Stabilization in Distribution by Delay Feedback Controls for Hybrid Stochastic Delay Differential Equations

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ABSTRACT
This article aims to design a linear delay feedback control to stabilize an unstable hybrid stochastic delay differential equation in distribution. Under the global Lipschitz condition, sufficient criteria are established to guarantee the stability of the controlled system. Then LMI techniques are employed to design the control law in two structure forms: state feedback and output injection.

KEYWORDS
Brownian motion; Markovian switching, delay feedback control; stability in distribution; stochastic delay differential equations

1. Introduction

In practice, hybrid systems have been widely used to model many scientific and industrial systems when these systems may experience abrupt changes in their structures and parameters (see e.g. (Athans, 1987; Luo & Mao, 2009; Sworder & Robinson, 1973)). When the futures of hybrid systems depend on their past states, hybrid stochastic differential delay equations(SDDEs), or SDDEs with Markovian switching, have been frequently applied as models. Basic theories and applications about hybird SDDEs can be found in (Mao, Matasov & Piunovskiy, 2000) and (Mao & Yuan, 2006).

From the view of automatic control, asymptotic stability and stabilization are two important issues. Toward those systems having trivial solutions (or equilibrium states), most of literatures have focused on stability in the sense of moment and almost sure. Rich results have been achieved on stability analysis (see e.g. (Ji & Chizeck, 1990; Lewis, 2000; Mao, 1999; Mao, Matasov & Piunovskiy, 2000; Shaikhet, 1996; Sun, Lam, Xu & Zou, 2007; Yue & Han, 2005; Zhu, Tian & Wang, 2015) and refereces therein) and stabilization (see e.g. (Cheng, Park & Wu, 2021; Cheng, Park, Yan & Wu, 2022; Jiang, Hu, Lu, Mao & Mao, 2021; Mao, Jiang, Hu & Mao, 2022; Mao, 2013; Mao, Lam & Huang, 2008; Mao, Liu, Hu, Luo & Lu, 2014; Song, Wang, Li & Chen, 2021; Song,

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Zhang, Zhu & Li, 2022)).

The case of stability in distribution arises when a system has no trivial solutions frequently met, for example, in fault tolerant control systems and multiple target tracking(Athans, 1987; Basak, Bisi & Ghosh, 1996; Luo & Mao, 2009; Mariton, 1990; Sworder & Robinson, 1973). In many practical applications, such as population models for some species (Bahar & Mao, 2004; Mao, 1997), it will be more appropriate to study the stability in distribution when the system is persistent (Mao, 2011; Yuan, Zou & Mao, 2003). There are fewer references on the stability in distribution for SDDEs. (Yuan, Zou & Mao, 2003) is the first article proposing sufficient conditions for stability in distribution of an SDDE. After that, some improved results have been proved by applying different techniques and arguments, such as in (Du, Dang & Dieu, 2014). While in (Hu & Wang, 2012), the stability in distribution of neutral stochastic functional delay equations(SFDEs) has been studied with the help of a suitable Lyapunov function. We also have seen articles by Bao and his coauthors (Bao, Yin & Yuan, 2014; Bao, Yin, Yuan & Wang, 2014; Bao, Yin & Yuan, 2017) about the existence and uniqueness of the invariant measure for different classes of SFDEs. In (Wang, Wu & Mao, 2019), stability-in-distribution criteria for SFDEs have been proposed, where the coefficients involving delay components are highly non-linear.

In recent years, we have seen plenty of references dealing with problems of stabilizing SDDEs in the sense of moment or almost sure with various controls. But to authors' knowledge, there are no reports on stabilization in distribution for SDDEs. This article aims to show that we can stabilize an SDDE in distribution even by the classical delay feedback control. In (You, Hu, Lu & Mao, 2022), a procedure has been proposed to design a delay feedback control for stabilization in distribution for an SDE. Although the problem discussed in this article can seen as an extension of that in (You, Hu, Lu & Mao, 2022), the rules for designing the control function are quite different, because SDDEs are infinite-dimensional dynamical systems with more complex structures.

Indeed, there are rich sets of control laws used for stabilizing SDDEs in the sense of moment and almost sure. Here we mention some of them. Delay feedback controls have been applied to stabilize SDEs in the sense of almost sure in (Mao, Lam & Huang, 2008). After discrete-time feedback controls first proposed in (Mao, 2013), they have been frequently applied for designing control laws even in SDDEs with highly non-linear coefficients as in (Mao, Liu, Hu, Luo & Lu, 2014; Song, Zhang, Zhu & Li, 2022). Quantized feedback stabilization with discrete-time observations has been analysed in non-linear hybrid SDDEs (Song, Wang, Li & Chen, 2021). (Cheng, Park, Yan & Wu, 2022) has studied the dynamic output feedback control issue for a non-homogeneous Markov switching system with an event-triggered round-robin protocol. To periodic systems with singular perturbations, hidden Markov model based controls are discussed in (Cheng, Park & Wu, 2021). Intermittent controls have attracted many researcher's attention, and been applied as controllers in (Jiang, Hu, Lu, Mao & Mao, 2021; Mao, Jiang, Hu & Mao, 2022).

As the first try on stabilization in distribution for SDDEs, analyses will be done toward equations satisfying the global Lipschitz condition. We will show that we can stabilize such system by applying the classical delay feedback control. Our contributions are mainly focused on

(1) giving a procedure for designing a linear delay feedback control to stabilize an unstable SDDE in distribution;
(2) proposing a suitable Lyapunov functional with simpler structures than those used for stabilization in the sense of moment or almost sure;

(3) applying LMI techniques to obtain the coefficient matrices of the linear controllers.

This article is arranged as follows. In section 2, some fundamental concepts and notations are listed. In section 3, main theorems are stated. LMIs are applied to calculate the matrices of linear controls as discussed in section 4. Two illustrative examples are given in section 5 to verify derived results and show effectiveness of our algorithm. Conclusions and future research aspects are made and discussed in the last section.

2. Preliminaries and Notations

Throughout this article, following notations will be used. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathcal{B}(\mathbb{R}^n)$ be the family of all Borel measurable sets in $\mathbb{R}^n$. For $\delta > 0$, $\mathcal{C}_\delta$ (or $C([-\delta,0];\mathbb{R}^n)$) denotes the family of continuous functions $\xi : [-\delta,0] \to \mathbb{R}^n$ with norm $\|\xi\|_\delta = \sup_{-\delta\le u\le 0} |\xi(u)|$. $\mathcal{B}(\mathcal{C}_\delta)$ denotes the family of all Borel measurable sets in $\mathcal{C}_\delta$. Denote by $|x|$ the Euclidean norm of a vector $x \in \mathbb{R}^n$. For a matrix A, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^TA)}$ and its operator norm is denoted by $\|A\| = \max\{|Ax| : |x| = 1\}$. For a symmetric matrix A ($A = A^T$), denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalues, respectively. By $A > 0$ and $A \ge 0$, we mean A is positive and non-negative definite, respectively. Denote $a\wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$ for $a,b \in \mathbb{R}$.

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge 0},P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions, and $B(t) = (B_1(t),B_2(t),...,B_m(t))^T$ be an m-dimensional Brownian Motion defined on this space. Also, there is a right continuous irreducible Markov chain $r(t)$, $t \ge 0$, taking values in a finite state space $\mathbb{S} = \{1,2,...,N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j|r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & i \ne j \\ 1 + \gamma_{ii}\Delta + o(\Delta), & i = j \end{cases}$$

where $\Delta > 0$, and $\gamma_{ij} > 0 (i \ne j)$ is the transition rate from state i to j with $\gamma_{ii} = -\sum_{j\ne i}\gamma_{ij}$. Assume that $r(t)$ and $B(t)$ are independent.

Consider a hybrid SDDE defined on $t \ge 0$,

$$dX(t) = f(X(t),X(t-\delta),r(t))dt + g(X(t),X(t-\delta),r(t))dB(t), \tag{1}$$

with the initial condition

$$\{X(t)| -\delta \le t \le 0\} = \xi \in \mathcal{C}_\delta, r(0) = i_0 \in \mathbb{S} \tag{2}$$

where $X(t)$ takes values in $\mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^{n\times m}$ are the drift and diffusion coefficients, respectively.

In order that the equation (1), with the initial condition (2), exists a unique solution, we make following traditional assumptions.

Assumption 2.1. *There is a pair of positive constants a_1 and a_2 such that*

$$|f(x,z_1,i) - f(y,z_2,i)|^2 \le a_1(|x-y|^2 + |z_1 - z_2|^2)$$

and

$$|g(x,z_1,i)-g(y,z_2,i)|^2 \le a_2(|x-y|^2+|z_1-z_2|^2)$$

for all $x,y,z_1,z_2\in\mathbb{R}^n$ *and* $i\in\mathbb{S}$.

It is easy to derive from Assumption 2.1 that

$$|f(x,z,i)|^2 \le 2a_1(|x|^2+|z|^2)+a_0, \tag{3}$$
$$|g(x,z,i)|^2 \le 2a_2(|x|^2+|z|^2)+a_0 \tag{4}$$

hold for all $(x,z,i)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}$, where $a_0=2\max_{i\in\mathbb{S}}(|f(0,0,i)|^2\vee|g(0,0,i)|^2)$.

It can be shown as in (Mao & Yuan, 2006) that the hybrid SDDE (1) has a unique global solution $X(t)$ on $t\ge-\delta$. Furthermore, (Yuan, Zou & Mao, 2003) has proposed some sufficient conditions leading to stability in distribution for the SDDE as (1). Now there comes a new question: if the original system (1) is unstable in distribution, can we design a control u such that the controlled system is stable in distribution?

Due to the lag between the time when the observation is made and the time when the control reaches the system, it will be more reasonable to use a delay feedback control for stabilization, which means that we need to design a delay control $u(X(t-\tau),r(t))$ such that the controlled system

$$\begin{aligned}dX(t)=&[f(X(t),X(t-\delta),r(t))+u(X(t-\tau),r(t))]dt\\&+g(X(t),X(t-\delta),r(t))dB(t)\end{aligned}\tag{5}$$

becomes stable in distribution.

