# WEAK SOLUTIONS TO THE EQUATIONS OF STATIONARY COMPRESSIBLE FLOWS IN ACTIVE LIQUID CRYSTALS

ZHILEI LIANG, APALA MAJUMDAR, DEHUA WANG, AND YIXUAN WANG

Abstract. The equations of stationary compressible flows of active liquid crystals are considered in a bounded three-dimensional domain. The system consists of the stationary Navier-Stokes equations coupled with the equation of Q-tensors and the equation of the active particles. The existence of weak solutions to the stationary problem is established through a two-level approximation scheme, compactness estimates and weak convergence arguments. Novel techniques are developed to overcome the difficulties due to the lower regularity of stationary solutions, a Moser-type iteration is used to deal with the strong coupling of active particles and fluids, and some weighted estimates on the energy functions are achieved so that the weak solutions can be constructed for all values of the adiabatic exponent $\gamma > 1$.

## 1. Introduction

Active hydrodynamics refer to dynamical systems that are continuously driven out of equilibrium state by injected energy effects on small scales and exhibit collective phenomenon on a large scale, for example, bacterial colonies, motor proteins, and living cells [2,32,33]. Active systems have natural analogies with nematic liquid crystals because the particles exhibit a orientational ordering at a high concentration due to the collective motion. In comparison with the passive nematic liquid crystals, the system of active hydrodynamics is usually unstable and has novel characteristics such as low Reynolds numbers and very different spatial and temporal patterns [22,42]. We refer the readers to [8,22,26,33,35,41,42] and their references for the physical background, applications and modeling of active hydrodynamics. Theoretical studies on active liquid crystals are relatively new and have attracted a lot of attention in recent years. For example, the evolutionary incompressible flows of active liquid crystals were studied in [9,27] and the evolutionary compressible flows were investigated in [10,36]. In this paper we are concerned with the stationary compressible flows of active liquid crystals, described by the following equations in a bounded domain $\mathcal{O} \subset \mathbb{R}^3$ :

$$\begin{cases} \operatorname{div}(\rho\mathbf{u}) = 0, \\ \mathbf{u}\cdot\nabla c - \triangle c = g_1, \\ \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\rho^\gamma - \operatorname{div}\left(\mathbb{S}_{ns}(\nabla\mathbf{u}) + \mathbb{S}_1(Q) + \mathbb{S}_2(c,Q)\right) = \rho g_2, \\ \mathbf{u}\cdot\nabla Q + Q\Omega - \Omega Q + c_*Q\operatorname{tr}(Q^2) + \dfrac{(c-c_*)}{2}Q - b\left(Q^2 - \dfrac{1}{3}\operatorname{tr}(Q^2)\mathbb{I}\right) - \triangle Q = g_3, \end{cases} \tag{1.1}$$

where $\rho$, $c$, $\mathbf{u}$ denote the total density, the concentration of active particles, and the velocity field, respectively; the nematic tensor order parameter $Q$ is a traceless and symmetric $3\times 3$

*Date*: May 3, 2022.
2010 *Mathematics Subject Classification.* 35Q35, 35Q30, 35D35, 76D05, 76A15.
*Key words and phrases.* Active liquid crystals, stationary compressible flows, Navier-Stokes equations, Q-tensor, weak solutions, weak convergence.

matrix, $\rho^\gamma$ is the pressure with adiabatic exponent $\gamma > 1$, and the functions $g_i\,(i = 1, 2, 3)$ are given external force terms. We denote the Navier-Stokes stress tensor by

$$\mathbb{S}_{ns}(\nabla\mathbf{u}) = \mu\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right) + \lambda \mathrm{div}\mathbf{u}\mathbb{I}, \tag{1.2}$$

where $(\nabla\mathbf{u})^\top$ denotes the transpose of $\nabla\mathbf{u}$, $\mathbb{I}$ is the identity matrix, and the constants $\mu$, $\lambda$ are viscous coefficients satisfying the following physical requirement:

$$\mu > 0, \quad \mu + 3\lambda \geq 0. \tag{1.3}$$

In (1.1), $\Omega = \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^\top)$, and the additional stress tensors are:

$$\mathbb{S}_1(Q) = -\nabla Q \odot \nabla Q + \frac{1}{2}|\nabla Q|^2\mathbb{I} + \frac{1}{2}\left(1 + \frac{c_*}{2}\mathrm{tr}(Q^2)\right)\mathrm{tr}(Q^2)\mathbb{I}, \tag{1.4}$$

and

$$\mathbb{S}_2(c, Q) = Q\triangle Q - \triangle QQ + \sigma_* c^2 Q, \tag{1.5}$$

where $c_* > 0$ and $\sigma_* \in \mathbb{R}$ are given constants. The corresponding evolutionary equations of compressible active liquid crystal flows can be found in [10]. The equations (1.1) can be regarded as the stationary version of the evolutionary equations in [10] through the time-discretization and play an important role in the long-time behavior of active hydrodynamics. However, the mathematical analysis of the stationary equations (1.1) remains open. The aim of this paper is to construct the weak solutions to the stationary equations (1.1) subject to the following structural conditions:

$$\mathbf{u} = 0, \ \frac{\partial c}{\partial n} = 0, \ \frac{\partial Q}{\partial n} = 0, \quad \text{on } \partial\mathcal{O}, \tag{1.6}$$

and

$$\int_{\mathcal{O}} \rho(x)dx = m_1 > 0 \quad \text{and} \quad \int_{\mathcal{O}} c(x)dx = m_2 > 0, \tag{1.7}$$

where $n$ denotes the outward unit normal vector of the boundary $\partial\mathcal{O}$, $m_1$ and $m_2$ are given constants. We remark that the two conditions on the total mass and the total active particles in (1.7) guarantee that the density function $\rho$ and the particle concentration $c$ are uniquely determined. For the modeling and analysis of the $Q$-tensor systems of nematic liquid crystals we refer the readers to [5, 7, 23, 30, 44] and references therein.

We now introduce some notation that will be frequently used throughout this article. For given symmetric matrices $A = (a_{ij})_{3\times3}$ and $B = (b_{ij})_{3\times3}$, denote $\mathrm{tr}(AB) = A : B = \sum_{i,j=1}^3 a_{ij}b_{ij}$, $\mathrm{tr}(A^2) = |A|^2$, and $S_0^3 := \{A = (a_{ij})_{3\times3} : \ a_{ij} = a_{ji}, \ \mathrm{tr}(A) = 0\}$. For two vectors $a$, $b \in \mathbb{R}^3$, denote $a \cdot b = \sum_{i=1}^3 a_i b_i$ and $a \otimes b = (a_i b_j)_{3\times3}$. Denote the Sobolev spaces (cf. [1]) by

$$W^{k,p} = W^{k,p}(\mathcal{O}), \quad L^p = W^{0,p}, \quad H^k = W^{k,2}, \quad p \in [1, \infty], \quad k \in \mathbb{N}_+.$$

Additionally, we use $W^{k,p}(\mathcal{O}, \mathbb{R}^3)$ and $W^{k,p}(\mathcal{O}, S_0^3)$ for the Sobolev spaces valued in $\mathbb{R}^3$ and $S_0^3$, respectively. We denote by $|\mathcal{O}|$ the measure of the domain $\mathcal{O}$, and write $\int_{\mathcal{O}} f(x)dx$ as $\int f$ for simplicity of notation.

