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Remaining lifetime of degrading systems continuously monitored by degrading sensors

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ABSTRACT

We consider degrading engineering systems monitored by degrading sensors. Since accurate information is crucial for predicting system health condition and the subsequent decision-making, considering the effect of sensor degradation is highly important to determine the justified reliability characteristics of systems such as the remaining useful life (RUL). Although the concept of sensor degradation has been introduced previously in the literature, the remaining useful life estimation in this case or parameter estimation in the presence of sensor degradation has not been studied in detail. To fill the gap, this study aims to estimate the RUL of a system that is continuously monitored by a degrading sensor. In this work, to distinguish sensor degradation from that of the main system, an additional calibration sensor is used to accurately inspect the system health condition at certain points of time. Subsequently, maximum-a-posteriori estimation technique is employed to estimate the parameters for the system degradation process and maximum likelihood estimation is used to estimate the parameters of sensor degradation. A Kalman filter is then used to estimate the system and sensor states, followed by system RUL evaluation. A numerical example with simulated data is employed to illustrate the effectiveness of the proposed method. It is shown through the numerical study that neglecting sensor degradation can result in significant errors in RUL estimation, which can further impact the subsequent maintenance decisions.

1. Introduction

1.1. Background

High requirements for reliability of complex engineering systems are essential since, in many instances, unexpected equipment failures result in economic losses and accidents that can even cause serious disasters. In addition, extensive amounts of money and time have been spent on preventive maintenance of items to decrease likelihood of the future unexpected failures. Therefore, to minimize potential losses, the system health should be properly managed and the appropriate maintenance actions should be effectively planned [1–10]. One of the emerging frameworks for monitoring and maintaining system's health is Prognostics and Health Management (PHM). PHM is usually associated with the entire life cycle management of systems, aiming to optimize all maintenance activities during the complete life cycle of assets. Most of the existing PHM studies require monitoring of the key parameters of systems that enable the prediction of future failures (prognostics) by estimating the Remaining Useful Life (RUL) [11–14].

In order to predict the RUL of the assets, the central task is to monitor a number of system health indicators and obtain information about the changes in magnitudes of these indicators. Prognostics is a difficult task that counts on precise, adaptive and intuitive models to predict the future health states of the systems [1]. The PHM techniques require sensor technologies for collecting long-term accurate in-situ information about the health indicators to provide anomaly or fault detection, fault isolation, degradation level assessment, and remaining useful life prediction. The higher the accuracy of the information collected, the more successful is the deployment of PHM, in improving system safety and reducing maintenance costs [4,15].

It is preferred that monitoring is executed during all life cycle stages of a product including manufacturing, shipment, storage, handling and operation, since failures may occur due to the abnormal operational or environmental conditions that might occur during any of these stages [16]. This makes the PHM highly reliant on numerous sensors that continuously monitor, record, analyse, and transfer a large amount of data during a life cycle of a product. Therefore, it is extremely

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Notations

$X(t)$	System degradation level at time t
$S(t)$	Sensor degradation level at time t
$Y(t)$	Reading from degrading sensor at time t
$B_1(t), B_2(t), B_3(t)$	Standard Brown motion
ϵ	Measurement error
α	Drift parameter of system degradation process
β	Drift parameter of sensor degradation process
σ	Diffusion parameter of system degradation process
η	Diffusion parameter of sensor degradation process
σ_ϵ^2	Variance of measurement error
θ_1	Set of parameters to be estimated from calibration data
θ_2	Set of parameters to be estimated from monitoring data
τ_i	ith calibration time point
$\Delta\tau$	Calibration interval
ΔX_i	ith increment of system degradation level at calibration
I	Number of calibration measurements
ΔY_j	jth increment of sensor reading
Δt_j	jth interval of sensor monitoring
Σ	Variance–covariance matrix of the increments of calibration measurements
Ω	Variance–covariance matrix of the increments of monitoring measurements

important in making effective risk-avert decisions to take account of sensors degradation that effects measurements in real-life situations.

However, much less attention was devoted to the influence of non-ideal sensors on monitoring of system health. As sensors play a vital role in monitoring the health of systems, it is important to monitor the performance of the sensors as well, and to evaluate the impact of the health condition of sensors on the observation data thereof [17]. Thus, existence of sensor degradation makes it difficult to obtain accurate information on the health of monitored systems.

1.2. Motivating examples

Sensor degradation is often observed when sensors operate for a long time in a severe environment. Two typical examples that are encountered in practice are the wind turbines and nuclear power plants. Degradation and malfunctioning of sensors in offshore wind systems are mostly due to environmental effects such as gusts of wind and thundering. Vibration of the whole structure also contributes to these effects. Similar to wind energy systems, anomalous behaviour of sensors have also been observed in nuclear power plants. Operating in a harsh environment (e.g., high temperature, radioactivity, vibration, etc.) performance of sensors worsen over time. Therefore, the preventive maintenance/replacement of sensors is usually performed when the whole plant or a specific unit is shut down for general maintenance.

1.3. Literature review

There exist numerous studies in the literature dealing with RUL estimation with respect to various products across a wide range of industries. For example, Lei et al. [18] conducted an excellent survey that provides in-depth reviews of the common RUL estimation techniques within different categories of prognostic approaches. The various existing works highlight the multitude of techniques utilized to estimate RUL. One of the recent methods for RUL estimation in presence of limited data availability has been proposed in [19]. The work developed a general delay-time based RUL framework in states of invisible crack growth in mechanical systems and developed a model based on hypothetical distribution with Kalman filter to predict visible cracks. Chang et al. [20] developed a long short-term memory network (LSTM) RUL prediction algorithm based on multi-layer grid search optimization.

Amongst the various methods of RUL estimation, Wiener processes have been extensively used for degradation modelling and RUL estimation in engineering systems. For instance, Wiener process has been used to model the continuous degradation paths of the quality characteristic of a contact image scanner [21]. This approach allowed the non-constant variance and the nonzero correlation among data collected at different time points. An adaptive Wiener process model developed by Wang et al. [22] achieved better performance than the standard Wiener process model. In this model, the drift parameter of the degradation process adapts to the history of monitored information. A Wiener process-based framework, which evolved in response to the time-varying operating conditions for predicting the RUL of a single unit by incorporating the results of both accelerated degradation testing and the in-situ condition monitoring was provided by Liao and Tian [23]. A general stochastic degradation model based on a Wiener process was applied by Si et al. [24], where parameters of the model were considered as deterministic and three sources of variability in observed historical data were simultaneously characterized to integrate their effects in the model for estimating the RUL of an individual system. In [25], two degradation models with Wiener diffusion processes were developed under the pre-specified periodical calibrations. A Wiener process model has been implemented by Gao et al. [26] to describe the degradation process of metallized film capacitors, to estimate the RUL of individual capacitors. The authors addressed the multi-stage degradation process of systems, which might occur due to the staged changes in internal failure mechanism and dynamic changes in external environmental conditions. A Wiener process model for complex systems which combines the historical data with online monitoring data has been proposed by Cai et al. [27], to overcome the drawback of the Markovian characteristic of independence on historical data to improve the accuracy of the RUL estimation. A dynamic Bayesian model was used to characterize the uncertainty caused by hidden or partially observed data. Zhang and Wu [28] developed a reliability model for systems that is repaired by an unreliable tool where the degradation process of the system is unobservable. Differently, we consider the scenario of an unreliable sensor in monitoring the system states.

