Discrete Fragmentation Equations with Time-Dependent Coefficients

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Dedicated to Jerry Goldstein on the occasion of his 80th birthday

Abstract

We examine an infinite, linear system of ordinary differential equations that models the evolution of fragmenting clusters, where each cluster is assumed to be composed of identical units. In contrast to previous investigations into such discrete-size fragmentation models, we allow the fragmentation coefficients to vary with time. By formulating the initial-value problem for the system as a non-autonomous abstract Cauchy problem, posed in an appropriately weighted ℓ^1 space, and then applying results from the theory of evolution families, we prove the existence and uniqueness of physically relevant, classical solutions for suitably constrained coefficients.

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1 Introduction

Fragmentation is a commonly observed phenomenon in various physical processes such as polymer degradation, liquid droplet breakup, and the crushing and grinding of rocks. Deterministic models of fragmentation are usually based on the simplifying assumption that the fragmenting objects can be distinguished by means of a single 'size' variable, such as mass. When this size variable is permitted to take any positive value, the resulting model describing the continuous time evolution of the system of fragmenting objects typically takes the form of a linear integro-differential equation which is referred to as the continuous (size) fragmentation equation; see [7, §2.2.2] and also the discussion in Section 5. However, when describing the fragmentation of clusters that are assumed to be comprised of a finite number of identical fundamental particles, it is clearly more appropriate to use a discrete-size variable. It is the discrete-size case that we examine in this paper, and, for convenience, we adopt the polymer-based terminology that is frequently used when dealing with discrete-size fragmentation. Consequently, the fundamental particle is referred to as a monomer, and an n-mer is then a cluster consisting of n monomers. By suitable scaling, a monomer can be assumed to have unit mass in which case an n-mer has mass n.

In terms of the number density, $u_n(t)$, of n-mers at time t, the evolution of the system of fragmenting clusters is described by the infinite-system of linear ordinary differential equations (ODEs)

$$\begin{aligned} u_n'(t) &= -a_n(t)u_n(t) + \sum_{j=n+1}^{\infty} a_j(t)b_{n,j}(t)u_j(t), \quad t \in (0,T],\ n \in \mathbb{N}; \\ u_n(0) &= \mathring{u}_n, \quad n \in \mathbb{N}, \end{aligned} \tag{1.1}$$

where $T > 0$, and $\mathring{u}$ is an initial density sequence. At each time $t \in [0,T]$, the coefficients $a_n(t)$ and $b_{n,j}(t)$ represent, respectively, the rate at which n-mers are lost due to fragmentation (when $n \geq 2$), and the average number of n-mers that are produced when a larger j-mer fragments. Throughout, we assume that the fragmentation coefficients satisfy the following natural physical constraints:

(A1) $a_n(t) \geq 0$, $\forall t \in [0,T]$ and $n \in \mathbb{N}$,

(A2) $b_{n,j}(t) \geq 0$, $\forall t \in [0,T]$ and $n, j \in \mathbb{N}$, with $b_{n,j}(t) = 0$ if $j \leq n$.

As monomers cannot fragment to produce smaller clusters, the case $a_1(t) > 0$ signifies a depletion in the number of monomers due to some other mechanism; see [8] and [20].

The total mass of all clusters at time $t \in [0,T]$ is given by the first moment, $M_1(u(t))$, of the sequence of densities $u(t) = (u_n(t))_{n=1}^{\infty}$, where

$$M_1\big(u(t)\big) := \sum_{n=1}^{\infty} n u_n(t). \tag{1.2}$$

On representing the total mass of daughter clusters produced from the fragmentation of a j-mer at time t by

$$\sum_{n=1}^{j-1} n b_{n,j}(t) = \big(1 - \lambda_j(t)\big) j, \qquad j = 2, 3, \ldots, \tag{1.3}$$

where each λ_j is a real-valued function, a formal calculation establishes that if $u(t) = (u_n(t))_{n=1}^{\infty}$ is a solution of (1.1), then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(M_1\big(u(t)\big)\Big) = -a_1(t)u_1(t) - \sum_{j=2}^{\infty} j\lambda_j(t)a_j(t)u_j(t), \quad t \in (0,T]. \tag{1.4}$$

The expression in (1.4) gives the rate at which mass may be lost or gained from the system of fragmenting clusters, and also shows, at least formally, that there is no change in the total mass during the fragmentation process when $a_1(t) = 0$ and $\lambda_j(t) = 0$ for all $j = 2, 3, \ldots$, and $t \in [0,T]$. Let us note that we do not impose any sign restrictions on $\lambda_j(t)$, i.e. we also allow mass to be gained in a fragmentation event, in which case $\lambda_j(t) < 0$.

In contrast to the continuous fragmentation equation, where several investigations, such as [15] and [18], have dealt with time-dependent coefficients, previous investigations into (1.1) appear to have considered only the case when all fragmentation coefficients are time-independent. The approach used in [3,14,20] to analyse the constant-coefficient fragmentation system is to formulate the initial-value problem as an autonomous abstract Cauchy problem (ACP), posed in an appropriate Banach lattice. Conditions on the coefficients are then determined under which the ACP has a unique classical solution that can be expressed in terms of a positive C_0-semigroup of contractions (i.e. a substochastic semigroup), usually referred to as the fragmentation semigroup.

In a recent paper [13], we also applied this semigroup-based strategy, working within the framework of weighted ℓ^1 spaces of the form

$$\ell_w^1 := \left\{ f = (f_n)_{n=1}^\infty : f_n \in \mathbb{R}, \forall n \in \mathbb{N}, \text{ and } \|f\|_w := \sum_{n=1}^\infty w_n |f_n| < \infty \right\}, \quad (1.5)$$

where $w_n > 0$, $n = 1, 2, \ldots$. Each space ℓ_w^1 is a real Banach lattice with positive cone

$$(\ell_w^1)_+ := \{ f = (f_n)_{n=1}^\infty \in \ell_w^1 : f_n \geq 0, \forall n \in \mathbb{N} \}. \quad (1.6)$$

Moreover, since $\|\cdot\|_w$ is additive on $(\ell_w^1)_+$, ℓ_w^1 is an *AL*-space; see [4, Definition 2.56]. By allowing general weights, $w = (w_n)_{n=1}^\infty$ with $w_n > 0$ for all $n \in \mathbb{N}$, and not just the specific cases of $w_n = n$ and $w_n = n^p$, $p > 1$, used, respectively, in [14, 20] and [3], we were able to establish the following results.

Firstly, given *any* fragmentation coefficients satisfying the time-independent versions of (A1) and (A2), it transpires that it is *always* possible to determine a weight which will guarantee the existence of a substochastic fragmentation semigroup on ℓ_w^1. More precisely, from [13, Theorem 3.4], there exists a substochastic fragmentation semigroup on ℓ_w^1 whenever $w_n \geq n$ for all $n \in \mathbb{N}$, and there exists $\kappa \in (0, 1]$ such that

$$\sum_{n=1}^{j-1} w_n b_{n,j} \leq \kappa w_j, \ \forall j = 2, 3, \ldots. \quad (1.7)$$

Secondly, under the more restrictive condition that $0 < \kappa < 1$, it is shown [13, Theorem 5.2] that the fragmentation semigroup is analytic when defined on the complexification of ℓ_w^1. Details on the process of complexification of a real Banach lattice can be found in [4, §2.2.5]. For the particular case of ℓ_w^1, the process leads simply to the complex Banach lattice defined as in (1.5) but now for complex sequences $f = (f_n)_{n=1}^\infty$, $f_n \in \mathbb{C}$. The partial order in this complex Banach lattice is given by

$$f = (f_n)_{n=1}^\infty \leq g = (g_n)_{n=1}^\infty \iff \operatorname{Re} f_n \leq \operatorname{Re} g_n \text{ and } \operatorname{Im} f_n = \operatorname{Im} g_n, \ \forall n \in \mathbb{N},$$

and this ensures that the corresponding positive cone is also given by (1.6). Clearly, for any given coefficients $b_{n,j}$, a sequence $(w_n)_{n=1}^\infty$ can be constructed iteratively such that $w_n \geq n$ and (1.7) is satisfied for some $\kappa \in (0, 1)$; see [13, Theorem 5.5].

Our aim in the current paper is to exploit the above result on the analyticity of the fragmentation semigroup in weighted ℓ^1 spaces to determine sufficient conditions under which the non-autonomous fragmentation system (1.1) is well posed. In keeping with the semigroup approach used for the autonomous system, the strategy we adopt involves the application of the theory of evolution families, an account of which can be found in the seminal books on semigroups of operators by Goldstein [11] and Pazy [19]. Such families have been employed in the analysis of a variety of linear, non-autonomous evolution equations, such as the time-dependent coefficient versions of the continuous integro-differential fragmentation equation [15] and the Black–Scholes equation [10].