We shall establish the existence of weak solutions to the problem (1.1)-(1.7) defined as follows.

**Definition 1.1.** The function $(\rho, c, \mathbf{u}, Q)$ is called a weak solution to the boundary-value problem (1.1)-(1.7) if there is some exponent $p > \frac{3}{2}$ such that

$$\rho \geq 0, \quad c \geq 0 \quad a.e. \text{ in } \mathcal{O},$$

$$\rho \in L^p(\mathcal{O}), \quad c \in H^2(\mathcal{O}), \quad \mathbf{u} \in H_0^1(\mathcal{O}, \mathbb{R}^3), \quad Q \in H^2(\mathcal{O}, S_0^3),$$

satisfying the following properties:

(i). The equations (1.1) are satisfied in the sense of distributions, (1.6) holds true in the trace sense, (1.7) holds true for given $m_1 > 0$ and $m_2 > 0$;

(ii). $(1.1)_1$ is satisfied in the sense of renormalized solutions, i.e., if $(\rho, \mathbf{u})$ is extended by zero outside $\mathcal{O}$, then

$$\operatorname{div}(b(\rho)\mathbf{u}) + \big(b'(\rho)\rho - b(\rho)\big)\operatorname{div}\mathbf{u} = 0, \quad \mathcal{D}'(\mathbb{R}^3),$$

where $b \in C^1([0,\infty))$ with $b'(z) = 0$ if $z$ is large,

(iii). $(1.1)_2$ and $(1.1)_4$ are satisfied almost everywhere in $\mathcal{O}$.

We are ready to state our main result.

**Theorem 1.1.** *Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Assume that the adiabatic exponent $\gamma > 1$, the constants $m_1 > 0$ and $m_2 > 0$, and the functions*

$$g_1 \in L^\infty(\mathcal{O}), \quad g_2 \in L^\infty(\mathcal{O}, \mathbb{R}^3), \quad g_3 \in L^\infty(\mathcal{O}, S_0^3) \tag{1.8}$$

*are given. Then there exists a small constant $\mathfrak{m}_2$ that depends on $m_1$, $c_*$, $\sigma_*$, $\mu, \lambda$, $\gamma$, $\|g_1\|_{L^\infty}$, $\|g_2\|_{L^\infty}$, $\|g_3\|_{L^\infty}$ and $|\mathcal{O}|$, such that if*

$$m_2 \in (0, \mathfrak{m}_2], \tag{1.9}$$

*the problem (1.1)-(1.7) admits a solution $(\rho, c, \mathbf{u}, Q)$ in the sense of Definition 1.1.*

**Remark 1.1.** The smallness assumption (1.9) is a technical condition that is mainly used to overcome the strong nonlinearity caused by the concentration $c$ of active particles.

**Remark 1.2.** In fact, Theorem 1.1 still holds true in the case when $c$ is any positive constant (hence $m_2 = c|\mathcal{O}|$).

We shall prove Theorem 1.1 by constructing approximate solutions and a two-level limiting procedure. The approximate solutions are constructed in light of time-discretization technique from the evolutionary equations in [10], and the limits are based on standard compactness theories developed in [10,16,31,37]. However, new difficulties arise due to the lower regularity of stationary solutions, strong nonlinearity and complex coupling of active particles and fluids. In order to make our ideas clear we comment on our approach and novelty below.

We begin in Section 2 with suitable linear equations to construct the approximations of system (1.1). For a given function $v$ in the set $\{v \in W^{1,\infty}(\mathcal{O}, \mathbb{R}^3),\ v = 0 \text{ on } \partial\mathcal{O}\}$, we impose the transport equation $(1.1)_1$ with the extra diffusion $\epsilon^2 \triangle \rho$ and obtain $\rho = \rho[v, \epsilon]$ in Lemma 2.1. With the force term $g_1$ given in $(1.1)_2$ we can solve $c = c[v, \epsilon]$. Having $\rho = \rho[v, \epsilon]$) and $c = c[v, \epsilon]$ in hand, for a given $v$ and a given function $\tilde{Q}$ in the set $\{\tilde{Q} \in W^{2,\infty}(\mathcal{O}, \mathbb{R}^3),\ \frac{\partial \tilde{Q}}{\partial n} = 0 \text{ on } \partial\mathcal{O}\}$ we are able to construct the solution to a linear system of $Q$ in terms of $v$ and $\tilde{Q}$. In the same manner, we consider the approximate momentum equations (2.8) and solve $\mathbf{u} = \mathbf{u}[v, \tilde{Q}, \epsilon]$. We should point out that the appearance of highest derivative of $Q$ due to (1.5) requires $W^{3,p}$ regularity for $Q$. Moreover, since $\nabla v$ and $g_3$ are only in $L^\infty$, we adopt the ideas in [11] and use a global mollification technique such that the above approximation is smooth.

The approximate equations (3.1) come from the linear equations in Section 2 and will be solved using the Schaefer Fixed Point Theorem (cf. [15]). The approximate solutions are constructed by a two-level approximation scheme involving the artificial viscosity and artificial pressure. However, the strong nonlinearity in the quantities

$$\int c^2 Q : \nabla \mathbf{u} \quad \text{and} \quad \int c\mathbf{u} \cdot \nabla c$$

causes new difficulties in closing the basic *a priori* estimates. To this end, we explore a Moser-type iteration such that $\|c\|_{L^\infty}$ can be bounded by $\|c\|_{L^1} = m_2$, and hence we are able to control the above mentioned nonlinear terms provided that some small assumption on $m_2$ is made. In this connection, we are allowed to close the energy estimates to obtain the existence of approximate solutions, and further improve the regularity of the solutions by a bootstrap argument.

Next we shall take the limit in the approximate solutions as first $\epsilon \to 0$ and then $\delta \to 0$ through the weak convergence arguments. We remark that the nonlinear coupling of $c$ and $Q$ in the momentum equation makes the limiting process much more subtle. For example, for the integral quantity

$$\int \left( Q_\epsilon^{ik} \triangle Q_\epsilon^{kj} - \triangle Q_\epsilon^{ik} Q_\epsilon^{kj} \right) \partial_j \phi \partial_i \triangle^{-1}(\rho_\epsilon),$$

the $\epsilon$-limit is not obvious because both $\triangle Q_\epsilon^{kj}$ and $\partial_i \triangle^{-1}(\rho_\epsilon)$ are only weakly convergent. Fortunately, we can overcome the difficulty using the integration by parts as well as the symmetry of $Q$; see (4.36) for a detailed explanation.

One disadvantage for the stationary problem is that it has no useful information on the density other than $\|\rho\|_{L^1}$, which is very different from the evolutionary equations for which the higher regularity $\|\rho\|_{L^\gamma}$ with $\gamma > 1$ is available. As a consequence we have extra difficulties in taking $\delta$-limit procedure (especially if $\gamma > 1$ is close to 1). Taking account of the ideas in [19, 25, 28, 29], we use the refined weighted estimates on both pressure and kinetic energy functions. However, the involvement of $c$ and $Q$ makes the proof much more complex and delicate. We utilize different weighted functions in dealing with the boundary case and interior case, and finally succeed in obtaining the uniform estimates for all adiabatic exponent $\gamma > 1$ under the smallness assumption (1.9). This is different from our previous papers [28, 29] for Cahn-Hilliard/Navier-Stokes equations where the restriction $\gamma > \frac{4}{3}$ seems to be critical because the pressure depends both on the density and the concentration. Once the Proposition 5.1 is obtained, we are able to use the standard compactness theories in [16, 31] to take $\delta$-limit and complete the proof of Theorem 1.1.