In addition, non-linear Wiener processes have also been investigated in existing studies. For example, RUL prediction for time-varying and non-linear cutting tool cutting tools using a non-linear Wiener-based model has been developed by Sun et al. [29]. Wang et al. [30] have used a nonlinear Wiener process model with unit-to-unit variability and measurement error to estimate system RUL. They carried out the study on the degradation data of aero-turbofan engines to understand the effect of the random failure threshold (RFT) on the prediction results for the RUL. Yu et al. [31] have proposed a generalized Wiener process-based degradation model with an adaptive drift to characterize the non-linear degradation behaviour with temporal uncertainty, item-to-item variability, and time-varying degradation processes. To provide a more robust solution, a LSTM neural network has been trained to obtain the degradation trend function in Wiener process based model in [32]. The model utilizes transfer learning to update the parameters of the LSTM neural network.

In the existing studies, the effect of measurement errors during the RUL prediction are usually considered as zero mean, Gaussian random variables [11,33–35]. Few studies consider the case of sensor degradation that exhibits deteriorating performance with the operating time.

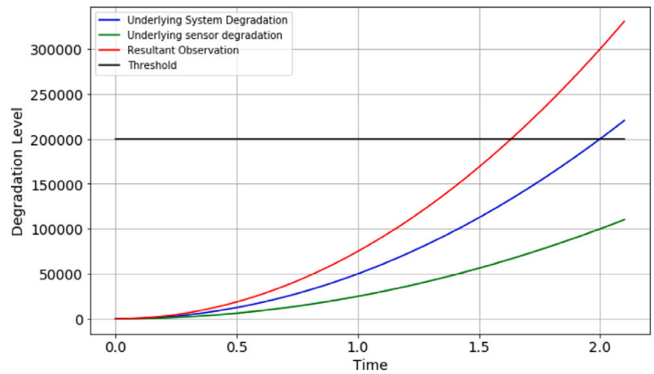

Fig. 1. Sketch of system and sensor degradation, and sensor readings.

1.4. Rationale and novelty of the proposed approach

To illustrate the effect of sensor degradation in system health monitoring, Fig. 1 sketches the degradation processes of a system and of an increasing measurement error (due to degradation) of a sensor. The latter, for convenience will be called degradation process of a sensor. Information from the degrading sensor is represented by the red curve, actual system degradation, by the blue curve and the sensor's degradation, by the green curve. The black line represents the failure threshold. The figure shows that the observed degradation value $Y(t)$, represented by the red line, crosses the threshold at a much earlier time point than the original degradation presented by the blue line, giving a false warning of system failure. This highlights the importance of considering sensor degradation in RUL estimation.

The concept of sensor degradation discussed in this paper has appeared in the literature. For instance, the measurement biases caused by degrading sensors and sensor failure during gas turbine operations, resulting in misleading diagnostic results were investigated in [36]. Sensor degradation has been described by Liu et al. [17] as one of the most serious damages in PHM technology. It was pointed out that this process can occur either due to the internal degradation as a result of chemical and physical interaction during the operation period or it can be caused by the uncontrollable operating environment, such as variations of temperature and humidity. Sensor fault types and diagnostics technique advancement have been reviewed by Li et al. [37]. However, although the concept of sensors degradation has been introduced previously in the literature, the RUL estimation in this case or parameter estimation in the presence of sensor degradation has not been studied in detail. Stochastic degradation in engineering systems, utilizing prognostics and health management framework to estimate remaining useful life has been studied in many publications, whereas the concept of sensor degradation in this framework has been recently introduced by Liu et al. [17]. In the current work, based on this concept, we develop a methodology and provide new estimation procedures for the estimation of degradation parameters of a system in the presence of sensor degradation. We also introduce the utilization of calibration sensors to address the issue of identifiability that arises when estimating the parameters. Although the maximum likelihood estimation and maximum a posterior are common approaches for parameter estimation, to the best of our knowledge, this is the first work to employ these methods in the presence of sensor degradation. To fill the gap, this study aims to estimate the RUL of a degrading system that is continuously monitored by a degrading sensor. Along with the proposed approach to modelling degradation in the described setup, this is the main innovative feature of our study.

It is assumed that both the system and the sensor degradation processes follow Wiener process. In this study, we consider the case that a calibration (ideal) sensor is used to distinguish the degradation level of the system and sensor at discrete time instants. Calibration is a common measure in industries such as nuclear to test the measurement accuracy of the sensors that are operating in harsh environment. Maximum a posteriori estimation is used to estimate and update the system degradation parameters with the arrival of new observations and maximum likelihood estimation is used to estimate the parameters associated with the sensor degradation process. Subsequently, the Kalman filter is employed to estimate the states of the system and the sensor, followed by RUL estimation. Finally, a numerical example is used to illustrate the developed model.

The remainder of the paper is organized as follows. Section 2 describes the degradation processes of the system and the sensor. Section 3 estimates the degradation parameters. Section 4 presents the system's and the sensor's state estimation using the Kalman filter, followed by the RUL estimation. The detailed numerical example is presented in Section 5 to illustrate the proposed approach. Finally, Section 6 concludes the paper.

2. System & sensor degradation modelling

We consider a degrading system being continuously monitored by a degrading sensor, where the degradation processes of the system and the sensor that follow a Wiener process are assumed to be independent of each other.

We denote the system degradation level at time t as $X(t)$. The degradation process of the system is expressed as:

$$X(t) = X(0) + \alpha t + \sigma B_1(t) \tag{1}$$

where α is the drift parameter, σ is the diffusion parameter, $B_1(t)$ is the standard Brownian motion representing the stochastic dynamics of the degradation process, $B_1(t) \sim N(0,t)$, and $X(0)$ represents the initial degradation level. Without loss of generality, $X(0)$ is assumed to be 0. Then the equation can be reduced to:

$$X(t) = \alpha t + \sigma B_1(t) \tag{2}$$

Since sensors are usually the micro-electromechanical systems, it is assumed that over time, due to ageing and severe working environment, they also degrade in a similar pattern to that of the main system. This degradation in health condition will impact the performance of the sensors and cause them to produce erroneous results. Thus, a Wiener process with linear drift will be used to describe the degradation process of the sensor. We denote the sensor degradation level at time t as $S(t)$. Without loss of generality, the initial degradation level of the sensor is assumed to be 0. Thus, we denote the degradation process of the sensor as:

$$S(t) = \beta t + \eta B_2(t) \tag{3}$$

where β is the drift parameter, η is the diffusion parameter of the sensor degradation process, and $B_2(t)$ is the standard Brownian motion.