In Section 2, we give some prerequisite information on non-autonomous ACPs and strongly continuous evolution families, and then express (1.1) in the form of a non-autonomous ACP posed in an ℓ_w^1 space. In Section 3, this abstract formulation of (1.1) is shown to be well posed when the weight w and the fragmentation coefficients are suitably constrained. In Section 4 we consider the asymptotic behaviour of solutions as $t \to \infty$. In particular, under the assumption of mass conservation we prove that solutions converge to a monomeric state with an explicit exponential rate. Finally, in Section 5, some potential extensions to the work presented here are discussed.

2 Preliminaries and Abstract Formulation

To enable an approach based on the theory of evolution families to be applied to the fragmentation system, the initial-value problem (1.1) must first be recast as a non-autonomous ACP. It turns out to be useful to allow arbitrary initial times. So, for fixed $T > 0$ we consider the family of initial-value problems

$$u'(t) = G(t)u(t), \quad t \in (s,T]; \qquad u(s) = \mathring{u}, \tag{2.1}$$

where $s \in [0,T)$. Moreover, for each $t \in [0,T]$, $G(t)$ is a linear operator that maps $D(G(t)) \subseteq X$ into X, where X is a Banach space, and $\mathring{u} \in X$ is an initial value. The aim is to determine conditions on $G(t)$ which ensure that (2.1) has a unique solution $u : [s,T] \to X$ that can be expressed in terms of a (strongly continuous) evolution family (or evolution system), which, from [19, Definition 5.5.3], is a two-parameter family of bounded linear operators $(U(t,s))_{0\le s\le t\le T}$, on X satisfying

(EF1) $U(s,s) = I$, $U(t,r)U(r,s) = U(t,s)$ for $0 \le s \le r \le t \le T$,

(EF2) $(t,s) \mapsto U(t,s)$ is strongly continuous for $0 \le s \le t \le T$.

A function u is a *classical solution* of (2.1) if $u \in C([s,T],X) \cap C^1((s,T],X)$, $u(t) \in D(G(t))$ for all $t \in (s,T]$, and (2.1) is satisfied. The following result, which is a slightly modified version of [19, Theorem 5.6.8] (see also [19, Theorem 5.6.1]), gives sufficient conditions on the operators $G(t)$, $0 \le t \le T$, for the existence of a unique classical solution to (2.1), and also highlights the key role played by evolution families and analytic semigroups.

Theorem 2.1. *Let X be a complex Banach space. For each $t \in [0,T]$, where $T > 0$, let $G(t)$ be the generator of an analytic semigroup, $(S_t(\tau))_{\tau\ge0}$, on X. Assume that the following conditions are satisfied.*

(P1) *The domain $D(G(t)) =: \mathcal{D}$ of $G(t)$ is independent of $t \in [0,T]$.*

(P2) *For $t \in [0,T]$, the resolvent $R(\lambda, G(t)) := (\lambda I - G(t))^{-1}$ exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge 0$ and there is a constant M such that*

$$\big\|R(\lambda, G(t))\big\| \le \frac{M}{|\lambda|+1} \qquad \text{for all } t \in [0,T], \text{ and } \lambda \in \mathbb{C} : \operatorname{Re}\lambda \ge 0.$$

(P3) *There exist constants L and $\sigma \in (0,1]$ such that*

$$\big\|\big(G(t) - G(s)\big)G(\tau)^{-1}\big\| \le L|t-s|^{\sigma} \qquad \text{for } s,t,\tau \in [0,T].$$

Then there exists an evolution family $(U(t,s))_{0\le s\le t\le T}$ *such that, for every* $s \in [0,T)$ *and* $\mathring{u} \in X$*, the non-autonomous ACP* (2.1) *has a unique classical solution which is given by* $u(t) = U(t,s)\mathring{u}$.

Note that assumption (P2) implies that the semigroup $(S_t(\tau))_{\tau\ge 0}$ is analytic. We included it explicitly in the formulation to emphasise the importance of analyticity. Further, note that $U(t,s)$ maps X into $\mathcal{D}$ if $s < t$ because $u(t) = U(t,s)\mathring{u}$ is a classical solution of (2.1).

The abstract formulation of the fragmentation system (1.1) is posed in the complex Banach lattice version of the space ℓ_w^1 from (1.5), which will also be denoted by ℓ_w^1. The weight $w = (w_n)_{n=1}^{\infty}$ is assumed to satisfy

(A3) $w_n \ge n$ for all $n \in \mathbb{N}$,

(A4) $\exists \kappa \in (0,1) : \sum_{n=1}^{j-1} w_n b_{n,j}(t) \le \kappa w_j$ for all $t \in [0,T]$ and $j = 2,3,\ldots$.

It follows from (A3) that ℓ_w^1 is continuously embedded in the space

$$X_{[1]} := \left\{ f = (f_n)_{n=1}^{\infty} : f_n \in \mathbb{R},\ \forall n \in \mathbb{N},\ \text{and } \|f\|_{[1]} := \sum_{n=1}^{\infty} n|f_n| < \infty \right\},$$

which is often referred to as the first moment space since

$$\|f\|_{[1]} = M_1(f) \qquad \text{for all } f \in (X_{[1]})_+.$$

Moreover, on defining the bounded linear functional ϕ_w on ℓ_w^1 by

$$\phi_w\big((f_n)_{n=1}^{\infty}\big) := \sum_{n=1}^{\infty} w_n f_n, \qquad (f_n)_{n=1}^{\infty} \in \ell_w^1, \tag{2.2}$$

it is clear that

$$\phi_w(f) = \|f\|_w \qquad \text{for all } f \in \big(\ell_w^1\big)_+,$$

and so ϕ_w coincides with the norm $\|\cdot\|_w$ on the positive cone.

Remark 2.2.

(i) If $b_{n,j}$ is bounded on $[0,T]$ for all $n,j \in \mathbb{N}$, $n < j$, then one can construct a sequence $(w_n)_{n=1}^{\infty}$ iteratively such that (A3) and (A4) are satisfied.

(ii) It can be shown in a similar way as in [13, Theorem 5.5 and Lemma 5.4] that, when
$$\sum_{n=1}^{j-1} n b_{n,j}(t) \le j, \qquad j = 2,3,\ldots, \tag{2.3}$$
(or equivalently $\lambda_j(t) \ge 0$ in (1.3)), then one can choose the sequence $(w_n)_{n=1}^{\infty}$ such that it grows at most exponentially. The condition (2.3) means that the total mass does not grow in each fragmentation event.

Example 2.3. Consider the case when

$$b_{n,j}(t) \equiv b_{n,j} = \beta_n\,\zeta_j, \qquad n,j \in \mathbb{N},\ n < j,\ t \in [0,T],$$

where $\beta_n = n^\nu$ with $\nu \geq -1$. Under the assumption that mass is conserved during each fragmentation event (i.e. $\sum_{n=1}^{j-1} n b_{n,j} = j$ for $j = 2, 3, \ldots$), we then obtain

$$\zeta_j = \frac{j}{\sum\limits_{l=1}^{j-1} l^{\nu+1}}.$$

In this case, we can show that (A4) is satisfied with $w_n = n^p$ for some $p \geq 1$. To this end, we consider

$$\frac{1}{w_j}\sum_{n=1}^{j-1} w_n b_{n,j} = \frac{1}{j^p} \cdot \frac{j}{\sum\limits_{l=1}^{j-1} l^{\nu+1}} \sum_{n=1}^{j-1} n^{p+\nu}$$

for $j = 2, 3, \ldots$. We can estimate the two sums with integrals:

$$\begin{aligned}
\sum_{n=1}^{j-1} n^{p+\nu} &\leq \int_1^j x^{p+\nu}\,\mathrm{d}x \leq \frac{1}{p+\nu+1} j^{p+\nu+1}, \\
\sum_{l=1}^{j-1} l^{\nu+1} &\geq \int_0^{j-1} x^{\nu+1}\,\mathrm{d}x = \frac{1}{\nu+2}(j-1)^{\nu+2} \geq \frac{1}{\nu+2}\Big(\frac{j}{2}\Big)^{\nu+2},
\end{aligned}$$

which yields

$$\frac{1}{w_j}\sum_{n=1}^{j-1} w_n b_{n,j} \leq \frac{(\nu+2)2^{\nu+2}}{p+\nu+1}. \tag{2.4}$$

For fixed $\nu \geq -1$ we can choose $p \geq 1$ such that the right-hand side of (2.4) is strictly less than 1, which shows that (A4) is satisfied. $\Diamond$

