|x_0|^p e^{-(i+1)\lambda\tilde{k}\tau} \le C(p,\tau)|x_0|^p e^{-\lambda t}, \end{aligned}$$

where $C(p,\tau) = e^{\lambda\tilde{k}\tau} e^{(\hat{\eta}+0.5\hat{k}^2)p\tilde{k}\tau}$. By the basic inequality, Assumptions 2.2 and 2.3, it is easy to show that

$$\begin{aligned} E\Big(\sup_{i\tilde{k}\tau\le s\le t}|x(s)|^2\Big) &\le 3E|x(i\tilde{k}\tau)|^2 + 3E\Big(\sup_{i\tilde{k}\tau\le s\le t}\Big|\int_{i\tilde{k}\tau}^{s}[-B(r(v))x(v) + A(r(v))g(x(v))]dv\Big|^2\Big) \\ &+ 3E\Big(\sup_{i\tilde{k}\tau\le s\le t}\Big|\int_{i\tilde{k}\tau}^{s}\sigma(x(\delta_v),(r(v))I(v)dw(v)\Big|^2\Big) \\ &\le 3E|x(i\tilde{k}\tau)|^2 + 3\tilde{k}\hat{C}\tau\int_{i\tilde{k}\tau}^{t}E|x(s)|^2ds + 12\hat{k}^2\int_{i\tilde{k}\tau}^{t}E|x(\delta_s)|^2ds \\ &\le 3E|x(i\tilde{k}\tau)|^2 + (3\tilde{k}\hat{C}\tau + 12\hat{k}^2)\int_{i\tilde{k}\tau}^{t}E\sup_{i\tilde{k}\tau\le v\le s}|x(v)|^2ds. \end{aligned}$$

Then the Gronwall inequality implies that

$$E\Big(\sup_{i\tilde{k}\tau\le t\le(i+1)\tilde{k}\tau}|x(t)|^2\Big) \le 3E|x(i\tilde{k}\tau)|^2 e^{(3\tilde{k}\hat{C}\tau+12\hat{k}^2)\tilde{k}\tau}.$$

By the Hölder inequality, we have

$$\begin{aligned} E\Big(\sup_{i\tilde{k}\tau\le t\le(i+1)\tilde{k}\tau}|x(t)|^p\Big) &\le E|x(i\tilde{k}\tau)|^p\Big(3e^{(3\tilde{k}\hat{C}\tau+12\hat{k}^2)\tilde{k}\tau}\Big)^{\frac{p}{2}} \\ &\le \Big(3e^{(3\tilde{k}\hat{C}\tau+12\hat{k}^2)\tilde{k}\tau}\Big)^{\frac{p}{2}}|x_0|^p e^{-i\lambda\tilde{k}\tau} \end{aligned} \tag{4.17}$$

for all $i \geq 1$. Using the Markov inequality and (??), we get

$$P\Big(\sup_{i\tilde{k}\tau \leq t \leq (i+1)\tilde{k}\tau} |x(t)|^p \geq e^{-0.5i\lambda\tilde{k}\tau}\Big) \leq (3e^{(3\tilde{k}\hat{C}\tau+12\hat{k}^2)\tilde{k}\tau}\Big)^{\frac{p}{2}}|x_0|^p e^{-0.5i\lambda\tilde{k}\tau}$$

for all $i \geq 1$. By the Borel-Cantelli lemma, we can obtain that for almost all $\omega \in \Omega$, there exists an integer $i_0 = i_0(\omega)$ such that $\sup_{i\tilde{k}\tau \leq t \leq (i+1)\tilde{k}\tau} |x(t)|^p < e^{-0.5i\lambda\tilde{k}\tau}$ for any $i > i_0(\omega)$. This implies that $\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) < -\frac{\lambda}{2p}$ for almost all $\omega \in \Omega$. The proof is therefore complete. □

Remark 4.9 *In fact, Lemmas ??, ?? and Theorem ?? enable us to design the periodically intermittent feedback control based on discrete-time state observations for stochastic stabilization problem (2.2) in two steps.*

Step 1. For an unstable hybrid system

$$dx(t)/dt = -B(r(t))x(t) + A(r(t))g(x(t)), \tag{4.18}$$

we design the control function $\sigma : R^n \times S \to R^{n\times m}$ for Eq.(3.2) to be pth moment exponentially stable.

Step 2. Find the unique root $\tau^* > 0$ to Eq.(??)

$$H_2(p,\tau)[e^{(2\hat{\eta}+\hat{k}^2)(\tau+\frac{\log(\frac{\psi_M}{\psi_m\varepsilon})}{\gamma})} - 1]^{\frac{p}{2}} e^{(\hat{\eta}+\hat{k}^2)p(\tau+\frac{\log(\frac{\psi_M}{\psi_m\varepsilon})}{\gamma})} = 1-\varepsilon$$

of Lemma ?? and make sure $\tau < \tau^*$. Then the periodically intermittent feedback control based on discrete-time state observations will stabilize Eq.(??) in the sense of the almost sure exponential stability.