It is assumed that the degradation processes of system and sensor are independent. The observed degradation data which we get from the sensor contains the sum of the sensor degradation and the system degradation. Hence, at a time t the resultant degradation measurement $Y(t)$ obtained from the degrading (monitoring) sensor can be written as:

$$Y(t) = X(t) + S(t) + \epsilon \tag{4}$$

where ϵ is the measurement noise, following a Gaussian distribution with $N(0, \sigma_\epsilon^2)$. ϵ is assumed to be independent of the degradation level of system and sensor. Substituting the degradation processes of Eqs. (2) and (3), $Y(t)$ can be rewritten as:

$$Y(t) = (\alpha + \beta)t + \sqrt{(\sigma^2 + \eta^2)} B_3(t) + \epsilon \tag{5}$$

where $B_3(t)$ is the standard Brownian motion. The resulting measurements $Y(t)$ consists of a Wiener process with linear drift $(\alpha + \beta)$ and diffusion coefficient $\sqrt{(\sigma^2 + \eta^2)}$, and the measurement noise ϵ.

The internal degradation of the sensor results in the stochastically increasing measurement error that is modelled by the defined Wiener process. However, our equations contain two different terms within this measurement error — the first one is due to the underlying degradation of the sensor and the second one is the measurement noise commonly used in the literature. Eq. (3) highlights the underlying degradation of the sensor, where the drift increases with time whereas Eq. (4) contains both the sensor degradation as well as the measurement noise. The measurement error of the sensor due to its degradation has an increasing mean and increasing variance, whereas the measurement noise is the random Gaussian noise with zero mean as commonly assumed in the literature (e.g., [11]). To the best of the authors' knowledge, this paper is the first work that considers the stochastically increasing measurement error in the defined way.

It is worth noting that in this study we focus on Wiener processes for degradation modelling. However, in addition to Wiener process, alternative stochastic processes such as Gamma and Inverse Gaussian have also been widely used for degradation modelling [38]. For illustrative purpose, we present in the Appendix the case of Gamma processes for modelling the system and sensor degradation processes.

Remark 1. The degradation process of each type of sensor differs due to the nature of the measurement mechanism. It is difficult to have a uniform degradation process to describe the degradation processes of all the sensors and their effects upon the overall measurements. Literature on sensor degradation and reliability analysis is rather limited. In the work of Yoo et al. [39], the authors discussed four types of sensor faults, among which our model fits the concept of drift fault. Also the study of Liu et al. [17] described an additive effect of the sensor degradation. As an early attempt, we use Wiener process to describe the sensor degradation and assume an additive effect on the overall observations. For future work, if more accurate degradation processes and mechanisms could be extracted, the proposed model could be extended to fit a specific system/sensor.

3. Parameter estimation of the degradation processes

After modelling the degradation processes, effective procedures are required to estimate parameters from the degradation data. These estimated parameters will then be channelled as inputs to the Kalman filter for estimation of the system and the sensor states at each time point. This section is devoted to estimation of the parameters of the system and sensor degradation processes.

The set of unknown parameters that requires estimation is denoted as $\{\alpha, \beta, \sigma, \eta, \sigma_\epsilon\}$. However, since these parameters are from a mix of two distributions, it is difficult or even impossible to estimate the system's and the sensor's parameters simultaneously using only one sensor. Though Liu et al. [17] have explained estimation of the sensor's degradation parameters, they have not considered the estimation process when both (a system and a sensor) sets of the parameters are unknown.

To distinguish the Wiener degradation processes of the system and the sensor, in this study, a calibration sensor is employed, which is considered as a perfect sensor that provides accurate measurements but is normally only deployed after long time intervals. For example, in the nuclear power plants, calibration is occasionally implemented to remedy the measurement errors due to sensor degradation. We will use the calibration data to estimate the parameters of the system degradation process and then update the sensor degradation parameters with the monitoring data.

We use calibration sensor to solve the identifiability issue while estimating the parameters. It would be difficult to distinguish the effect of sensor and system degradation on the overall observations without the calibration sensor. In addition, although calibration is a common practice in engineering settings, it only provides information at discrete time epochs. Compared with the continuous monitoring sensor, calibration is usually more costly and cannot provide real-time monitoring. Therefore, we use both the calibration and real-time monitoring sensors. The focus of the calibration sensor is to provide accurate reading at specific epochs to improve the estimation of system degradation level subject to a degrading sensor. The calibration sensor discussed in the paper is different from "directly calibrating the sensor". Rather, the calibration sensor is referred to an additional sensor that is used to provide an accurate reading. We do not directly calibrate or repair the sensor at calibration points.

Denote the calibration time points as $\{\tau_i, i = 1, 2, \dots\}$. The degradation value observed from the calibration sensor can be described as

$$X(\tau_i) = \alpha\tau_i + \sigma B_1(\tau_i) \tag{6}$$

The values of α and σ^2 will be estimated from calibration data. The next subsection describes the estimation process.

3.1. Estimation of system degradation parameters from the calibration sensor

The first set of parameters will be estimated from the calibration data $X(\tau_i)$, which is available at calibration points. It is assumed that these parameters are fixed but unknown and hence modelled by a prior distribution. The maximum-a-posteriori (MAP) method will be employed to estimate the parameters from the calibration data. The MAP provides an estimate of the unknown quantities that equals the mode of the posteriori distribution [40,41].

The parameters to be estimated from the calibration data are denoted as $\theta_1 = (\alpha, \sigma)$, where α and σ are independent of each other. To estimate θ_1, we assume that the prior distribution of θ_1 follows the bivariate Gaussian distribution, $\theta_1 \sim N(\mu, \Sigma_1)$, where $\mu = [\alpha_0, \sigma_\mu]^T$ and

$$\Sigma_1 = \begin{bmatrix} \sigma_0^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix}$$

The calibration intervals are denoted as $\Delta\tau_i = \tau_i - \tau_{i-1}$, and the associated measurement increments as $\Delta X_i = X(\tau_i) - X(\tau_{i-1})$. The set of measurement increments is written as $\Delta X = \{\Delta X_i : i = 1, 2, \dots, I\}$ and the set of calibration points can be expressed as $\Delta\tau = \{\Delta\tau_i : i = 1, 2, \dots, I\}$, where I represents the number of calibration measurements.

As a Wiener process poses the property of independent increments, the set of measurement increments ΔX follow a multivariate Gaussian distribution $\Delta X \sim N(\alpha\Delta\tau, \Sigma)$, where $\Sigma = diag(\sigma^2\Delta\tau_1, \sigma^2\Delta\tau_2, \dots, \sigma^2\Delta\tau_I)$.

Denote the posterior distribution of θ_1 as $p(\theta_1|\Delta X)$. Based on the MAP, the estimate of θ_1 is given as:

$$\hat{\theta}_1 = \arg\max_{\theta_1} p(\theta_1|\Delta X)$$

In addition, we have

$$p(\theta_1|\Delta X) \propto p(\Delta X|\theta_1)p(\theta_1)$$

where $p(\Delta X|\theta_1)$ is the likelihood function of the degradation increments ΔX provided by the calibration sensor, $p(\theta_1)$ is the probability density function (pdf) of the prior distribution.