Remark 2.4. It is worth noting that analogous separable coefficients which take the form $b(x, y) = \beta(x)\,\zeta(y)$, $0 < x < y$, have been considered in investigations into the continuous, mass-conserving, autonomous fragmentation equation. In particular, the case $\beta(x) = x^\nu$ and

$$\zeta(y) = (\nu+2)y^{-\nu-1} = \frac{y}{\int_0^y y^{\nu+1}\,\mathrm{d}y}$$

is examined in [6], and results are obtained on the analyticity of associated fragmentation semigroups defined on the weighted spaces $L^1(\mathbb{R}_+, (1 + x^m)\,\mathrm{d}x)$; see [6, Theorems 2.1 and 2.3]. $\Diamond$

Motivated by the terms in (1.1), we introduce, for each $t \in [0, T]$, the formal expressions

$$\mathcal{A}(t) : (f_n)_{n=1}^\infty \mapsto \big(-a_n(t) f_n\big)_{n=1}^\infty$$

and

$$\mathcal{B}(t) : (f_n)_{n=1}^\infty \mapsto \left(\sum_{j=n+1}^\infty a_j(t) b_{n,j}(t) f_j\right)_{n=1}^\infty.$$

Operator realisations, $A(t)$ and $B(t)$, of $\mathcal{A}(t)$ and $\mathcal{B}(t)$ respectively, are then defined in ℓ_w^1 by

$$A(t)f = \mathcal{A}(t)f, \qquad D\big(A(t)\big) = \big\{f \in \ell_w^1 : \mathcal{A}(t)f \in \ell_w^1\big\}, \tag{2.5}$$

$$B(t)f = \mathcal{B}(t)f, \qquad D\big(B(t)\big) = D\big(A(t)\big). \tag{2.6}$$

That $B(t)$ is well defined on $D(A(t))$, for each $t \in [0,T]$, can be seen as follows. Assumption (A4) implies that, for $f = (f_n)_{n=1}^\infty \in (D(A(t)))_+$, we have

$$\begin{aligned}\phi_w\big(\mathcal{B}(t)f\big) &= \sum_{n=1}^\infty w_n \sum_{j=n+1}^\infty a_j(t)b_{n,j}(t)f_j = \sum_{j=2}^\infty \left(\sum_{n=1}^{j-1} w_n b_{n,j}(t)\right) a_j(t)f_j \\ &\le \sum_{j=2}^\infty \kappa w_j a_j(t) f_j \le -\kappa \sum_{j=1}^\infty w_j\big(-a_j(t)\big)f_j = -\kappa\phi_w\big(A(t)f\big);\end{aligned} \tag{2.7}$$

the change in the order of summation in the calculation above is justified since each term is positive. Now let $f = (f_n)_{n=1}^\infty \in D(A(t))$. Then $|f| = (|f_n|)_{n=1}^\infty \in D(A(t))_+$, and we obtain from (2.7) that

$$\begin{aligned}\big\|\mathcal{B}(t)f\big\|_w &= \sum_{n=1}^\infty w_n \left| \sum_{j=n+1}^\infty a_j(t)b_{n,j}(t)f_j \right| \le \phi_w\big(\mathcal{B}(t)|f|\big) \\ &\le -\kappa\phi_w\big(A(t)|f|\big) = -\kappa \sum_{n=1}^\infty w_n\big(-a_n(t)\big)|f_n| = \kappa\big\|A(t)f\big\|_w < \infty,\end{aligned}$$

which yields $f \in D(B(t))$ and

$$\|B(t)f\|_w \le \kappa\|A(t)f\|_w \qquad \text{for all } f \in D\big(A(t)\big). \tag{2.8}$$

On setting $G(t) = A(t) + B(t)$, we now write (1.1) as the non-autonomous ACP

$$u'(t) = G(t)u(t), \quad s < t \le T; \qquad u(s) = \mathring{u} \tag{2.9}$$

with $\mathring{u} \in \ell_w^1$.

The following results on the operators $G(t)$, $t \in [0,T]$, will be required in the next section.

Lemma 2.5. *Let assumptions* (A1)–(A4) *be satisfied. For each* $t \in [0,T]$,

(a) *the operator* $G(t)$ *is the generator of an analytic, substochastic* C_0*-semigroup,* $(S_t(\tau))_{\tau\ge 0}$*, on* ℓ_w^1*;*

(b) *for* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > 0$*, the resolvent operator* $R(\lambda, G(t))$ *can be factorised as*

$$R\big(\lambda, G(t)\big) = R\big(\lambda, A(t)\big)\Big[I - B(t)R\big(\lambda, A(t)\big)\Big]^{-1} \tag{2.10}$$

where the factors on the right-hand side satisfy

$$\big\|R\big(\lambda, A(t)\big)\big\| \le \frac{1}{|\lambda|}, \qquad \left\|\Big[I - B(t)R\big(\lambda, A(t)\big)\Big]^{-1}\right\| \le \frac{1}{1-\kappa}. \tag{2.11}$$

Proof. Part (a) is an immediate consequence of [13, Theorem 5.2].

For (b) let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$. Since $a_n(t) \ge 0$, we have $\lambda \in \rho(A(t))$ and, for $f \in \ell_w^1$,

$$\big\|R\big(\lambda, A(t)\big)f\big\|_w = \sum_{n=1}^\infty w_n \frac{1}{|\lambda + a_n(t)|}|f_n| \le \frac{1}{|\lambda|}\|f\|_w,$$

which yields the first inequality in (2.11). Observe that $D(B(t)) = D(A(t))$ and hence

$$\begin{aligned}\lambda I - G(t) = \lambda I - A(t) - B(t) &= \Big(I - B(t)\big(\lambda I - A(t)\big)^{-1}\Big)\big(\lambda I - A(t)\big) \\ &= \Big(I - B(t)R\big(\lambda, A(t)\big)\Big)\big(\lambda I - A(t)\big). \end{aligned} \tag{2.12}$$

For $f \in \ell^1_w$, we use (2.8) to obtain

$$\begin{aligned}\big\|B(t)R\big(\lambda, A(t)\big)f\big\|_w &\le \kappa\big\|A(t)R\big(\lambda, A(t)\big)f\big\| \\ &= \kappa\sum_{n=1}^{\infty} w_n\frac{a_n(t)}{|\lambda + a_n(t)|}|f_n| \le \kappa\|f\|_w,\end{aligned}$$

which implies $\|B(t)R(\lambda, A(t))\| \le \kappa < 1$. Consequently, $I - B(t)R(\lambda, A(t))$ is invertible and

$$\left\|\Big[I - B(t)R\big(\lambda, A(t)\big)\Big]^{-1}\right\| = \left\|\sum_{n=0}^{\infty}\Big[B(t)R\big(\lambda, A(t)\big)\Big]^n\right\| \le \sum_{n=0}^{\infty}\kappa^n = \frac{1}{1-\kappa}\,,$$

which proves the second inequality in (2.11). Taking inverses on both sides of (2.12) we obtain (2.10). □

3 Well-Posedness

We now establish sufficient conditions on the fragmentation coefficients and the weight $w = (w_n)_{n=1}^{\infty}$ for the non-autonomous ACP (2.9) to be well posed in the complex Banach lattice ℓ^1_w. In addition to requiring (A1)–(A4) to hold, we also assume that constants $C_1 \ge 0$, $C_2 \ge 0$ and $\sigma \in (0, 1]$ exist such that

(A5) for all $n \in \mathbb{N}$ and $s, t, \tau \in [0, T]$,

$$\frac{|a_n(t) - a_n(s)|}{1 + a_n(\tau)} \le C_1|t - s|^{\sigma};$$

(A6) for all $j \in \{2, 3, \ldots\}$ and $s, t, \tau \in [0, T]$,

$$\frac{1}{1 + a_j(\tau)}\sum_{n=1}^{j-1} w_n\big|a_j(t)b_{n,j}(t) - a_j(s)b_{n,j}(s)\big| \le C_2 w_j|t - s|^{\sigma}.$$

Example 3.1. Let $c_n, d_n \ge 0$ for $n \in \mathbb{N}$ and let $\varphi : [0, T] \to [K_1, \infty)$ be a function such that $|\varphi(t) - \varphi(s)| \le K_2|t - s|^{\sigma}$ with $K_1, K_2 > 0$. Then

$$a_n(t) = c_n\varphi(t) + d_n$$

satisfies (A5), which can be seen as follows: for $t, s, \tau \in [0, T]$ we have

$$\begin{aligned}\frac{|a_n(t) - a_n(s)|}{1 + a_n(\tau)} &= \frac{c_n|\varphi(t) - \varphi(s)|}{c_n\varphi(\tau) + d_n + 1} \le \frac{K_2 c_n|t - s|^{\sigma}}{c_n K_1 + 1} \\ &= \frac{K_2}{K_1}\cdot\frac{c_n}{c_n + \frac{1}{K_1}}|t - s|^{\sigma} \le \frac{K_2}{K_1}|t - s|^{\sigma}.\end{aligned}$$