In this article, we will look for a linear feedback control $u(X(t-\tau),r(t)) = A(r(t))X(t-\tau)$, where $A(i)=A_i\in\mathbb{R}^{n\times n}$ for all $i\in\mathbb{S}$. The underlying controlled system (5) therefore becomes

$$\begin{aligned}dX(t)=&\big(f(X(t),X(t-\delta),r(t))+A(r(t))X(t-\tau)\big)dt\\&+g(X(t),X(t-\delta),r(t))dB(t).\end{aligned}\tag{6}$$

It is valuable to mention here that other control modes as listed in the introduction section may also be applied to design the control laws for stabilization in distribution. A direct proof is (Li, Liu, Luo & Mao, 2022), in which a control law based on discrete-time observation has been built for stabilization in distribution for an SDE. But we should note that the Markov property of the controlled system should be guaranteed for further discussion. When we use a feedback control with a constant delay, the segment of the solution for the controlled system (6) will form a time homogeneous Markov process, as explained below.

Let $\bar\delta=\delta\vee\tau$. Redefine $\xi(t)=\xi(-\delta)$ for $t\in[-\bar\delta,-\delta)$ and set an initial condition suitable for (6) as

$$\{X(t)|-\bar\delta\le t\le 0\}=\xi\in\mathcal{C}_{\bar\delta}, r(0)=i_0 \tag{7}$$

based on (2) for the initial system (1). It is obvious that $\|\xi\|_{\bar\delta}=\|\xi\|_\delta$. In following discussion, we will ignore the subscript when we use such norms and their meanings can be seen from contexts.

It is known (see, e.g., (Mao & Yuan, 2006)) that under Assumption 2.1, the controlled SDDE (6), with the initial data (7), has a unique global solution $X^{\xi,i_0}(t)$ on $t \geq -\bar{\delta}$. Moreover, define $X_t^{\xi,i_0} = \{X^{\xi,i_0}(t+u) : -\bar{\delta} \leq u \leq 0\}$ for $t \geq 0$, which is a $\mathcal{C}_{\bar{\delta}}$-valued process. It has been shown (see, e.g., (Mao & Yuan, 2006)) that for any $t \geq 0$,

$$\mathbb{E}\|X_t^{\xi,i_0}\|^2 \leq \gamma_t(1 + \|\xi\|^2) \tag{8}$$

where γ_t is a positive constant dependent on t but independent of (ξ, i_0).

Denote $Y^{\xi,i_0}(t) = (X_t^{\xi,i_0}, r(t))$ be a $\mathcal{C}_{\bar{\delta}} \times \mathbb{S}$ valued process. Then $Y(t)$ is a time homogeneous Markov process and denote $p(t, \xi, i_0, \mathrm{d}\zeta \times \{j\})$ the transition probability of $Y^{\xi,i_0}(t)$. The process $Y^{\xi,i_0}(t)$ is said to be stable in distribution, if there exists a probability measure $\pi(\cdot)$ on $\mathcal{C}_{\bar{\delta}} \times \mathbb{S}$ such that $p(t, \xi, i_0, \mathrm{d}\zeta \times \{j\})$ converges weakly to $\pi(\mathrm{d}\zeta \times \{j\})$ as $t \to \infty$ for any initial data $(\xi, i_0) \in \mathcal{C}_{\bar{\delta}} \times \mathbb{S}$. Let $\mathcal{P}(\mathcal{C}_{\bar{\delta}})$ be the space of all probability measures on $\mathcal{C}_{\bar{\delta}}$. As explained in (Anderson, 1991), the law of the irreducible Markov chain discussed in this article will converge to its unique stationary distribution. So we only need to show that the probability measure $\mathcal{L}(X_t^{\xi,i_0})$, generated by $X_t^{\xi,i_0} \in \mathcal{C}_{\bar{\delta}}$, converges to some probability measure $\mu_{\bar{\delta}} \in \mathcal{P}(\mathcal{C}_{\bar{\delta}})$.

For two probability measures $P_1, P_2 \in \mathcal{P}(\mathcal{C}_{\bar{\delta}})$, define following distance d between P_1 and P_2 as

$$d(P_1, P_2) = \sup_{\psi \in L} \left| \int \psi(\xi) P_1(d\xi) - \int \psi(\xi) P_2(d\xi) \right|$$

where

$$L = \{\psi : \mathcal{C}_{\bar{\delta}} \to \mathbb{R} \big| |\psi(\xi) - \psi(\eta)| \leq \|\xi - \eta\| \text{ and } |\psi(\cdot)| \leq 1 \text{ for any } (\xi, \eta) \in \mathcal{C}_{\bar{\delta}} \times \mathcal{C}_{\bar{\delta}}\}.$$

We will use following equivalent definition on stability in distribution of (6) for discussion.

Definition 2.2. The controlled system (6) is said to be stable in distribution, if there exists a probability measure $\mu_{\bar{\delta}} \in \mathcal{P}(\mathcal{C}_{\bar{\delta}})$ such that

$$\lim_{t \to \infty} d(\mathcal{L}(X_t^{\xi,i_0}), \mu_{\bar{\delta}}) = 0$$

holds for any initial data $(\xi, i_0) \in \mathcal{C}_{\bar{\delta}} \times \mathbb{S}$.

The procedure for designing a linear delay feedback control $A(r(t))X(t-\tau)$ has two steps:

(1) looking for N matrices $A_i, i \in \mathbb{S}$ such that the linear control without delay, $u = A(r(t))X(t)$, can stabilize the unstable system (1), which means that the auxiliary controlled system

$$\begin{aligned} dX(t) = \big(f(X(t), X(t-\delta), r(t)) + A(r(t))X(t)\big)dt \\ + g(X(t), X(t-\delta), r(t))dB(t) \end{aligned} \tag{9}$$

is stable in distribution.

(2) applying suitable Lyapunov functionals to get the bound for the delay τ such that the delay feedback controlled system (6) is stable in distribution.

In the first step, $A_i, i \in \mathbb{S}$ will be found based on the stability of the auxiliary system (9). Although we have used a control without delay, (9) is still an infinite dimensional system. We will apply the argument in (Yuan, Zou & Mao, 2003) on stability in distribution for hybrid SDDEs. And we need following Assumption 2.3 that will be more complex than Assumption 3.1 in (You, Hu, Lu & Mao, 2022) for an SDE.

Assumption 2.3. *There exist two positive numbers λ_1, λ_2 with $\lambda_1 > \lambda_2$ and N symmetric positive definite matrices W_i $(1 \le i \le N)$ such that*

$$\begin{aligned}\Psi(x, y, z_1, z_2, i) &:= 2(x-y)^T W_i[f(x, z_1, i) - f(y, z_2, i) + A_i(x-y)] \\ &\quad + \text{trace}[(g(x, z_1, i) - g(y, z_2, i))^T W_i(g(x, z_1, i) - g(y, z_1, i))] \\ &\quad + \sum_{j=1}^{N} \gamma_{ij}(x-y)^T W_j(x-y) \\ &\le -\lambda_1|x-y|^2 + \lambda_2|z_1 - z_2|^2 \end{aligned} \tag{10}$$

holds for all $(x, y, z_1, z_2, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}$.

As shown in (Yuan, Zou & Mao, 2003), under Assumption 2.1 and 2.3, the controlled system with feedback control $u = A(r(t))X(t)$, i.e. equation (9), is stable in distribution. So we have completed the first step in above procedure. In following discussion, we will show that the delay feedback controlled system (6) will remain stable in distribution if the time lag τ is less than τ^* as defined in (21) later.

It is straightforward to show from Assumptions 2.1 and 2.3 that

$$\begin{aligned}\Phi(x, z, i) :=& 2x^T W_i[f(x, z, i) + A_i x] + \text{trace}[g(x, z, i)^T W_i g(x, z, i)] + \sum_{j=1}^{N} \gamma_{ij} x^T W_j x \\ \le& -\lambda_1|x|^2 + \lambda_2|z|^2 + \lambda_3|x| + \lambda_4|z| + \lambda_0 \end{aligned} \tag{11}$$

holds for all $(x, z, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}$, where λ_3, λ_4 and λ_0 are positive numbers. Set

$$\alpha_1 = \max_{i \in \mathbb{S}} \|A_i\|^2 \text{ and } \alpha_2 = \max_{i \in \mathbb{S}} \|W_i A_i\| \tag{12}$$

for subsequent discussion.

3. Main Results

In the second step of the procedure, the constant delay τ will be determined. In those references on stabilization in the sense of moment and almost sure with delay feedback controls, two methods are usually used to obtain sufficient bounds for delay sizes. One is estimating $\mathbb{E}|x(t) - x(t-\tau)|$ as a function of τ, and setting an upper bound for the expectation, such as in (Li, Liu, Luo & Mao, 2022; Mao, 2013; Mao, Lam & Huang, 2008; Mao, Liu, Hu, Luo & Lu, 2014; Sun, Lam, Xu & Zou, 2007). The other one is applying Lyapunov method with suitable Lyapunov functions or functionals, as in (Cheng, Park & Wu, 2021; Cheng, Park, Yan & Wu, 2022; Li & Mao, 2020; Lu, Hu

& Mao, 2019; Song, Wang, Li & Chen, 2021; Song, Zhang, Zhu & Li, 2022). The frequently used Lyapunov functional has the form of

$$U(x(t)) + \theta \int_{-\omega}^{0} \int_{t+s}^{t} [\omega |f(x(v), x(v-\delta), r(v))|^2 + |g(x(v), x(v-\delta), r(v))|^2] dv ds.$$

For details about such Lyapunov functional, one can refer to (You, Liu, Lu, Mao & Qiu, 2015). This type of Lyapunov functional has been proved to be efficient specially in SDDEs with highly nonlinear coefficients. Because the equation studied in this article satisfies the global Lipschitz condition, we won't need such complex functional but apply functionals of the form (13) with simpler structures.