As the degradation increments are independent and follow Gaussian distributions, $p(\Delta X|\theta_1)$ is expressed as:

$$p(\Delta X|\theta_1) = \prod_{i=1}^{I} \frac{1}{\sqrt{2\pi\sigma^2\Delta\tau_i}} \exp\left(-\frac{1}{2\sigma^2\Delta\tau_i}(\Delta X_i - \alpha\Delta\tau_i)^2\right)$$

and $p(\theta_1)$ is given as:

$$p(\theta_1) = \frac{1}{2\pi\sigma_0\sigma_1} \exp\left(-\frac{1}{2}\left(\frac{(\alpha-\alpha_0)^2}{\sigma_0^2} + \frac{(\sigma-\sigma_\mu)^2}{\sigma_1^2}\right)\right)$$

It follows

$$p(\theta_1|\Delta X) \propto \prod_{i=1}^{I} \frac{1}{\sqrt{2\pi\sigma^2\Delta\tau_i}} \exp\left(-\frac{1}{2\sigma^2\Delta\tau_i}(\Delta X_i - \alpha\Delta\tau)^2\right) \times \frac{1}{2\pi\sigma_0\sigma_1} \exp\left(-\frac{1}{2}\left(\frac{(\alpha-\alpha_0)^2}{\sigma_0^2} + \frac{(\sigma-\sigma_\mu)^2}{\sigma_1^2}\right)\right)$$

Taking logarithms of the above expression leads to:

$$\log(p(\theta_1|\Delta X)) = \sum_{i=1}^{I}\Big[-\ln(\sqrt{2\pi\Delta\tau_i\sigma^2}) - \frac{(\Delta X_i-\alpha\Delta\tau_i)^2}{2\Delta\tau_i\sigma^2}\Big] - \frac{(\alpha-\alpha_0)^2(\sigma-\sigma_\mu)^2}{2\sigma_0^2\sigma_1^2} + const$$

Taking derivative with respect to α, we have:

$$\frac{\partial\log(p(\theta_1|\Delta X))}{\partial\alpha} = -\frac{(\alpha-\alpha_0)(\sigma-\sigma_\mu)^2}{\sigma_0^2\sigma_1^2} + \sum_{i=1}^{I}\frac{\Delta X_i-\alpha\Delta\tau_i}{\sigma^2}$$

Setting the right-hand-side (RHS) term to 0 and after simple algebra, it follows

$$\alpha = \left(\sum_{i=1}^{I}\frac{\Delta X_i}{\sigma^2}+\frac{\alpha_0}{\sigma_0^2}\right)\left(\frac{(\sigma^2\sigma_0^2)}{\sigma^2+k\sigma_0^2\Delta\tau_i}\right) \tag{7}$$

However, α cannot be readily obtained without knowing the estimate of σ. Taking derivative of $\ln(p(\theta_1)|\Delta X)$ with respect to σ and setting it to 0 leads to

$$\sum_{i=1}^{I}\Big(\frac{(\Delta X_i-\alpha\Delta\tau_i)^2}{\Delta\tau_i\sigma^3}-\frac{1}{\sigma}\Big) = \frac{(\alpha-\alpha_0)^2(\sigma-\sigma_\mu)}{\sigma_0^2\sigma_1^2} \tag{8}$$

Estimate of σ can be obtained by substituting the value of α obtained in Eq. (7).

However, the closed-form solution cannot be obtained and numerical methods will be applied instead to find the MAP estimate of σ and α.

3.2. Estimation of sensor degradation parameters from the monitoring data

We will use the maximum likelihood estimation (MLE) to estimate parameters for the sensor degradation using the monitoring data. MLE is used here instead of the MAP as we have a large set of monitoring data that is collected continuously. The general idea is to treat the monitoring process as a whole and estimate the parameters related to $Y(t)$. The sensor degradation parameters will then be estimated by plugging the estimated values obtained using calibration data. As the system degradation parameters are updated at the arrival of a new calibration, this subsection will focus on the period within two calibrations.

Denote $\Delta t_i = t_i - t_{i-1}$ as the ith monitoring interval and $\Delta Y_i = Y(t_i) - Y(t_{i-1})$ as the ith increment of monitoring measurement, where t_i is the ith monitoring epoch. Let $\Delta Y = \{\Delta Y_1, \Delta Y_2, \dots, \Delta Y_J\}$ be the set of measurement increments, and $\Delta t = \{\Delta t_1, \Delta t_2, \dots, \Delta t_J\}$ the set of monitoring intervals, where J is the number of monitoring intervals within two calibrations. Based on the evolvement of monitoring measurements from Eq. (4) and that the measurement noise is independent from the degradation processes, we can have that the measurement increments ΔY follows a multi-variate Gaussian distribution, i.e., $\Delta Y \sim N(\omega\Delta t, \Omega)$, where $\omega = \alpha+\beta$ and Ω are the variance–covariance matrices. Since the system, the sensor degradation and the measurement errors, are independent random variables following the normal distributions, the variance–covariance matrix Ω can be obtained as:

$$\Omega_{j,k} = \mathrm{cov}\left(\Delta Y_j, \Delta Y_k|\theta_2\right) = \begin{cases}(\sigma^2+\eta^2)\Delta t_j + 2\sigma_\epsilon^2, & j=k\\ 0, & \text{otherwise}\end{cases} \tag{9}$$

where $\theta_2 = \{\beta, \eta, \sigma_\epsilon\}$ is the set of parameters to be estimated from the monitoring data.

The likelihood function can be expressed as:

$$L(\theta_2) = p(\Delta Y|\theta_2) = \prod_{j=1}^{J}\frac{1}{\sqrt{2\pi|\Omega|}} \tag{10}$$

$$\times\exp\Big(-\frac{1}{2}(\Delta Y_i-\omega\Delta t_j)^T\Omega^{-1}(\Delta Y_i-\omega\Delta t_j)\Big) \tag{11}$$

Taking logarithms of the above expression leads to

$$\log(p(\theta_2|\Delta Y)) = -\frac{J}{2}log|\Omega| - \frac{1}{2}\sum_{j=1}^{J}(\Delta Y_j-\omega\Delta t_j)^T\Omega^{-1}(\Delta Y_j-\omega\Delta t_j) + const$$

Taking derivative with respect to ω and setting to 0, it follows:

$$\hat{\omega} = \frac{1}{J}\sum_{j=1}^{J}\frac{\Delta Y_j}{\Delta t_j} \tag{12}$$

Since $\omega = \alpha+\beta$ and α have been estimated from the calibration data, we have

$$\hat{\beta} = \frac{1}{J}\sum_{j=1}^{J}\frac{\Delta Y_j}{\Delta t_j} - \hat{\alpha}$$

For the system under continuous monitoring, the monitoring intervals are usually small (smaller compared to the time between two calibrations) but identical, i.e., $\Delta t_i = \Delta t_j$, for all $i, j \in \{1, 2, \dots, J\}$. Then all the diagonal elements of variance–covariance matrix Ω are identical and can be estimated as

$$\hat{\Omega}_{i,i} = \frac{1}{J}\sum_{j=1}^{J}(\Delta Y_j-\omega\Delta t_j)(\Delta Y_j-\omega\Delta t_j)^T, \forall i\in\{1,2,\dots,J\}. \tag{13}$$

However, the solution of Eq. (13) alone cannot distinguish the estimates of η and σ, as shown in the expression of Ω in Eq. (9). A natural way to solve the problem of "identifiability" is to estimate the parameters with measurements sampled at a different interval. Together with the solution of Eq. (13), the estimates of η and σ_ϵ can then be obtained.