◊

Remark 3.2. If each $b_{n,j}$ is constant on $[0,T]$, say $b_{n,j}(t) \equiv b_{n,j}$, then (A6) follows from (A4) and (A5):

$$\frac{1}{1+a_j(\tau)}\sum_{n=1}^{j-1} w_n\big|a_j(t)b_{n,j}(t) - a_j(s)b_{n,j}(s)\big| = \sum_{n=1}^{j-1} w_n \frac{|a_j(t)-a_j(s)|}{1+a_j(\tau)} b_{n,j}$$

$$\le C_1|t-s|^\sigma \sum_{n=1}^{j-1} w_n b_{n,j} \le C_1\kappa w_j|t-s|^\sigma.$$

The case of constant $b_{n,j}$ corresponds to the situation when the outcome of the fragmentation of an n-mer ($n \ge 2$) does not depend on the time at which it occurs. For example, this arises in the Becker–Döring model of a coagulation–fragmentation process in which the break-up of an n-mer always results in a monomer and an $(n-1)$-mer; see [7, §2.2.1]. Note also that the coefficients $b_{n,j}$ in Example 2.3 are constant on $[0,T]$. ♢

Lemma 3.3. *Let assumptions* (A1)–(A6) *hold. Then,*

(a) $D(G(t)) = D(G(0)) =: \mathcal{D}$ *for all* $t \in [0,T]$*;*

(b) *for all* $s,t,\tau \in [0,T]$ *we have*

$$\big\|\big(G(t)-G(s)\big)R\big(1,A(\tau)\big)\big\| \le (C_1+C_2)|t-s|^\sigma. \tag{3.1}$$

Proof. (a) Let $s,t \in [0,T]$ and assume that $f \in D(G(s)) = D(A(s))$. It follows from (A5) with $\tau = s$ that

$$\begin{aligned}\sum_{n=1}^\infty w_n a_n(t)|f_n| &\le \sum_{n=1}^\infty w_n\big|a_n(t)-a_n(s)\big|\,|f_n| + \sum_{n=1}^\infty w_n a_n(s)|f_n| \\ &\le \sum_{n=1}^\infty w_n C_1|t-s|^\sigma\big(1+a_n(s)\big)|f_n| + \sum_{n=1}^\infty w_n a_n(s)|f_n| \\ &= \big(C_1|t-s|^\sigma+1\big)\|A(s)f\|_w + C_1|t-s|^\sigma\|f\|_w < \infty,\end{aligned}$$

which implies that $f \in D(A(t)) = D(G(t))$. Since s and t were arbitrary, it follows that the domain of $G(t)$ is independent of t.

(b) Let $t,s,\tau \in [0,T]$ and $f \in \ell^1_w$. Then

$$\begin{aligned}\big\|\big(G(t)-G(s)\big)R\big(1,A(\tau)\big)f\big\|_w &\le \big\|\big(A(t)-A(s)\big)\big(I-A(\tau)\big)^{-1}f\big\|_w \\ &\quad + \big\|\big(B(t)-B(s)\big)\big(I-A(\tau)\big)^{-1}f\big\|_w.\end{aligned} \tag{3.2}$$

Let us estimate each term separately. From (A5) we obtain

$$\big\|\big(A(t)-A(s)\big)\big(I-A(\tau)\big)^{-1}f\big\|_w = \sum_{n=1}^\infty w_n\frac{|a_n(t)-a_n(s)|}{1+a_n(\tau)}|f_n|$$

$$\le \sum_{n=1}^\infty w_n C_1|t-s|^\sigma|f_n| = C_1|t-s|^\sigma\|f\|_w. \tag{3.3}$$

For the second term on the right-hand side of (3.2) we can use (A6) to deduce that

$$
\begin{aligned}
&\big\|\big(B(t)-B(s)\big)\big(I-A(\tau)\big)^{-1}f\big\|_w \\
&= \sum_{n=1}^{\infty} w_n \left| \sum_{j=n+1}^{\infty} \Big(a_j(t)b_{n,j}(t)-a_j(s)b_{n,j}(s)\Big)\frac{1}{1+a_j(\tau)}\, f_j \right| \\
&\le \sum_{n=1}^{\infty} w_n \sum_{j=n+1}^{\infty} \frac{\big|a_j(t)b_{n,j}(t)-a_j(s)b_{n,j}(s)\big|}{1+a_j(\tau)}\,|f_j| \\
&= \sum_{j=2}^{\infty}\sum_{n=1}^{j-1} w_n \frac{\big|a_j(t)b_{n,j}(t)-a_j(s)b_{n,j}(s)\big|}{1+a_j(\tau)}\,|f_j| \\
&\le \sum_{j=2}^{\infty} C_2 w_j |t-s|^{\sigma}|f_j| \le C_2|t-s|^{\sigma}\|f\|_w. \qquad (3.4)
\end{aligned}
$$

Combining (3.2), (3.3) and (3.4) we arrive at (3.1). □

To enable Theorem 2.1 to be applied, we rescale each semigroup $(S_t(\tau))_{\tau\ge 0}$ that is generated by $G(t)$ by setting

$$T_t(\tau) = e^{-\tau}S_t(\tau), \qquad \tau \ge 0,\ t\in[0,T],$$

and consider the associated non-autonomous ACPs

$$v'(t) = H(t)v(t), \quad s<t\le T; \qquad v(s)=\mathring{v}, \qquad (3.5)$$

for $s\in[0,T)$, where $H(t)=G(t)-I$ is the generator of $(T_t(\tau))_{\tau\ge0}$ for $t\in[0,T]$.

Proposition 3.4. *Let assumptions* (A1)–(A6) *hold. Then there exists an evolution family,* $(V(t,s))_{0\le s\le t\le T}$ *on* ℓ_w^1*, with the following properties:*

(a) $v(t)=V(t,s)\mathring{v}$ *is the unique classical solution in* ℓ_w^1 *of* (3.5) *for any* $\mathring{v}\in\ell_w^1$ *and* $s\in[0,T)$*;*

(b) *if* $\mathring{v}\ge 0$*, then* $v(t)=V(t,s)\mathring{v}\ge 0$ *for* $t\in[s,T]$*;*
if, in addition, $\mathring{v}\ne 0$*, then* $v(t)\ne 0$ *for* $t\in[s,T]$*.*

Proof. (a) We show that the operators $H(t)$, $t\in[0,T]$, satisfy the assumptions (P1)–(P3) of Theorem 2.1. As each $G(t)$ generates an analytic substochastic C_0-semigroup on ℓ_w^1, it follows immediately that each $H(t)$ is also the generator of an analytic C_0-semigroup on ℓ_w^1. Moreover, from Lemma 3.3 (a), $D(H(t)) = D(G(t)) = \mathcal{D}$ for all $t\in[0,T]$. Hence (P1) is satisfied.

For (P2), we use Lemma 2.5 (b) to obtain

$$\big\|R\big(\lambda,H(t)\big)\big\| \le \frac{1}{(1-\kappa)|\lambda+1|} \qquad \text{for } \lambda\in\mathbb{C}:\operatorname{Re}\lambda>-1. \qquad (3.6)$$

Let $\lambda=\alpha+i\beta$, where $\alpha\ge0$ and $\beta\in\mathbb{R}$. On applying the arithmetic mean–quadratic mean inequality we deduce that

$$
\begin{aligned}
|\lambda+1| &= \sqrt{2}\cdot\sqrt{\frac{(\alpha+1)^2+\beta^2}{2}} \ge \sqrt{2}\cdot\frac{\alpha+1+|\beta|}{2} \\
&= \frac{\sqrt{\alpha^2+2\alpha|\beta|+|\beta|^2}+1}{\sqrt{2}} \ge \frac{\sqrt{\alpha^2+\beta^2}+1}{\sqrt{2}} = \frac{|\lambda|+1}{\sqrt{2}}.
\end{aligned}
$$

Together with (3.6) we obtain

$$\big\|R\big(\lambda, H(t)\big)\big\| \le \frac{\sqrt{2}}{1-\kappa}\cdot\frac{1}{|\lambda|+1}$$

for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge 0$, which shows (P2).