The Lyapunov functionals used in this article will be of the form

$$\begin{aligned} V(\hat{X}_t, r(t), t) &:= X^T(t) W(r(t)) X(t) \\ &+ \int_{t-\tau}^{t} \int_{s}^{t} \big[\theta_1 |X(v)|^2 + \theta_2 |X(v-\delta)|^2 + \theta_3 |X(v-\tau)|^2\big] dv ds \end{aligned} \tag{13}$$

for $t \geq \tau$. Here $W_i, i \in \mathbb{S}$ are the matrices specified in Assumption 2.3, while θ_1, θ_2 and θ_3 are three free positive numbers.

It is useful to observe that

$$c_1 |X(t)|^2 \leq V(\hat{X}_t, r(t), t) \leq c_2 |X(t)|^2 + c_3 \int_{t-\tau-\bar{\delta}}^{t} |X(v)|^2 dv, \tag{14}$$

where $c_3 = \tau(\theta_1 \vee \theta_2 \vee \theta_3)$, $c_1 = \min_{i \in \mathbb{S}} \lambda_{\min}(W_i)$ and $c_2 = \max_{i \in \mathbb{S}} \lambda_{\max}(W_i)$.

Applying the generalized Itô formula to $V(\hat{X}_t, r(t), t)$, we have

$$dV(\hat{X}_t, r(t), t) = LV(\hat{X}_t, r(t), t) dt + dM(t) \tag{15}$$

for $t \geq \tau$, where

$$\begin{aligned} &LV(\hat{X}_t, r(t), t) \\ &= \Phi(X(t), X(t-\delta), r(t)) - 2X^T(t) W_{r(t)} A_{r(t)} (X(t) - X(t-\tau)) \\ &+ \theta_1 \tau |X(t)|^2 - \theta_1 \int_{t-\tau}^{t} |X(s)|^2 ds \\ &+ \theta_2 \tau |X(t-\delta)|^2 - \theta_2 \int_{t-\tau}^{t} |X(s-\delta)|^2 ds \\ &+ \theta_3 \tau |X(t-\tau)|^2 - \theta_3 \int_{t-\tau}^{t} |X(s-\tau)|^2 ds \end{aligned} \tag{16}$$

and $M(t)$ is a martingale with $M(0) = 0$ (whose form is of no use in this paper). Making use of (11) and introducing the fourth free positive number $\theta_4 \in (0, \lambda_1/\alpha_2)$, we get, from (16), that

$$
\begin{aligned}
&LV(\hat{X}_t, r(t), t)\\
&\le -(\lambda_1 - \alpha_2\theta_4 - \theta_1\tau)|X(t)|^2 + \lambda_3|X(t)|\\
&+ (\lambda_2 + \theta_2\tau)|X(t-\delta)|^2 + \lambda_4|X(t-\delta)| + \lambda_0\\
&+ \theta_3\tau|X(t-\tau)|^2 + (\alpha_2/\theta_4)|X(t) - X(t-\tau)|^2\\
&- \theta_1\int_{t-\tau}^t |X(s)|^2 ds - \theta_2\int_{t-\tau}^t |X(s-\delta)|^2 ds - \theta_3\int_{t-\tau}^t |X(s-\tau)|^2 ds \qquad (17)
\end{aligned}
$$

for $t \ge \tau$.

Applying the Hölder inequality and the martingale inequality, it can be derived from (3) that

$$
\begin{aligned}
&\mathbb{E}|X(t) - X(t-\tau)|^2\\
\le& 2\tau\mathbb{E}\int_{t-\tau}^t |f(X(s), X(s-\delta), r(s)) + A(r(s))X(s-\tau)|^2 ds + 2\mathbb{E}\int_{t-\tau}^t |g(X(s), X(s-\delta), r(s))|^2 ds\\
\le& 4\tau\left(\mathbb{E}\int_{t-\tau}^t (2a_1(|X(s)|^2 + |X(s-\delta)|^2) + a_0)ds + \mathbb{E}\int_{t-\tau}^t \alpha_1|X(s-\tau)|^2 ds\right)\\
&+ 2\mathbb{E}\int_{t-\tau}^t \big(2a_2(|X(s)|^2 + |X(s-\delta)|^2) + a_0\big)\, ds\\
\le& (4a_0\tau^2 + 2a_0\tau) + (8\tau a_1 + 4a_2)\int_{t-\tau}^t \mathbb{E}|X(s)|^2 ds\\
&+ (8\tau a_1 + 4a_2)\int_{t-\tau}^t \mathbb{E}|X(s-\delta)|^2 ds + 4\tau\alpha_1\int_{t-\tau}^t \mathbb{E}|X(s-\tau)|^2 ds. \qquad (18)
\end{aligned}
$$

We therefore obtain that

$$
\begin{aligned}
&\mathbb{E}(LV(\hat{X}_t, r(t), t))\\
&\le -(\lambda_1 - \alpha_2\theta_4 - \theta_1\tau)\mathbb{E}|X(t)|^2 + \lambda_3\mathbb{E}|X(t)|\\
&+ (\lambda_2 + \theta_2\tau)E|X(t-\delta)|^2 + \lambda_4\mathbb{E}|X(t-\delta)|\\
&+ \theta_3\tau\mathbb{E}|X(t-\tau)|^2 + \big(\lambda_0 + (\alpha_2/\theta_4)(4a_0\tau^2 + 2a_0\tau)\big)\\
&- \left(\theta_1 - \frac{\alpha_2}{\theta_4}(8\tau a_1 + 4a_2)\right)\int_{t-\tau}^t \mathbb{E}|X(s)|^2 ds\\
&- \left(\theta_2 - \frac{\alpha_2}{\theta_4}(8\tau a_1 + 4a_2)\right)\int_{t-\tau}^t \mathbb{E}|X(s-\delta)|^2 ds\\
&- \left(\theta_3 - \frac{\alpha_2}{\theta_4}4\tau\alpha_1\right)\int_{t-\tau}^t \mathbb{E}|X(s-\tau)|^2 ds \qquad (19)
\end{aligned}
$$

holds for $t \ge \tau$.

Let four free positive parameters $\theta_1 \sim \theta_4$ take values in the set

$$
\Theta = \{(\theta_1, \theta_2, \theta_3, \theta_4) | \theta_1\theta_4 > 4a_2\alpha_2, \theta_2\theta_4 > 4a_2\alpha_2, \theta_3 > 0, \alpha_2\theta_4 < \lambda_1 - \lambda_2\} \qquad (20)
$$

and set

$$\tau^* = \sup_{(\theta_1,\theta_2,\theta_3,\theta_4)\in\Theta} \left(\frac{\lambda_1 - \lambda_2 - \alpha_2\theta_4}{\theta_1+\theta_2+\theta_3} \wedge \frac{\theta_1\theta_4 - 4a_2\alpha_2}{8a_1\alpha_2} \wedge \frac{\theta_2\theta_4 - 4a_2\alpha_2}{8a_1\alpha_2} \wedge \frac{\theta_3\theta_4}{4\alpha_1\alpha_2} \right). \quad (21)$$

Lemma 3.1. *For $\tau < \tau^*$, the solution for equation (6) satisfies*

$$\mathbb{E}\|X_t^{\xi,i_0}\|^2 \le C_1(1+\|\xi\|^2) \quad (22)$$

for any $t>0$, where C_1 is a positive constant independent of (ξ, i_0).

Proof. Set $\beta_0 = \lambda_0 + (\alpha_2/\theta_4)(4a_0\tau^2 + 2a_0\tau)$, $\beta_1 = \lambda_1 - \alpha_2\theta_4 - \theta_1\tau$, $\beta_2 = \lambda_2 + \theta_2\tau$, $\beta_3 = \theta_3\tau$ and

$$\beta_4 = \min\left(\theta_1 - \frac{\alpha_2}{\theta_4}(8\tau a_1 + 4a_2), \theta_2 - \frac{\alpha_2}{\theta_4}(8\tau a_1 + 4a_2), \theta_3 - \frac{\alpha_2}{\theta_4}4\tau\alpha_1\right).$$

From (19), we have

$$\mathbb{E}(LV(\hat{X}_t, r(t), t)) \le \beta_0 - \beta_1\mathbb{E}|X(t)|^2 + \lambda_3\mathbb{E}|X(t)| + \beta_2\mathbb{E}|X(t-\delta)|^2 + \lambda_4\mathbb{E}|X(t-\delta)| + \beta_3\mathbb{E}|X(t-\tau)|^2 - \beta_4\int_{t-\tau-\bar{\delta}}^t \mathbb{E}|X(s)|^2 ds. \quad (23)$$

Set

$$\hat{\beta} = \frac{\beta_1 - \beta_2 - \beta_3}{3} > 0,$$

and rewrite above inequality (23) as

$$\begin{aligned}\mathbb{E}(LV(\hat{X}_t, r(t), t)) &\le \beta_0 - (\beta_1 - \hat{\beta})\mathbb{E}|X(t)|^2 + (\beta_2 + \hat{\beta})\mathbb{E}|X(t-\delta)|^2 + \beta_3\mathbb{E}|X(t-\tau)|^2 \\ &+ (-\hat{\beta}\mathbb{E}|X(t)|^2 + \lambda_3\mathbb{E}|X(t)|) + (-\hat{\beta}\mathbb{E}|X(t-\delta)|^2 + \lambda_4\mathbb{E}|X(t-\delta)|) \\ &- \beta_4\int_{t-\tau-\bar{\delta}}^t \mathbb{E}|X(s)|^2 ds. \end{aligned} \quad (24)$$

Obviously, we have $-\hat{\beta}\mathbb{E}|X(t)|^2 + \lambda_3\mathbb{E}|X(t)| \le \frac{\lambda_3^2}{4\hat{\beta}^2}$ and $-\hat{\beta}\mathbb{E}|X(t-\delta)|^2 + \lambda_4\mathbb{E}|X(t-\delta)| \le \frac{\lambda_4^2}{4\hat{\beta}^2}$.