The rest of paper is organized as follows. In Section 2, we introduce some linear equations and their preliminary existence results that will be used in the construction of approximate solutions. In Section 3, we construct the approximate solutions by a two-level approximation scheme involving the artificial viscosity coefficient $\epsilon > 0$ and the parameter $\delta > 0$ in the artificial pressure, and prove the existence using the fixed point argument. In Section 4, we take the limit as $\epsilon \to 0$ of the approximate solutions for any fixed $\delta > 0$, and finally in Section 5 we take the limit as $\delta \to 0$ for the vanishing of the artificial pressure and conclude the existence of weak solutions.

## 2. Preliminary Results on Linear Equations

In this section we present some linear equations, in preparation for constructing the approximate solutions to the problem (1.1)-(1.7) in (3.1) next section.

Define the following function spaces:

$$W_0^{1,\infty}(\mathcal{O},\mathbb{R}^3) := \{v \in W^{1,\infty}(\mathcal{O},\mathbb{R}^3),\ \ v = 0 \ \text{ on } \partial\mathcal{O}\},$$
$$W_n^{2,\infty}(\mathcal{O},S_0^3) := \left\{Q \in W^{2,\infty}(\mathcal{O},S_0^3),\ \ \frac{\partial Q}{\partial n} = 0 \text{ on } \partial\mathcal{O}\right\},$$
$$\mathcal{W} := W_0^{1,\infty}(\mathcal{O},\mathbb{R}^3) \times W_n^{2,\infty}(\mathcal{O},S_0^3).$$

Now let $\epsilon \in (0,1)$ and $p \in (1,\infty)$ be fixed. Recall $m_1$ and $m_2$ defined in (1.7). The first lemma is for the solvability of a relaxed transport equation with dissipation from [37].

**Lemma 2.1.** [37, Proposition 4.29] *For any given $v \in W_0^{1,\infty}(\mathcal{O},\mathbb{R}^3)$, there exists a unique solution $\rho = \rho[v] \in W^{2,p}(\mathcal{O})$ to the following problem*

$$\epsilon\rho + \operatorname{div}(\rho v) = \epsilon^2 \triangle\rho + \epsilon\frac{m_1}{|\mathcal{O}|}, \quad \frac{\partial\rho}{\partial n} = 0 \ \text{ on } \ \partial\mathcal{O}, \tag{2.1}$$

*such that*

$$\epsilon^2 \int \nabla\rho\cdot\nabla\eta - \int \rho v\cdot\nabla\eta + \epsilon\int(\rho-\rho_0)\eta = 0, \quad \eta \in C^\infty(\overline{\mathcal{O}}). \tag{2.2}$$

*Moreover,*

$$\rho \geq 0 \ \textit{a.e.} \text{ in } \mathcal{O}, \quad \|\rho\|_{L^1} = m_1, \quad \|\rho\|_{W^{2,p}} \leq C(\epsilon, p, m_1, \mathcal{O}, \|v\|_{W^{1,\infty}}). \tag{2.3}$$

**Lemma 2.2.** *For any given $g_1 \in L^\infty(\mathcal{O})$ and $v \in W_0^{1,\infty}(\mathcal{O},\mathbb{R}^3)$, the following problem*

$$v\cdot\nabla c = \triangle c + g_1, \quad \int c = m_2, \quad \frac{\partial c}{\partial n} = 0 \ \text{ on } \ \partial\mathcal{O}, \tag{2.4}$$

*has a unique nonnegative solution $c = c[v] \in W^{2,p}(\mathcal{O})$.*

*Proof.* Consider the approximate equation

$$\alpha c + v\cdot\nabla c = \triangle c + g_1 + \alpha\frac{m_2}{|\mathcal{O}|}, \quad \alpha \in (0,1). \tag{2.5}$$

Following the proof of [37, Proposition 4.29], we see that (2.5) has a solution $c_\alpha = c_\alpha(v) \in W^{2,p}(\mathcal{O})$ $(1 < p < \infty)$, satisfying

$$c_\alpha \geq 0 \ \ a.e., \ \ \mathcal{O}, \quad \|c_\alpha\|_{L^1} = m_2, \quad \|c_\alpha\|_{W^{2,p}} \leq C(p, m_2, \mathcal{O}, \|v\|_{W^{1,\infty}}).$$

Then, taking the limit $\alpha \to 0^+$, we complete the proof. □

The following two lemmas can be obtained from the elliptic theory (see [21]).

**Lemma 2.3.** *For any given $(v,\tilde{Q}) \in \mathcal{W}$, the following problem*

$$\begin{cases} \triangle Q - \epsilon Q = F^1(v,\tilde{Q}) := v\cdot\nabla\tilde{Q} + \dfrac{(c-c_*)}{2}\tilde{Q} + c_*\tilde{Q}\operatorname{tr}(\tilde{Q}^2) - b\left(\tilde{Q}^2 - \dfrac{1}{3}\operatorname{tr}(\tilde{Q}^2)\mathbb{I}\right) \\ \qquad\qquad\qquad + \tilde{Q}\langle\tilde{\Omega}\rangle - \langle\tilde{\Omega}\rangle\tilde{Q} - \langle g_3\rangle, \\ \dfrac{\partial Q}{\partial n} = 0 \quad \text{on } \partial\mathcal{O}, \end{cases} \tag{2.6}$$

*has a unique solution* $Q = Q[v, \tilde{Q}]$ *satisfying*

$$\|Q\|_{W^{3,p}} \leq C\|F^1\|_{W^{1,p}} < \infty, \tag{2.7}$$

*where* $c = c[v]$ *is solved in Lemma 2.2, and* $\langle\tilde{\Omega}\rangle = \langle\nabla v\rangle - (\langle\nabla v\rangle)^\top$ *with* $\langle f\rangle$ *being a smooth approximation of function* $f$ *globally in* $\mathcal{O}$.

**Remark 2.1.** We use the smooth approximations $\langle\Omega\rangle$ and $\langle g_3\rangle$ to guarantee that $F^1$ belongs to $W^{1,p}$. Such approximations can be obtained through the global mollification $\langle f\rangle = \eta_\epsilon * f$ with $\eta_\epsilon$ the Friedrichs mollifier (see e.g., [15]). Due to the Neumann boundary condition, we impose $\epsilon Q$ to guarantee that the values of function $Q$ is uniquely determined.

**Lemma 2.4.** *For any given* $(v, \tilde{Q}) \in \mathcal{W}$*, the following problem*

$$\begin{cases} \mathrm{div}\mathbb{S}_{ns}(\nabla\mathbf{u}) = F^2(v, \tilde{Q}) := \epsilon\rho v + \mathrm{div}(\rho v \otimes v) + \nabla(\delta\rho^4 + \rho^\gamma) + \epsilon^2\nabla\rho\cdot\nabla v \\ \qquad\qquad - \mathrm{div}\left(-\nabla Q\otimes\nabla Q + \frac{1}{2}|\nabla Q|^2\mathbb{I} + \frac{1}{2}tr(Q^2)\mathbb{I} + \frac{c_*}{4}(tr(Q^2))^2\mathbb{I}\right) \\ \qquad\qquad - \mathrm{div}\left(Q\triangle Q - \triangle QQ + \sigma_* c^2 Q\right) - \rho g_2 \\ \mathbf{u} = 0, \quad \text{on } \partial\mathcal{O}, \end{cases} \tag{2.8}$$

*has a unique solution* $\mathbf{u} = \mathbf{u}[v, \tilde{Q}]$ *satisfying*

$$\|\mathbf{u}\|_{W^{2,p}} \leq C\|F^2\|_{L^p} < \infty. \tag{2.9}$$

*where both* $\epsilon \in (0,1)$ *and* $\delta \in (0,1)$ *are fixed constants;* $\rho = \rho[v]$, $c = c[v]$ *and* $Q = Q[v, \tilde{Q}]$ *are determined in Lemmas 2.1-Lemma 2.3.*

**Remark 2.2.** The artificial pressure $\delta\rho^4$ is used to improve the integrability of density, which will be used in subsequent analysis.

