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Stabilisation in Distribution of Hybrid Ordinary Differential Equations by Periodic Noise

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Abstract: Many systems in the real world are periodic due to periodic phenomena in nature. Periodic hybrid stochastic differential equations are often used to model them. In many situations, it is inappropriate to study whether the solutions of periodic hybrid stochastic differential equations will converge to an equilibrium state (say, 0 or the trivial solution) but more appropriate to discuss whether the probability distributions of the solutions will converge to a stationary distribution, known as stability in distribution. This paper aims to determine whether or not a periodic stochastic state feedback control can make a given nonlinear periodic hybrid differential equation, which is not stable in distribution, to become stable in distribution. We will refer to this problem as stabilisation in distribution by periodic noise. There is little known on this problem so far. This paper initiates the study in this direction.

1 Introduction

Many practical systems may experience abrupt changes in their structure and parameters. These practical systems include electric power systems, the control system of a solar thermal central receiver, manufacturing systems, financial systems. Hybrid ordinary differential equations (ODEs) and stochastic differential equations (SDEs) have been widely used to model these systems (see, e.g., [5, 15, 16, 40, 42, 48, 49]). On the other hand, periodicity (e.g., seasonal changes) is a natural phenomena which occurs in many practical systems too. Naturally, many authors have devoted their interests to the study of periodic hybrid ODEs and SDEs (see, e.g., [4, 9, 13, 23, 25, 38, 44, 45]).

Since systems in the real world often need to run for a long period of time, their stability is one of the major concerns. On the asymptotic stability of SDE systems, there are two fundamental categories: (ASE) asymptotic stability of an equilibrium state; (ASD) asymptotic stability in distribution. ASE is to study whether the solutions of a given SDE system will tend to the equilibrium state (e.g., 0 as in most papers) in moment or in probability; while ASD is to study whether the probability distributions of the solutions of the given SDE system will converge to a probability distribution, known as stationary distribution. There is an intensive literature on ASE (see, e.g., [6, 11, 19, 30, 34, 37] and many others). The literature on ASD is much less than ASE but has been growing quickly for the past 10 years (see, e.g., [46, 51, 52]), in particular, several recent papers [21, 22, 50]. The reason why there are fewer papers on ASD than ASE is because the mathematics involved is much more complicated than that used for the study of ASE but certainly not because ASD is less important. In fact, it is inappropriate to study ASE for many stochastic hybrid systems in the real world but more appropriate to study ASD. For example, for many population systems under random environment, the stochastic permanence is a more desired control objective than the extinction (see, e.g., [7, 8, 17]). In this situation it is useful to investigate whether or not the probability distribution of the solutions will converge to a probability distribution (i.e., ASD), but not to zero (i.e., ASE) (see, e.g., [17, 31, 43]). The two stability categories can also be illustrated by the control of Covid-19. There are essentially two control strategies: one is to suppress infected to 0 but the other is to live with Covid-19. The former is to stabilise the infected to 0 with probability 1 (i.e., ASE), while the latter is to stabilise the distribution of the infected to a stationary distribution (i.e., ASD).

We have here just mentioned the concept of control. It is a normal practice that a feedback control is used to make the controlled system to be stable if a given system is not and this is known as stabilisation by feedback controls. For SDE systems, most of papers on the stabilisation use the feedback controls in the drift term, referred to as deterministic feedback controls for convenience (see, e.g., [19, 41, 47]). Nevertheless, there are some papers where feedback controls driven by Brownian motions, referred to as stochastic feedback controls for convenience, are used (i.e., controls are in the diffusion term). Comparison between deterministic and stochastic feedback controls, in particular, some advantages of the latter can be found in, e.g., [32]. In particular, stochastic feedback controls have been used or observed in many real world systems. For example, the stochastic volatility stabilise the financial markets (see, e.g., [14] and the control here is the volaility); the environmental noise suppresses explosion in population dynamics (see, e.g., [33] and here the control is the natural environmental noise), noise suppresses or expresses exponential growth in biological and ecological systems (see, e.g., [12] and here the control is noise again). The pioneering work on the latter was due to Hasminskii [20, p.229], who investigated how an ODE system could be stabilised by using two white noise sources. The theory on stabilisation driven by Brownian motion has since then been developed by several authors (see, e.g., [2, 3, 10, 26, 28, 35, 39]). It is noted that all of the existing papers in this area aim to make the solutions of stochastically controlled SDEs to tend to the equilibrium state (i.e., 0 by default) with probability 1 (i.e., in the area of category ASE).

However, there is so far little known on the problem: if a stochastic feedback control can make a given unstable system stable in distribution. The aim of this paper is to address this problem. To explain more precisely in mathematics, we assume that the given unstable system is described by a periodic hybrid ODE driven by a continuous-time Markov chain of the form

$$\dot{x}(t) = f(x(t), r(t), t), \tag{1.1}$$

where $x(t)$ is in general referred to as the state and $r(t)$ is regarded as the mode and is modelled by a Markov chain on a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$. (The notation used in this section will be explained in more detail in the next section). We assume that the coefficient f is periodic in t with period h, that is

$$f(x, i, t) = f(x, i, h + t) \quad \forall (x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+.$$

Our aim is to discuss if we can design a periodic feedback control driven by an m-dimensional Brownian motion $B(t)$ of the form $u(x(t), r(t), t)dB(t)$ so that the stochastically controlled system

$$dX(t) = f(X(t), r(t), t)dt + u(X(t), r(t), t)dB(t) \tag{1.2}$$

becomes stable in distribution, where $u : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ and is periodic in t with period h, that is

$$u(x, i, t) = u(x, i, h + t), \quad \forall (x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+.$$

Please also note that we have replaced the state process $x(t)$ by $X(t)$ to highlight the state $X(t)$ of the controlled system differs from the state $x(t)$ of the given system. For convenience, we will call the problem above as the stabilisation in distribution by periodic noise. Before we develop our theory, let us highlight some special features of this paper to close this section:

- This paper is the first to study the stabilisation in distribution by periodic noise.
- The challenge lies in the fact that it is much harder mathematically to study if the probability distributions of the solutions to the periodic controlled SDE will converge to a stationary distribution periodically.
- The usefulness of this paper is because it is more desired to have the property of stability in distribution for many systems in the real world, e.g., the control of Covid-19.

2 Notation and Definition

Throughout this paper, unless otherwise specified, we let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathcal{B}(\mathbb{R}^n)$ denote the family of all Borel measurable sets in $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, then $|x|$ is its Euclidean norm. Let $\mathbb{R}^{2n}_0 = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$ while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$. If A is a symmetric matrix, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ its largest and smallest eigenvalue, respectively. By $A > 0$ and $A \geq 0$, we mean A is positive and non-negative definite, respectively. If both a, b are real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Let $\mathbb{N}_+$ denote the set of nonnegative integers. If G is a set, $I_G(\cdot)$ denotes its indicator function, that is $I_G(x) = 1$ for $x \in G$ and 0 otherwise. We set $\inf \emptyset = \infty$, where $\emptyset$ denotes the empty set. Moreover, $x := y$ means x is defined by y while $y =: x$ means y is denoted by x.

We let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \geq 0$, be a right-continuous irreducible Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$.

For a positive number h, denote by $\mathcal{K}_h$ the family of càdlàg (right continuous with left limits) periodic functions κ from $\mathbb{R}_+$ to $[0, 1]$ with period h. If $\kappa \in \mathcal{K}_h$, we set $\kappa^{(\nu)} = (1/h)\int_0^h \kappa^\nu(s)ds$ for $\nu = 1, 2$, where throughout this paper we write $(\kappa(s))^\nu = \kappa^\nu(s)$. Denote by $\mathcal{C}_h$ the family of continuous functions ξ from $[0, h]$ to $\mathbb{R}^n$ with norm $\|\xi\|_h = \sup_{s\in[0,h]} |\xi(s)|$. Denote by $\mathcal{P}(\mathcal{C}_h)$ the family of probability measures on $\mathcal{C}_h$. For $P_1, P_2 \in \mathcal{P}(\mathcal{C}_h)$, define the Kantorovich metric d_Φ by

$$d_\Phi(P_1, P_2) = \sup_{\phi\in\Phi} \left| \int_{\mathcal{C}_h} \phi(\xi)P_1(d\xi) - \int_{\mathcal{C}_h} \phi(\xi)P_2(d\xi) \right|$$

where

$$\begin{aligned} \Phi = &\{\phi : \mathcal{C}_h \to \mathbb{R} \text{ satisfying } |\phi(\xi) - \phi(\zeta)| \leq \|\xi - \zeta\|_h \\ &\text{and } |\phi(\xi)| \leq 1 \text{ for } \xi, \zeta \in \mathcal{C}_h\}. \end{aligned}$$

It is known that $(\mathcal{C}_h, d_\Phi)$ is a complete metric space (see, e.g., [18] for the details on the Kantorovich metric).

Let us consider the stochastically controlled system (1.2). For it to be well defined, we impose the following assumption.