And then (24) gives

$$\begin{aligned}&\mathbb{E}(LV(\hat{X}_t, r(t), t)) \\ &\le \left(\beta_0 + \frac{\lambda_3^2 + \lambda_4^2}{4\hat{\beta}^2}\right) - (\beta_1 - \hat{\beta})\mathbb{E}|X(t)|^2 + (\beta_2 + \hat{\beta})\mathbb{E}|X(t-\delta)|^2 + \beta_3\mathbb{E}|X(t-\tau)|^2 \\ &\quad - \beta_4\int_{t-\tau-\bar{\delta}}^t \mathbb{E}|X(s)|^2 ds. \end{aligned} \quad (25)$$

By the definition of τ^*, we can find a constant $\kappa_1 > 0$ such that $c_3\tau\kappa_1 < \beta_4$ and $c_2\kappa_1 - (\beta_1 - \hat{\beta}) + (\beta_2 + \hat{\beta})e^{\kappa_1\delta} + \beta_3 e^{\kappa_1\tau} < 0$ hold at the same time. Applying Ito's lemma to $e^{\kappa_1 t}V(\hat{X}_t, r(t), t)$, we will have

$$
\begin{aligned}
& e^{\kappa_1 t}\mathbb{E}(V(\hat{X}_t, r(t), t)) - e^{\kappa_1 \tau}\mathbb{E}(V(\hat{X}_\tau, r(\tau), \tau)) \\
=& \int_\tau^t \left(\kappa_1 e^{\kappa_1 s}\mathbb{E}V(\hat{X}_s, r(s), s) + e^{\kappa_1 s}\mathbb{E}LV(\hat{X}_s, r(s), s)\right) ds \\
\le& \int_\tau^t \left(\beta_0 + \frac{\lambda_3^2+\lambda_4^2}{4\hat{\beta}^2}\right) e^{\kappa_1 s} ds \\
& + \int_\tau^t e^{\kappa_1 s}\left(-(\beta_1 - \hat{\beta} - \kappa_1 c_2)\mathbb{E}|X(s)|^2 + (\beta_2+\hat{\beta})\mathbb{E}|X(s-\delta)|^2 + \beta_3\mathbb{E}|X(s-\tau)|^2\right) ds \\
& - (\beta_4 - c_3\tau\kappa_1)\int_\tau^t e^{\kappa_1 s}\int_{s-\tau-\bar{\delta}}^s \mathbb{E}|X(v)|^2 dv ds \\
\le& \frac{1}{\kappa_1}\left(\beta_0 + \frac{\lambda_3^2+\lambda_4^2}{4\hat{\beta}^2}\right) e^{\kappa_1 t} - (\beta_1 - \hat{\beta} - \kappa_1 c_2)\int_\tau^t e^{\kappa_1 s}\mathbb{E}|X(s)|^2 ds \\
& + (\beta_2+\hat{\beta})e^{\kappa_1\delta}\int_{\tau-\delta}^{t-\delta} e^{\kappa_1 s}\mathbb{E}|X(s)|^2 ds + \beta_3 e^{\kappa_1\tau}\int_0^{t-\tau} e^{\kappa_1 s}\mathbb{E}|X(s)|^2 ds \\
\le& \frac{1}{\kappa_1}\left(\beta_0 + \frac{\lambda_3^2+\lambda_4^2}{4\hat{\beta}^2}\right) e^{\kappa_1 t} + (\beta_2+\hat{\beta})e^{\kappa_1(\delta+\tau)}\int_{\tau-\delta}^{\tau}\mathbb{E}|X(s)|^2 ds + \beta_3 e^{2\kappa_1\tau}\int_0^\tau \mathbb{E}|X(s)|^2 ds.
\end{aligned}
\tag{26}
$$

By (8) and (11), we can conclude that as $t \ge \tau$,

$$
\mathbb{E}|X(t)|^2 \le C_2(1 + \|\xi\|^2)
$$

holds for some positive constant C_2 independent on (ξ, i_0).

Applying the well-known BDG inequality, we can derive following bound for $\mathbb{E}\|X_t\|^2$ as $t \ge \tau + \bar{\delta}$, which is

$$
\begin{aligned}
\mathbb{E}\|X_t\|^2 \le& 3\mathbb{E}|X(t-\bar{\delta})|^2 + 3\mathbb{E}\left(\sup_{t-\bar{\delta}\le s\le t}\left|\int_{t-\bar{\delta}}^s [f(X(v), X(v-\delta), r(v)) + A(r(v))X(v-\tau)]dv\right|^2\right) \\
& + 3\mathbb{E}\left(\sup_{t-\bar{\delta}\le s\le t}\left|\int_{t-\bar{\delta}}^s g(X(v), X(v-\delta), r(v))dB_v\right|^2\right) \\
\le& 3C_2(1+\|\xi\|^2) + 6\bar{\delta}\mathbb{E}\left(\int_{t-\bar{\delta}}^t 2\big(2a_1(|X(v)|^2 + |X(v-\delta)|^2)\big) + a_0 + a_3|X(v-\tau)|^2 dv\right) \\
& + 12\mathbb{E}\left(\int_{t-\bar{\delta}}^t [2a_2\big(|X(v)|^2 + |X(v-\delta)|^2\big) + a_0]dv\right) \\
\le& C_1(1+\|\xi\|^2)
\end{aligned}
\tag{27}
$$

with C_1 independent on the initial data. Then the required assertion (22) can be verified from (8) and (27) together.

Lemma 3.2. *If $\tau < \tau^*$, then for any $(\xi, \eta, i_0) \in \mathcal{C}_{\bar{\delta}} \times \mathcal{C}_{\bar{\delta}} \times \mathbb{S}$,*

$$
\mathbb{E}\|X_t^{\xi,i_0} - X_t^{\eta,i_0}\| \le C_3\|\xi - \eta\|^2 e^{-\kappa_2 t} \tag{28}
$$

for all $t \geq \tau + \bar{\delta}$, *where* C_3 *and* κ_2 *are positive constants independent of* (ξ, η, i_0).

Proof. Set $H(t) = X^{\xi,i_0}(t) - X^{\eta,i_0}(t)$ and $\hat{H}_t = \{H(t+u)| -\tau - \bar{\delta} \leq u \leq 0\}$ as $t \geq \tau + \bar{\delta}$.

Obviously, we have following differential rule for $H(t)$,

$$\begin{aligned}dH(t) =&\big[f(X^{\xi,i_0}(t), X^{\xi,i_0}(t-\delta), r(t)) - f(X^{\eta,i_0}(t), X^{\eta,i_0}(t-\delta), r(t)) + A(r(t))H(t-\tau)\big]dt\\&+ \big[g(X^{\xi,i_0}(t), X^{\xi,i_0}(t-\delta), r(t)) - g(X^{\eta,i_0}(t), X^{\eta,i_0}(t-\delta), r(t))\big]dB_t.\end{aligned}$$

By this rule, we can estimate

$$\begin{aligned}&\mathbb{E}|H(t) - H(t-\tau)|^2\\\leq&2\tau\mathbb{E}\int_{t-\tau}^{t} \Big|f(X^{\xi,i_0}(s), X^{\xi,i_0}(s-\delta), r(s)) - f(X^{\eta,i_0}(s), X^{\eta,i_0}(s-\delta), r(s)) + A(r(s))H(s-\tau)\Big|^2 ds\\&+ 2\mathbb{E}\int_{t-\tau}^{t} \Big|g(X^{\xi,i_0}(s), X^{\xi,i_0}(s-\delta), r(s)) - g(X^{\eta,i_0}(s), X^{\eta,i_0}(s-\delta), r(s))\Big|^2 ds\\\leq&(4a_1\tau + 2a_2)\mathbb{E}\int_{t-\tau}^{t} \big(|H(s)|^2 + |H(s-\delta)|^2\big)ds + 4\alpha_1\tau\mathbb{E}\int_{t-\tau}^{t} |H(s-\tau)|^2 ds \qquad (29)\end{aligned}$$

and use the same symbols as in Lemma 3.1 to get

$$\begin{aligned}LV(\hat{H}_t, r(t), t) =&\Psi\big(X^{\xi,i_0}(t), X^{\eta,i_0}(t), X^{\xi,i_0}(t-\delta), X^{\eta,i_0}(t-\delta), r(t)\big)\\&- 2H^T(t)W(r(t))A(r(t))\big(H(t) - H(t-\tau)\big)\\&+ \theta_1\tau|H(t)|^2 - \theta_1\int_{t-\tau}^{t}|H(s)|^2 ds + \theta_2\tau|H(t-\delta)|^2 - \theta_2\int_{t-\tau}^{t}|H(s-\delta)|^2 ds\\&+ \theta_3\tau|H(t-\tau)|^2 - \theta_3\int_{t-\tau}^{t}|H(s-\tau)|^2 ds\\\leq& -(\lambda_1 - \alpha_2\theta_4 - \theta_1\tau)|H(t)|^2 + (\lambda_2 + \theta_2\tau)|H(t-\delta)|^2 + \theta_3\tau|H(t-\tau)|^2\\&- \theta_1\int_{t-\tau}^{t}|H(s)|^2 ds - \theta_2\int_{t-\tau}^{t}|H(s-\delta)|^2 ds - \theta_3\int_{t-\tau}^{t}|H(s-\tau)|^2 ds\\&+ \alpha_2/\theta_4|H(t) - H(t-\tau)|^2.\end{aligned}$$

Taking expectation on both sides and using (29) yields

$$\begin{aligned}\mathbb{E}LV(\hat{H}_t, r(t), t) \leq& -(\lambda_1 - \alpha_2\theta_4 - \theta_1\tau)\mathbb{E}|H(t)|^2 + (\lambda_2 + \theta_2\tau)\mathbb{E}|H(t-\delta)|^2 + \theta_3\tau\mathbb{E}|H(t-\tau)|^2\\&- \big(\theta_1 - \alpha_2/\theta_4(4a_1\tau + 2a_2)\big)\mathbb{E}\int_{t-\tau}^{t}|H(s)|^2 ds\\&- \big(\theta_2 - \alpha_2/\theta_4(4a_1\tau + 2a_2)\big)\mathbb{E}\int_{t-\tau}^{t}|H(s-\delta)|^2 ds\\&- \big(\theta_3 - 4\alpha_1\alpha_2\tau/\theta_4\big)\mathbb{E}\int_{t-\tau}^{t}|H(s-\tau)|^2 ds.\end{aligned}$$