## 3. Approximate Solutions

In this section we construct the approximate solutions to the problem (1.1)-(1.7). Have the existence results for the lineaized problems in Lemmas 2.1-2.4, we consider the following nonlinear approximate system:

$$\begin{cases} \epsilon\rho_\epsilon + \mathrm{div}(\rho_\epsilon\mathbf{u}_\epsilon) = \epsilon^2\triangle\rho_\epsilon + \epsilon\rho_0, \\ \mathbf{u}_\epsilon\cdot\nabla c_\epsilon = \triangle c_\epsilon + g_1, \\ \mathrm{div}\mathbb{S}_{ns}(\nabla\mathbf{u}_\epsilon) = F^2(\mathbf{u}_\epsilon, Q_\epsilon), \\ \triangle Q_\epsilon = \epsilon Q_\epsilon + F^1(\mathbf{u}_\epsilon, Q_\epsilon), \\ \frac{\partial\rho_\epsilon}{\partial n} = 0, \quad \mathbf{u}_\epsilon = 0, \quad \frac{\partial c_\epsilon}{\partial n} = 0, \quad \frac{\partial Q_\epsilon}{\partial n} = 0 \quad \text{on } \partial\mathcal{O}, \\ \int \rho_\epsilon = m_1, \quad \int c_\epsilon = m_2, \end{cases} \tag{3.1}$$

where $\rho_0 = \frac{m_1}{|\mathcal{O}|}$, the functions $F^1$ and $F^2$ are taken from (2.6) and (2.8) respectively.

The theorem below states the existence of solutions to problem (3.1).

**Theorem 3.1.** *Assume that* (1.8) *holds true and* $\epsilon$ *is sufficiently small. Then there is a small constant* $\mathfrak{m}_2$ *depending on* $m_1$, $c_*$, $\mu, \lambda$, $\gamma$, $\epsilon$, $\delta$, $|\mathcal{O}|$, $\|g_1\|_{L^\infty}$, $\|g_2\|_{L^\infty}$, $\|g_3\|_{L^\infty}$, *such that if* $m_2 \le \mathfrak{m}_2$, *the problem* (3.1) *admits a solution* $(\rho_\epsilon, c_\epsilon, \mathbf{u}_\epsilon, Q_\epsilon)$ *satisfying, for any* $p \in (1, \infty)$,

$$0 \le \rho_\epsilon \in W^{2,p}(\mathcal{O}), \quad \|\rho_\epsilon\|_{L^1(\mathcal{O})} = m_1, \tag{3.2}$$

$$0 \le c_\epsilon \in W^{2,p}(\mathcal{O}), \quad \|c_\epsilon\|_{L^1(\mathcal{O})} = m_2, \tag{3.3}$$

$$\mathbf{u}_\epsilon \in W^{2,p}(\mathcal{O}, \mathbb{R}^3), \quad Q_\epsilon \in W^{3,p}(\mathcal{O}, S_0^3). \tag{3.4}$$

*Proof.* The proof is based on the Schaefer Fixed Point Theorem (see, e.g., Chapter 9, Theorem 4 in [15]). Thanks to Lemmas 2.1-2.4, for any given $(v, \tilde{Q}) \in \mathcal{W}$, we have

$$(\mathbf{u}_\epsilon, Q_\epsilon) = A[v, \tilde{Q}] := (\mathbf{u}[v, \tilde{Q}], Q[v, \tilde{Q}]). \tag{3.5}$$

By (2.7) and (2.9), it is clear that the operator $A : \mathcal{W} \to \mathcal{W}$ is compact. A straightforward computation shows that $A$ is continuous; see, e.g., [10]. In order to apply the Schaefer Fixed Point Theorem, we need to prove the following proposition:

**Proposition 3.1.** *Assume that* $(\mathbf{u}_\epsilon, Q_\epsilon)$ *is a solution to the equations* (2.6) *and* (2.8). *Then the set*

$$\left\{ (\mathbf{u}_\epsilon, Q_\epsilon) \in \mathcal{W} \,\middle|\, \begin{array}{l} (\mathbf{u}_\epsilon, Q_\epsilon) = tA[\mathbf{u}_\epsilon, Q_\epsilon] \\ \text{for some } \ t \in [0,1], \ \text{and} \quad \rho_\epsilon = \rho[\mathbf{u}_\epsilon], \ \ c_\epsilon = c[\mathbf{u}_\epsilon] \end{array} \right\} \tag{3.6}$$

*is bounded.*

From Proposition 3.1, we may use the Schaefer Fixed Point Theorem to conclude that $(\mathbf{u}_\epsilon, Q_\epsilon) = A[\mathbf{u}_\epsilon, Q_\epsilon]$ with $\rho_\epsilon = \rho[\mathbf{u}_\epsilon]$ and $c_\epsilon = c[\mathbf{u}_\epsilon]$. This together with Lemma 2.1 and Lemma 2.2 guarantee the existence of the solution $(\rho_\epsilon, c_\epsilon, \mathbf{u}_\epsilon, Q_\epsilon)$ to the problem (3.1) for any fixed $\epsilon > 0$. Consequently, the estimates (3.2)-(3.3) follow directly from (2.3) and Lemma 2.2.

We now prove Proposition 3.1 as well as (3.4), leading to the complete proof of Theorem 3.1. We will drop the subscript $\epsilon$ and use $(\rho, c, \mathbf{u}, Q)$ to denote $(\rho_\epsilon, c_\epsilon, \mathbf{u}_\epsilon, Q_\epsilon)$ for the sake of simplicity. Observe that $(\rho, c, \mathbf{u}, Q)$ solves

$$\begin{cases} \epsilon\rho + \mathrm{div}(\rho\mathbf{u}) = \epsilon^2 \triangle\rho + \epsilon\rho_0, \\ \mathbf{u} \cdot \nabla c = \triangle c + g_1, \\ \triangle Q = \epsilon Q + tF^1(\mathbf{u}, Q), \\ \mathrm{div}\mathbb{S}_{ns}(\nabla\mathbf{u}) = tF^2(\mathbf{u}, Q), \\ \dfrac{\partial\rho}{\partial n} = 0, \quad \mathbf{u} = 0, \quad \dfrac{\partial c}{\partial n} = 0, \quad \dfrac{\partial Q}{\partial n} = 0, \quad \text{on } \partial\mathcal{O}, \\ \displaystyle\int \rho = m_1, \qquad \int c = m_2. \end{cases} \tag{3.7}$$