Assumption 2.1. *The coefficients $f(x, i, t)$ and $u(x, i, t)$ are mappings from $\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$ to $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, respectively, and they are continuous and periodic in t with period h (> 0) and locally Lipschtiz in x. There are moreover periodic functions $\kappa_1, \kappa_2 \in \mathcal{K}_h$ and non-negative numbers a_i, b_i, c_i $(i \in \mathbb{S})$*

such that

$$(x-y)^T(f(x,i,t)-f(y,i,t)) \le a_i\kappa_1(t)|x-y|^2,$$
$$|u(x,i,t)-u(y,i,t)| \le b_i\kappa_2(t)|x-y|,$$
$$|(x-y)^T(u(x,i,t)-u(y,i,t))| \ge c_i\kappa_2(t)|x-y|^2$$

for all $(x,y,i,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

Set $K_1 := \max_{i\in\mathbb{S}}(a_i \vee b_i)$. It then follows from Assumption 2.1 that

$$(x-y)^T(f(x,i,t)-f(y,i,t)) \le K_1|x-y|^2, \quad (2.1)$$
$$|u(x,i,t)-u(y,i,t)| \le K_1|x-y|. \quad (2.2)$$

It is hence well known (see, e.g., [30, 36]) that under Assumption 2.1, for any given initial data $X(0)=\hat{x}\in\mathbb{R}^n$ and $r(0)=\hat{i}\in\mathbb{S}$ at time 0, the SDE (1.2) has a unique global solution on $t\ge 0$, which will be denoted by $X_{\hat{x},\hat{i}}(t)$ in this paper in order to highlight the role of the initial data, though we often write it as $X(t)$ for convenience. We also denote by $r_{\hat{i}}(t)$ the Markov chain starting from $\hat{i}$ at time 0. It is also known that the second moment of the solution $X_{\hat{x},\hat{i}}(t)$ is finite for all $t\ge 0$.

To discuss the stability in distribution, we need the time-homogeneous Markov property (see, e.g., [1]). It is known that the joint process $(X_{\hat{x},\hat{i}}(t), r_{\hat{i}}(t))$ is a Markov process on $t\ge 0$ (see, e.g., [36]) but not time-homogeneous. Fortunately, the coefficients are periodic with period h. This enables us to form two time-homogeneous Markov processes for the use of this paper:

- For any fixed number $\bar{h}\in[0,h)$, $\{(X_{\hat{x},\hat{i}}(\bar{h}+kh), r_{\hat{i}}(\bar{h}+kh))\}_{k\in\mathbb{N}_+}$ forms a discrete-time $\mathbb{R}^n\times\mathbb{S}$-valued time-homogeneous Markov process.
- For $k\in\mathbb{N}_+$, define $\widetilde{X}_{\hat{x},\hat{i}}(kh)=\{X_{\hat{x},\hat{i}}(kh+s): 0\le s\le h\}$ which is $\mathcal{C}_h$-valued. Then $\{(\widetilde{X}_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh))\}_{k\in\mathbb{N}_+}$ forms a discrete-time $\mathcal{C}_h\times\mathbb{S}$-valued time-homogeneous Markov process.

In fact, the time-homogeneous property for both processes defined above follows clearly from the periodic property of the coefficients. So we only need to explain their Markov property. It is easy to see that the first process is Markov by the known fact that $(X_{\hat{x},\hat{i}}(t), r_{\hat{i}}(t))$ is a Markov process on $t\in\mathbb{R}_+$. This first process with $\bar{h}=0$ will play its important role in this paper and we denote by $P(k,\hat{x},\hat{i}; dy\times\{j\})$ its k-step transition probability measure, namely

$$P(k,\hat{x},\hat{i}; B\times S) = \mathbb{P}((X_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh)) \in B\times S)$$

for any $B\in\mathcal{B}(\mathbb{R}^n)$ and $S\subset\mathbb{S}$. To see why the second process is Markov, we observe that once $(\widetilde{X}_{\hat{x},\hat{i}}(k_1h), r_{\hat{i}}(k_1h))$ for some $k_1\in\mathbb{N}_+$ is given, $(X_{\hat{x},\hat{i}}(k_1h), r_{\hat{i}}(k_1h))$ is known and then $(X_{\hat{x},\hat{i}}(t), r_{\hat{i}}(t))$ for all $t\ge k_1h$, namely $(\widetilde{X}_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh))$ for all $k\ge k_1$, can be uniquely determined by solving the SDE (1.2) with initial data $(X_{\hat{x},\hat{i}}(k_1h), r_{\hat{i}}(k_1h))$ at time k_1h, but the information on how the process reaches $(\widetilde{X}_{\hat{x},\hat{i}}(k_1h), r_{\hat{i}}(k_1h))$ starting from $(\hat{x},\hat{i})$ at time 0 is of no further use. These do not only explain the Markov property but also show the following important property that

$$\mathbb{E}\phi(\widetilde{X}_{\hat{x},\hat{i}}((k+q)h)) = \sum_{j\in\mathbb{S}}\int_{\mathbb{R}^n}\phi(\widetilde{X}_{y,j}(qh))P(k,\hat{x},\hat{i}; dy\times\{j\}) \quad (2.3)$$

for $\phi\in\Phi$ and $k,q\in\mathbb{N}_+$. It should be emphasised that the formula above uses the transition probability measure of $\{(X_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh))\}_{k\in\mathbb{N}_+}$ but not that of $\{(\widetilde{X}_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh))\}_{k\in\mathbb{N}_+}$. This formula will play a critical role in the proof of our main theorem in this paper.

Denote by $\mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}(t))$ the probability measure on $\mathcal{C}_h$ generated by $\widetilde{X}_{\hat{x},\hat{i}}(t)$. (Please see, e.g., [18], for more details about probability measures generated by stochastic processes and Definition 2.2 below.) We can now give the definition of the stability in distribution.

Definition 2.2. *The controlled SDE* (1.2) *is said to be asymptotically stable in distribution if there exists a probability measure* $\mu_h\in\mathcal{P}(\mathcal{C}_h)$ *such that*

$$\lim_{k\to\infty} d_\Phi(\mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}(kh)), \mu_h) = 0$$

for all $(\hat{x},\hat{i})\in\mathbb{R}^n\times\mathbb{S}$.

It should be pointed out that in the literature (see, e.g., [51]), the asymptotic stability in distribution is in general defined on the joint process $(\widetilde{X}_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh))$. On the other hand, given the known fact that the probability distribution of the Markov chain $r_{\hat{i}}(t)$ converges to its unique stationary distribution (see, e.g., [1]), our definition here only on $\widetilde{X}_{\hat{x},\hat{i}}(kh)$ is consistent with that in the literature.

3 Stabilisation by Periodic Noise

In this section, we shall impose some additional conditions on the coefficient f and the control function u so that the controlled SDE (1.2) will be stable in distribution. However, we will only address in Section 5 the issue how to design the control function u to meet these additional conditions given that f satisfies its corresponding conditions. Let us begin with a lemma which will play a fundamental role in this section.

Lemma 3.1. *Under Assumption* 2.1*,*

$$\mathbb{P}(X_{\hat{x},\hat{i}}(t) - X_{\hat{y},\hat{i}}(t) \ne 0 \textit{ for all } t\ge 0) = 1 \quad (3.1)$$

for any $\hat{x},\hat{y}\in\mathbb{R}^n$ *with* $\hat{x}\ne\hat{y}$ *and* $\hat{i}\in\mathbb{S}$.

Proof. If (3.1) were false, there would exist some $(\hat{x},\hat{y},\hat{i})\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}$ with $\hat{x}\ne\hat{y}$ such that $\mathbb{P}(\tau<\infty)>$

0, where

$$\tau = \inf\{t \geq 0 : X_{\hat{x},\hat{i}}(t) - X_{\hat{y},\hat{i}}(t) = 0\}.$$

We can then find a pair of positive numbers R and T such that $\mathbb{P}(\Omega_1) > 0$, where

$$\Omega_1 = \{\omega \in \Omega : \tau(\omega) \leq T \text{ and } \sup_{0\leq t\leq \tau(\omega)} (X_{\hat{x},\hat{i}}(t,\omega)| \vee |X_{\hat{y},\hat{i}}(t,\omega)|) \leq R-1\}.$$

Recall (2.1), (2.2) and let $K_1 = 2K_1 + 4K_1^2$. Define the Lyapunov function

$$V_1(z,t) = e^{-K_2 t}|z|^{-2}$$

for $(z,t) \in (\mathbb{R}^n - \{0\}) \times \mathbb{R}_+$. For any $\varepsilon \in (0, |\hat{x} - \hat{y}|)$, define a stopping time

$$\tau_\varepsilon = \inf\{t \geq 0 : |X_{\hat{x},\hat{i}}(t) - X_{\hat{y},\hat{i}}(t)| \leq \varepsilon \text{ or } |X_{\hat{x},\hat{i}}(t)| \wedge |X_{\hat{y},\hat{i}}(t)| \geq R\}.$$

Set $Z(t) = X_{\hat{x},\hat{i}}(t) - X_{\hat{y},\hat{i}}(t)$. Applying the Itô formula (see, e.g., [29]), we can show that

$$\begin{aligned}&\mathbb{E}V_1(Z(\tau_\varepsilon \wedge T), \tau_\varepsilon \wedge T) - |Z(0)|^{-2}\\ =&\mathbb{E}\int_0^{\tau_\varepsilon\wedge T} e^{-K_2 s} LV_1(X_{\hat{x},\hat{i}}(s), X_{\hat{y},\hat{i}}(s), r_{\hat{i}}(s), s)ds \end{aligned} \quad (3.2)$$

where $LV_1 : \mathbb{R}_0^{2n} \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$\begin{aligned}LV_1(x,y,i,t) &= -K_2|z|^{-2}\\ &- |z|^{-4}(2z^T\bar{f} + |\bar{u}|^2) + 4|z|^{-6}|z^T\bar{u}|^2,\end{aligned}$$

in which $z = x - y$, $\bar{f} = f(x,i,t) - f(y,i,t)$ and $\bar{u} = u(x,i,t) - u(y,i,t)$. Applying (2.1) and (2.2) yields

$$\begin{aligned}&LV_1(x,y,i,t)\\ &\leq -K_2|z|^{-2} + (2K_1 + 4K_1^2)|z|^{-2} = 0.\end{aligned}$$

It then follows from (3.2) that

$$\mathbb{E}\big[e^{-K_2(\tau_\varepsilon\wedge T)}|Z(\tau_\varepsilon \wedge T)|^{-2}\big] \leq |\hat{x} - \hat{y}|^{-2}.$$

Noting that $\tau_\varepsilon \leq T$ and $|Z(\tau_\varepsilon)| = \varepsilon$ whenever $\omega \in \Omega_1$, we see from the inequality above that

$$\mathbb{E}\big[e^{-K_2 T}\varepsilon^{-2} I_{\Omega_1}\big] \leq |\hat{x} - \hat{y}|^{-2}.$$

This implies

$$\mathbb{P}(\Omega_1) \leq \varepsilon^2 |\hat{x} - \hat{y}|^{-2} e^{K_2 T}.$$

Letting $\varepsilon \to 0$ yields that $\mathbb{P}(\Omega_1) = 0$. This is in contradiction with $\mathbb{P}(\Omega_1) > 0$. The required assertion (3.1) must hence hold. The proof is complete. □

The following is the first technical assumption. We will, in Section 5, explain how the control function u can be designed to satisfy it plus Assumption (3.3) below.