As $\lambda_1 - \lambda_2 - \alpha_2\theta_4 - (\theta_1 + \theta_2 + \theta_3)\tau > 0$, $\theta_1 - \alpha_2(4a_1\tau + 2a_2)/\theta_4 > 0$, $\theta_2 - \alpha_2(4a_1\tau +$

$2a_2)/\theta_4 > 0$ and $\theta_3 - 4\alpha_1\alpha_2\tau/\theta_4 > 0$, we have

$$\mathbb{E}LV(\hat{H}_t, r(t), t) \le -\beta_1\mathbb{E}|H(t)|^2 + \beta_2\mathbb{E}|H(t-\delta)|^2 + \beta_3\mathbb{E}|H(t-\tau)|^2 - \beta_5\int_{t-\tau-h}^{t}\mathbb{E}|X(s)|^2ds$$

where $\beta_1, \beta_2, \beta_3$ are defined as in Lemma 3.1 with $\beta_1 > \beta_2 + \beta_3$ and

$$\beta_5 = \min\left(\theta_1 - \frac{\alpha_2}{\theta_4}(4\tau a_1 + 2a_2), \theta_2 - \frac{\alpha_2}{\theta_4}(4\tau a_1 + 2a_2), \theta_3 - \frac{\alpha_2}{\theta_4}4\tau\alpha_1\right) > 0.$$

Now following the same argument as in Lemma 3.1, and fixing some $0 < \kappa_2 < \min\left(\frac{\beta_1-\beta_2-\beta_3}{c_2}, \frac{\beta_5}{c_3\tau}\right)$, we can show that $\mathbb{E}|H(t)|^2 \le C_3\|\xi - \eta\|^2 e^{-\kappa_2 t}$ for some positive constant C_3 independent on (ξ, η) and i_0. And then (28) can be derived similarly as (22).

Lemma 3.3. *If $\tau < \tau^*$, then for any $R > 0$,*

$$\lim_{t\to\infty} d(\mathcal{L}(X_t^{\xi,i}), \mathcal{L}(X_t^{\eta,j})) = 0$$

holds uniformly for any $(\xi, i, \eta, j) \in K \times \mathbb{S} \times K \times \mathbb{S}$, where $K = \{\zeta | \|\zeta\| \le R\}$.

Proof. By the ergodic property of the Markov chain, the stopping time $\tau_{ij} = \inf\{t | r^i(t) = r^j(t)\}$ is finite.

Given any $\epsilon \in (0, 1)$, there is a $T_1 > 0$ such that $\mathbb{P}(\tau_{ij} \le T_1) > 1 - \frac{\epsilon}{6}$ holds for any $i, j \in \mathbb{S}$ because of the finiteness of $\mathbb{S}$.

As shown in (Mao & Yuan, 2006), the solution of equation (6) satisfies

$$\sup_{(\xi,i)\in K\times\mathbb{S}} \mathbb{E}\left(\sup_{-\bar{\delta}\le t\le T_1}|X^{\xi,i}(t)|^2\right) < \infty,$$

so that there exists a large number R_1 such that

$$\mathbb{P}(\Omega_{\xi,i}) > 1 - \frac{\epsilon}{12}$$

holds for any $(\xi, i) \in K \times \mathbb{S}$, where $\Omega_{\xi,i}$ is defined by

$$\Omega_{\xi,i} = \{\omega \in \Omega | \sup_{-h\le t\le T_1}|X^{\xi,i}(t, \omega)| \le R_1\}.$$

Fix any $(\xi, i, \eta, j) \in K \times \mathbb{S} \times K \times \mathbb{S}$. For any $\psi \in L$ and $t > T_1$, it will be held that

$$\begin{aligned}
&|\mathbb{E}\psi(X_t^{\xi,i}) - \mathbb{E}\psi(X_t^{\eta,j})| \\
\le&\mathbb{E}|\psi(X_t^{\xi,i}) - \psi(X_t^{\eta,j})| \\
=&\mathbb{E}\big(I_{\{\tau_{ij}>T_1\}}|\psi(X_t^{\xi,i}) - \psi(X_t^{\eta,j})|\big) + \mathbb{E}\big(I_{\{\tau_{ij}\le T_1\}}|\psi(X_t^{\xi,i}) - \psi(X_t^{\eta,j})|\big) \\
\le&2\mathbb{P}\big(\tau_{ij} > T_1\big) + \mathbb{E}\big(I_{\{\tau_{ij}\le T_1\}}|\psi(X_t^{\xi,i}) - \psi(X_t^{\eta,j})|\big) \\
\le&\frac{\epsilon}{3} + \mathbb{E}\big(I_{\{\tau_{ij}\le T_1\}}|\psi(X_t^{\xi,i}) - \psi(X_t^{\eta,j})|\big).
\end{aligned} \tag{30}$$

Let $\bar{\xi} = X^{\xi,i}_{\tau_{ij}}, \bar{\eta} = X^{\eta,j}_{\tau_{ij}}$ and $\bar{\tau} = r^i(\tau_{ij}) = r^j(\tau_{ij})$. Notice that

$$
\begin{aligned}
&\mathbb{E}\Big(I_{\{\tau_{ij}\le T_1\}}|\psi(X^{\xi,i}_t) - \psi(X^{\eta,j}_t)|\Big)\\
=&\mathbb{E}\Big(I_{\{\tau_{ij}\le T_1\}}\mathbb{E}\big(|\psi(X^{\xi,i}_t) - \psi(X^{\eta,j}_t)|\big)|\mathcal{F}_{\tau_{ij}}\Big)\\
=&\mathbb{E}\Big(I_{\{\tau_{ij}\le T_1\}}\mathbb{E}\big(|\psi(X^{\bar{\xi},\bar{\tau}}_{t-\tau_{ij}}) - \psi(X^{\bar{\eta},\bar{\tau}}_{t-\tau_{ij}})|\big)\Big)\\
\le&\mathbb{E}\Big(I_{\{\{\tau_{ij}\le T_1\}\cap\Omega_{\bar{\xi},\bar{\tau}}\cap\Omega_{\bar{\eta},\bar{\tau}}\}}\mathbb{E}\big(|\psi(X^{\bar{\xi},\bar{\tau}}_{t-\tau_{ij}}) - \psi(X^{\bar{\eta},\bar{\tau}}_{t-\tau_{ij}})|\big)\Big) + \mathbb{E}\Big(I_{\bar{\Omega}_{\bar{\xi},\bar{\tau}}\cup\bar{\Omega}_{\bar{\eta},\bar{\tau}}}\mathbb{E}\big(|\psi(X^{\bar{\xi},\bar{\tau}}_{t-\tau_{ij}}) - \psi(X^{\bar{\eta},\bar{\tau}}_{t-\tau_{ij}})|\big)\Big).
\end{aligned}
$$

By Lemma 3.2, for any $\omega \in \{\tau_{ij} \le T_1\} \cap \Omega_{\bar{\xi},\bar{\tau}} \cap \Omega_{\bar{\eta},\bar{\tau}}$, we see $\|\bar{\xi}\| \le R_1$ and $\|\bar{\eta}\| \le R_1$, so that there exists another $T_2 > 0$ such that for any $t > T_1 + T_2$,

$$\mathbb{E}\|X^{\bar{\xi},\bar{\tau}}_{t-\tau_{ij}} - X^{\bar{\eta},\bar{\tau}}_{t-\tau_{ij}}\| < \frac{\epsilon}{3},$$

and then

$$\mathbb{E}\Big(I_{\{\{\tau_{ij}\le T_1\}\cap\Omega_{\bar{\xi},\bar{\tau}}\cap\Omega_{\bar{\eta},\bar{\tau}}\}}\mathbb{E}\big(|\psi(X^{\bar{\xi},\bar{\tau}}_{t-\tau_{ij}}) - \psi(X^{\bar{\eta},\bar{\tau}}_{t-\tau_{ij}})|\big)\Big) \le \frac{\epsilon}{3} \tag{31}$$

holds by the definition of ψ.

On the other hand, we have

$$\mathbb{E}\Big(I_{\bar{\Omega}_{\bar{\xi},\bar{\tau}}\cup\bar{\Omega}_{\bar{\eta},\bar{\tau}}}\mathbb{E}\big(|\psi(X^{\bar{\xi},\bar{\tau}}_{t-\tau_{ij}})-\psi(X^{\bar{\eta},\bar{\tau}}_{t-\tau_{ij}})|\big)\Big) \le 2\mathbb{P}\big(\bar{\Omega}_{\bar{\xi},\bar{\tau}}\cup\bar{\Omega}_{\bar{\eta},\bar{\tau}}\big) \le 2\big(\mathbb{P}(\bar{\Omega}_{\bar{\xi},\bar{\tau}})+\mathbb{P}(\bar{\Omega}_{\bar{\eta},\bar{\tau}})\big) \le \frac{\epsilon}{3}, \tag{32}$$

Substituting (31) and (32) into (30), it is directly derived that for any $t > T_1 + T_2$,

$$|\mathbb{E}\psi(X^{\xi,i}_t) - \mathbb{E}\psi(X^{\eta,j}_t)| \le \epsilon.$$

Subsequently, we have

$$\lim_{t\to\infty} d(\mathcal{L}(X^{\xi,i}_t), \mathcal{L}(X^{\eta,j}_t)) = 0.$$

for any $(\xi, i, \eta, j) \in K \times S \times K \times S$.

Theorem 3.4. *If $\tau < \tau^*$, there exists a unique probability measure $\mu_{\bar{\delta}} \in \mathcal{P}(\mathcal{C}_{\bar{\delta}})$ such that*

$$\lim_{t\to\infty} d(\mathcal{L}(X^{\xi,i_0}_t), \mu_{\bar{\delta}}) = 0$$

for any $(\xi, i_0) \in \mathcal{C}_{\bar{\delta}} \times \mathbb{S}$.

Proof. Firstly, we show that for fixed $(\xi, i_0) \in \mathcal{C}_{\bar{\delta}} \times \mathbb{S}$, $\{\mathcal{L}(X^{\xi,i_0}_t)|t \ge 0\}$ is Cauchy in the metric space $\mathcal{P}\big(\mathcal{C}_{\bar{\delta}}\big)$ with the metric measure d.