To prove Proposition 3.1, it suffices to show that there is a constant $M < \infty$ independent of $t$ such that

$$\|(\mathbf{u}, Q)\|_{\mathcal{W}} < M. \tag{3.8}$$

3.1. **Basic inequalities.** Multiplying $(3.7)_1$ by $\frac{t}{2}|\mathbf{u}|^2$ and $(3.7)_4$ by $\mathbf{u}$ respectively, we get

$$\begin{aligned}
&\frac{\epsilon t}{2}\int(\rho+\rho_0)|\mathbf{u}|^2+t\int\mathbf{u}\cdot\nabla\left(\delta\rho^4+\rho^\gamma\right)+\mu\int|\nabla\mathbf{u}|^2+(\lambda+\mu)\int|\mathrm{div}\mathbf{u}|^2\\
&=t\int\rho g_2\cdot\mathbf{u}-t\int\triangle Q:(\mathbf{u}\cdot\nabla)Q+\frac{1}{2}\mathrm{div}\mathbf{u}\,tr(Q^2)\left(1+\frac{c_*}{2}tr(Q^2)\right)\\
&\quad+t\int\mathrm{div}(Q\triangle Q-\triangle QQ)\mathbf{u}-t\sigma_*\int c^2Q:\nabla\mathbf{u},
\end{aligned}\tag{3.9}$$

where we have used (1.4) and the following computation

$$\begin{aligned}
&\int\mathrm{div}\left(-\nabla Q\odot\nabla Q+\frac{1}{2}|\nabla Q|^2\right)\mathbf{u}\\
&=-\int\partial_i(\partial_iQ^{kl}\partial_jQ^{kl})\mathbf{u}^j+\frac{1}{2}\int\mathbf{u}^j\partial_j|\partial_iQ|^2=-\int\triangle Q:(\mathbf{u}\cdot\nabla)Q.
\end{aligned}$$

By $(3.7)_1$, one deduces

$$\begin{aligned}
&\int\mathbf{u}\cdot\nabla\left(\delta\rho^4+\rho^\gamma\right)=\int\rho\mathbf{u}\cdot\nabla\left(\frac{4\delta}{3}\rho^3+\frac{\gamma}{\gamma-1}\rho^{\gamma-1}\right)\\
&=\epsilon\int\left(\frac{4\delta}{3}\rho^3+\frac{\gamma}{\gamma-1}\rho^{\gamma-1}\right)(\rho-\rho_0)+\epsilon^2\int\nabla\left(\frac{4\delta}{3}\rho^3+\frac{\gamma}{\gamma-1}\rho^{\gamma-1}\right)\cdot\nabla\rho\\
&\geq\epsilon\int\left(\frac{\delta}{3}\rho^4+\frac{1}{\gamma-1}\rho^\gamma\right)-\epsilon\int\left(\frac{\delta}{3}\rho_0^4+\frac{1}{\gamma-1}\rho_0^\gamma\right)+\epsilon^2\int\left(\delta|\nabla\rho^2|^2+\frac{4}{\gamma}|\nabla\rho^{\frac{\gamma}{2}}|^2\right).
\end{aligned}$$

Then substituting the above estimate into (3.9) gives

$$\begin{aligned}
&\frac{\epsilon t}{2}\int(\rho+\rho_0)|\mathbf{u}|^2+\epsilon t\int\left(\frac{\delta}{3}\rho^4+\frac{1}{\gamma-1}\rho^\gamma\right)+\epsilon^2t\int\left(\delta|\nabla\rho^2|^2+\frac{4}{\gamma}|\nabla\rho^{\frac{\gamma}{2}}|^2\right)\\
&\quad+\mu\int|\nabla\mathbf{u}|^2+(\lambda+\mu)\int|\mathrm{div}\mathbf{u}|^2\\
&\leq\epsilon t\int\left(\frac{\delta}{3}\rho_0^4+\frac{1}{\gamma-1}\rho_0^\gamma\right)+t\int\rho g_2\cdot\mathbf{u}\\
&\quad-t\int\triangle Q:(\mathbf{u}\cdot\nabla)Q+\frac{1}{2}\mathrm{div}\mathbf{u}\,\mathrm{tr}(Q^2)\left(1+\frac{c_*}{2}\mathrm{tr}(Q^2)\right)\\
&\quad+t\int\mathrm{div}(Q\triangle Q-\triangle QQ)\mathbf{u}-t\sigma_*\int c^2Q:\nabla\mathbf{u}\\
&=:\epsilon t\int\left(\frac{\delta}{3}\rho_0^4+\frac{1}{\gamma-1}\rho_0^\gamma\right)+\sum_{i=1}^4I_i.
\end{aligned}\tag{3.10}$$

Next, following [10] we multiply $(3.7)_3$ by $-\triangle Q + Q + c_* Q\mathrm{tr}(Q^2)$ to obtain

$$
\begin{aligned}
&\int |\triangle Q|^2 + (1+\epsilon)\int |\nabla Q|^2 + \epsilon \int \left(|Q|^2 + c_*|Q|^4\right) + tc_* \int \left(|Q|^4 + c_*|Q|^6\right) \\
&= 2tc_* \int \triangle Q : Qtr(Q^2) + t\int \frac{(c-c_*)}{2} Q : (\triangle Q - Q - c_* Qtr(Q^2)) \\
&\quad + t\int b\left(Q^2 - \frac{1}{3}\mathbb{I}tr(Q^2)\right) : \left(-\triangle Q + Q + c_* Qtr(Q^2)\right) + t\int (Q\langle\Omega\rangle - \langle\Omega\rangle Q) : \triangle Q \\
&\quad - t\int (Q\langle\Omega\rangle - \langle\Omega\rangle Q) : (Q + c_* Qtr(Q^2)) + t\int \mathbf{u}\cdot\nabla Q : \left(\triangle Q - Q - c_* Qtr(Q^2)\right) \\
&\quad + t\int \langle g_3\rangle : \left(-\triangle Q + Q + c_* Qtr(Q^2)\right) \\
&=: \sum_{j=1}^{7} J_j.
\end{aligned}
\tag{3.11}
$$

3.2. **Uniform in $\epsilon$ and $t$ estimates.** Now we estimate the terms on the right-hand side in (3.10) and (3.11). In this subsection, the generic constant $C$ may rely on $\lambda$, $\mu$, $m_1$, $\delta$, $\gamma$, $|\mathcal{O}|$, $c_*$, $\sigma_*$, $\|g_1\|_{L^\infty}$, $\|g_2\|_{L^\infty}$, $\|g_3\|_{L^\infty}$, but not on $t$ and $\epsilon$.