The procedure and expressions are similar to those presented, and therefore the details are omitted to avoid repetition.

4. State estimation and RUL evaluation

After estimating the parameters of the degradation processes using calibration measurements and real-time monitoring data, the next step is to estimate the system and sensor degradation states at each measurement point. The Kalman filtering technique will be applied to estimate the states. In our case, as shown in Eqs. (2) and (3), the system and sensor exhibit linear degrading processes and the measurement noise follow a Gaussian distribution which facilitates the application of Kalman filtering [17].

The sets of the system and the sensor degradation data are denoted as $X_{1:J} = \{x_1, x_2, \dots, x_J\}$ and $S_{1:J} = \{s_1, s_2, \dots, s_J\}$, respectively. Following the properties of the Wiener process, the state-space model can be written as ;

$$x_j = x_{j-1} + \alpha_j(t_j - t_{j-1}) + \varpi_j \tag{14}$$

$$s_j = s_{j-1} + \beta_k(t_j - t_{j-1}) + \upsilon_j \tag{15}$$

$$y_j = x_j + s_j + \epsilon_j \tag{16}$$

where $\varpi_j = \sigma[B_1(j) - B_1(j-1)]$ and $\upsilon_j = \eta[B_2(j) - B_2(j-1)]$.

$\{u_j, j\geq 0\}, \{\upsilon_j, j\geq 0\}$, and $\{\epsilon_j, j\geq 0\}$ follow non-identical normal distributions, that is, $u_j \sim N(0, \sigma^2(t_j - t_{j-1})), \upsilon_j \sim N(0, \eta^2(t_j - t_{j-1}))$, and $\epsilon_j \sim N(0, \sigma_\epsilon^2)$.

The observed data set obtained from the degrading sensor, $Y_{1:j}$, will be used for the state space modelling as it is not possible to obtain $X_{1:j}$, which is contaminated by the degrading effect of the sensor.

According to the Kalman filtering technique, the state-space Equations can be written as:

$$\mathbf{z}_j = \mathbf{A}\mathbf{z}_{j-1} + \mathbf{B}_j + \mathbf{w}_j \tag{17}$$

$$y_j = \mathbf{H}_{z_j} + \epsilon \tag{18}$$

where

$$\mathbf{z}_j = \begin{bmatrix} x_j \\ s_j \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{B}_j = \begin{bmatrix} \alpha(t_j - t_{j-1}) \\ \beta(t_j - t_{j-1}) \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \mathbf{w}_j \in R^{2\times 1}, \mathbf{w}_j \sim N(0, \mathbf{Q}_j),$$

$$\mathbf{Q}_j = \begin{bmatrix} \sigma^2(t_j - t_{j-1}) & 0 \\ 0 & \eta^2(t_j - t_{j-1}) \end{bmatrix}$$

The expectation and variance of the estimators z_j conditional on the observation history $Y_{1:j}$, are given as:

$$\hat{\mathbf{z}}_j|_j = \begin{bmatrix} \hat{x}_j|_j \\ \hat{s}_j|_j \end{bmatrix} = E(z_j|Y_{1:j})$$

$$\mathbf{P}_j|_j = \begin{bmatrix} \chi^2_{x,j} & \chi^2_{x_{s,j}} \\ \chi^2_{x_{s,j}} & \chi^2_{s,j} \end{bmatrix} = cov(z_j|Y_{1:j})$$

where $\hat{x}_{j|j} = E(x_j|Y_{1:j})$, $\hat{s}_{j|j} = E(s_j|Y_{1:j})$, $\chi^2_{x,j} = var(x_j|Y_{1:j})$, and $\chi^2_{x_{s,j}} = cov(x_{js_j}|Y_{1:j})$.

At the jth monitoring epoch, the variance and estimator of z_j are given as :

$$\hat{\mathbf{z}}_{j|j-1} = \begin{bmatrix} \hat{x}_{j|j-1} \\ \hat{s}_{j|j-1} \end{bmatrix} = E(z_j|Y_{1:j-1})$$

$$\mathbf{P}_{j|j-1} = \begin{bmatrix} \chi^2_{x,j|j-1} & \chi^2_{x_s,j|j-1} \\ \chi^2_{x_{s,j}|j-1} & \chi^2_{s,j|j-1} \end{bmatrix} = cov(z_j|Y_{1:j-1})$$

At the jth monitoring epoch t_j, the following steps are taken to estimate and update system and sensor states:

(1) State Estimation State Prediction

$$\hat{\mathbf{z}}_{j|j-1} = \mathbf{A}\hat{\mathbf{z}}_{j-1|j-1} + \mathbf{B}_j \tag{19}$$

Updated State Estimation

$$\hat{\mathbf{z}}_{j|j} = \hat{\mathbf{z}}_{j|j-1} + \mathbf{K}(j)(y_j - \mathbf{H}\hat{\mathbf{z}}_{j|j-1}) \tag{20}$$

(2) State Covariance Estimation Covariance Prediction

$$\mathbf{P}_{j|j-1} = \mathbf{A}\mathbf{P}_{j-1|j-1}\mathbf{A}^T + Q_j \tag{21}$$

The filter gain can be obtained by:

$$\mathbf{K}(j) = \mathbf{P}_{j|j-1}\mathbf{H}^T[\mathbf{H}\mathbf{P}_{j|j-1}\mathbf{H}^T + \epsilon]^{-1} \tag{22}$$

The updated state covariance can be obtained by

$$\mathbf{P}_{j|j} = \mathbf{P}_{j|j-1} - \mathbf{K}(j)\mathbf{H}\mathbf{P}_{j|j-1} \tag{23}$$