Finally, for (P3), let $s, t, \tau \in [0,T]$. From Lemmas 2.5 (b) and 3.3 (b) we can deduce that

$$\begin{aligned}
&\big\|\big(H(t)-H(s)\big)H(\tau)^{-1}\big\| = \big\|\big(G(t)-G(s)\big)R\big(1,G(\tau)\big)\big\| \\
&= \left\|\big(G(t)-G(s)\big)R\big(1,A(\tau)\big)\Big[I - B(\tau)R\big(1,A(\tau)\big)\Big]^{-1}\right\| \\
&\le \left\|\big(G(t)-G(s)\big)R\big(1,A(\tau)\big)\right\| \left\|\Big[I - B(\tau)R\big(1,A(\tau)\big)\Big]^{-1}\right\| \\
&\le (C_1+C_2)|t-s|^{\sigma}\cdot\frac{1}{1-\kappa}.
\end{aligned}$$

Therefore (P3) is also satisfied. Now the assertion follows from Theorem 2.1.

(b) To establish that the unique classical solution, $v(t) = V(t,s)\mathring{v}$, is nonnegative for all $t \in [s,T]$ whenever $\mathring{v} \in (\ell_w^1)_+$, we determine an infinite matrix representation of $V(t,s)$, $0 \le s \le t \le T$, with respect to the natural Schauder basis $(e_n)_{n=1}^{\infty}$ for ℓ_w^1 that is given by

$$(e_n)_m = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Let $s \in [0,T)$ be fixed. We begin by considering, for each fixed $n \in \mathbb{N}$, the finite system of linear ODEs

$$\begin{aligned}
&\frac{\partial}{\partial t}v_{m,n}(t,s) = -\big(1+a_m(t)\big)v_{m,n}(t,s) + \sum_{j=m+1}^{n} a_j(t)b_{m,j}(t)v_{j,n}(t,s), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in (s,T], \quad m = 1,2,\dots,n; \\
&v_{n,n}(s,s) = 1; \qquad v_{m,n}(s,s) = 0, \quad m = 1,\dots,n-1,
\end{aligned} \tag{3.8}$$

where we set $\sum\limits_{j=n+1}^{n} a_j(t)b_{m,j}(t)v_{j,n}(t,s) = 0$. It follows from assumptions (A5) and (A6) that all the coefficient functions, $1 + a_m$ and $a_j b_{m,j}$, in (3.8) are continuous on $[0,T]$. Standard ODE theory [12, §III.1] then establishes that, for each s and n, the system (3.8) has a unique solution. If we now define

$$v_{m,n}(t,s) \equiv 0, \qquad \text{for } m > n, \tag{3.9}$$

then the resulting infinite sequence, $(v_{m,n}(t,s))_{m=1}^{\infty}$, is a classical solution of the ACP (3.5), with $\mathring{v} = e_n$. By uniqueness of classical solutions to (3.5), we can deduce that $V(t,s)e_n = (v_{m,n}(t,s))_{m=1}^{\infty}$. Moreover, as $(e_n)_{n=1}^{\infty}$ is a Schauder basis for ℓ_w^1, and each operator $V(t,s)$ is linear and continuous on ℓ_w^1, we obtain

$$V(t,s)f = \sum_{n=1}^{\infty} f_n V(t,s)e_n, \qquad f = (f_n)_{n=1}^{\infty} \in \ell_w^1,$$

and hence, for each $m \in \mathbb{N}$,

$$\big(V(t,s)f\big)_m = \left(\sum_{n=1}^{\infty} f_n V(t,s)e_n\right)_m = \sum_{n=1}^{\infty} v_{m,n}(t,s)f_n,$$

which can be interpreted as a matrix multiplication of the infinite matrix $\mathbb{V}(t,s) = (v_{m,n}(t,s))_{m,n\in\mathbb{N}}$ and $f = (f_n)_{n=1}^{\infty}$ written as a column vector. It follows from (3.9) that $\mathbb{V}(t,s)$ has the form

$$\mathbb{V}(t,s) = \begin{bmatrix} v_{1,1}(t,s) & v_{1,2}(t,s) & v_{1,3}(t,s) & \cdots \\ 0 & v_{2,2}(t,s) & v_{2,3}(t,s) & \cdots \\ 0 & 0 & v_{3,3}(t,s) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3.10}$$

Since $v_{m,n}$ are solutions of (3.8), the entries of $\mathbb{V}(t,s)$ do not depend on the weight w. In the following we show that all these entries are non-negative. Let us start with the main diagonal. For $m = n \in \mathbb{N}$ the differential equation in (3.8) is

$$\frac{\partial}{\partial t} v_{n,n}(t,s) = -\big(1 + a_n(t)\big)v_{n,n}(t,s), \quad t \in (s,T]; \qquad v_{n,n}(s,s) = 1,$$

and therefore the terms in the leading diagonal of $\mathbb{V}(t,s)$ are given by

$$v_{n,n}(t,s) = \exp\!\left(-\int_s^t \big(1 + a_n(\tau)\big)\,\mathrm{d}\tau\right) > 0. \tag{3.11}$$

Next consider the case when $n > 1$ and $m = n-1$. Suppose that $v_{n-1,n}(t,s) < 0$ for t in some maximal interval $(\varepsilon_{n-1}, \hat{\varepsilon}_{n-1})$, where $s \le \varepsilon_{n-1} < \hat{\varepsilon}_{n-1} \le T$. From (3.8), we have

$$\frac{\partial}{\partial t} v_{n-1,n}(t,s) = -\big(1 + a_{n-1}(t)\big)v_{n-1,n}(t,s) + a_n(t)b_{n-1,n}(t)v_{n,n}(t,s), \qquad t \in (s,T]. \tag{3.12}$$

Since $v_{n,n}(t,s) \ge 0$ for $t \in [s,T]$, the right-hand side of (3.12) is positive on $(\varepsilon_{n-1}, \hat{\varepsilon}_{n-1})$. On the other hand, by continuity, $v_{n-1,n}(\varepsilon_{n-1}, s) = 0$, and therefore, by the Mean Value Theorem, there exists $\varepsilon \in (\varepsilon_{n-1}, \hat{\varepsilon}_{n-1})$ such that $\frac{\partial}{\partial t} v_{n-1}(t,s)\big|_{t=\varepsilon} < 0$. This is a contradiction, and so $v_{n-1,n}(t,s) \ge 0$ for all $t \in [s,T]$.

If $n > 2$ and $m = n-2$, a similar argument shows that $v_{n-2,n}(t,s) \ge 0$ for $t \in [s,T]$, and continuing in this way we obtain $v_{m,n}(t,s) \ge 0$ for all $t \in [s,T]$ and $m \le n$. Since $v_{m,n}(t,s) \equiv 0$ for all $m > n$, it follows that $v_{m,n}(t,s) \ge 0$ for all $m,n \in \mathbb{N}$ and $t \in [s,T]$. From this it is immediate that $v(t) \ge 0$ if $\mathring{v} \ge 0$.

To prove the last statement, let $\mathring{v} \ne 0$ so that $\mathring{v}_m > 0$ for some $m \in \mathbb{N}$. Then

$$v_m(t) = v_{m,m}(t,s)\mathring{v}_m + \sum_{n=m+1}^{\infty} v_{m,n}(t,s)\mathring{v}_n > 0$$

for $t \in [s,T]$ since the first term is strictly positive by (3.11) and the infinite series is non-negative. $\square$

Proposition 3.4 leads immediately to the main result of the paper, namely the existence in ℓ_w^1 of a unique, classical solution of (2.9) and its positivity for positive initial conditions. We also establish mass conservation under an additional assumption. To this end, let us recall that the total mass is given by the first moment, M_1, where

$$M_1(f) := \sum_{n=1}^{\infty} n f_n, \qquad f = (f_n)_{n=1}^{\infty} \in \ell_w^1. \tag{3.13}$$

Theorem 3.5. *Let assumptions* (A1)–(A6) *hold. Then there exists an evolution family $(U(t,s))_{0\le s\le t\le T}$ on ℓ_w^1 such that the following statements are true.*

(a) *$u(t) = U(t,s)\mathring{u}$ is the unique classical solution in ℓ_w^1 of* (2.9) *for any $\mathring{u} \in \ell_w^1$.*

(b) *If $\mathring{u} \ge 0$, then the solution from* (a) *satisfies $u(t) \ge 0$ for $t \in [s,T]$; if, in addition, $\mathring{u} \ne 0$, then $u(t) \ne 0$ for $t \in [s,T]$.*

(c) *If*

$$a_1(t) = 0, \quad \sum_{n=1}^{j-1} n b_{n,j}(t) = j \qquad \text{for all } j = 2,3,\dots,\ t \in [0,T], \tag{3.14}$$

then $M_1(u(t)) = M_1(\mathring{u})$ for $t \in [s,T]$ and $\mathring{u} \in (\ell_w^1)_+$.