Assumption 3.2. *There is a constant $p \in (0,1)$ such that*

$$\mathcal{A} := \text{diag}(\sigma_1 - pa_1, \cdots, \sigma_N - pa_N) - \Gamma \quad (3.3)$$

is a nonsingular M-matrix, where

$$\sigma_i = 0.5p[(2-p)c_i^2 - b_i^2],\ i \in \mathbb{S}. \quad (3.4)$$

In the Appendix, we will give a couple of easy-to-check sufficient criteria for Assumption 3.2 to hold. We need a number of new notations. Define

$$(\theta_1, \cdots, \theta_N)^T = \mathcal{A}^{-1}(1, \cdots, 1)^T. \quad (3.5)$$

By the theory of M-matrices (see, e.g., [36, Theorem 2.10 on page 68]), $\theta_i > 0$ for all $i \in \mathbb{S}$. Set

$$\hat{\theta} = \min_{1\leq i\leq N} \theta_i,\ \ \check{\theta} = \max_{1\leq i\leq N} \theta_i, \quad (3.6)$$

$$\hat{a} = \min_{1\leq i\leq N} a_i,\ \ \check{b} = \max_{1\leq i\leq N} b_i,\ \ \check{\sigma} = \max_{1\leq i\leq N} \sigma_i. \quad (3.7)$$

It should be pointed out that we must have $\check{\sigma} > 0$. If not, $\sigma_i \leq 0$ for all $i \in \mathbb{S}$ and hence, by Proposition 8.3 in the Appendix, $\mathcal{A}$ can never be a nonsingular M-matrix. Let us now state our second assumption.

Assumption 3.3. *With the notations above, assume*

$$\bar{\beta} := \frac{1}{\check{\theta}} + p\hat{a}\big(1 - \kappa_1^{(1)}\big) - \check{\sigma}\big(1 - \kappa_2^{(2)}\big) > 0. \quad (3.8)$$

It is useful to observe that under Assumption 3.2, if $\kappa_2(\cdot) \equiv 1$, then Assumption 3.3 always holds. Let us present four lemmas in order to show our main theorem.

Lemma 3.4. *Let Assumptions* 2.1, 3.2 *and* 3.3 *hold. Define*

$$\beta(t) = \frac{1}{\check{\theta}} + p\hat{a}(1 - \kappa_1(t)) - \check{\sigma}(1 - \kappa_2^2(t)) \quad (3.9)$$

for $t \geq 0$. Then

$$\Big|\int_0^t \beta(s)ds - \bar{\beta}t\Big| \leq (p\hat{a} + \check{\sigma})h. \quad (3.10)$$

Proof. Let k be the integer part of t/h, whence $kh \leq t < (k+1)h$. By the properties of the $\mathcal{K}_h$-class functions κ_1

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and κ_2, we derive

$$
\begin{aligned}
&-\int_0^t \beta(s)ds \\
&= \int_0^t \Big(-\frac{1}{\breve{\theta}} - p\hat{a}(1-\kappa_1(s)) + \check{\sigma}(1-\kappa_2^2(s))\Big) ds \\
&\le -\frac{t}{\breve{\theta}} - p\hat{a}(1-\kappa_1^{(1)})hk + \check{\sigma}(1-\kappa_2^{(2)})h(k+1) \\
&\le -\frac{t}{\breve{\theta}} - p\hat{a}(1-\kappa_1^{(1)})(t-h) + \check{\sigma}(1-\kappa_2^{(2)})(t+h) \\
&= -\bar{\beta}t + (p\hat{a} + \check{\sigma})h. \qquad (3.11)
\end{aligned}
$$

Similarly, we can show

$$
-\bar{\beta}t - (p\hat{a} + \check{\sigma})h \le -\int_0^t \beta(s)ds. \qquad (3.12)
$$

Combining both (3.11) and (3.12) together gives the assertion. The proof is complete. □

Lemma 3.5. *Let Assumptions* 2.1*,* 3.2 *and* 3.3 *hold. Then for any* $(\hat{x}, \hat{i}) \in \mathbb{R}_n \times \mathbb{S}$*,*

$$
\mathbb{E}|X_{\hat{x},\hat{i}}(t)|^p \le C_1(1 + |\hat{x}|^p) \qquad (3.13)
$$

for all $t \ge 0$*, where* C_1 *is a positive number independent of the initial data* $(\hat{x}, \hat{i})$*.*

Proof. Fix $(\hat{x}, \hat{i}) \in \mathbb{R}_n \times \mathbb{S}$ arbitrarily and write $X_{\hat{x},\hat{i}}(t) = X(t)$ and $r_{\hat{i}}(t) = r(t)$ for convenience. It is easy to show from Assumption 2.1 that there is a positive constant K_3 such that

$$
\begin{aligned}
2x^T f(x,i,t) &\le 2\kappa(t)a_i|x|^2 + K_3|x|, \\
|u(x,i,t)|^2 &\le \kappa^2(t)b_i^2|x|^2 + K_3(|x|+1), \qquad (3.14) \\
|x^T u(x,i,t)|^2 &\ge \kappa^2(t)c_i^2|x|^4 - K_3(|x|^3 + |x|^2)
\end{aligned}
$$

for all $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{S}$. Define a Lyapunov function

$$
V_2(x,i,t) = \theta_i(1+|x|^2)^{0.5p} e^{\int_0^t \beta(s)ds}
$$

for $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where $\beta(\cdot)$ was defined by (3.9). Applying the generalised Itô formula (see, e.g., [36, Theorem 1.45 on page 48]), we can easily show that

$$
\begin{aligned}
&\mathbb{E}V_2(X(t),r(t),t) - \theta_{\hat{i}}(1+|\hat{x}|^2)^{0.5p} \\
&= \mathbb{E}\int_0^t e^{\int_0^s \beta(v)dv}\Big(\beta(s)\theta_{r(s)}(1+|X(s)|^2)^{0.5p} \\
&\quad + LV_2(X(s),r(s),s)\Big)ds \qquad (3.15)
\end{aligned}
$$

for $t \ge 0$, where $LV_2 : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$
\begin{aligned}
&LV_2(x,i,s) \\
&= 0.5p\theta_i(1+|x|^2)^{0.5p-1}[2x^T f(x,i,s) + |u(x,i,s)|^2] \\
&- 0.5p(2-p)\theta_i(1+|x|^2)^{0.5p-2}|x^T u(x,i,s)|^2 \\
&+ \sum_{j=1}^N \gamma_{ij}\theta_j(1+|x|^2)^{0.5p}. \qquad (3.16)
\end{aligned}
$$

Recalling $p \in (0,1)$, we observe that $(1+|x|^2)^{0.5p-1}(|x|+1)$ and $(1+|x|^2)^{0.5p-2}(|x|^3+|x|^2+1)$ are bounded for $x \in \mathbb{R}^n$. Using (3.14) and (3.5), we then derive that

$$
\begin{aligned}
&LV_2(x,i,s) \\
&\le K_4 + (1+|x|^2)^{-0.5p}\Big(pa_i\theta_i\kappa_1(s) + 0.5pb_i^2\kappa_2^2(s) \\
&- 0.5p(2-p)c_i^2\theta_i\kappa_2^2(s) + \sum_{j=1}^N \gamma_{ij}\theta_j\Big) \\
&=\le K_4 + (1+|x|^2)^{-0.5p}\Big(-(\sigma_i - pa_i)\theta_i + \sum_{j=1}^N \gamma_{ij}\theta_j \\
&- pa_i\theta_i\big(1-\kappa_1(s)\big) + \sigma_i\theta_i\big(1-\kappa_2^2(s)\big)\Big) \\
&\le K_4 + (1+|x|^2)^{-0.5p}\Big(-1 - p\hat{a}\theta_i\big(1-\kappa_1(s)\big) \\
&\quad + \check{\sigma}\theta_i\big(1-\kappa_2^2(s)\big)\Big), \qquad (3.17)
\end{aligned}
$$

where K_4, the following K_5, K_6, K_7 and K_8, are all positive numbers independent of the initial data. On the other hand,

$$
\begin{aligned}
\beta(s)\theta_{r(s)} &= \Big(\frac{1}{\breve{\theta}} + p\hat{a}(1-\kappa_1(s)) - \check{\sigma}(1-\kappa_2^2(s))\Big)\theta_{r(s)} \\
&\le 1 + p\hat{a}\theta_{r(s)}(1-\kappa_1(s)) - \check{\sigma}\theta_{r(s)}(1-\kappa_2^2(s)). \quad (3.18)
\end{aligned}
$$

Applying (3.17) and (3.18) to (3.15), we obtain

$$
\mathbb{E}V_2(X(t),r(t),t) - \theta_{\hat{i}}(1+|\hat{x}|^2)^{0.5p} \le K_4\int_0^t e^{\int_0^s \beta(v)dv}ds.
$$