The initial values of the degradation states are given as:

$$\hat{\mathbf{z}}_{0|0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{P}_{j|j} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since the diffusion process and the measurement error are normal distributed, the posterior estimate of the system and the sensor state conditional on observation history $Y_{1:j}$, follows a bivariate Gaussian distribution $\mathbf{z}_j|\mathbf{Y}_{1:j} \sim N(\hat{\mathbf{z}}_{j|j}, \mathbf{P}_{j|j})$ [17]. Thus we have

$$x_j|\mathbf{Y}_{1:j} \sim N(\hat{x}_{j|j}, \chi^2_{x,j})$$

$$s_j|\mathbf{Y}_{1:j} \sim N(\hat{s}_{j|j}, \chi^2_{s,j})$$

Failure of the system is defined as the epoch when its degradation process hits a pre-specified threshold ξ for the first time (the first passage time (FPT)) [17]. The remaining useful life at time δ_j can be expressed as:

$$L_j = inf\{l_j : X(l_j + \delta_j) \geq \xi\} \tag{24}$$

where L_j is the remaining lifetime, ξ is the failure threshold. Using Kalman filtering, based on the observation history $Y_{1:j}$, the distribution of the RUL could be written as:

$$f_{L_{j|Y_{1:j}}}(t) = \frac{(\xi - \hat{x}_{j|j})\sigma^2 + \chi^2_{x,j}\alpha}{\sqrt{2\pi(\chi^2_{x,j} + \sigma^2 t)^3}} exp\left(-\frac{(\xi - \hat{x}_{j|j} - \alpha t)^2}{2(\chi^2_{x,j} + \sigma^2 t)}\right) \tag{25}$$

and

$$\begin{aligned} F_{L_j|Y_{1:j}}(t) = 1 - \phi\left(\frac{\xi - \hat{x}_{j|j} - \alpha t}{\sqrt{\chi^2_{x,j} + \sigma^2 t}}\right) \\ + \exp\left(-\frac{2\alpha(\xi - \hat{x}_{j|j})}{\sigma^2} + \left(\frac{\sqrt{2}\alpha\chi_{x,j}}{\sigma^2}\right)^2\right) \\ \times \phi\left(\frac{-\xi + \hat{x}_{j|j} - \alpha t - 2\alpha\frac{\chi^2_{x,k}}{\sigma^2}}{\sqrt{\chi^2_{x,j} + \sigma^2 t}}\right) \end{aligned} \tag{26}$$

where $f_{L_{j|Y_{1:j}}}$ is the conditional probability density function and $F_{L_j|Y_{1:j}}$ the cumulative distribution function of the RUL.

5. Numerical example

This section presents a numerical example to illustrate the proposed approach. We will first generate the simulated data to describe the system and sensor degradation processes following Eqs. (2) to (5). Subsequently, we will illustrate the parameter estimation process and demonstrate the accuracy of the proposed approach.

5.1. Data simulation for the degradation processes

Since Wiener process has been used to describe the degradation processes of the system and the sensor, the sum of the two Wiener processes along with measurement error leads to the resultant observations $Y(t)$, which is a deviation from the actual underlying system degradation $X(t)$.

To simulate the degradation data, the drift parameter of the system degradation is set as $\alpha = 10$, and the diffusion parameter of the system degradation as $\sigma = 5$. In addition, the sensor degradation parameters are set as $\beta = 5$, $\eta = 3$. The measurement error has a mean 0 and the standard deviation $\sigma_\epsilon = 0.45$. We consider the case of an identical monitoring interval with the sampling time step $\Delta t = 0.0001$. We simulate 21,000 data points describing the monitoring process, shown in Fig. 2, which plots the underlying system (blue line) and sensor (green line) degradation processes, and also the resultant observations (red line). Regarding the calibration data, since calibration is conducted less frequently, the number of calibration data is limited. Therefore we assume there are 21 calibration points with the first one being at time 0 and with the interval $\Delta\tau = 0.1$. We select 21 data points from the simulated underlying system degradation to reflect the case that calibration data is accurate. The calibration data is sampled at every 1000 measurement points, shown as the blue line plot with the red starred marking showing the calibration points in subplot 2 of Fig. 2. After simulating the degradation model, the next step is to estimate the unknown parameters.

5.2. Parameter estimation of the degradation processes

As previously discussed, the parameters to be estimated are α, β, σ, η, and σ_ϵ. We employ the MAP estimation to estimate the system degradation parameters α and σ based on the calibration data, and the MLE is used to estimate the remaining parameters.

For the estimation of α and σ, i.e., the system degradation parameters, the calibration data is used as inputs and the MAP estimation is used for the estimation purpose. According to the MAP mechanism,

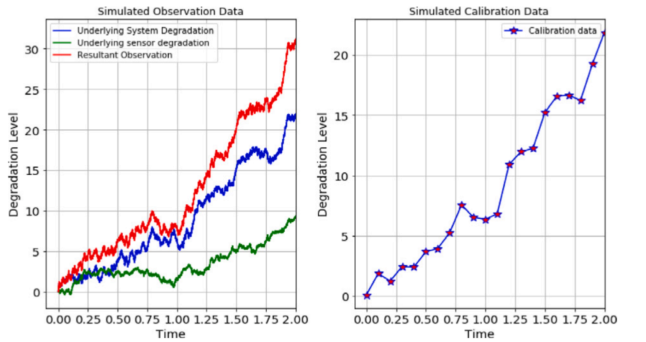

Fig. 2. System and sensor degradation and resultant observation including measurement error, using $\alpha = 10$, $\sigma = 5$, $\beta = 5$, $\eta = 3$, $\sigma_\epsilon = 0.45$.

Table 1
True values vs. estimated values for system degradation parameters.

	True value	Prior mean	Prior variance	Estimated value
α	10	9.95	1	9.98
σ	5	4	1	4.3

the prior of the parameters is assumed to follow normal distributions. Hence, the prior mean and the standard deviation of the system drift α are set as $\alpha_0 = 9.95$, and $\sigma_0 = 1$. The prior mean and the standard deviation of the system diffusion σ are set as $\sigma_\mu = 4$, and $\sigma_1 = 1$.

The system degradation parameters α and σ can be estimated based on Eqs. (7) and (8). As previously discussed, σ needs to be estimated first in order to estimate the updated α. However, since the estimated value of σ cannot be obtained analytically, numerical method is used instead. Estimation of α can be achieved by putting the estimated value of σ in Eq. (7). Fig. 3 presents the estimated α with the calibration data points.

From Fig. 3 it can be observed that with 21 calibration data points, the estimated value of α ranges from 9.95 to 9.98, with the final value at the last calibration point being 9.98. This value is close to the actual value of $\alpha = 10$, i.e., the value of α which has been used for simulating the degradation data. The dotted red line in the graph represents the actual α whereas the blue line represents the estimated α at different time points. The estimated value of α being close to 10 also highlights that MAP estimation technique works well with less number of data points, which is suitable for the scenario being modelled here, where calibration data is collected very less frequently.

In addition, we present the true values, prior means and estimated values in Table 1, to straightforwardly show the effectiveness of the proposed estimation approach. Table 1 shows the true values of α and σ used to generate system degradation (calibration data), the prior mean and variances of the unknown parameters α and σ and finally the values that could be estimated by using the maximum-a-posteriori estimation algorithm. The prior has higher influences on the estimate when the number of data points are lesser. This fact can be established from the table which shows that when the input prior σ_0 is 4, the estimated α is 4.3 which is close to the prior value.

The rest of the parameters β, η and σ_ϵ are estimated using MLE. These parameters are assumed to be updated whenever the system degradation parameters α and σ are updated at every calibration epoch. Therefore we focus on one period between two calibrations. In the following context, we will present the results of estimating the remaining parameters using the monitoring data between the 10th and 11th calibration points, where the system degradation parameters are estimated as $\alpha = 9.98$ and $\sigma = 4.3$ and there are 1000 monitoring data between two calibration points as per our numerical example (see Table 2).