5 Two examples

Example 5.1 *Let $w(t)$ be a scalar Brownian motion and $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with the generator*

$$\Gamma = (\gamma_{ij})_{2\times2=} \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix}.$$

Consider the scalar hybrid neural networks

$$\frac{dx(t)}{dt} = -b(r(t))x(t) + a(r(t))g(x(t)), \tag{5.1}$$

where

$$b(1) = -0.2, b(2) = -0.3, a(1) = 0.1, a(2) = 0.25, g(x) = ReLu(x) = max\{x, 0\}.$$

Obviously, the Markov chain $r(t)$ *has the stationary distribution* $\pi = (\pi_1, \pi_2) = (\frac{4}{5}, \frac{1}{5})$. *Then, we have that*

$$\sum_{i=1}^{2} \pi_i \eta_i = \sum_{i=1}^{2} \pi_i[-\lambda_{min}(B(i)) + |G| \|A(i)\|] = 0.35 > 0.$$

*It is obvious that hybrid neural networks (**??**) is unstable.*

*To make this given hybrid neural networks (**??**) stable, we use the periodic intermittent control. Consider* $\sigma(r(t))x(t)I(t)dw(t)$ *as the stochastic perturbation, then the intermittently scalar hybrid stochastic neural networks can be described by*

$$dx(t) = [-b(r(t))x(t) + a(r(t))g(x(t))]dt + \sigma(r(t))x(t)I(t)dw(t), \tag{5.2}$$

where $\sigma(1) = 1, \sigma(2) = 1.2$. *We assume* $w(t)$ *and* $r(t)$ *are assumed to be independent. It is easy to see that* $k_1 = \beta_1 = 1$ *and* $k_2 = \beta_2 = 1.2$. *Then we get that*

$$\sum_{i=1}^{2} \pi_i[\eta_i + \theta(0.5k_i^2 - \beta_i^2)] = 0.35 - 0.544\theta.$$

By Theorem 3.2, we can conclude that if we choose $\theta \in (0.6434, 1)$, *then intermittently hybrid stochastic neural networks (**??**) is almost surely exponentially stable. That is, hybrid neural networks (**??**) can be stabilized by intermittent stochastic perturbation* $\sigma(r(t))x(t)I(t)dw(t)$ *with* $\theta \in (0.6434, 1)$. *Given the initial value* $x(0) = 2$ *and* $r(0) = 1$, *Figure 1 shows that the intermittently hybrid stochastic neural networks (**??**) is stable when* $\theta = 0.9$. □

Remark 5.2 *If* $\theta = 1$, *we can have* $\sum_{i=1}^{2} \pi_i[\eta_i + \theta(0.5k_i^2 - \beta_i^2)] = -0.143 < 0$. *This implies that hybrid neural networks (**??**) can be stabilized by continuous time stochastic perturbation* $\sigma(r(t))x(t)dw(t)$. *In order to reduce the control cost, the intermittent stochastic control is naturally selected to stabilize the above neural network.*

Remark 5.3 *In fact, whether the controlled stochastic neural networks (**??**) is stable depends not only on parameter* θ, *but also on the intensity of noise. If we choose* $\sigma(1) = 0.5, \sigma(2) = 1$, *we get* $\sum_{i=1}^{2} \pi_i[\eta_i + \theta(0.5k_i^2 - \beta_i^2)] = 0.35 - 0.2\theta$. *In this case, if* $\theta = 1$, *then* $\sum_{i=1}^{2} \pi_i[\eta_i + \theta(0.5k_i^2 - \beta_i^2)] = 0.15$ *is not less than zero. In other words, it is difficult to stabilize the neural network (**??**) even if the continuous feedback control strategy is adopted.*

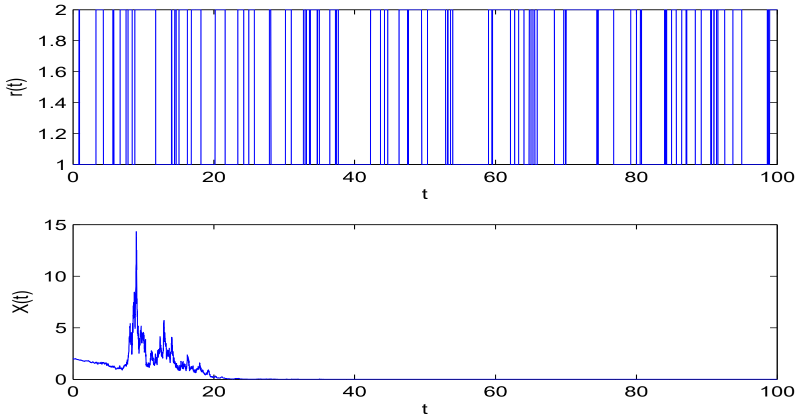

Figure 1: The sample paths of intermittently hybrid stochastic neural networks (**??**).