Direct calculations show

$$
J_1 = -2tc_* \int |\nabla Q|^2 tr(Q^2) - tc_* \int |\nabla tr(Q^2)|^2 \le 0, \tag{3.12}
$$

$$
\begin{aligned}
J_6 &= t\int \mathbf{u}\cdot\nabla Q : (\triangle Q - Q - c_* Qtr(Q^2)) \\
&= t\int \mathbf{u}\cdot\nabla Q : \triangle Q - t\int \mathbf{u}\cdot\nabla\left(\frac{1}{2}(trQ^2) + \frac{c_*}{4}((trQ^2))^2\right) = -I_2,
\end{aligned}
\tag{3.13}
$$

and by the fact that $Q$ is symmetric and $\Omega$ is skew-symmetric, one has

$$
J_5 = t\int (Q\Omega - \Omega Q) : (Q + c_* Qtr(Q^2)) = 0. \tag{3.14}
$$

Moreover, we have the following computation:

$$
\begin{aligned}
I_3 + J_4 &= t\int \mathrm{div}(Q\triangle Q - \triangle Q Q)\mathbf{u} + t\int (Q\langle\Omega\rangle - \langle\Omega\rangle Q) : \triangle Q \\
&= t\int \mathrm{div}(Q\triangle Q - \triangle Q Q)\mathbf{u} + t\int (Q\Omega - \Omega Q) : \triangle Q \\
&\quad + t\int [(Q\langle\Omega\rangle - \langle\Omega\rangle Q) - (Q\langle\Omega\rangle - \langle\Omega\rangle Q)] : \triangle Q \\
&= t\int [(Q\langle\Omega\rangle - \langle\Omega\rangle Q) - (Q\langle\Omega\rangle - \langle\Omega\rangle Q)] : \triangle Q \\
&\le tC\|\langle\nabla\mathbf{u}\rangle - \nabla\mathbf{u}\|_{L^2}\|Q\|_{L^\infty}\|\triangle Q\|_{L^2} \\
&\le tC\|\langle\nabla\mathbf{u}\rangle - \nabla\mathbf{u}\|_{L^2}\left(\|Q\|_{L^4}^2 + \|\triangle Q\|_{L^2}^2\right),
\end{aligned}
\tag{3.15}
$$

where the last equality is from [9, Lemma A1], and the last inequality is from the interpolation inequality.

As a result of (3.12)-(3.15), inequalities (3.10) and (3.11) provide us

$$
\begin{aligned}
&\frac{\epsilon t}{2}\int(\rho+\rho_0)|\mathbf{u}|^2+\epsilon t\int\left(\frac{\delta}{3}\rho^4+\frac{1}{\gamma-1}\rho^\gamma\right)+\epsilon^2 t\int\left(\delta|\nabla\rho^2|^2+\frac{4}{\gamma}|\nabla\rho^{\frac{\gamma}{2}}|^2\right)\\
&\quad+\mu\int|\nabla\mathbf{u}|^2+\int|\triangle Q|^2+\int|\nabla Q|^2+tc_*\int\left(|Q|^4+c_*|Q|^6\right)\\
&\le Ct+Ct\|\langle\nabla\mathbf{u}\rangle-\nabla\mathbf{u}\|_{L^2}\left(\|Q\|_{L^4}^2+\|\triangle Q\|_{L^2}^2\right)+I_1+I_4+J_2+J_3+J_7.
\end{aligned}
\tag{3.16}
$$

By (1.8), we have

$$
\begin{aligned}
&|I_1|+|J_3|+|J_7|\\
&\le Ct\|g_2\|_{L^\infty}\|\rho\|_{L^{\frac{6}{5}}}\|\mathbf{u}\|_{L^6}+C\sigma(1+\|g_3\|_{L^2})\left(\|\triangle Q\|_{L^2}+\|Q\|_{L^6}^5+1\right)\\
&\le Ct+Ct\|\rho\|_{L^{\frac{6}{5}}}^2+\frac{tc_*^2}{4}\int|Q|^6+\frac{\mu}{4}\int|\nabla\mathbf{u}|^2+\frac{1}{4}\int|\triangle Q|^2,
\end{aligned}
\tag{3.17}
$$

and

$$
\begin{aligned}
&|I_4|+|J_2|\\
&\le Ct\|c\|_{L^\infty}^2\|Q\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}+Ct(1+\|c\|_{L^\infty})\left(\|Q\|_{L^2}\|\triangle Q\|_{L^2}+\|Q\|_{L^6}^4+1\right)\\
&\le Ct+Ct\|c\|_{L^\infty}^6+\frac{tc_*^2}{4}\int|Q|^6+\frac{\mu}{4}\int|\nabla\mathbf{u}|^2+\frac{1}{4}\int|\triangle Q|^2.
\end{aligned}
\tag{3.18}
$$

Substituting (3.17) and (3.18) into (3.16) leads to

$$
\begin{aligned}
&\epsilon\int(\rho+\rho_0)|\mathbf{u}|^2+\epsilon\int\left(\frac{\delta}{3}\rho^4+\frac{1}{\gamma-1}\rho^\gamma\right)+\epsilon^2\int\left(\delta|\nabla\rho^2|^2+\frac{4}{\gamma}|\nabla\rho^{\frac{\gamma}{2}}|^2\right)\\
&\quad+\int|\nabla\mathbf{u}|^2+\int|\triangle Q|^2+\int|\nabla Q|^2+\int\left(|Q|^4+|Q|^6\right)\\
&\le C+C\|\rho\|_{L^{\frac{6}{5}}}^2+C\|c\|_{L^\infty}^6+C\|\langle\nabla\mathbf{u}\rangle-\nabla\mathbf{u}\|_{L^2}\left(\|Q\|_{L^4}^2+\|\triangle Q\|_{L^2}^2\right).
\end{aligned}
\tag{3.19}
$$

Observe that the constant $C$ in (3.19) is independent of $\epsilon$, then we may choose $\epsilon$ sufficiently small and use the standard properties of mollification such that $C\|\langle\nabla\mathbf{u}\rangle-\nabla\mathbf{u}\|_{L^2}\le\frac{1}{2}$ to obtain

$$
\begin{aligned}
&\epsilon\int(\rho+\rho_0)|\mathbf{u}|^2+\epsilon\int\left(\frac{\delta}{3}\rho^4+\frac{1}{\gamma-1}\rho^\gamma\right)+\epsilon^2\int\left(\delta|\nabla\rho^2|^2+\frac{4}{\gamma}|\nabla\rho^{\frac{\gamma}{2}}|^2\right)\\
&\quad+\int|\nabla\mathbf{u}|^2+\int|\triangle Q|^2+\int|\nabla Q|^2+\int\left(|Q|^4+|Q|^6\right)\\
&\le C+C\|\rho\|_{L^{\frac{6}{5}}}^2+C\|c\|_{L^\infty}^6.
\end{aligned}
\tag{3.20}
$$

3.3. **$\epsilon$-dependent regularity.** Thanks to $\|\rho\|_{L^1}=m_1$ and the interpolation inequalities, it follows from (3.20) that

$$
\begin{aligned}
&\epsilon\int(\rho+\rho_0)|\mathbf{u}|^2+\epsilon\int\left(\frac{\delta}{3}\rho^4+\frac{1}{\gamma-1}\rho^\gamma\right)+\epsilon^2\int\left(\delta|\nabla\rho^2|^2+\frac{4}{\gamma}|\nabla\rho^{\frac{\gamma}{2}}|^2\right)\\
&\quad+\int|\nabla\mathbf{u}|^2+\int|\triangle Q|^2+\int|\nabla Q|^2+\int\left(|Q|^4+|Q|^6\right)\\
&\le C\left(1+\|c\|_{L^\infty}^6\right),
\end{aligned}
\tag{3.21}
$$

where and in the rest of this subsection, the constant $C$ may rely on $\epsilon$.