We will show that for any $\varepsilon \in (0, 1)$, there exists a positive number $\bar{T}_1 = \bar{T}_1(\varepsilon) > 0$ such that for any $t > \bar{T}_1$ and $s > 0$,

$$d(\mathcal{L}(X^{\xi,i_0}_{t+s}), \mathcal{L}(X^{\xi,i_0}_t)) < \varepsilon. \tag{33}$$

By Lemma 3.1, there exists a $\bar{R} > 0$ such that for any $s > 0$,

$$\mathbb{P}\{\omega \in \Omega \big| \|X_s^{\xi,i_0}\| \le \bar{R}\} \ge 1 - \varepsilon/4. \tag{34}$$

For any $\psi \in L$, it can be derived that

$$\begin{aligned} & |\mathbb{E}\psi(X_{t+s}^{\xi,i_0}) - \mathbb{E}\psi(X_t^{\xi,i_0})| \\ = \; & |\mathbb{E}\Big(\mathbb{E}\big(\psi(X_{t+s}^{\xi,i_0})|\mathcal{F}_s\big)\Big) - \mathbb{E}\psi(X_t^{\xi,i_0})| \\ = \; & \Big| \sum_{i\in\mathbb{S}} \int_{\mathcal{C}_{\bar{\delta}}} \mathbb{E}\psi(X_t^{\zeta,j}) p(s,\xi,i_0;d\zeta\times\{j\}) - \mathbb{E}\psi(X_t^{\xi,i_0})\Big| \\ \le \; & \sum_{i\in\mathbb{S}} \int_{\mathcal{C}_{\bar{\delta}}} \Big|\mathbb{E}\psi(X_t^{\zeta,j}) - \mathbb{E}\psi(X_t^{\xi,i_0})\Big| p(s,\xi,i_0;d\zeta\times\{j\}). \end{aligned} \tag{35}$$

Define a compact set $K = \{\zeta \big| \|\zeta\| \le \bar{R}\}$ and denote its complementary set as K^c. By (34) and the definition of ψ, we can get

$$\sum_{i\in\mathbb{S}} \int_{K^c} \Big|\mathbb{E}\psi(X_t^{\zeta,j}) - \mathbb{E}\psi(X_t^{\xi,i_0})\Big| p(s,\xi,i_0;d\zeta\times\{j\}) \le 2\mathbb{P}\{\omega \in \bar{K}\} \le \varepsilon/2. \tag{36}$$

While by Lemma 3.3, there exists another positive number $\bar{T}_2$ such that for any $t > \bar{T}_2$, $d(\mathcal{L}(X_t^{\zeta,j}), \mathcal{L}(X_t^{\xi,i_0})) < \varepsilon/2$. Then by the definition of d, we see

$$\sum_{i\in\mathbb{S}} \int_{K} \Big|\mathbb{E}\psi(X_t^{\zeta,j}) - \mathbb{E}\psi(X_t^{\xi,i_0})\Big| p(s,\xi,i_0;d\zeta\times\{j\}) \le d(\mathcal{L}(X_t^{\zeta,j}), X_t^{\xi,i_0})) < \varepsilon/2. \tag{37}$$

From (36) and (37), (33) will be true for $t > \bar{T}_1 \vee \bar{T}_2$.

Taking $\xi = 0$ and $i_0 = 1$ as the initial data, there exists a unique probability measure $\mu_{\bar{\delta}} \in \mathcal{P}(C_{\bar{\delta}})$ such that $\lim_{t\to\infty} d(\mathcal{L}(X_t^{0,1}), \mu_{\bar{\delta}}) = 0$ holds as $\{\mathcal{L}(X_t^{0,1}) | t \ge 0\}$ is a Cauchy sequence.

Then for any $(\xi, i_0) \in \mathcal{C}_{\bar{\delta}} \times \mathbb{S}$, applying the triangle inequality

$$d(\mathcal{L}(X_t^{\xi,i_0}), \mu_{\bar{\delta}}) \le d(\mathcal{L}(X_t^{\xi,i_0}), \mathcal{L}(X_t^{0,1})) + d(\mathcal{L}(X_t^{0,1}), \mu_{\bar{\delta}}),$$

and Lemma 3.3, we get $\lim_{t\to\infty} d(\mathcal{L}(X_t^{\xi,i_0}), \mu_{\bar{\delta}}) = 0$ as required.

4. LMIs for calculating A_is

In this section, we will apply LMI techniques to seek matrices $A_i, i \in \mathbb{S}$ for stabilization.

Generally, choose N positively definitely symmetric matrices $W_i > 0, i \in \mathbb{S}$ and express coefficients f and g as

$$\begin{aligned} & 2(x-y)^T W_i[f(x,z_1,i) - f(y,z_2,i)] \\ & + \text{trace}[(g(x,z_1,i) - g(y,z_2,i))^T W_i (g(x,z_1,i) - g(y,z_2,i))] \\ \le & (x-y)^T W_i^1 (x-y) + (z_1-z_2)^T W_i^2 (z_1-z_2), \end{aligned} \tag{38}$$

where W_i^1 and W_i^2, $i \in \mathbb{S}$ are N pairs of symmetric matrices. Then we look for N matrices $A_i, i \in \mathbb{S}$ and a constant $a > 0$ such that following LMIs hold

$$W_i^1 + W_i A_i + A_i^T W_i + \sum_{j=1}^{N} \gamma_{ij} W_j + aI < 0$$
$$aI - W_i^2 > 0. \tag{39}$$

If $A_i, i \in \mathbb{S}$ make (38) and (39) hold, we can see that (10) will be held with

$$\lambda_1 = -\max_{i \in \mathbb{S}} \lambda_{\max}\big(W_i^1 + W_i A_i + A_i^T W_i + \sum_{j=1}^{N} \gamma_{ij} W_j\big)$$

and

$$\lambda_2 = \max_{i \in \mathbb{S}} \lambda_{\max}\big(W_i^2\big),$$

because we have $\lambda_1 > a > \lambda_2$.

In the case of a linear system, we can find W_i and $A_i, i \in \mathbb{S}$ by LMIs simultaneously. Let us discuss that in detail. Suppose f and g have following linear forms

$$\begin{aligned} f(x,z,i) =& k_i + K_i^1 x + K_i^2 z, \\ g(x,z,i) =& (l_{i1} + L_{i1}^1 x + L_{i1}^2 z, l_{i2} + L_{i2}^1 x + L_{i2}^2 z, \cdots, l_{im} + L_{im}^1 x + L_{im}^2 z), i \in \mathbb{S}, \end{aligned} \tag{40}$$

where $k_i, l_{ij} \in \mathbb{R}^n$ and $K_i^1, K_i^2, L_{ij}^1, L_{ij}^2 \in \mathbb{R}^{n\times n}$. It can be directly verified that $\Psi(x, y, z_1, z_2, i)$ in Assumption 2.1 has the form of

$$\Psi(x, y, z_1, z_2, i) \le (x-y)^T U_i (x-y) + (z_1 - z_2)^T V_i (z_1 - z_2) \tag{41}$$

with

$$U_i = W_i(K_i^1 + A_i) + (K_i^1 + A_i)^T W_i + W_i + 2\sum_{l=1}^{m} (L_{il}^1)^T W_i L_{il}^1 + \sum_{j=1}^{N} \gamma_{ij} W_j$$

and

$$V_i = (K_i^2)^T W_i K_i^2 + 2\sum_{l=1}^{m} (L_{il}^2)^T W_i L_{il}^2.$$

Set $D_i = W_i A_i$. Now we can state following corollary for a linear system.

Corollary 4.1. *If there exist N pairs of matrices $(W_i, D_i), i = 1, 2, \cdots, N$ and a* *positive number $a > 0$ such that following linear matrix inequalities are held*

$$
\begin{aligned}
-W_i < 0, i = 1, 2 \cdots, N \\
W_i K_i^1 + (K_i^1)^T W_i + D_i + D_i^T + W_i + 2\sum_{l=1}^{m} (L_{il}^1)^T W_i L_{il}^1 + \sum_{j=1}^{N} \gamma_{ij} W_j + aI < 0 \\
-aI + (K_i^2)^T W_i K_i^2 + 2\sum_{l=1}^{m} ((L_{il}^2)^T W_i L_{il}^2) < 0
\end{aligned} \tag{42}
$$

then the controlled system (6) *with linear coefficients f, g as in* (40) *will be stable in distribution, where the controller is given by $A_i = W_i^{-1} D_i$ and $\tau < \tau^*$ calculated by* (21)*.*

The frequently used linear controllers have two structure forms: state feedback and output injection. In the case of state feedback, A_i has the form $A_i = F_i G_i$ with given $G_i \in \mathbb{R}^{l \times n}$ and unknown $F_i \in \mathbb{R}^{n \times l}$, where l is a fixed integer. Substituting $A_i = F_i G_i$ into (42), we can easily get following rules for designing F_i.

Corollary 4.2. *If the following LMIs*

$$
\begin{aligned}
W_i > 0, i = 1, 2 \cdots, N \\
W_i K_i^1 + (K_i^1)^T W_i + E_i G_i + G_i^T E_i^T + W_i + 2\sum_{l=1}^{m} (L_{il}^1)^T W_i L_{il}^1 + \sum_{j=1}^{N} \gamma_{ij} W_j + aI < 0 \\
-aI + (K_i^2)^T W_i K_i^2 + 2\sum_{l=1}^{m} ((L_{il}^2)^T W_i L_{il}^2) < 0
\end{aligned} \tag{43}
$$

have the solution $(W_i, E_i), i = 1, 2, \cdots, N$ and $a > 0$, then the controller can be expressed by $A_i = W_i^{-1} E_i G_i$ and τ^* given by (21)*.*

In the second case of output injection, A_i has the same form of $A_i = F_i G_i$, but with given F_i and unknown G_i. In order to get LMIs by applying Schur complement, we add two auxiliary positive constants $a, b > 0$ satisfying $ab < 1$ and rewrite (42), so that G_i can be given by solving LMIs.