Using MLE the estimated value of $\omega = \alpha + \beta$ varies from 14.75 with the final estimated value at the 11 000th data point estimating to 15.26. The actual value of ω used to generate the data was 15 ($\alpha = 10$, $\beta = 5$) and the value of estimated α using MAP is 9.98, so plugging that value in the estimated $\alpha + \beta$, β comes out to be 5.28. Fig. 4 depicts the estimated values of $\alpha + \beta$ between 10 000th and 11 000th monitoring point or 10th and 11th calibration point. The estimation is performed using Eq. (12).

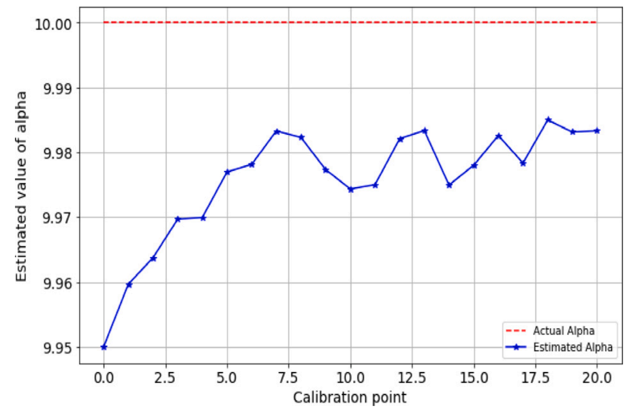

Fig. 3. Estimated α vs the number of calibration data points when prior is 9.95.

Table 2
True values vs. estimated values for the sensor degradation parameters.

Parameter	True value	Estimated value
β	5	5.28
η	3	3
σ_ϵ	0.45	0.3

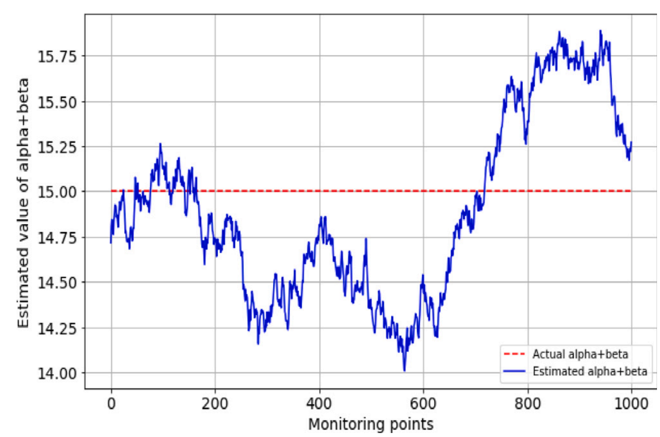

Fig. 4. Estimated $\omega = \alpha + \beta$ vs the number of data points between 10 000 and 11 000 monitoring points (here 0 in the X axis corresponds to the 10 000th monitoring point).

5.3. State estimation and RUL evaluation

This section demonstrates the numerical example of the system state estimation using Kalman filter. By the 'state', as previously discussed in this paper, we refer to the value of the corresponding degradation process level.

Fig. 5 compares the degradation obtained from simulated data and the state estimation using the Kalman filter and shows that the curves are very close, indicating the effectiveness of the developed approach. We believe that the tiny difference between the two curves is due to the difference of the true and estimated parameters. The considered example illustrates the accurate performance of the approach in this setting.

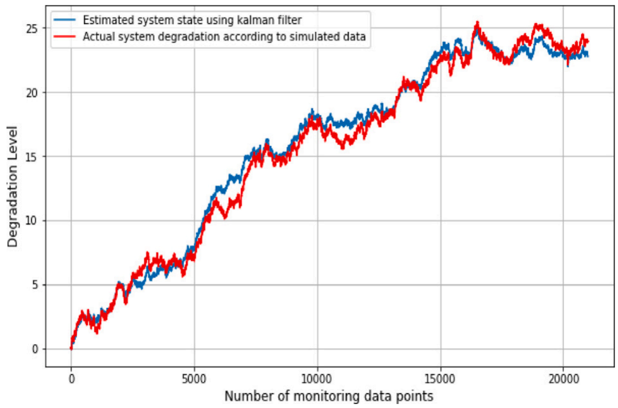

Fig. 5. Comparison of state estimation using Kalman filter and actual degradation.

6. Conclusion

In this paper, we develop a novel model to investigate the impact of a deteriorating sensor on monitoring the health condition of a degrading system. Neglecting the effect of sensor degradation may lead to erroneous RUL estimation and subsequent sub-optimal decision-making. We consider the case the both the system and sensor suffer degradation processes, which are modelled by Wiener processes with linear drift. Since both the system and the sensor are degrading and have their own sets of unknown parameters, readings from one sensor alone are not sufficient for the estimation process. To estimate the unknown parameters and attack the issue of "identifiability", the concept of a calibration sensor is used which can accurately inspect system states at certain periods. The unknown parameters are estimated using MAP at calibration time points and MLE for monitoring data. The Kalman filter is then used to estimate the system and sensor states, followed by RUL estimation. Our results show that the RUL estimated from data with sensor degradation can vary significantly from those without sensor degradation. Therefore, the corresponding deviation in readings should not only be attributed to measurement errors (as it is usually done in the literature) but can also be due to sensor degradation, which is more likely to occur when the sensors are embedded in the system or working in severe environmental conditions.

For future research directions, the following two extensions can be conducted to enrich the research on the sensor degradation. First,this paper focuses the scenario with a single system being monitored by a single sensor. However in complex engineering systems there are often multiple components which degrade at different rates and due to interaction might impact the degradation processes of each other. Also, critical systems are often monitored by multiple sensors where each sensor might undergo their own degradation due to ageing or working conditions. Hence, it is important to consider other related scenarios with multiple components and multiple sensors. In addition, in this work we use a numerical example to illustrate the impact of sensor degradation on RUL estimation. The findings of the study aids to raising the attention on sensor degradation which is often neglected in current practice. Although in the real world several engineering settings such as automotive, wind farms, nuclear power plants, and transformer systems have faced and have raised initial discussions about sensor degradation, it has not been addressed in much detail so far. Hence, gathering real-world data acts as a limitation for us. Our numerical example sets out a solution and supports our initial model. Through our work, we aim to raise more awareness about sensor degradation which was not being considered in most settings. However, the sensors themselves being Micro-electro-mechanical systems and working under the same environment as the systems are also expected to show degradation over time. It is crucial to address sensor degradation, even though we do not have detailed real data in real-world applications. With the increasing prevalence of sensor monitoring and data-driven decisions, we would like to raise the awareness of sensor degradation for industries to improve system reliability. In future we will use a real case study to further illustrate the impact of sensor degradation on PHM and risk management.

CRediT authorship contribution statement

Koushiki Mukhopadhyay: Writing – review & editing, Writing – original draft, Visualization, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Bin Liu:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Tim Bedford:** Writing – review & editing, Validation, Supervision, Methodology, Investigation, Funding acquisition, Conceptualization. **Maxim Finkelstein:** Writing – review & editing, Validation, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix

Gamma process based degradation modelling

Consider the case that the degradation processes of the system and sensor are modelled by two independent Gamma processes. Different from Wiener processes, Gamma processes hold the property of monotonically increasing sample paths.