In order to bound $\|c\|_{L^\infty}$ in (3.21), we need the following lemma:

**Lemma 3.1.** *There exist constants $C$ and $C_1$ depending only on $|\mathcal{O}|$ such that*

$$\|c\|_{L^\infty} \le C(1+\|\mathbf{u}\|_{L^6})^{C_1}(1+\|g_1\|_{L^\infty})m_2. \tag{3.22}$$

We will continue the proof of Theorem 3.1 and postpone the proof of Lemma 3.1 to the end of this section. With the help of (3.22) and (1.8), we estimate (3.21) as

$$\begin{aligned}
&\epsilon\int(\rho+\rho_0)|\mathbf{u}|^2+\epsilon\int\left(\frac{\delta}{3}\rho^4+\frac{1}{\gamma-1}\rho^\gamma\right)+\epsilon^2\int\left(\delta|\nabla\rho^2|^2+\frac{4}{\gamma}|\nabla\rho^{\frac{\gamma}{2}}|^2\right)\\
&\quad+\int|\nabla\mathbf{u}|^2+\int|\triangle Q|^2+\int|\nabla Q|^2+\int\left(|Q|^4+|Q|^6\right)\\
&\le C\left(1+\|c\|_{L^\infty}^6\right)\\
&\le C\left(1+m_2^6\|\nabla\mathbf{u}\|_{L^2}^{6C_1}\right)\\
&\le 2C,
\end{aligned}\tag{3.23}$$

where the last inequality is valid if

$$m_2 \le (2C)^{-\frac{C_1}{2}}. \tag{3.24}$$

We remark that, by (3.24), the choice of $m_2$ depends only on $m_1$, $c_*$, $\sigma_*$, $\mu,\lambda$, $\gamma$, $\epsilon$, $\delta$, $|\mathcal{O}|$, $\|g_1\|_{L^\infty}$, $\|g_2\|_{L^\infty}$, $\|g_3\|_{L^\infty}$.

Having (3.23) obtained, we multiply $(3.7)_2$ by $c$ and utilize (1.8) to deduce

$$\begin{aligned}
\int|\nabla c|^2 &\le \int|g_1c|+\int|\mathbf{u}\cdot\nabla c||c|\\
&\le \|c\|_{L^\infty}(\|g_1\|_{L^\infty}+\|\nabla c\|_{L^2}\|\nabla\mathbf{u}\|_{L^2})\\
&\le \frac{1}{2}\|\nabla c\|_{L^2}^2+C.
\end{aligned}\tag{3.25}$$

If we multiply $(3.7)_2$ by $-\triangle c$, we obtain

$$\begin{aligned}
\int|\triangle c|^2 &\le \int|g_1\triangle c|+|\mathbf{u}\cdot\nabla c\triangle c|\\
&\le C\|\triangle c\|_{L^2}(1+\|\nabla c\|_{L^3}\|\nabla\mathbf{u}\|_{L^2})\\
&\le C\|\triangle c\|_{L^2}(1+\|\nabla c\|_{L^2}^{\frac12}\|\triangle c\|_{L^2}^{\frac12}\|\nabla\mathbf{u}\|_{L^2})\\
&\le \frac{1}{2}\|\triangle c\|_{L^2}^2+C\|\nabla c\|_{L^2}.
\end{aligned}\tag{3.26}$$

The last two estimates (3.25) and (3.26) guarantee that, for small $m_2$,

$$\|\nabla c\|_{L^2}^2+\|\triangle c\|_{L^2}^2 \le C. \tag{3.27}$$

We next consider the Neumann boundary problem:

$$\triangle\rho=\mathrm{div}b \quad\text{with}\quad \frac{\partial\rho}{\partial n}\Big|_{\partial\mathcal{O}}=0. \tag{3.28}$$

**Lemma 3.2.** [37, Lemma 4.27] *Let $p \in (1,\infty)$ and $b \in L^p(\mathcal{O},\mathbb{R}^3)$. Then the problem* (3.28) *admits a solution $\rho \in W^{1,p}(\mathcal{O})$, satisfying*

$$\int \nabla\rho\cdot\nabla\phi = \int b\cdot\nabla\phi, \quad \forall\ \phi\in C^\infty(\overline{\mathcal{O}}),$$

*and the estimates*

$$\|\nabla\rho\|_{L^p} \le C(p,|\mathcal{O}|)\|b\|_{L^p} \quad \text{and} \quad \|\nabla\rho\|_{W^{1,p}} \le C(p,|\mathcal{O}|)(\|b\|_{L^p} + \|\mathrm{div} b\|_{L^p}).$$

**Lemma 3.3.** [37, Lemma 3.17] *There is a linear operator $\mathcal{B} = (\mathcal{B}^1,\mathcal{B}^2,\mathcal{B}^3)$ which satisfies*

(i) *Let $\overline{L^p} := \left\{ f\in L^p \mid \int f = 0\right\}$ with $p\in(1,\infty)$. Then,*

$$\mathcal{B}(f) : \overline{L^p} \mapsto \left(W_0^{1,p}\right)^3, \quad \mathrm{div}\mathcal{B}(f) = f \text{ a.e. in } \mathcal{O}, \quad \forall\ f\in\overline{L^p}.$$

(ii) *For any $g\in L^p(\mathcal{O},\mathbb{R}^3)$ with $g\cdot n|_{\partial\mathcal{O}} = 0$,*

$$\|\nabla\mathcal{B}(f)\|_{L^p} \le C\|f\|_{L^p}, \quad \|\mathcal{B}(\mathrm{div} g)\|_{L^p} \le C\|g\|_{L^p},$$

*where the constant $C$ depends only on $p$ and $|\mathcal{O}|$.*

Rewrite $(3.7)_1$ as

$$\epsilon^2\triangle\rho = \mathrm{div}(\rho\mathbf{u} + \epsilon\mathcal{B}(\rho-\rho_0)). \tag{3.29}$$

Applying Lemma 3.2 to (3.29), and using (3.23), Lemma 3.3, we find

$$\begin{aligned}\|\nabla\rho\|_{L^4} &\le C\|\rho\mathbf{u} + \mathcal{B}(\rho-\rho_0)\|_{L^4}\\ &\le C\|\rho\mathbf{u}\|_{L^4} + C\|\nabla\mathcal{B}(\rho-\rho_0)\|_{L^4}\\ &\le C\|\mathbf{u}\|_{L^6}\|\rho^2\|_{L^6}^{\frac12} + C\|\rho-\rho_0\|_{L^4} \le C,\end{aligned}$$

then using $L^p$ estimate on (3.29) yields

$$\begin{aligned}\|\rho\|_{H^2} &\le C\|\mathrm{div}(\rho\mathbf{u} + \epsilon\mathcal{B}(\rho-\rho_0))\|_{L^2}\\ &\le C\|\mathbf{u}\cdot\nabla\rho + \rho\mathrm{div}\mathbf{u}\|_{L^2} + C\|\mathrm{div}\mathcal{B}(\rho-\rho_0)\|_{L^2} \le C.\end{aligned} \tag{3.30}$$

By virtue of (3.23) and (3.27), one has $\|\mathbf{u}\cdot\nabla Q + tF^1(\mathbf{u},Q)\|_{W^{1,\frac32}} \le C$, and hence

$$\|Q\|_{W^{3,\frac32}} \le C \tag{3.31}$$

from $(3.7)_4$. By (3.30) and (3.31), we deduce $\|tF^2(\mathbf{u},Q)\|_{L^{\frac32}} \le C$, which together with $L^p$ regularity and $(3.7)_3$ imply

$$\|\mathbf{u}\|_{W^{2,\frac32}} \le C. \tag{3.32}$$

Finally, using (3.27), (3.31), (3.32), and $L^p$ regularity, we obtain from $(3.7)_2$ that

$$\|c\|_{W^{2,p}} \le C \quad (p<6). \tag{3.33}$$

As a result of (3.30)-(3.33), using bootstrap procedure generates, for $p\in(1,\infty)$,

$$\|(\mathbf{u},c)\|_{W^{2,p}} \le C, \quad \|Q\|_{W^{3,p}} \le C.$$

We have completed the proof of Proposition 3.1 and (3.4), except that we still need to prove Lemma 3.1. □

The last part of this section is to give a proof of Lemma 3.1.