This implies

$$
\begin{aligned}
&\hat{\theta}\mathbb{E}(1+|X(t)|^2)^{0.5p}e^{\int_0^t \beta(s)ds} \\
&\le \check{\theta}(1+|\hat{x}|^2)^{0.5p} + K_4\int_0^t e^{\int_0^s \beta(v)dv}ds.
\end{aligned}
$$

Applying Lemma 3.4, we then have

$$
\begin{aligned}
&\hat{\theta}\mathbb{E}(1+|X(t)|^2)^{0.5p}e^{\bar{\beta}t - (p\hat{a}+\check{\sigma})h)} \\
&\le \check{\theta}(1+|\hat{x}|^2)^{0.5p} + K_4\int_0^t e^{\bar{\beta}s + (p\hat{a}+\check{\sigma})h)}ds \\
&\le \check{\theta}(1+|\hat{x}|^2)^{0.5p} + (K_4/\bar{\beta})e^{\bar{\beta}t + (p\hat{a}+\check{\sigma})h}.
\end{aligned}
$$

This implies that

$$
\mathbb{E}(1+|X(t)|^2)^{0.5p} \le K_5(1+|\hat{x}|^p), \qquad (3.19)
$$

hence the assertion (3.13) follows. The proof is complete. □

Lemma 3.6. *Let Assumptions* 2.1*,* 3.2 *and* 3.3 *hold. Then for any* $(\hat{x}, \hat{y}, \hat{i}) \in \mathbb{R}_0^{2n} \times \mathbb{S}$*,*

$$
\mathbb{E}\|\widetilde{X}_{\hat{x},\hat{i}}(kh) - \widetilde{X}_{\hat{y},\hat{i}}(kh)\|_h^p \le C_2|\hat{x} - \hat{y}|^p e^{-\bar{\beta}kh} \qquad (3.20)
$$

for all $k \in \mathbb{N}_+$*, where* C_2 *is a positive constant independent of the initial data* $(\hat{x}, \hat{y}, \hat{i})$*.*

Proof. Fix $(\hat{x}, \hat{y}, \hat{i}) \in \mathbb{R}_0^{2n} \times \mathbb{S}$ arbitrarily. Set $Z(t) = X_{\hat{x},\hat{i}}(t) - X_{\hat{y},\hat{i}}(t)$ and write $r_{\hat{i}}(t) = r(t)$ simply. By Lemma 3.1, $Z(t) \neq 0$ for all $t \geq 0$ with probability 1. Define a Lyapunov function

$$V_3(z, i, t) = \theta_i |z|^p e^{\int_0^t \beta(s)ds}$$

for $(z, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where $\beta(\cdot)$ was defined by (3.9). Applying the generalised Itô formula, we can show that

$$\begin{aligned}
&\mathbb{E}V_3(Z(t), r(t), t) - \theta_{\hat{i}}|Z(0)|^p \\
=&\mathbb{E}\int_0^t e^{\int_0^s \beta(v)dv}\Big(\beta(s)\theta_{r(s)}|Z(s)|^p \\
&+ LV_3(X_{\hat{x},\hat{i}}(s), X_{\hat{y},\hat{i}}(s), r(s), s)ds \qquad (3.21)
\end{aligned}$$

for $t \geq 0$, where $LV_3 : \mathbb{R}_0^{2n} \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$\begin{aligned}
LV_3(x, y, i, s) &= p\theta_i|z|^{p-2}z^T\bar{f} + 0.5p\theta_i|z|^{p-2}|\bar{u}|^2 \\
&- 0.5p(2-p)\theta_i|z|^{p-4}|z^T\bar{u}|^2 + \sum_{j=1}^N \gamma_{ij}\theta_j|z|^p, \qquad (3.22)
\end{aligned}$$

in which $z = x - y$, $\bar{f} = f(x, i, s) - f(y, i, s)$ and $\bar{u} = u(x, i, s) - u(y, i, s)$. By Assumption 2.1 as well as (3.5), we derive

$$\begin{aligned}
&LV_3(x, y, i, s)|z|^{-p} \\
&\leq p\theta_i a_i\kappa_1(s) + 0.5p\theta_i b_i^2\kappa_2^2(s) \\
&- 0.5p(2-p)\theta_i c_i^2\kappa_2^2(s) + \sum_{j=1}^N \gamma_{ij}\theta_j \\
&= -(\sigma_i - pa_i)\theta_i + \sum_{j=1}^N \gamma_{ij}\theta_j \\
&- pa_i\theta_i(1-\kappa_1(s)) + \sigma_i\theta_i(1-\kappa_2^2(s)) \\
&\leq -1 - p\hat{a}\theta_i\big(1-\kappa_1(s)\big) + \breve{\sigma}\theta_i\big(1-\kappa_2^2(s)\big).
\end{aligned}$$

Applying this and (3.18) to (3.21) we obtain

$$\mathbb{E}V_3(Z(t), r(t), t) - \theta_{\hat{i}}|Z(0)|^p \leq 0,$$

which yields

$$\hat{\theta}\mathbb{E}|Z(t)|^p \leq \breve{\theta}|Z(0)|^p e^{-\int_0^t \beta(s)ds} \qquad (3.23)$$

for all $t \geq 0$. This, together with Lemma 3.4, yields

$$\mathbb{E}|Z(t)|^p \leq K_6|Z(0)|^p e^{-\bar{\beta}t} \qquad (3.24)$$

for all $t \geq 0$, where $K_6 = (\breve{\theta}/\hat{\theta})e^{(p\hat{a}+\breve{\sigma})h}$.

Now, for any $k \in \mathbb{N}_+$, set $\widetilde{Z}(kh) = \{Z(kh+s) : 0 \leq s \leq h\}$. By the Itô formula (see, e.g., [29]) and (2.1), (2.2), it is easy to show that

$$\begin{aligned}
\mathbb{E}\|\widetilde{Z}(kh)\|_h^p &\leq \mathbb{E}|Z(kh)|^p + \mathbb{E}\Big(\sup_{0\leq s\leq h} H_1(s)\Big) \\
&+ K_7\int_{kh}^{(k+1)h} \mathbb{E}|Z(t)|^p dt, \qquad (3.25)
\end{aligned}$$

where

$$H_1(s) = \int_{kh}^{kh+s} p|Z(t)|^{p-2}Z^T(t)\hat{u}(t)dB(t)$$

in which $\hat{u}(t) = u(X_{\hat{x},\hat{i}}(t), r(t)) - u(X_{\hat{y},\hat{i}}(t), r(t))$. By the Burkholder-Davis-Gundy inequality (see, e.g., [36, page 76]) and (2.2), we can derive

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq s\leq h} J_1(s)\Big) \\
&\leq 3\mathbb{E}\Big(\int_{kh}^{(k+1)h} p^2K_1^2|Z(t)|^{2p}dt\Big)^{1/2} \\
&\leq 3pK_1\mathbb{E}\Big(\|\widetilde{Z}(kh)\|_h^p\int_{kh}^{(k+1)h}|Z(t)|^p dt\Big)^{1/2} \\
&\leq 0.5\mathbb{E}\|\widetilde{Z}(kh)\|_h^p + 4.5p^2K_1^2\int_{kh}^{(k+1)h}\mathbb{E}|Z(t)|^p dt.
\end{aligned}$$

Substituting this into (3.25) yields

$$\mathbb{E}\|\widetilde{Z}(kh)\|_h^p \leq 2\mathbb{E}|Z(kh)|^p + K_8\int_{kh}^{(k+1)h}\mathbb{E}|Z(t)|^p dt.$$

Making use of (3.24), we obtain the required assertion (3.20). The proof is complete. □

Lemma 3.7. *Let Assumptions* 2.1, 3.2 *and* 3.3 *hold. Then for any compact subset* G *of* $\mathbb{R}^n$,

$$\lim_{k\to\infty} d_\Phi(\mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}(kh)), \mathcal{L}(\widetilde{X}_{\hat{y},\hat{j}}(kh))) = 0 \qquad (3.26)$$

uniformly in $\hat{x}, \hat{y} \in G$ *and* $\hat{i}, \hat{j} \in \mathbb{S}$.

Proof. Note that $\{r(kh)\}_{k\in\mathbb{N}_+}$ is a discrete-time ergodic Markov chain with its one-step transition probability matrix $e^{h\Gamma}$. Define the stopping time

$$\kappa_{\hat{i}\hat{j}} = \inf\{kh : r_{\hat{i}}(kh) = r_{\hat{j}}(kh),\ k \in \mathbb{N}_+\}.$$

Then $\kappa_{\hat{i}\hat{j}} < \infty$ a.s. (see, e.g., [1]). Hence, for any $\varepsilon \in (0, 1)$, there is a positive number $T_1 > 0$ such that

$$\mathbb{P}(\kappa_{\hat{i}\hat{j}} \leq T_1) > 1 - \varepsilon/6 \quad \forall \hat{i}, \hat{j} \in \mathbb{S}. \qquad (3.27)$$

Recalling a known result ([36, p. 99, Theorem 3.24]) that

$$\sup_{(\hat{x},\hat{i})\in G\times\mathbb{S}} \mathbb{E}\Big(\sup_{0\leq t\leq T_1}|X_{\hat{x},\hat{i}}(t)|^2\Big) < \infty,$$

we see there is a sufficiently large $\rho > 0$ such that

$$\mathbb{P}(\Omega_{\hat{x},\hat{i}}) > 1 - \varepsilon/12 \quad \forall(\hat{x}, \hat{i}) \in G \times \mathbb{S}, \qquad (3.28)$$

where $\Omega_{\hat{x},\hat{i}} = \big\{\omega \in \Omega : \sup_{0\leq t\leq T_1}|X_{\hat{x},\hat{i}}(t, \omega)| \leq \rho\big\}$. We now fix $\hat{x}, \hat{y} \in G$ and $\hat{i}, \hat{j} \in \mathbb{S}$ arbitrarily. For any $\phi \in \Phi$ and