Proof. Let $(V(t,s))_{0\le s\le t\le T}$ be the evolution family on ℓ_w^1 that is associated with (3.5), and define

$$U(t,s) := e^{t-s} V(t,s), \qquad 0 \le s \le t \le T.$$

A routine argument shows that $(U(t,s))_{0\le s\le t\le T}$ is also an evolution family on ℓ_w^1. Further, for $\mathring{u} \in \ell_w^1$, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(U(t,s)\mathring{u}\Big) &= e^{t-s} V(t,s)\mathring{u} + e^{t-s}\frac{\partial}{\partial t}\Big(V(t,s)\mathring{u}\Big) \\
&= e^{t-s}\Big[V(t,s)\mathring{u} + H(t)V(t,s)\mathring{u}\Big] \\
&= e^{t-s}\big(I + H(t)\big)V(t,s)\mathring{u} = G(t)U(t,s)\mathring{u}.
\end{aligned}$$

Hence $u(t) = U(t,s)\mathring{u}$ is a classical solution of (2.9); it is clearly non-negative on $[s,T]$ whenever $\mathring{u} \in (\ell_w^1)_+$, and it is non-zero when $\mathring{u} \ne 0$. To establish that $u(t) = U(t,s)\mathring{u}$ is the unique classical solution in ℓ_w^1, we simply note that if another classical solution, say $\tilde{u}$, exists, then (3.5) has a second classical solution given by $\tilde{v}(t) = e^{s-t}\tilde{u}(t)$, and this contradicts Proposition 3.4 (a).

Finally, let us prove (c). Since M_1 is a bounded linear functional on ℓ_w^1 and the solution $u(t)$ and the coefficients $a_n(t)$, $b_{n,j}(t)$ are non-negative, we obtain from (3.14) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big[M_1\big(u(t)\big)\Big] &= M_1\big(u'(t)\big) = M_1\big(G(t)u(t)\big) \\
&= \sum_{n=1}^{\infty} n\left(-a_n(t)u_n(t) + \sum_{j=n+1}^{\infty} a_j(t)b_{n,j}(t)u_j(t)\right)
\end{aligned}$$

$$= -\sum_{n=1}^{\infty} na_n(t)u_n(t) + \sum_{j=2}^{\infty}\left(\sum_{n=1}^{j-1} nb_{n,j}(t)\right)a_j(t)u_j(t)$$

$$= -\sum_{n=2}^{\infty} na_n(t)u_n(t) + \sum_{j=2}^{\infty} ja_j(t)u_j(t) = 0.$$

From this we can deduce that $M_1(u(t)) = M_1(\mathring{u})$ for $t \in [s,T]$. $\square$

The classical solution that exists by Theorem 3.5 satisfies the original infinite systems of equations (1.1) since the mth component of $G(t)u(t)$ is the right-hand side of the mth equation of (1.1). Note, however, that solutions of (1.1) are not unique in general; see, for example, the discussion at the end of Section 4 in [20] for the autonomous case.

4 Asymptotic Behaviour of Solutions

We now turn our attention to the long-time behaviour of classical solutions to the non-autonomous ACP (2.9), focussing on the mass-conserving case. When (3.14) holds, and the coefficients $a_n(t)$, $n \geq 2$, are strictly positive for all t, it is expected, from physical considerations, that, if the unique solution $u(t) = U(t,s)\mathring{u}$ exists for all $t \geq s$, then $u(t)$ should converge to the monomeric state $M_1(\mathring{u})e_1$ as $t \to \infty$. There have been several related investigations into the asymptotic behaviour of classical solutions to the autonomous ACP formulation of the constant-coefficient, mass-conserving fragmentation system. In particular, the expected convergence to $M_1(\mathring{u})e_1$ is established in the first moment space $X_{[1]}$ for constant-coefficient binary fragmentation in [9], and for constant-coefficient multiple fragmentation in [2]. The case of convergence in spaces ℓ^1_w for more general weights w is discussed in [5], where $w(x) = x^p$, $p > 1$, and also in our recent paper [13]. In both [5] and [13], it is shown that the convergence of solutions to the monomeric state is at an exponential rate, which is given explicitly in [13]. Our aim now is to adapt the arguments we used in [13] to prove that, under suitable conditions on the time-dependent coefficients, solutions to the mass-conserving non-autonomous fragmentation ACP, also converge to a monomeric state at an explicitly defined exponential rate. We begin with the following proposition.

Proposition 4.1. *Let assumptions* (A1)–(A6) *hold, let* $s \in [0,T)$ *and set*

$$\widetilde{a}_{s,T} \coloneqq \inf_{\tau\in[s,T]} \inf_{n\in\mathbb{N}} a_n(\tau).$$

Then

$$\|U(t,s)\| \leq \exp\bigl[-\widetilde{a}_{s,T}(1-\kappa)(t-s)\bigr], \qquad t \in [s,T], \tag{4.1}$$

where $(U(t,s))_{0\leq s\leq t\leq T}$ *is the evolution family on* ℓ^1_w *whose existence is established in Theorem 3.5.*

Proof. Let $\mathring{u} \in (\ell^1_w)_+ \setminus \{0\}$ be arbitrary, and let $u(t) = U(t,s)\mathring{u}$ be the classical solution of (2.1) from Theorem 3.5. Further, let ϕ_w be defined as in (2.2).

Since ϕ_w is a bounded linear functional on ℓ^1_w, we obtain from (2.7) that, for $\tau \in (s,T)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\phi_w\big(u(\tau)\big) = \phi_w\big(u'(\tau)\big) &= \phi_w\big(A(\tau)u(\tau)\big) + \phi_w\big(B(\tau)u(\tau)\big) \\ &\le (1-\kappa)\phi_w\big(A(\tau)u(\tau)\big) = -(1-\kappa)\sum_{n=1}^{\infty} w_n a_n(\tau)u_n(\tau) \\ &\le -(1-\kappa)\widetilde{a}_{s,T}\sum_{n=1}^{\infty} w_n u_n(\tau) = -(1-\kappa)\widetilde{a}_{s,T}\phi_w\big(u(\tau)\big). \qquad (4.2)\end{aligned}$$

By Theorem 3.5 (b), $u(\tau) \ge 0$ and $u(\tau) \ne 0$, and hence $\phi_w(u(\tau)) = \|u(\tau)\| > 0$. Dividing both sides of (4.2) by $\phi_w(u(\tau))$ and integrating over τ from s to t for $t \in (s,T]$ we deduce that

$$\phi_w\big(u(t)\big) \le \phi_w\big(u(s)\big)\exp\big[-\widetilde{a}_{s,T}(1-\kappa)(t-s)\big].$$

Since $u(t) \ge 0$ and $u(s) = \mathring{u} \ge 0$, this yields

$$\|U(t,s)\mathring{u}\|_w = \|u(t)\|_w \le \|\mathring{u}\|_w \exp\big[-\widetilde{a}_{s,T}(1-\kappa)(t-s)\big].$$

It follows from the positivity of $U(t,s)$ and [4, Proposition 2.67] that

$$\|U(t,s)\| = \sup_{\mathring{u}\ge 0,\, \|\mathring{u}\|_w \le 1} \|U(t,s)\mathring{u}\|_w \le \exp\big[-\widetilde{a}_{s,T}(1-\kappa)(t-s)\big],$$

which is (4.1). □

Remark 4.2. Note that, in particular, $\|U(t,s)\| \le 1$ for $s \le t$, since $\widetilde{a}_{s,T} \ge 0$ by assumption (A1). ◊

In Theorem 4.5 below we prove that, under certain assumptions, the solution converges to a pure monomeric state as $t \to \infty$, i.e. the state where only the first component is non-zero. Let us therefore consider a decomposition of the space ℓ^1_w into the span of e_1, where e_1 is defined in (3.7), and a complement. Define the space

$$Y_w := \left\{(f_n)_{n=2}^{\infty} : \sum_{n=2}^{\infty} w_n|f_n| < \infty\right\} \quad \text{with norm} \quad \big\|(f_n)_{n=2}^{\infty}\big\|_{Y_w} := \sum_{n=2}^{\infty} w_n|f_n|,$$

and let

$$\begin{aligned} J &: Y_w \to \ell^1_w, \qquad J\big[(f_n)_{n=2}^{\infty}\big] = (0, f_2, f_3, \ldots), \\ P &: \ell^1_w \to Y_w, \qquad P\big[(f_n)_{n=1}^{\infty}\big] = (f_2, f_3, f_4, \ldots),\end{aligned}$$

be the embedding of Y_w into ℓ^1_w and the projection from ℓ^1_w onto Y_w respectively. Then $\ell^1_w = \mathrm{span}\{e_1\} \oplus JY_w$. Let us start with a little lemma.