Example 5.4 *Let $w(t)$ be a scalar Brownian motion and $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with the generator*

$$\Gamma = (\gamma_{ij})_{2\times 2=} \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}.$$

Consider the following two dimensional hybrid neural networks

$$\frac{dx(t)}{dt} = -B(r(t))x(t) + A(r(t))g(x(t)), \tag{5.3}$$

where

$$B(1) = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.8 \end{pmatrix}, \quad B(2) = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.4 \end{pmatrix}, \; A(1) = \begin{pmatrix} 0.8 & 0.4 \\ -0.6 & -0.3 \end{pmatrix},$$

and

$$A(2) = \begin{pmatrix} 1 & 0.5 \\ -0.5 & -0.2 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0.25(|x_1+1| - |x_1-1|) \\ 0.25(|x_2+1| - |x_2-1|) \end{pmatrix}.$$

Simple computations show that

$$\lambda_{min}(B(1)) = 0.5, \quad \lambda_{min}(B(2)) = 0.4, \quad \|A(1)\| = 1.118, \quad \|A(2)\| = 1.24.$$

Meantime, we obtain that the Markov chain has the stationary distribution $\pi = (\pi_1, \pi_2) = (\frac{1}{3}, \frac{2}{3})$. Then, we have that

$$\sum_{i=1}^{2} \pi_i \eta_i = \sum_{i=1}^{2} \pi_i[-\lambda_{min}(B(i)) + |G|\|A(i)\|] = 0.166333 > 0.$$

*It is obvious that hybrid neural networks (**??**) is unstable.*

*Now, we will use the intermittent feedback control based on discrete-time state observations to stablize the hybrid neural networks (**??**)*

$$dx(t) \quad = [-B(r(t))x(t) + A(r(t))g(x(t))]dt + \sigma(r(t))x(\lfloor t/\tau \rfloor \tau)I(t)dw(t), \tag{5.4}$$

where

$$\sigma(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma(2) = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}.$$

Choose $p = 0.5$*, then* $\rho_1(0.5) = 0.0955$ *and* $\rho_2(0.5) = 0.1713$*. The matrix* $\mathcal{A}_{0.5}$ *defined in (**??**) becomes*

$$\mathcal{A}_{0.5} = \begin{bmatrix} 2.0955 & -2 \\ -1 & 1.1713 \end{bmatrix}$$

which is a nonsingular M-matrix. Then, we can determine $\psi_1 = 6.9798, \psi_2 = 6.8130$ *and hence,* $\psi_M^{-1} = 0.1433$ *and* $\bar{\psi} = 0.1345$*. By Lemma **??**, we can conclude that if* $\theta \in (0.4842, 1)$*, then intermittently hybrid stochastic neural networks*

$$dx(t) \quad = [-B(r(t))x(t) + A(r(t))g(x(t))]dt + \sigma(r(t))x(t)I(t)dw(t) \tag{5.5}$$

has the property that

$$\limsup_{t\to\infty} \frac{1}{t} \log E|x(t)|^p \le 0.1345 - 0.2778\theta < 0.$$

We further choose $\theta = 0.75$*,* $\varepsilon = 0.9$*, then Eq.(**??**) becomes*

$$H_2(0.5, \tau)[e^{2.69(\tau+0.762)} - 1]^{0.25} \times e^{1.235(\tau+0.762)} = 0.1$$

which has the unique positive root $\tau^* = 3.08 \times 10^{-8}$ *(which is about 0.97 seconds if the time unit is of year). By Theorem **??**, we can conclude that Eq.(**??**) is almost surely exponentially stable provided* $\tau < 3.08 \times 10^{-8}$*. Given the initial value* $x_1(0) = 2, x_2(0) = 3$ *and* $r(0) = 2$*, Figure 2 showed hybrid neural networks (**??**) can be stabilized by intermittently feedback control based on discrete-time state observations when* $\theta = 0.75$*.* $\square$

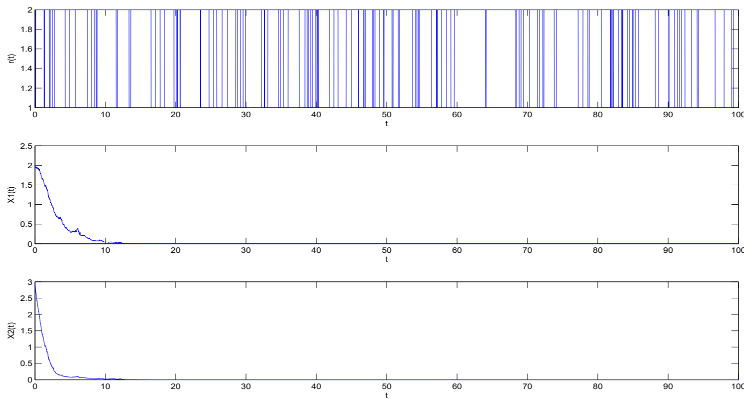

Figure 2: The sample paths of intermittently hybrid stochastic neural networks (**??**).

6 Conclusion

This paper is devoted to the stablization of hybrid neural networks by intermittent stochastic feedback control based on discrete-time observations. The exponential martingale inequality and ergodic property of the Markov chain are used to establish sufficient stability criterion on hybrid neural networks by intermittent control based on continuous-time state observations. Meantime, by M-matrix theory and comparison principle, we obtain that hybrid neural networks can be stabilized by intermittent control based on discrete-time state observations as long as $\tau < \tau^*$.

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