$k \in \mathbb{N}_+$ with $kh \geq T_1$, we have

$$|\mathbb{E}\phi(\widetilde{X}_{\hat{x},\hat{i}}(kh)) - \mathbb{E}\phi(\widetilde{X}_{\hat{y},\hat{j}}(kh))| \leq \frac{\varepsilon}{3} + H_2(kh), \tag{3.29}$$

where

$$H_2(kh) = \mathbb{E}\Big(I_{\{\kappa_{\hat{i}\hat{j}} \leq T_1\}}|\phi(\widetilde{X}_{\hat{x},\hat{i}}(kh)) - \phi(\widetilde{X}_{\hat{y},\hat{j}}(kh))|\Big).$$

Set $\Omega_1 = \Omega_{\hat{x},\hat{i}} \cap \Omega_{\hat{y},\hat{j}} \cap \{\kappa_{\hat{i}\hat{j}} \leq T_1\}$. By the time-homogeneous Markov property (please recall the paragraph containing (2.3)), we derive

$$\begin{aligned}
&H_2(kh)\\
=&\mathbb{E}\Big(I_{\{\kappa_{\hat{i}\hat{j}} \leq T_1\}}\mathbb{E}\big(|\phi(\widetilde{X}_{\hat{x},\hat{i}}(kh)) - \phi(\widetilde{X}_{\hat{y},\hat{j}}(kh))|\big|\mathcal{F}_{\kappa_{\hat{i}\hat{j}}}\big)\Big)\\
=&\mathbb{E}\Big(I_{\{\kappa_{\hat{i}\hat{j}} \leq T_1\}}\\
&\times \mathbb{E}|\phi(\widetilde{X}_{w,l}(kh-\kappa_{\hat{i}\hat{j}})) - \phi(\widetilde{X}_{z,l}(kh-\kappa_{\hat{i}\hat{j}}))|\Big)\\
\leq&\frac{\varepsilon}{3} + 2\mathbb{E}\Big(I_{\Omega_1}\\
&\times \mathbb{E}\big[1 \wedge (0.5\|\widetilde{X}_{w,l}(kh-\kappa_{\hat{i}\hat{j}}) - \widetilde{X}_{z,l}(kh-\kappa_{\hat{i}\hat{j}})\|_h)\big]\Big)\\
\leq&\frac{\varepsilon}{3} + 2\mathbb{E}\Big(I_{\Omega_1}\mathbb{E}\|\widetilde{X}_{w,l}(kh-\kappa_{\hat{i}\hat{j}}) - \widetilde{X}_{z,l}(kh-\kappa_{\hat{i}\hat{j}})\|_h^p\Big),
\end{aligned} \tag{3.30}$$

where $w = X_{\hat{x},\hat{i}}(\kappa_{\hat{i}\hat{j}})$, $z = X_{\hat{y},\hat{j}}(\kappa_{\hat{i}\hat{j}})$ and $l = r_{\hat{i}}(\kappa_{\hat{i}\hat{j}}) = r_{\hat{j}}(\kappa_{\hat{i}\hat{j}})$. Observing that for any given $\omega \in \Omega_1$, $|w| \vee |z| \leq \rho$, we can apply Lemma 3.6 to see that there is another positive constant T_2 such that

$$\mathbb{E}\|\widetilde{X}_{w,l}(kh-\kappa_{\hat{i}\hat{j}}) - \widetilde{X}_{z,l}(kh-\kappa_{\hat{i}\hat{j}})\|_h^p \leq \frac{\varepsilon}{6}$$

for $kh \geq T_1 + T_2$. Substituting this into (3.30) yields that $H_2(kh) \leq 2\varepsilon/3$ for all $kh \geq T_1 + T_2$. This, together with (3.29), implies that

$$|\mathbb{E}\phi(\widetilde{X}_{\hat{x},\hat{i}}(kh) - \mathbb{E}\phi(\widetilde{X}_{\hat{y},\hat{j}}(kh))| \leq \varepsilon \tag{3.31}$$

for $kh \geq T_1 + T_2$. Since ϕ is arbitrary, we must have

$$d_\Phi(\mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}(kh)), \mathcal{L}(\widetilde{X}_{\hat{y},\hat{j}}(kh))) \leq \varepsilon, \quad \forall kh \geq T_1 + T_2$$

for all $\hat{x}, \hat{y} \in G$ and $\hat{i}, \hat{j} \in \mathbb{S}$. This proves (3.26). The proof is complete. □

We can now establish our main theorem in this paper.

Theorem 3.8. *Let Assumptions* 2.1, 3.2 *and* 3.3 *hold. Then there exists a unique probability measure $\mu \in \mathcal{P}(\mathcal{C}_h)$ such that*

$$\lim_{k\to\infty} d_\Phi(\mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}(kh)), \mu) = 0 \tag{3.32}$$

for all $(\hat{x}, \hat{i}) \in \mathbb{R}^n \times \mathbb{S}$. In other words, the SDE (1.2) *is asymptotically stable in distribution.*

Proof. We claim that $\{\mathcal{L}(\widetilde{X}_{0,1}(kh))\}_{k\in\mathbb{N}_+}$ is a Cauchy sequence in $\mathcal{P}(\mathcal{C}_h)$ with metric d_Φ. In other words, we need to show that for any $\varepsilon > 0$, there is an integer $k_0 > 0$ such that

$$d_\Phi(\mathcal{L}(\widetilde{X}_{0,1}((v+q)h)), \mathcal{L}(\widetilde{X}_{0,1}(qh))) \leq \varepsilon \tag{3.33}$$

for all integers $q \geq k_0$ and $v \geq 1$. Let $\varepsilon \in (0,1)$ be arbitrary. By Lemma 3.5, there is a $\rho > 0$ such that

$$\mathbb{P}\{\omega \in \Omega : |X_{0,1}(vh, \omega)| \leq \rho\} > 1 - \varepsilon/4 \tag{3.34}$$

for any integer $v \geq 1$. For any $\phi \in \Phi$, we can then derive, using (2.3) and (3.34), that

$$\begin{aligned}
&|\mathbb{E}\phi(\widetilde{X}_{0,1}((v+q)h)) - \mathbb{E}\phi(\widetilde{X}_{0,1}(qh))|\\
=&|\mathbb{E}(\mathbb{E}[\phi(\widetilde{X}_{0,1}((v+q)h))|\mathcal{F}_{vh}]) - \mathbb{E}\phi(\widetilde{X}_{0,1}(qh))|\\
=&\Big|\sum_{j\in\mathbb{S}}\int_{\mathbb{R}^n}\mathbb{E}\phi(\widetilde{X}_{y,j}(qh))P(v,0,1;dy\times\{j\})\\
&\quad - \mathbb{E}\phi(\widetilde{X}_{0,1}(qh))\Big|\\
\leq&\sum_{j\in\mathbb{S}}\int_{\mathbb{R}^n}|\mathbb{E}\phi(\widetilde{X}_{y,j}(qh)) - \mathbb{E}\phi(\widetilde{X}_{0,1}(qh))|\\
&\quad \times P(v,0,1;dy\times\{j\})\\
\leq&\frac{\varepsilon}{2} + \sum_{j\in\mathbb{S}}\int_{B_\rho}d_\Phi(\mathcal{L}(\widetilde{X}_{y,j}(qh)), \mathcal{L}(\widetilde{X}_{0,1}(qh)))\\
&\quad \times P(v,0,1;dy\times\{j\}),
\end{aligned}$$

where $B_\rho = \{x \in \mathbb{R}^n : |x| \leq \rho\}$. By Lemma 3.7, there is a positive integer k_0 such that

$$d_\Phi(\mathcal{L}(\widetilde{X}_{\hat{y},\hat{j}}(qh)), \mathcal{L}(\widetilde{X}_{0,1}(qh))) \leq \frac{\varepsilon}{2} \quad \forall q \geq k_0$$

whenever $(y, j) \in B_\rho \times \mathbb{S}$. We therefore obtain

$$|\mathbb{E}\phi(\widetilde{X}_{0,1}((v+q)h)) - \mathbb{E}\phi(\widetilde{X}_{0,1}(qh))| \leq \varepsilon$$

for $q \geq k_0$ and $v \geq 1$. As this holds for any $\phi \in \Phi$, we must have (3.33) as claimed. Consequently, there is a unique $\mu \in \mathcal{P}(\mathcal{C}_h)$ such that

$$\lim_{k\to\infty} d_\Phi(\mathcal{L}(\widetilde{X}_{0,1}(kh)), \mu) = 0.$$

This, together with Lemma 3.7, implies that

$$\begin{aligned}
&\lim_{k\to\infty} d_\Phi(\mathcal{L}(\widetilde{X}_{0,1}(kh)), \mu)\\
\leq&\lim_{k\to\infty} d_\Phi(\mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}(kh)), \mathcal{L}(\widetilde{X}_{0,1}(kh)))\\
&+ \lim_{k\to\infty} d_\Phi(\mathcal{L}(\widetilde{X}_{0,1}(kh)), \mu) = 0
\end{aligned}$$

for all $(\hat{x}, \hat{i}) \in \mathbb{R}^n \times \mathbb{S}$, which is assertion (3.32). The proof is complete. □

Note that $\mathcal{C}_h$ is an infinite space and $\mathcal{P}(\mathcal{C}_h)$ is huge. It may therefore be hard to numerically approximate μ, not mentioning to obtain its probability distribution theoretically. Fortunately, in practice, we are more concerned with the probability distribution of $X_{\hat{x},\hat{i}}(t)$ in long term. For this purpose, let us return to the discrete-time $\mathbb{R}^n \times \mathbb{S}$-valued time-homogeneous Markov process $\{(X_{\hat{x},\hat{i}}(\bar{h}+kh), r_{\hat{i}}(\bar{h}+kh))\}_{k\in\mathbb{N}_+}$, where $\bar{h}\in[0,h)$. Accordingly, let us denote by $\mathcal{P}(\mathbb{R}^n)$ the family of probability measures on $\mathbb{R}^n$. For $P_1, P_2 \in \mathcal{P}(\mathbb{R}^n)$, define the Kantorovich metric d_Ψ by

$$d_\Psi(P_1,P_2) = \sup_{\psi\in\Psi}\Big|\int_{\mathbb{R}^n}\psi(x)P_1(dx) - \int_{\mathbb{R}^n}\psi(x)P_2(dx)\Big|$$

where

$$\Psi = \{\psi : \mathbb{R}^n \to \mathbb{R} \text{ satisfying } |\psi(x)-\psi(y)| \le |x-y| \text{ and } |\phi(x)| \le 1 \text{ for } x,y\in\mathbb{R}^n\}.$$

Denote also by $\mathcal{L}(X_{\hat{x},\hat{i}}(\bar{h}+kh))$ the probability measure on $\mathbb{R}^n$ generated by $X_{\hat{x},\hat{i}}(\bar{h}+kh)$. Noting that $\mathcal{L}(X_{\hat{x},\hat{i}}(\bar{h}+kh))$ is a marginal probability measure of $\mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}(kh))$, we obtain the following useful corollary (which can be proved directly in the same fashion as Theorem 3.8 was proved).