Lemma 4.3. *Let $w = (w_n)_{n=1}^{\infty}$ with $w_n \ge n$ for $n \in \mathbb{N}$, and let M_1 be defined as in* (3.13)*. Further, let $g \in \ell^1_w$ and assume that $M_1(g) = 0$. Then*

$$\|g\|_w \le (w_1 + 1)\big\|Pg\big\|_{Y_w}. \qquad (4.3)$$

Proof. We can decompose $g = (g_n)_{n=1}^\infty$ as

$$g = g_1 e_1 + J\widehat{g} \qquad \text{with } \widehat{g} = Pg = (g_n)_{n=2}^\infty.$$

The assumption $0 = M_1(g) = M_1(g_1 e_1) + M_1(J\widehat{g})$ implies that

$$|g_1| = \bigl|M_1(g_1 e_1)\bigr| = \bigl|-M_1(J\widehat{g})\bigr| \le \sum_{n=2}^\infty n|g_n| \le \sum_{n=2}^\infty w_n|g_n| = \bigl\|\widehat{g}\bigr\|_{Y_w},$$

which yields

$$\|g\|_w = w_1|g_1| + \bigl\|\widehat{g}\bigr\|_{Y_w} \le (w_1+1)\bigl\|\widehat{g}\bigr\|_{Y_w};$$

this proves (4.3). □

The next proposition provides an explicit estimate for the distance of the solution from a monomeric state on finite time intervals. It is used in the proof of Theorem 4.5 below.

Proposition 4.4. *Let assumptions* (A1)–(A6) *and relation* (3.14) *hold. Further, let $s \in [0, T)$ and set*

$$\widehat{a}_{s,T} := \inf_{\tau\in[s,T]} \inf_{n\ge 2} a_n(\tau).$$

Let $\mathring{u} \in (\ell_w^1)_+$ and let $u(t) = U(t,s)\mathring{u}$ be the classical solution of (2.1) *from Theorem 3.5. With M_1 defined as in* (3.13) *we have*

$$\bigl\|u(t) - M_1(\mathring{u})e_1\bigr\|_w \le (w_1+1)\|\mathring{u}\|_w \exp\bigl[-\hat{a}_{s,T}(1-\kappa)(t-s)\bigr], \qquad t \in [s,T]. \tag{4.4}$$

Proof. Let us consider the matrix representation of $U(t,s)$ for $t \in [s,T]$, which is obtained by multiplying the matrix in (3.10) with e^{t-s}, and splitting it according to the decomposition of the space into $\operatorname{span}\{e_1\}$ and JY_w,

$$\mathbb{U}(t,s) = \left[\begin{array}{c|ccc} u_{1,1}(t,s) & u_{1,2}(t,s) & u_{1,3}(t,s) & \cdots \\ \hline 0 & u_{2,2}(t,s) & u_{2,3}(t,s) & \cdots \\ 0 & 0 & u_{3,3}(t,s) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}\right] =: \begin{bmatrix} u_{1,1}(t,s) & U_{(12)}(t,s) \\ 0 & U_{(22)}(t,s) \end{bmatrix}.$$

Let $\mathring{u} = (\mathring{u}_n)_{n=1}^\infty \in (\ell_w^1)_+$ and fix $t \in (s,T]$. The matrix representation $\mathbb{U}(t,s)$ yields

$$\begin{aligned} g &:= U(t,s)\mathring{u} - M_1(\mathring{u})e_1 \\ &= \Bigl[u_{1,1}(t,s)\mathring{u}_1 + U_{(12)}(t,s)P\mathring{u} - M_1(\mathring{u})\Bigr]e_1 + JU_{(22)}(t,s)P\mathring{u}. \end{aligned}$$

By Theorem 3.5 (c) we have

$$M_1(g) = M_1\bigl(U(t,s)\mathring{u}\bigr) - M_1(\mathring{u}) = 0,$$

which allows us to apply Lemma 4.3 and obtain

$$\|g\|_w \le (w_1+1)\|Pg\|_{Y_w} = (w_1+1)\bigl\|U_{(22)}(t,s)P\mathring{u}\bigr\|_{Y_w}. \tag{4.5}$$

In order to estimate the right-hand side, let us set

$$\widehat{w}_n := w_{n+1}, \qquad \widehat{a}_n(t) := a_{n+1}(t), \qquad \widehat{b}_{n,j}(t) := b_{n+1,j+1}(t)$$

for $n, j \in \mathbb{N}$. It is easy to see that $\widehat{w}_n$, $\widehat{a}_n(t)$, $\widehat{b}_{n,j}(t)$ satisfy assumptions (A1)–(A6); for instance, (A6) can be checked as follows:

$$\begin{aligned}
&\frac{1}{1+\widehat{a}_j(\tau)} \sum_{n=1}^{j-1} \widehat{w}_n \big|\widehat{a}_j(t)\widehat{b}_{n,j}(t) - \widehat{a}_j(s)\widehat{b}_{n,j}(s)\big| \\
&= \frac{1}{1+a_{j+1}(\tau)} \sum_{n=2}^{j} w_n \big|a_j(t)b_{n,j}(t) - a_j(s)b_{n,j}(s)\big| \\
&\le \frac{1}{1+a_{j+1}(\tau)} \sum_{n=1}^{j} w_n \big|a_j(t)b_{n,j}(t) - a_j(s)b_{n,j}(s)\big| \\
&\le C_2 w_{j+1}|t-s|^\sigma = C_2 \widehat{w}_j |t-s|^\sigma .
\end{aligned}$$

Since u solves (2.1), the component $Pu(\cdot) = U_{(22)}(\,\cdot\,, s)P\mathring{u}$ solves (2.1) with $G(t)$ obtained by replacing w_n, $a_n(t)$ and $b_{n,j}(t)$ by $\widehat{w}_n$, $\widehat{a}_n(t)$ and $\widehat{b}_{n,j}(t)$ respectively. Applying Proposition 4.1 to $U_{(22)}(t, s)$ and using (4.5) we obtain

$$\begin{aligned}
\big\|u(t) - M_1(\mathring{u})e_1\big\|_w = \|g\|_w &\le (w_1+1)\|U_{(22)}(t,s)\|\,\|P\mathring{u}\|_{Y_w} \\
&\le (w_1+1)\|\mathring{u}\|_w \exp\big[-\widehat{a}_{s,T}(1-\kappa)(t-s)\big],
\end{aligned}$$

which proves (4.4). □

In the next theorem, which is the main result of this section, we consider solutions of the non-autonomous ACP

$$u'(t) = G(t)u(t), \quad t \in (0,\infty); \qquad u(0) = \mathring{u}, \tag{4.6}$$

where the operator $G(t)$ is defined for all $t \in [0,\infty)$. We assume that $(w_n)_{n=1}^\infty$ and κ are fixed so that assumptions (A1)–(A6) hold for all $T \in (0,\infty)$. It follows from Theorem 3.5 that (4.6) has a unique classical solution in ℓ^1_w when $\mathring{u} \in \ell^1_w$.

Theorem 4.5. *Let a_n and $b_{n,j}$ be defined on $(0,\infty)$ for $n, j \in \mathbb{N}$ and let $w_n > 0$, $n \in \mathbb{N}$, and $\kappa \in (0,1)$ be such that assumptions* (A1)–(A6) *hold for every $T > 0$* (*the constants C_1, C_2, σ in* (A5), (A6) *may depend on T*). *Further assume that* (3.14) *holds, let $\mathring{u} \in (\ell^1_w)_+$, and let u be the unique classical solution of* (4.6).

(a) *If*

$$\widehat{a}_{0,\infty} := \inf_{t\in(0,\infty)} \inf_{n\ge 2} a_n(t) > 0,$$

then

$$\big\|u(t) - M_1(\mathring{u})e_1\big\|_w \le (w_1+1)\|\mathring{u}\|_w \exp\big[-\widehat{a}_{0,\infty}(1-\kappa)t\big], \qquad t \in [0,\infty).$$

(b) *If*

$$\widehat{a} := \liminf_{t\to\infty} \inf_{n\ge 2} a_n(t) > 0,$$

then, for every $c < \widehat{a}(1-\kappa)$ there exists $M > 0$ such that

$$\big\|u(t) - M_1(\mathring{u})e_1\big\|_w \le Me^{-ct}, \qquad t \in [0,\infty). \tag{4.7}$$

Proof. The assertion in (a) follows directly from Proposition 4.4.