Following Gamma process, the system degradation is expressed as

$$X_1(t) \sim Gamma(t; \alpha_1, \sigma_g)$$

where α_1 is the shape parameter and σ_g the scale parameter representing the degradation characteristics of the system, α_1, $\sigma_g > 0$

The sensor degradation is given as

$$S_1(t) \sim Gamma(t; \beta_1, \sigma_g)$$

where β_1 is the shape parameter and σ_g is the scale parameter representing the degradation characteristics of the sensor, $\beta_1 > 0$.

For simplicity, we assume that the scale parameters are the same. Therefore the resultant reading $Y_1(t)$ obtained from the degraded sensor follows an "imperfect" Gamma process that is contaminated by measurement noise, i.e.,

$$Y_1(t) = X_1(t) + S_1(t) + \epsilon_g$$

where $[X(t) + S(t)] \sim Gamma(t; \alpha_1 + \beta_1, \sigma_g)$

Assume that the measurement noise ϵ_g is independent of the system and sensor degradation process and follows Gaussian distribution with 0 mean and $\sigma_{\epsilon_g}^2$ variance.

In this scenario, the parameters to be estimated are $\{\alpha_1, \beta_1, \sigma_g, \sigma_{\epsilon_g}\}$. Similar to the procedure of the Wiener case, we will use the calibration data to obtain the system degradation parameters by employing MAP, and use MLE to estimate the remaining parameters with the monitoring data.

Let $\Theta = (\alpha_1, \sigma_g)$, $\Delta X_{1i} = X_1(\tau_i) - X_1(\tau_{i-1})$ be the system degradation increments between two calibration intervals, and ΔX_{1i} the set of degradation increments. The MAP estimate is given as:

$$\hat{\Theta} = \arg\max_{\Theta} f(\Theta|\Delta X_1)$$

where

$$f((\Theta)|\Delta X_1) \propto f(\Delta X_1|(\Theta))p(\Theta)$$

and

$$f(\Delta X_1|(\Theta)) = \prod_{i=1}^{k} \frac{\sigma_g^{\alpha_1} x_{1i}^{\alpha_1\tau_i - 1} e^{-\sigma_g x_{1i}}}{\Gamma(\alpha_1\tau_i)}$$

where $\Gamma(\cdot)$ is the Gamma function.

Assuming that the unknown priors in this case follow the bivariate normal distribution, with the mean $[\alpha_{g0}, \sigma_{g0}]^T$, and variance

$$\begin{bmatrix} \sigma_{\mu g}^2 & 0 \\ 0 & \sigma_{1g}^2 \end{bmatrix}$$

The pdf of $p(\Theta)$ is expressed as:

$$p(\alpha_1, \sigma_g) = \frac{1}{2\pi\sigma_{\mu g}\sigma_{1g}} \exp\left(-\frac{1}{2}\left(\frac{(\alpha_1 - \alpha_{g0})^2}{\sigma_{\mu g}^2} + \frac{(\sigma_g - \sigma_{g0})^2}{\sigma_{1g}^2}\right)\right)$$

Taking logarithm algebra, it follows

$$\begin{aligned} \log f((\Theta)|\Delta X_1) = (\alpha_1\tau_i - 1)\sum_{i=1}^{k} \ln(\Delta X_1) - \sum_{i=1}^{k} \alpha_1\tau_i \ln(\sigma_g) \\ -\sigma_g \sum_{i=1}^{k} \Delta X_1 - \sum_{i=1}^{k} \ln\Gamma(\alpha_1\tau_i) \\ \times \frac{1}{2\pi\sigma_{0g}\sigma_{1g}} \exp\left(-\frac{1}{2}\left(\frac{(\alpha_1 - \alpha_{g0})^2}{\sigma_{g0}^2} + \frac{(\sigma_{g0} - \sigma_{\mu g})^2}{\sigma_{1g}^2}\right)\right) \end{aligned}$$

Taking derivative with respect to α_1 leads to

$$\begin{aligned} \frac{\partial \log(f(\Theta)|X_1)}{\partial \alpha_1} = \ln(\sigma_g)\sum_{i=1}^{k} \tau_i + \sum_{i=1}^{k} \tau_i \ln(\Delta X_1) - \\ \sum_{i=1}^{k} \varphi(\alpha_1\tau_i)\tau_i - \frac{1}{2}\left(\frac{2(\alpha_1 - \alpha_{g0})}{\sigma_{g0}^2}\right) \end{aligned}$$

$$\frac{\partial \log(f(\Theta)|X_1)}{\partial \sigma_g} = \frac{k\tau_i\alpha_1}{\sigma_g} - \sum_{i=1}^{k} x_{1i} - \frac{1}{2}\left(\frac{2(\sigma_g - \sigma_{\mu g})}{\sigma_{1g}^2}\right) \tag{27}$$

where $\varphi(\cdot)$ is the digamma function. However, since the Gaussian distribution is not a conjugate prior for the Gamma distribution, we cannot achieve the closed form of the mode of the posterior distribution. As an alternative, numerical methods, such as Newton's method will be used.

Let $g(\Theta) = (g_1(\Theta), g_2(\Theta))^T$, where

$$g_1 = \frac{\partial \log(f(\Theta)|X_1)}{\partial \alpha_1}, g_2 = \frac{\partial \log(f(\Theta)|X_1)}{\partial \sigma_g}$$

Using Newton's method, at the jth iteration it follows:

$$\Theta_{(j+1)} = \Theta_k - J(\Theta_j)^{-1} g(\Theta_j)$$

where $J(\Theta)$ is the Jacobian matrix,

$$J(\Theta) = \begin{bmatrix} \frac{\partial(g_1(\Theta))}{\partial\alpha_1} & \frac{\partial(g_1(\Theta))}{\partial\sigma_g} \\ \frac{\partial(g_2(\Theta))}{\partial\alpha_1} & \frac{\partial(g_2(\Theta))}{\partial\sigma_g} \end{bmatrix}$$

$$\frac{\partial(g_1(\Theta))}{\partial\alpha_1} = -\sum_{i=1}^{k} \varphi(\alpha_1\tau_i)\tau_i^2 - \frac{1}{\sigma_{g0}^2}$$

$$\frac{\partial(g_2(\Theta))}{\partial\alpha} = -\frac{\sum_{i=1}^{k} \tau_i}{\sigma_g}$$

$$\frac{\partial(g_1(\Theta))}{\partial\sigma_g} = \frac{\partial(g_2(\Theta))}{\partial\alpha_1}$$

$$\frac{\partial(g_2(\Theta))}{\partial\sigma_g} = -\frac{\alpha_1\sum_{i=1}^{k}}{\sigma_g} - \frac{1}{\sigma_{1g}^2}$$

For the Gamma case, filtering algorithms such as unscented Kalman filter can be employed for estimation of the system and sensor state [34].

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