Corollary 3.9. *Let Assumptions* 2.1, 3.2 *and* 3.3 *hold. Then, for every* $\bar{h}\in[0,h)$, *there exists a unique probability measure* $\mu_{\bar{h}} \in \mathcal{P}(\mathbb{R}^n)$ *such that*

$$\lim_{k\to\infty} d_\Psi(\mathcal{L}(X_{\hat{x},\hat{i}}(\bar{h}+kh)), \mu_{\bar{h}}) = 0 \tag{3.35}$$

for all $(\hat{x},\hat{i}) \in \mathbb{R}^n\times\mathbb{S}$.

4 Special but Important Cases

In this section we will demonstrate that our theory established in the previous section can be applied to several special but important cases.

4.1 Time-homogeneous ODE

Let us first consider the case where the given unstable system is described by a time homogeneous ODE

$$\dot{x}(t) = f(x(t), r(t)), \tag{4.1}$$

where $f : \mathbb{R}^n\times\mathbb{S}\to\mathbb{R}^n$. Assume that $f(x,i)$ is locally Lipschtiz in x and there are non-negative numbers a_i ($i\in\mathbb{S}$) such that

$$(x-y)^T(f(x,i)-f(y,i)) \le a_i|x-y|^2 \tag{4.2}$$

for all $(x,y,i)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}$. We first consider to design a time-homogeneous control function $u : \mathbb{R}^n\times\mathbb{S}\to\mathbb{R}^{n\times m}$ for which we can find non-negative numbers b_i, c_i ($i\in\mathbb{S}$) such that

$$|u(x,i)-u(y,i)| \le b_i|x-y|, \tag{4.3}$$

$$|(x-y)^T(u(x,i)-u(y,i))| \ge c_i|x-y|^2 \tag{4.4}$$

for all $(x,y,i)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}$. The corresponding controlled SDE is

$$dX(t) = f(X(t), r(t))dt + u(X(t),t)dB(t). \tag{4.5}$$

If we regard both f and u as a periodic function with period $h=0$, this SDE is a special case of our underlying SDE (1.2). In this case, $\mathcal{C}_h$ reduces to $\mathbb{R}^n$ and Assumption 2.1 holds with $\kappa_1(\cdot)=\kappa_2(\cdot)\equiv 1$. If Assumption 3.2 holds, then, by (3.8), $\bar{\beta} = 1/\breve{\theta}$ which is always positive, whence Assumption 3.3 must hold. The following useful corollary follows therefore from Theorem 3.8.

Corollary 4.1. *Let conditions* (4.2), (4.3) *and* (4.4) *as well as Assumption* 3.2 *hold. Then there exists a unique probability measure* $\mu_0\in\mathcal{P}(\mathbb{R}^n)$ *such that for every initial data* $(\hat{x},\hat{i})\in\mathbb{R}^n\times\mathbb{S}$, *the solution of the SDE* (4.5) *satisfies*

$$\lim_{k\to\infty} d_\Psi(\mathcal{L}(X_{\hat{x},\hat{i}}(kh)), \mu_0) = 0. \tag{4.6}$$

4.2 Intermittent control

Although it is natural to design a time-homogeneous control function given that the ODE (4.1) is time homogeneous, it may be necessary to design a periodic control function. For example, a controller needs a rest periodically and an intermittent control is required to be used (see, e.g., [27, 53]). A typical intermittent control function has the form $\kappa_2(t)u(x,i)$, where $u(x,i)$ is the same as in Section 4.1 and

$$\kappa_2(t) = \sum_{k=0}^{\infty} I_{[kh,(k+1-\delta)h)}(t),\ \ t\ge 0, \tag{4.7}$$

in which $\delta\in[0,1)$ is a positive constant. The corresponding controlled SDE is

$$dX(t) = f(X(t), r(t))dt + \kappa_2(t)u(X(t),t)dB(t). \tag{4.8}$$

In operation, the stochastic control is switched on and off periodically. That is, on during time periods $[0,(1-\delta)h))$, $[h,(2-\delta)h)$, $[2h,(3-\delta)h)$, $\cdots$, while off during $[(1-\delta)h,h)$, $[(2-\delta)h,2h)$, $[(3-\delta)h,3h)$, $\cdots$. The parameter δ is the proportion of rest in one period of h or in long term. Under conditions (4.2)-(4.4), Assumption 2.1 is satisfied with $\kappa_1(\cdot)\equiv 1$. If Assumption 3.2 holds, then Assumption 3.3 becomes

$$\frac{1}{\breve{\theta}} - \frac{\breve{\sigma}\delta}{h} > 0. \tag{4.9}$$

We hence have the following useful corollary.

Corollary 4.2. *Let conditions* (4.2), (4.3) *and* (4.4) *as well as Assumption* 3.2 *hold. If* (4.9) *is satisfied, then there exists a unique probability measure* $\mu\in\mathcal{P}(\mathcal{C}_h)$ *such that for every initial data* $(\hat{x},\hat{i})\in\mathbb{R}^n\times\mathbb{S}$, *the solution of the SDE* (4.8) *satisfies*

$$\lim_{k\to\infty} d_\Phi(\mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}(kh)), \mu) = 0. \tag{4.10}$$

4.3 Worst case

Let us return to the underlying ODE (1.1). Given the periodic coefficient f, it is easy to identify non-negative numbers a_i ($i\in\mathbb{S}$) such that

$$(x-y)^T(f(x,i,t)-f(y,i,t)) \le a_i|x-y|^2 \tag{4.11}$$

for $(x,y,i,t)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}\times\mathbb{R}_+$. On the other hand, it may be hard to identify a common $\kappa_1\in\mathcal{K}_h$ for all $i\in\mathbb{S}$ so

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that the first inequality in Assumption 2.1 holds. In this worst case, we just simply let $\kappa_1(\cdot) \equiv 1$. However, we could still design the periodic control function u as required by Assumption 2.1. In particular, it is unnecessary to choose $\kappa_2(\cdot) \equiv 1$, which is only one of many possible choices. Let us form another corollary to cope with this worst case.

Corollary 4.3. *Let Assumption* 2.1 *hold with* $\kappa_1(\cdot) \equiv 1$. *If, moreover, Assumption* 3.3 *holds and*

$$\frac{1}{\breve{\theta}} - \breve{\sigma}\big(1 - \kappa_2^{(2)}\big) > 0,$$

then the conclusion of Theorem 3.8 *holds.*

5 Design of Control Function

The use of Theorems 3.8 depends on whether the control function $u(x, i, t)$ can be designed to meet Assumptions 2.1, 3.2 and 3.3, given that the coefficient $f(x, i, t)$ satisfies the first inequality in Assumption 2.1. In this section we will demonstrate how to design the control function in various situations. Due to the page limit, we will only design the linear periodic control function in the form

$$u(x, i, t) = \kappa_2(t)(A_{1i}x, A_{2i}x, \cdots, A_{mi}x) \tag{5.1}$$

for $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where $A_{ki} \in \mathbb{R}^{n\times n}$ is symmetric and nonnegative definite for $i \in \mathbb{S}$ and $k = 1, 2, \cdots, m$ and $\kappa_2 \in \mathcal{K}_h$. For $(x, y, i, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, we have

$$|u(x, i, t) - u(y, i, t)|^2 = \kappa_2^2(t)\sum_{k=1}^m |A_{ki}(x - y)|^2$$

$$\leq \kappa_2^2(t)\Big(\sum_{k=1}^m \|A_{ki}\|^2\Big)|x - y|^2$$

and

$$|(x - y)^T(u(x, i, t) - u(y, i, t))|^2$$

$$=\kappa_2^2(t)\sum_{k=1}^m |(x - y)^T A_{ki}(x - y)|^2$$

$$\geq \kappa_2^2(t)\Big(\sum_{k=1}^m \lambda_{\min}^2(A_{ki})\Big)|x - y|^2.$$

These imply u satisfies Assumption 2.1 with

$$b_i = \sqrt{\sum_{k=1}^m \|A_{ki}\|^2} \text{ and } c_i = \sqrt{\sum_{k=1}^m \lambda_{\min}^2(A_{ki})}. \tag{5.2}$$

What we need to do is: (I) to refine the choices of A_{ki} for Assumption 3.2 to hold, (II) to design $\kappa_2(\cdot)$ for Assumption 3.3 to hold.