By multiplying $\hat{W}_i = (W_i)^{-1}$ from both the left and the right hands, the second inequality in (42) is equivalent to

$$
\begin{aligned}
K_i^1 \hat{W}_i + \hat{W}_i (K_i^1)^T + F_i G_i \hat{W}_i + \hat{W}_i G_i^T F_i^T + \hat{W}_i \\
+2\sum_{l=1}^{m} \hat{W}_i (L_{il}^1)^T (\hat{W}_i)^{-1} L_{il}^1 \hat{W}_i + \sum_{j=1}^{N} \gamma_{ij} \hat{W}_i ((\hat{W}_j)^{-1}) \hat{W}_i + \frac{1}{a} \hat{W}_i \hat{W}_i < 0
\end{aligned}
$$

and the third is then rewritten as

$$
-bI + (K_i^2)^T (\hat{W}_i)^{-1} K_i^2 + 2\sum_{l=1}^{m} (L_{il}^2)^T (\hat{W}_i)^{-1} L_{il}^2 < 0.
$$

Now we can conclude as following corollary.

Corollary 4.3. *If there exist two positive constants a, b with $ab < 1$ and matrices $\hat{W}_i, \hat{E}_i, i = 1, 2, \cdots, N$ such that the following LMIs hold:*

$$
\begin{aligned}
&-\hat{W}_i < 0, i = 1, 2 \cdots, N \\
&\begin{bmatrix} U^i_{11} & U^i_{12} & U^i_{13} & \hat{W}_i \\ (U^i_{12})^T & U^i_{22} & & \\ (U^i_{13})^T & & U^i_{33} & \\ \hat{W}_i & & & -aI \end{bmatrix} < 0, i = 1, 2 \cdots, N \\
&\begin{bmatrix} -bI & (K^2_i)^T & V^i_{13} \\ K^2_i & -\hat{W}_i & \\ (V^i_{13})^T & & V^i_{33} \end{bmatrix} < 0, i = 1, 2 \cdots, N
\end{aligned} \tag{44}
$$

where

$$
\begin{aligned}
U^i_{11} =& K^1_i \hat{W}_i + \hat{W}_i (K^1_i)^T + F_i \hat{E}_i + \hat{E}_i F^T_i + (1 + \gamma_{ii}) \hat{W}_i \\
U^i_{12} =& [\sqrt{2}\hat{W}_i (L^1_{il})^T, \cdots, \sqrt{2}\hat{W}_i (L^1_{im})^T] \\
U^i_{13} =& [\sqrt{\gamma_{i1}}\hat{W}_i, \cdots, \sqrt{\gamma_{i(i-1)}}\hat{W}_i, \sqrt{\gamma_{i(i+1)}}\hat{W}_i, \cdots, \sqrt{\gamma_{iN}}\hat{W}_i] \\
U^i_{22} =& \mathrm{diag}\big(-\hat{W}_i, \cdots, -\hat{W}_i\big) \\
U^i_{33} =& \mathrm{diag}\big(-\hat{W}_1, \cdots, -\hat{W}_{i-1}, -\hat{W}_{i+1}, \cdots, -\hat{W}_N\big) \\
V^i_{13} =& [\sqrt{2}\hat{W}_i (L^2_{il})^T, \cdots, \sqrt{2}\hat{W}_i (L^2_{im})^T] \\
V^i_{33} =& U^i_{22}
\end{aligned}
$$

then the controller can be expressed by $A_i = F_i \hat{E}_i \hat{W}_i^{-1}$ and τ^* given by (21)*.*

5. Illustrative examples

Example 5.1. As the first example, consider following SDDE defined in $\mathbb{R}^2$,

$$
\begin{aligned}
dx(t) =& [k(r(t)) + K^1(r(t))x(t) + K^2(r(t))x(t-\delta)]dt \\
&+ [l(r(t)) + L^1(r(t))x(t) + L^2(r(t))x(t-\delta)]dB(t)
\end{aligned} \tag{45}
$$

where $B(t)$ is a one dimensional Brownian motion, and $r(t)$ is an independent Markov chain defined in $\mathbb{S} = \{1, 2\}$ with the transition matrix $\Gamma = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$. The matrices are given by

$$
\begin{aligned}
k_1 &= \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix}, \quad K^1_1 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad K^2_1 = \begin{pmatrix} 0.1 & 0.1 \\ 0.2 & 0.3 \end{pmatrix}, \\
k_2 &= \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, \quad K^1_2 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad K^2_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
l_1 &= \begin{pmatrix} 0.3 \\ 0.1 \end{pmatrix}, \quad L^1_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad L^2_1 = \begin{pmatrix} 0.1 & 0 \\ 0.1 & 0.2 \end{pmatrix}, \\
l_2 &= \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}, \quad L^1_2 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad L^2_2 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix},
\end{aligned}
$$

and $\delta = 0.5$.

Computer simulation is performed to show that the system (45) is unstable in distribution, as illustrated in Figure 1. The middle and down sub-graphs show the trajectories of $E(x_1(t))$ with different constant initial conditions $(4,4)^T$ and $(-4,-4)^T$, which show that $E(x_1(t))$ tends to $+\infty$ and $-\infty$, respectively.

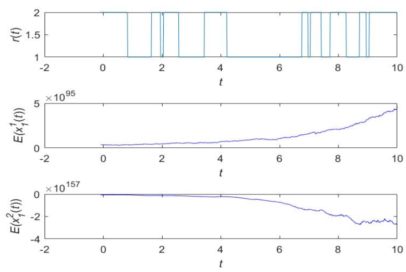

Figure 1. Up: the trajectory of $r(t)$. Middle: the trajectory of $E(x_1(t))$ with the constant initial condition $\xi = (4,4)^T$. Down: the trajectory of $E(x_1(t))$ with the constant initial condition $\xi = (-4,-4)^T$.

Now we try to seek linear controls to stabilize (45) in distribution.

(1) In the first case of state feedback, set $G_1 = (1,1)$ and $G_2 = (0,1)$. The feasible matrices can be calculated from LMIs in (43) as

$$W_1 = \begin{pmatrix} 3.2568 & -5.6426 \\ -5.6426 & 24.5533 \end{pmatrix}, W_2 = \begin{pmatrix} 22.8685 & 15.0330 \\ 15.0330 & 11.9438 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} -29.2424 \\ -9.7990 \end{pmatrix}, E_2 = \begin{pmatrix} 23.0111 \\ -29.8024 \end{pmatrix},$$

and $a = 8.0339$. And then the controllers are given by

$$A_1 = \begin{pmatrix} -16.0677 & -16.0677 \\ -4.0916 & -4.0916 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 15.3329 \\ 0 & -21.7941 \end{pmatrix}.$$

In order to calculate τ, we firstly use above matrices to get $\lambda_1 = 10.8920$, $\lambda_2 = 4.9040$, $\alpha_1 = 710.0798$, $\alpha_2 = 43.6150$, $a_1 = 0.2987$ and $a_2 = 0.1047$. Then setting $(\theta_1, \theta_2, \theta_3, \theta_4) = (133.5766, 133.5766, 40, 0.137)$ lying in Θ as defined in (20), we will have a feasible choice $\tau = 4.15 \times 10^{-5}$.

(2) In the case of output injection, set $F_1 = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix}$ and $F_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Solve LMIs in (44) to give

$$\hat{W}_1 = \begin{pmatrix} 1.1226 & 1.6032 \\ 1.6032 & 3.3268 \end{pmatrix}, \hat{W}_2 = \begin{pmatrix} 0.3202 & 0.3795 \\ 0.3795 & 3.8245 \end{pmatrix},$$

$$\hat{E}_1 = \begin{pmatrix} -4.3214 & -6.1808 \end{pmatrix}, \hat{E}_2 = \begin{pmatrix} 0.4786 & -10.6519 \end{pmatrix}.$$

Subsequently, the definitely positive matrices in the Lyapunov functional are

$$W_1 = \begin{pmatrix} 2.8571 & -1.3768 \\ -1.3768 & 0.9641 \end{pmatrix}, W_2 = \begin{pmatrix} 3.5395 & -0.3513 \\ -0.3513 & 0.2963 \end{pmatrix},$$

and the coefficient matrices of the controller are

$$A_1 = \begin{pmatrix} -3.8368 & -0.0089 \\ -5.7552 & -0.0134 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 \\ 5.4355 & -3.3246 \end{pmatrix},$$

and as in the first case to get $\lambda_1 = 0.1487$, $\lambda_2 = 0.1099$, $\alpha_1 = 47.8441$, $\alpha_2 = 9.3012$ with a_1 and a_2 unchanged. Finally, we can have a choice of $\tau = 4.1917 \times 10^{-6}$ as $(\theta_1, \theta_2, \theta_3, \theta_4) = (1298.5, 1298.5, 2.5, 0.003) \in \Theta$.