To prove (b), let $c < \widehat{a}(1-\kappa)$. There exists $s \in [0,\infty)$ such that

$$\frac{c}{1-\kappa} \le \widehat{a}_{s,\infty} := \inf_{t\in[s,\infty)} \inf_{n\ge 2} a_n(t).$$

It follows from Theorem 3.5 (c) and Proposition 4.4 that, for $t \in [s,\infty)$,

$$\begin{aligned}
\big\|u(t) - M_1(\mathring{u})e_1\big\|_w &= \big\|u(t) - M_1\big(u(s)\big)e_1\big\|_w \\
&\le (w_1+1)\|u(s)\|_w \exp\big[-\widehat{a}_{s,\infty}(1-\kappa)(t-s)\big] \\
&\le (w_1+1)\|\mathring{u}\|_w \exp[-c(t-s)] \\
&= (w_1+1)\|\mathring{u}\|_w e^{cs}e^{-ct};
\end{aligned} \tag{4.8}$$

note that $\|u(s)\|_w \le \|\mathring{u}\|_w$ by Remark 4.2. For $t \in [0,s)$ we have

$$\begin{aligned}
\big\|u(t) - M_1(\mathring{u})e_1\big\|_w &\le (w_1+1)\|\mathring{u}\|_w \exp\big[-\widehat{a}_{0,s}(1-\kappa)t\big] \\
&\le (w_1+1)\|\mathring{u}\|_w \le (w_1+1)\|\mathring{u}\|_w e^{cs}e^{-ct},
\end{aligned}$$

which, together with (4.8) proves (4.7) with $M = (w_1+1)\|\mathring{u}\|_w e^{cs}$. □

5 Concluding Remarks

To summarise, in this paper we have used the theory of evolution families to analyse the non-autonomous fragmentation system (1.1). By writing (1.1) as an ACP in an appropriately weighted ℓ^1 space, and exploiting results on the analyticity of semigroups associated with autonomous fragmentation systems, obtained in our earlier paper [13], we have proved the existence and uniqueness of classical solutions to the non-autonomous problem, for time-dependent fragmentation coefficients that satisfy the assumptions (A1)–(A6). Properties of these solutions such as non-negativity and, under the additional assumption (3.14), mass conservation have been established. Moreover, results on the asymptotic behaviour of solutions have been obtained.

As mentioned in the Introduction, evolution families have also featured in investigations into the non-autonomous continuous fragmentation equation, which is given by

$$\begin{aligned}
\frac{\partial}{\partial t}u(x,t) &= -a(x,t)u(x,t) + \int_x^\infty a(y,t)b(x,y,t)u(y,t)\,\mathrm{d}y, \\
&\qquad\qquad x\in(0,\infty),\ t\in(0,T], \\
u(x,0) &= \mathring{u}(x),
\end{aligned} \tag{5.1}$$

where $u(x,t)$ represents the density of particles of size $x \in (0,\infty)$ at time t, and the coefficients $a(x,t)$ and $b(x,y,t)$ are interpreted in an analogous manner to $a_n(t)$ and $b_{n,j}(t)$ in the discrete system (1.1). For the sake of comparison, we discuss briefly the key results that these investigations have produced.

In [15], a slightly different, but equivalent, formulation of the initial-value problem (5.1) is posed as a non-autonomous ACP in the space $L^1(\mathbb{R}_+, x\,\mathrm{d}x)$ (denoted by $L_{1,-1}$ in [15]). Only mass-conserving fragmentation is considered, and the fragmentation coefficients are assumed to satisfy the following conditions:

(i) for every $n > 0$ there exists a function $C_n : [0, T] \to (0, \infty)$ such that

$$a(x, t) \le C_n(t), \qquad x \in (0, n],\ t \in [0, T]; \tag{5.2}$$

(ii) there exists a function $G : (0, \infty) \times (0, \infty) \to (0, \infty)$ such that

$$\begin{aligned} \big|a(y, t)b(x, y, t) - a(y, \tau)b(x, y, \tau)\big| \le |t - \tau| G(x, y), \\ x, y \in (0, \infty),\ t, \tau \in [0, T], \end{aligned} \tag{5.3}$$

where, for every $n > 0$, G is bounded on $(0, n] \times (0, n]$.

Under these assumptions, the existence of a strongly continuous evolution family $(U(t, s))_{0 \le s \le t \le T}$, consisting of non-negative isometries on $L^1(\mathbb{R}_+, x\,\mathrm{d}x)$, is established. Each operator $U(t, s)$ is defined as the strong limit, as $n \to \infty$, of operators $U_n(t, s)$, $n > 0$, where, for each n, $(U_n(t, s))_{0 \le s \le t \le T}$, is a uniformly continuous evolution family that is associated with an appropriately truncated version of (5.1), where the truncation is with respect to x to the interval $(0, n]$. In the case of restricted initial data satisfying $\mathring{u}(x) \equiv 0$ on $[n, \infty)$, for some $n > 0$, it is shown that $u(t) = U(t, s)\mathring{u}$ is the unique classical solution of the non-autonomous ACP version of (5.1). However, there is no corresponding result for a general $\mathring{u} \in L^1(\mathbb{R}_+, x\,\mathrm{d}x)$. Instead, the function $u(t) = U(t, s)\mathring{u}$ is interpreted as a 'generalised' solution of the non-autonomous ACP, and, provided $C_n \in L^\infty([0, T])$, where C_n is the function in (5.2), the associated scalar-valued function $u(x, t) = [U(t, s)\mathring{u}](x)$ is shown to be a solution of the following integral version of (5.1)

$$u(x, t) = \mathring{u}(x) - \int_0^t a(x, \tau)u(x, \tau)\,\mathrm{d}\tau + \int_0^t \int_x^\infty a(y, \tau)b(x, y, \tau)u(y, \tau)\,\mathrm{d}y\,\mathrm{d}\tau.$$

Some partial results on the uniqueness of the solution $u(t) = U(t, s)\mathring{u}$, for the case when $\mathring{u} \in L^1(\mathbb{R}_+, x\,\mathrm{d}x)$ does not vanish on (n, ∞) for some $n > 0$, can be found in [16], where the notion of a weak solution is used. In particular, it is shown that $u(t) = U(t, s)\mathring{u}$ is the unique, non-negative, mass-conserving, weak solution of the non-autonomous ACP for any given non-negative initial data $\mathring{u} \in L^1(\mathbb{R}_+, x\,\mathrm{d}x)$, provided that the function b is independent of time, and $a(x, t) = a_0(x)\alpha(t)$, with $a_0(x) \le C_n$ on $(0, n]$, and α a Lipschitz continuous function on $[0, T]$.

More recently, evolution families, together with associated evolution semigroups, have also been used in [1] to establish the existence of a solution to the above integral version of (5.1), still under the assumption that each fragmentation event conserves mass, but with the milder restriction that the fragmentation rate a only has to be locally integrable with respect to time and locally bounded with respect to x. As in [15], the solution is given by $u(x, t) = [U(t, s)\mathring{u}](x)$, where $(U(t, s))_{0 \le s \le t \le T}$, is a strongly continuous evolution family of non-negative contractive operators on $L^1(\mathbb{R}_+, x\,\mathrm{d}x)$. Moreover, when a is bounded on $[0, M] \times [0, T]$, for any $M, T \in (0, \infty)$, each $U(t, s)$ is shown to be an isometry.

We believe that the approach we have used in this paper could also prove fruitful if applied to appropriately posed ACP versions of (5.1), and, in particular, may lead to new results concerning the existence and uniqueness of physically meaningful classical solutions. A first step would clearly be that of identifying weighted spaces, $L^1(\mathbb{R}_+, w(x)\,\mathrm{d}x)$, such that the semigroup associated with the autonomous, continuous fragmentation equation is analytic when defined on $L^1(\mathbb{R}_+, w(x)\,\mathrm{d}x)$. In connection with this, it is worth noting that sufficient conditions for the fragmentation coefficients are stated in [7, Section 5.1.7] which guarantee the analyticity of the continuous fragmentation semigroup for the cases $w(x) = x^m$ and $w(x) = 1 + x^m$, where $m > 1$.

Finally, a natural extension of the work presented here is to incorporate coagulation into the model, and then examine the full, non-linear, discrete coagulation–fragmentation (C–F) system, in which both the coagulation and the fragmentation coefficients are time-dependent. Although an approach based on evolution families has been used in [17], for continuous C–F equations in which the coagulation and fragmentation coefficients are both permitted to be time-dependent, we are unaware of similar investigations into the discrete case.

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