Let us first explain (II) should (I) has been done. Compute $\kappa_1^{(1)}$ by definition and $\hat{a}, \breve{\sigma}, \breve{\theta}$ by (3.5)-(3.7). For Assumption 3.3 to hold, all we need is to design $\kappa_2(\cdot)$ for

$$\breve{\sigma}\big(1 - \kappa_2^{(2)}\big) < \frac{1}{\breve{\theta}} + p\hat{a}\big(1 - \kappa_1^{(1)}\big) \tag{5.3}$$

to hold. There are lots of choices $\kappa_2 \in \mathcal{K}_h$ to make this happen. For example, $\kappa_2 \equiv 1$. Another example is the κ_2 defined by (4.7). In this case, $\kappa_2^{(2)} = 1 - \delta/h$ and hence, by (5.3), all we need is to set

$$\delta < \frac{h}{\breve{\sigma}}\Big(\frac{1}{\breve{\theta}} + p\hat{a}\big(1 - \kappa_1^{(1)}\big)\Big). \tag{5.4}$$

We leave the other choices of κ_2 to the reader but explain (I) in two useful situations.

Case 1. Consider the situation where the state $X(t)$ can be observed in every mode $i \in \mathbb{S}$ at any time and the stochastic feedback control can be applied in every mode as well. In this case, for $i \in \mathbb{S}$ and $1 \leq k \leq m$, choose symmetric matrices $\bar{A}_{ki}$ such that

$$\sqrt{2}\lambda_{\min}(\bar{A}_{ki}) > \|\bar{A}_{ki}\|. \tag{5.5}$$

Obviously, there are lots of such matrices. Choose a positive number α sufficiently large so that

$$0.5\alpha^2\Big(2\sum_{k=1}^m \lambda_{\min}^2(\bar{A}_{ki}) - \sum_{k=1}^m \|\bar{A}_{ki}\|^2\Big) > a_i \tag{5.6}$$

for all $i \in \mathbb{S}$. This guarantees that there is a $p \in (0, 1)$ sufficiently small for which

$$0.5\alpha^2\Big((2 - p)\sum_{k=1}^m \lambda_{\min}^2(\bar{A}_{ki}) - \sum_{k=1}^m \|\bar{A}_{ki}\|^2\Big) > a_i \tag{5.7}$$

for all $i \in \mathbb{S}$. Let us now set $A_{ki} = \alpha\bar{A}_{ki}$. Noting that σ_i defined by (3.4) has the form

$$\sigma_i = 0.5\alpha^2 p\Big((2 - p)\sum_{k=1}^m \lambda_{\min}^2(\bar{A}_{ki}) - \sum_{k=1}^m \|\bar{A}_{ki}\|^2\Big),$$

we see

$$\sigma_i > pa_i, \ \ \forall i \in \mathbb{S}. \tag{5.8}$$

By the theory of M-matrices (see, e.g., [36, Theorem 2.10 on page 68]), we see that $\mathcal{A}$ defined by (3.3) is a nonsigular M-matrix. In other words, Assumption 2.1 holds if A_{ki}'s are defined as above.

Observe that the arguments above still hold as long as

$$2\sum_{k=1}^m \lambda_{\min}^2(\bar{A}_{ki}) > \sum_{k=1}^m \|\bar{A}_{ki}\|^2, \ \ \forall i \in \mathbb{S} \tag{5.9}$$

but it is unnecessary for (5.5) to hold for every $i \in \mathbb{S}$ and $1 \leq k \leq m$. This gives us an opportunity to design the control function to fit into various situations in the real world. For example, we may let $\bar{A}_{ki} = 0$ for all $k = 2, \cdots, m$ but only need

$$0.5\alpha^2\Big((2 - p)\lambda_{\min}^2(\bar{A}_{1i}) - \|\bar{A}_{1i}\|^2\Big) > a_i$$

for all $i \in \mathbb{S}$. This is equivalent to the situation when $m = 1$. In other words, we may only use a scalar Brownian motion as the noise source to achieve the stochastic stabilisation in distribution.

The observation above also reveals another useful situation, where a different and independent scalar Brownian motion is used in different mode $i \in \mathbb{S}$. In terms of mathematics, we have that $m = N$ and $A_{ki} = 0$ in (5.1) for all $k \neq i$. In this situation, we may choose $\bar{A}_{ii}$'s and α for which

$$0.5\alpha^2\Big((2-p)\lambda^2_{\min}(\bar{A}_{ii}) - \|\bar{A}_{ii}\|^2\Big) > a_i$$

and then set $A_{ii} = \alpha\bar{A}_{ii}$ for all $i \in \mathbb{S}$.

Case 2. We now consider a situation where the state $X(t)$ cannot be observed in some modes, whence the stochastic control cannot be used in these modes. Without loss of any generality, we let $\mathbb{S}_1 = \{1, 2, \cdots, N_1\}$ contain these modes ($1 \le N_1 < N$). Mathematically speaking, we are forced to set

$$A_{ki} = 0 \text{ for } i \in \mathbb{S}_1,\ 1 \le k \le m,$$

whence $b_i = c_i = 0$ for $i \in \mathbb{S}_1$. What we need to do is to design matrices A_{ki} for $i \in \mathbb{S}_2 = \{N_1+1, \cdots, N\}$ and $1 \le k \le m$. To establish a simple criterion, we impose an additional condition: there is some $\hat{j} \in \mathbb{S}$ for which

$$\gamma_{i\hat{j}} > 0 \text{ for all } i \in \mathbb{S} \text{ but } i \neq \hat{j}. \tag{5.10}$$

Moreover, let $\upsilon = (\upsilon_1, \upsilon_2, \cdots, \upsilon_N) \in \mathbb{R}^{1\times N}$ denote the unique stationary distribution of the Markov chain. It is known that all υ_i's are positive. (Please see the appendix below for further details). Choose symmetric positive definite matrices $\bar{A}_{ki}$ for $N_1+1 \le i \le N$ and $1 \le k \le m$ so that

$$\sum_{k=1}^m \lambda^2_{\min}(\bar{A}_{ki}) > 0.5\sum_{k=1}^m \|\bar{A}_{ki}\|^2.$$

Then choose a positive number α so large that

$$\sum_{i=N_1+1}^N \upsilon_i\alpha^2 \sum_{k=1}^m \big(0.5\|\bar{A}_{ki}\|^2 - \lambda^2_{\min}(\bar{A}_{ki})\big) + \sum_{i=1}^N \upsilon_i a_i < 0. \tag{5.11}$$

Now set $A_{ki} = \alpha\bar{A}_{ki}$. Recalling (5.2), we see

$$b_i^2 = \alpha^2\sum_{k=1}^m \|\bar{A}_{ki}\|^2 \text{ and } c_i^2 = \alpha^2\sum_{k=1}^m \lambda^2_{\min}(\bar{A}_{ki})$$

for $N_1+1 \le i \le N$. Consequently, it follows from (5.11) that

$$\sum_{i=1}^N \upsilon_i a_i + \sum_{i=N_1+1}^N \upsilon_i(0.5b_i^2 - c_i^2) < 0.$$

That is

$$\sum_{i=1}^N \upsilon_i(a_i + 0.5b_i^2 - c_i^2) < 0$$

if we recall that $b_i = c_i = 0$ for $i \in \mathbb{S}_1$. By Proposition 8.2 in the appendix below, we see Assumption 3.2 is satisfied as long as A_{ki}'s are designed as above.

6 Example

Due to the page limit we will only discuss an example to illustrate our new theory.

Linear hybrid ODEs of the form $\dot{x}(t) = (\zeta_{r(t)} + F_{r(t)})x(t)$ have been used widely in many branches of science and industry (see, e.g., [6, 11, 19]), where $F_i \in \mathbb{R}^{n\times n}$ and $\zeta_i \in \mathbb{R}^n$ for $i \in \mathbb{S}$. Taking into account the natural phenomena of periodicity, e.g., seasonal changes (see, e.g., [4, 9, 23, 38]), we arrive at the periodic hybrid ODEs of the form

$$\dot{x}(t) = [\alpha_{r(t)}(t)\zeta_{r(t)} + \beta_{r(t)}(t)F_{r(t)}]x(t). \tag{6.1}$$

We assume that $\alpha_i : \mathbb{R}_+ \to \mathbb{R}$ and $\beta_i : \mathbb{R}_+ \to (0, 1]$ are all continuous and periodic with period h while $F_i + F_i^T > 0$ for all $i \in \mathbb{S}$. If we define $f(x, i, t) = [\alpha_i(i)\zeta_i + \beta_i(t)F_i]x$ for $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, the ODE (6.1) is in the form of (1.1). It is obvious that f is continuous and periodic in t with period h (> 0) and globally Lipschtiz in x. Moreover, define

$$a_i = 0.5\lambda_{\max}(F_i + F_i^T) \text{ and } \kappa_1(t) = \max_{i\in\mathbb{S}} \beta_i(t) \tag{6.2}$$

or let $\kappa_1(t) \equiv 1$ to make it simple (see the worst case in Section 4). Then, for all $(x, y, i, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$,

$$\begin{aligned}&(x-y)^T(f(x,i,t) - f(y,i,t))\\ =&0.5\beta_i(t)(x-y)^T(F_i + F_i^T)(x-y) \le a_i\kappa_1(t)|x-y|^2.\end{aligned}$$

In other words, f satisfies Assumption 2.1. Let $x_1(t)$ and $x_2(t)$ be two solutions of (6.1) with different initial states (i.e., $x_1(0) \neq x_2(0)$) but the same initial mode (i.e., the same $r(0)$). It is straightforward to show that

$$\frac{d}{dt}(|x_1(t) - x_2(t)|^2) \ge \hat{\lambda}\hat{\beta}|x_1(t) - x_2(t)|^2,$$

where $\hat{\lambda} := \min_{i\in\mathbb{S}} \lambda_{\min}(F_i + F_i^T) > 0$ and $\hat{\beta} := \min_{i\in\mathbb{S}} \min_{0\le t\le h} \beta_i(t) > 0$. This implies immediately that $|x_1(t) - x_2(t)| \to \infty$ with probability 1. Hence the ODE (6.1) is not stable in distribution.