Example 5.2. A high dimensional system will be used for illustrating effectiveness of our algorithm. Consider a system with the same structure as (45) defined in $\mathbb{R}^4$. As in the first example, $B(t)$ is a one dimensional Brownian motion, while $r(t)$ is an independent Markov chain taking values in $\mathbb{S} = \{1, 2, 3, 4\}$ with the transition matrix

$$\Gamma = \begin{pmatrix} -4 & 2 & 1 & 1 \\ 1 & -5 & 2 & 2 \\ 3 & 1 & -6 & 2 \\ 1 & 1 & 1 & -3 \end{pmatrix}.$$

Set $\delta = 1$. The coefficient matrices of the system are listed as follows, where we set $k(i) = k_i, K^j(i) = K_i^j, l(i) = l_i$ and $L^j(i) = L_i^j$ for $j = 1, 2, i = 1, 2, 3, 4$.

$$k_1 = \begin{pmatrix} 0.1 \\ 0.1 \\ 0.4 \\ -0.2 \end{pmatrix}, \quad k_2 = \begin{pmatrix} 1 \\ 0.5 \\ -0.5 \\ -0.2 \end{pmatrix}, \quad k_3 = \begin{pmatrix} 0 \\ 0.5 \\ 0 \\ -0.5 \end{pmatrix}, \quad k_4 = \begin{pmatrix} -1 \\ -0.5 \\ 0 \\ 0 \end{pmatrix}$$
$$l_1 = \begin{pmatrix} -0.1 \\ -0.2 \\ 0 \\ 0.4 \end{pmatrix}, \quad l_2 = \begin{pmatrix} -0.1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad l_3 = \begin{pmatrix} 0.3 \\ -0.5 \\ -0.1 \\ 0 \end{pmatrix}, \quad l_4 = \begin{pmatrix} -0.1 \\ 0 \\ 0.2 \\ 0.1 \end{pmatrix}$$

$$K_1^1 = \begin{pmatrix} -0.1 & 0 & 0.1 & -0.3 \\ -0.3 & 0.2 & -0.2 & 0.1 \\ 0.5 & -0.1 & -0.1 & -0.5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad K_2^1 = \begin{pmatrix} 0.1 & 0.1 & 0 & 0 \\ 0.2 & 0.3 & 0 & 0 \\ 0 & 0 & 0.1 & 0.1 \\ 0 & 0 & 0.2 & 0.3 \end{pmatrix},$$
$$K_3^1 = \begin{pmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{pmatrix}, \quad K_4^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$K_1^2 = \begin{pmatrix} 0 & 0 & 0 & -0.3 \\ 0 & 0 & -0.3 & 0.2 \\ 0 & -0.3 & 0.1 & 0.1 \\ -0.3 & 0 & 0 & 0 \end{pmatrix}, \quad K_2^2 = \begin{pmatrix} 0.1 & 0.1 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0 & 0.2 \\ -0.1 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & -0.5 \end{pmatrix},$$
$$K_3^2 = \begin{pmatrix} 0.1 & 0 & -0.1 & -0.3 \\ -0.5 & -0.2 & 0 & -0.1 \\ 0 & 0.1 & 0.1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}, \quad K_4^2 = \begin{pmatrix} -0.1 & 0 & 0.1 & -0.3 \\ -0.3 & 0.2 & -0.2 & 0.1 \\ 0.5 & -0.1 & -0.1 & -0.5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L_1^1 = \begin{pmatrix} 0.2 & -0.1 & 0.2 & -0.1 \\ 0 & 0 & 0 & 0 \\ 0.1 & 0.1 & 0.2 & 0.2 \\ -0.2 & -0.2 & -0.1 & -0.1 \end{pmatrix}, \quad L_2^1 = \begin{pmatrix} 0.1 & 0.1 & 0 & 0 \\ 0 & 0.1 & 0.1 & 0 \\ 0 & 0 & 0.1 & 0.1 \\ 0.1 & 0 & 0 & 0.1 \end{pmatrix},$$
$$L_3^1 = \begin{pmatrix} -0.1 & 0 & 0 & 0 \\ 0 & -0.1 & 0 & 0 \\ 0 & 0 & -0.1 & 0 \\ 0 & 0 & 0 & -0.1 \end{pmatrix}, \quad L_4^1 = \begin{pmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0.1 & 0.1 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0.1 \end{pmatrix}$$

$$L_1^2 = \begin{pmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{pmatrix}, \quad L_2^2 = \begin{pmatrix} -0.1 & 0 & 0 & 0 \\ -0.1 & -0.2 & 0 & 0 \\ -0.1 & -0.1 & -0.2 & 0 \\ -0.1 & -0.1 & -0.1 & -0.1 \end{pmatrix},$$
$$L_3^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_4^2 = \begin{pmatrix} -0.1 & 0 & 0 & 0 \\ 0 & -0.1 & 0 & 0 \\ 0 & 0 & -0.1 & 0 \\ 0 & 0 & 0 & -0.1 \end{pmatrix}$$

Here we use the state feedback control for stabilization with

$$G_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$G_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad G_4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

We can directly apply LMIs (43) to produce one set of feasible solutions:

$$W_1 = \begin{pmatrix} 4.0928 & -0.1761 & 2.3207 & -3.1409 \\ -0.1761 & 1.1037 & -1.2724 & 0.2173 \\ 2.3207 & -1.2724 & 4.5765 & -0.4927 \\ -3.1409 & 0.2173 & -0.4927 & 4.1171 \end{pmatrix} \quad E_1 = \begin{pmatrix} -6.0538 & -25.9843 \\ 35.4331 & -41.7372 \\ 36.1974 & -37.8158 \\ -3.2591 & -30.5235 \end{pmatrix}$$

$$W_2 = \begin{pmatrix} 5.5101 & -0.6837 & -0.0655 & 1.9670 \\ -0.6837 & 3.7078 & 1.2528 & -0.1613 \\ -0.0655 & 1.2528 & 0.9906 & 0.3422 \\ 1.9670 & -0.1613 & 0.3422 & 1.2313 \end{pmatrix} \quad E_2 = \begin{pmatrix} 9.1423 & -10.6372 \\ 2.2402 & 0.6844 \\ 5.3025 & -18.0000 \\ -4.1924 & 2.9955 \end{pmatrix}$$

$$W_3 = \begin{pmatrix} 3.7156 & -1.7555 & 1.9321 & -0.0307 \\ -1.7555 & 2.0838 & 0.2395 & -0.4869 \\ 1.9321 & 0.2395 & 6.4631 & -1.2916 \\ -0.0307 & -0.4869 & -1.2916 & 0.7623 \end{pmatrix} \quad E_3 = \begin{pmatrix} -4.9064 & 52.3325 \\ -2.3717 & 45.7642 \\ -44.4054 & 2.6805 \\ -46.1512 & -8.0896 \end{pmatrix}$$

$$W_4 = \begin{pmatrix} 3.3035 & -0.2948 & -2.7857 & 0.5552 \\ -0.2948 & 2.2537 & 1.1668 & -1.2070 \\ -2.7857 & 1.1668 & 3.7545 & -0.0958 \\ 0.5552 & -1.2070 & -0.0958 & 1.4683 \end{pmatrix} \quad E_4 = \begin{pmatrix} -2.7797 & 5.8412 \\ -4.5255 & -1.6292 \\ -8.7301 & 4.6770 \\ -4.9080 & -3.5401 \end{pmatrix}$$

and $a = 1.8399$. Subsequently, the coefficients $A_i, i \in \mathbb{S}$ will be given as

$$A_1 = \begin{pmatrix} -114.3054 & -104.4025 & -104.4025 & -114.3054 \\ 141.1668 & 65.2918 & 65.2918 & 141.1668 \\ 96.0826 & 61.7054 & 61.7054 & 96.0826 \\ -83.9446 & -83.9134 & -83.9134 & -83.9446 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & -7.2716 & 12.1328 \\ 0 & 0 & 13.3108 & -7.3515 \\ 0 & 0 & -38.0702 & 26.1715 \\ 0 & 0 & 22.9686 & -31.0232 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 2.3198 & 2.3198 & 60.1909 & 60.1909 \\ -26.6591 & -26.6591 & 77.0599 & 77.0599 \\ -33.3528 & -33.3528 & -18.4974 & -18.4974 \\ -133.9942 & -133.9942 & 9.6937 & 9.6937 \end{pmatrix},$$

and

$$A_4 = \begin{pmatrix} -15.0105 & 159.3179 & -15.0105 & 159.3179 \\ 8.9606 & -167.8571 & 8.9606 & -167.8571 \\ -16.0265 & 166.7797 & -16.0265 & 166.7797 \\ 8.6543 & -189.7630 & 8.6543 & -189.7630 \end{pmatrix},$$

respectively. Following steps to calculate the delay size, some key quantities are listed:

$$\lambda_1 = 2.1829, \lambda_2 = 1.7092, \alpha_1 = 23591.6583, \alpha_2 = 99.2097, a_1 = 12.7446, a_2 = 0.3966.$$

Now after checking inequalities in (20), we get a feasible set of $\theta_1 \sim \theta_4$ as $(\theta_1, \theta_2, \theta_3, \theta_4) = (1.6\times10^5, 1.6\times10^5, 10^4, 10^{-3})$, and then we can take $\tau = 1.0681\times10^{-7}$ as the delay size by (21).

6. Conclusions and discussions

In this article, we have considered the problem of stabilizing a class of hybrid SDDEs in distribution by delay feedback controls. Under the global Lipschitz condition, We have designed the control law with the help of a suitable Lyapunov functional. Toward an unstable linear hybrid SDDE, LMIs can be applied to get coefficients of the control efficiently, which has been proved by a high-dimensional system as in the second illustrative example. Two special delay feedback structures, including state feedback and output injection, have been discussed.

Although we can tackle the problem by a complete set of arguments, the derived results are conservative in some aspects. We discuss them for improvement and point out some future research directions.

The first is on the delay size τ^*. In practice, we will expect a largest value for the delay size, so that we will have more time to input the control into the original system. In this article, the delay size has been calculated by the supreme defined in (21). We can find that four free parameters $\theta_1 \sim \theta_4$ will affect the value of τ^*. Can we have efficient methods to give the best $\theta_1 \sim \theta_4$ so that τ^* gets the largest? On the other hand, we have made use of the Lyapunov functional (13) to derive the rules for τ^*. Is there another choice of Lyapunov functional? We have seen many different Lyapunov functions or functional used for stabilizing in the sense of moment or almost sure. Now we have no idea if those Lyapunov functions or functionals are feasible for stabilization in distribution.

The second is the choice of the control law. As listed in the introduction section, we have seen various of control laws with more practical significances. It has been reported as in (Li, Liu, Luo & Mao, 2022) that feedback controls based on discrete-time state observations have been applied for stabilizing an unstable SDE in distribution. We can look forward to other control laws to be used for stabilizing SDDEs in distribution. Of course, there will arise some new technique questions, e.g. the time-homogeneous Markov property of the controlled system and new Lyapunov functionals for discussion.

The final one is on the equation under discussed. We have discussed the problem for the equation satisfying the global Lipschitz condition. For an equation satisfying the local Lipschitz condition and the linear growth condition, we find that the Lyapunov functional (38) will be no longer efficient for the discussion. New arguments and rules are necessary for discussion. Meanwhile, for other types of delay equations, such as neutral stochastic delay differential equations, we can also discuss the same problem. We will leave above unsolved questions for future researches.

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Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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