Let us now design a stochastic feedback control to stabilise it. To make it simple, we will use a scalar Brownian motion $B(t)$ as the stochastic source, while look to design the control function in the form $u(x, i, t) = \kappa_2(t)A_i x$ for $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, namely, we need to design $\kappa_2 \in \mathcal{K}_h$ and matrices $A_i \in \mathbb{R}^{n\times n}$ for $i \in \mathbb{S}$. So the stochastically controlled system is

$$\begin{aligned}dX(t) =& [\alpha_{r(t)}(t)\zeta_{r(t)} + \beta_{r(t)}(t)F_{r(t)}]x(t)dt\\ &+ \kappa_2(t)A_{r(t)}X(t)dB(t).\end{aligned} \tag{6.3}$$

We will consider the situation described in Case 2 in Section 5 and use the notations there, bearing in mind that $m = 1$ in this example. In particular, we also assume (5.10). We hence set $A_i = 0$ for $i \in \mathbb{S}_1$. For each $i \in \mathbb{S}_2$, choose a symmetric $n \times n$ matrix $\bar{A}_i$ such that $\lambda_{\min}(\bar{A}_i) > \sqrt{0.5}\|\bar{A}_i\|$. Then choose a

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positive number α so large that

$$\sum_{i\in\mathbb{S}_2} \upsilon_i \alpha^2 \big(0.5\|\bar{A}_i\|^2 - \lambda_{\min}^2(\bar{A}_i)\big) + \sum_{i\in\mathbb{S}} \upsilon_i a_i < 0. \quad (6.4)$$

Now set $A_i = \alpha \bar{A}_i$. Recalling Section 5, we see $u(x,i,t)$ satisfies Assumption 2.1 with $b_i = c_i = 0$ for $i \in \mathbb{S}_1$ and $b_i = \alpha\|A_i\|$ and $c_i = \alpha\lambda_{\min}(\bar{A}_i)$ for $i \in \mathbb{S}_2$. Consequently, it follows from (6.4) that

$$\sum_{i\in\mathbb{S}} \upsilon_i a_i + \sum_{i\in\mathbb{S}_2} \upsilon_i (0.5 b_i^2 - c_i^2) < 0.$$

That is

$$\sum_{i\in\mathbb{S}} \upsilon_i (a_i + 0.5 b_i^2 - c_i^2) < 0.$$

By Proposition 8.2 in the appendix below, we see Assumption 3.2 is satisfied. In other words, we can find $p \in (0,1)$ for matrix $\mathcal{A}$ defined by (3.3) to be a nonsingular M-matrix. Finally, we can design κ_2 for Assumption 3.3 to hold as explained in the paragraph just before Case 1 in Section 5 or simply let $\kappa_2 \equiv 1$. In the latter case, Assumption 3.3 holds automatically and there is no need to determine p.

To perform computer simulation, we consider the 2-dimensional ODE (6.1), where the Markov chain $r(t)$ has its state space $\mathbb{S} = \{1,2\}$ and generator $\Gamma = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$ and the others are: $h = 0.1$,

$$a_1(t) = 0.5 + \cos(2\pi t/h),\ \beta_1(t) = 0.5 + 0.4\sin(2\pi t/h),$$

$$a_2(t) = 0.5 + \sin(2\pi t/h),\ \beta_2(t) = 0.5 + 0.4\cos(2\pi t/h),$$

$$\zeta_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \zeta_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

$$F_1 = \begin{pmatrix} 0.2 & 0.5 \\ -0.5 & 0.1 \end{pmatrix},\ F_2 = \begin{pmatrix} 0.1 & -0.2 \\ 0.3 & 0.1 \end{pmatrix}.$$

By (6.2), we have that $a_1 = 0.2$, $a_2 = 0.1309$ but we let $\kappa_1(t) \equiv 1$ to make it simple. Consider the case where $\mathbb{S}_1 = \{2\}$ and $\mathbb{S}_2 = \{1\}$. Accordingly, $A_2 = 0$ but we need to design A_1. We choose $A_1 = \alpha\bar{A}_1$ with $\bar{A}_1 = \text{diag}(1,1)$. Noting that $r(t)$ has its stationary distribution $\upsilon = (0.75, 0.25)$, (6.4) becomes $-0.375\alpha^2 + 0.182725 < 0$, i.e., $\alpha > 0.6980$ and we choose $\alpha = 2$ so the existence of $p \in (0,1)$ is guaranteed for Assumption 3.2 to hold. We finally choose $\kappa_2(t) \equiv 1$ so Assumption 3.3 holds as well and there is no need to determine p. In other words, the controlled system (6.3) is stable in distribution with the system coefficients specified above.

We apply the well-known Euler-Maruyama method (see, e.g., [29]) with the stepsize 0.001 to perform the simulation of three sample paths of the solution with 3 different initial values $(0,0)^T$, $(5,-2)^T$ and $(-2,5)^T$ for $X(0)$ but the same initial value 1 for $r(0)$, which are corresponding to Sample 1, 2 and 3 in Fig. 1, respectively. The simulation does not only show that three sample paths approach to each other very quickly but also that three sample paths look like stationary sequences. We observe that most of these sample paths are within $[-1,1]$ but some outside. This is significantly different from the given ODE—any of its two different solutions will diverge to infinity with probability 1. In other words, the simulation illustrates clearly that the stochastic feedback control stabilise the given ODE in distribution.

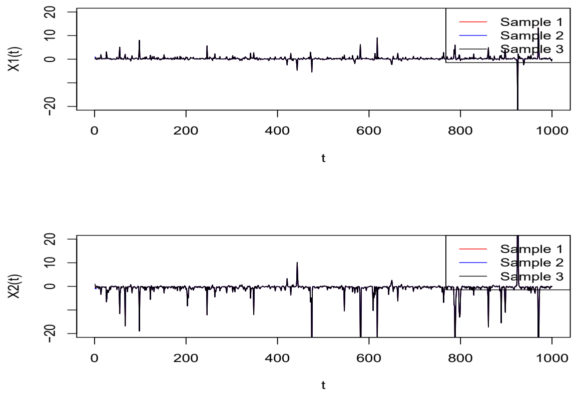

Fig. 1: Three sample paths of the controlled SDE.

7 Conclusion

In this paper we proposed a new problem of stabilisation in distribution by periodic noise: whether or not a periodic stochastic state feedback control can make a given nonlinear periodic hybrid differential equation, which is not stable in distribution, to become stable in distribution. We pointed out that there is little known on this problem so far but also explained why such a problem is required to be addressed from real applications including the control of Covid-19. We did not only investigated the problem successfully but also demonstrate how periodic stochastic feedback controls could be designed to stabilise given systems in distribution. A linear multi-dimensional example was discussed with computer simulation to illustrate our new theory on stabilisation in distribution by periodic noise.

8 Appendix

In this Appendix, we first give a couple of easy-to-check sufficient criteria for Assumption 3.2 to hold. The first one is [36, Theorem 5.13 on page 174] which we cite here as a proposition.

Proposition 8.1. *Assumption* 3.2 *holds if*

$$\begin{vmatrix} c_1^2 - 0.5b_1^2 - a_1, & -\gamma_{12}, & \cdots, & -\gamma_{1N} \\ c_2^2 - 0.5b_2^2 - a_2, & -\gamma_{22}, & \cdots, & -\gamma_{2N} \\ \vdots & & & \vdots \\ c_N^2 - 0.5b_N^2 - a_N, & -\gamma_{N2}, & \cdots, & -\gamma_{NN} \end{vmatrix} > 0$$

and, moreover, there is some $\hat{j} \in \mathbb{S}$ for which

$$\gamma_{i\hat{j}} > 0 \text{ for all } i \in \mathbb{S} \text{ but } i \neq \hat{j}. \quad (8.1)$$

To state another useful criterion, we recall that $r(t)$ is an irreducible Markov chain in the finite state space $\mathbb{S}$. Hence, it has a unique stationary distribution $\upsilon = (\upsilon_1, \upsilon_2, \cdots, \upsilon_N) \in \mathbb{R}^{1\times N}$ which can be determined by solving the linear equation $\upsilon\Gamma = 0$ subject to $\sum_{i\in\mathbb{S}} \upsilon_i = 1$ and $\upsilon_i > 0$ for all $i \in \mathbb{S}$.

Proposition 8.2. *Assumption* 3.2 *holds if* (8.1) *holds and*

$$\sum_{i\in\mathbb{S}} \upsilon_i(a_i + 0.5b_i^2 - c_i^2) < 0. \quad (8.2)$$

This Proposition was proved in the appendix of [35]. We next give another proposition which has been used to show why $\breve{\sigma} > 0$ in Section 3.

Proposition 8.3. *If $D = \text{diag}(d_1, \cdots, d_N)$ with all $d_i \leq 0$, then $D - \Gamma$ is not a nonsingular M-matrix.*

Proof. Assume that $D - \Gamma$ is a nonsingular M-matrix. Set $\mathbf{1}_N = (1, \cdots, 1)^T \in \mathbb{R}^N$ and $z = (z_1, \cdots, z_N)^T = (D - \Gamma)^{-1}\mathbf{1}_N$. Then all $z_i > 0$ and $(D - G)z = \mathbf{1}_N$. Multiplying it from the left by the stationary distribution υ of the Markov chain (see the paragraph before Proposition 8.2) gives

$$\upsilon(D - \Gamma)z = \upsilon\mathbf{1}_N = 1.$$

But

$$\upsilon(D - \Gamma)z = \upsilon Dz - \upsilon\Gamma z = \sum_{i\in\mathbb{S}} \upsilon_i d_i \leq 0.$$

We see a contradiction. The proposition must hold. □

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