*Proof of Lemma 3.1.* The proof of Lemma 3.1 is based on a Moser-type iteration technique. Fix $x_0 \in \mathcal{O}$. Let $B_R = B_R(x_0) \subset \mathcal{O}$ be a ball centered in $x_0$ with radius $R \le 1$, and let $\eta(x)$ be a smooth cut-off such that, for all $\frac{R}{2} \le r < r' \le R$,

$$\eta(x) \equiv 1 \text{ if } x \in B_r, \quad \eta(x) \equiv 0 \text{ if } x \notin B_{r'}, \quad |\nabla \eta| \le \frac{2}{(r'-r)}.$$

In the sequel, we assume $\|g_1\|_{L^\infty} \le \frac{1}{2}$. Otherwise, we will multiply $(3.7)_2$ by $(2\|g_1\|_{L^\infty})^{-1}$ and consider $\frac{c}{2\|g_1\|_{L^\infty}}$.

A simple computation shows

$$\begin{aligned} -\int \eta^2 c^p \triangle c &= \frac{4p}{(p+1)^2}\int \eta^2 |\nabla c^{\frac{p+1}{2}}|^2 + \frac{4}{p+1}\int \eta c^{\frac{p+1}{2}} \nabla\eta \nabla c^{\frac{p+1}{2}}\, dx \\ &\ge \frac{2p}{(p+1)^2}\int \eta^2|\nabla c^{\frac{p+1}{2}}|^2 - \frac{2}{p}\int |\nabla\eta|^2 c^{p+1}, \end{aligned}$$

and

$$\begin{aligned} \int \eta^2 c^p \mathbf{u}\cdot\nabla c &= \frac{2}{p+1}\int \eta^2 \mathbf{u}\cdot \nabla c^{\frac{p+1}{2}} c^{\frac{p+1}{2}} \\ &\le \frac{p}{(p+1)^2}\int \eta^2 |\nabla c^{\frac{p+1}{2}}|^2 + \frac{1}{p}\int \eta^2|\mathbf{u}|^2 c^{p+1}. \end{aligned}$$

With the above two inequalities, and the fact that $p^2\|g_1\|_{L^\infty}^{p+1}$ is uniformly bounded for any $p \ge 1$, we multiply $(3.7)_1$ by $\eta^2 c^p$ $(p \ge 1)$ to obtain

$$\begin{aligned} \int \eta^2 \left|\nabla c^{\frac{p+1}{2}}\right|^2 dx &\le C\int \left(|\nabla\eta|^2 + \eta^2|\mathbf{u}|^2\right) c^{p+1} + Cp^2\int \eta^2 c^p g_1 \\ &\le C\|\mathbf{u}\|_{L^6}^2 \left(\int_{B_{r'}} c^{\frac{3(p+1)}{2}}\right)^{\frac{2}{3}} + C\int |\nabla\eta|^2 c^{p+1} + Cp^2\|g_3\|_{L^\infty}^{p+1} \qquad (3.34) \\ &\le C\|\mathbf{u}\|_{L^6}^2 \left(\int_{B_{r'}} c^{\frac{3(p+1)}{2}}\right)^{\frac{2}{3}} + C\int |\nabla\eta|^2 c^{p+1} + C, \end{aligned}$$

where the constant $C$ may rely on $R$ and $|\mathcal{O}|$ but not on $p$.

Owing to the Sobolev embeddings (cf. [1]), for $f \in H_0^1(B_R)$ one has $\|f\|_{L^6} \le C\|\nabla f\|_{L^2}$. Thus,

$$\begin{aligned} \left(\int_{B_r} c^{3(p+1)}\right)^{\frac{1}{3}} &\le \left(\int \left|\eta c^{\frac{p+1}{2}}\right|^6\right)^{\frac{1}{3}} \\ &\le C\int \left|\nabla(\eta c^{\frac{p+1}{2}})\right|^2 \\ &\le C\left(\int \eta^2|\nabla c^{\frac{p+1}{2}}|^2 + \int c^{p+1}|\nabla\eta|^2\right), \end{aligned}$$

which together with (3.34) give us the following estimate:

$$\begin{aligned}\left(\int_{B_r} c^{3(p+1)}\right)^{\frac{1}{3}} &\leq C\|\mathbf{u}\|_{L^6}^2 \left(\int_{B_{r'}} c^{\frac{3(p+1)}{2}}\right)^{\frac{2}{3}} + C\int |\nabla\eta|^2 c^{p+1} + C \\ &\leq C\left(\|\mathbf{u}\|_{L^6}^2 + \frac{1}{|r'-r|^2}\right)\left(\int_{B_{r'}} c^{\frac{3(p+1)}{2}}\right)^{\frac{2}{3}} + C.\end{aligned} \tag{3.35}$$

Choosing

$$r' = r_{k-1} \quad \text{and} \quad r_k = \frac{R}{2}\left(1+\frac{1}{2^k}\right), \quad k = 1, 2, ....,$$

we obtain from (3.35) that

$$\left(\int_{B_{r_k}} c^{3(p+1)}\right)^{\frac{1}{2}} \leq C(1+\|\mathbf{u}\|_{L^6}^3)2^{3(k+1)}\left(\int_{B_{r_{k-1}}} c^{\frac{3(p+1)}{2}}\right) + C. \tag{3.36}$$

If $\int c^{\frac{3(p+1)}{2}}$ is bounded uniformly in $p$, then (3.22) follows directly by taking $p \to \infty$, subject to a subsequence. Otherwise, $\int c^{\frac{3(p+1)}{2}} \to \infty$ as $p$ goes to infinity. Hence, without loss of generality we may assume that $\int c^{\frac{3(p+1)}{2}} \geq C$ for all $p \geq 1$ and rewrite (3.36) as

$$\left(\int_{B_{r_k}} c^{3(p+1)}\right)^{\frac{1}{2}} \leq C\left(1+\|\mathbf{u}\|_{L^6}^3\right)2^{3(k+1)}\left(\int_{B_{r_{k-1}}} c^{\frac{3(p+1)}{2}}\right). \tag{3.37}$$

Selecting $\frac{3(p+1)}{2} = 2^{k-1}$ in (3.37), one has

$$\left(\int_{B_{r_k}} c^{2^k}\right)^{\frac{1}{2}} \leq C\left(1+\|\mathbf{u}\|_{L^6}^3\right)2^{3(k+1)}\left(\frac{1}{R^3}\int_{r_{k-1}} c^{2^{k-1}}\right),$$

which yields by the deduction argument

$$\left(\int_{B_{r_k}} c^{2^k}\right)^{\frac{1}{2^k}} \leq C\left(1+\|\mathbf{u}\|_{L^6}^3\right)^a 2^b \int_{B_R} c^2 \leq C\left(1+\|\mathbf{u}\|_{L^6}^3\right)^a \int_{B_R} c^2, \tag{3.38}$$

where

$$a = \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty, \quad b = \sum_{k=1}^{\infty} \frac{3(k+1)}{2^k} < \infty.$$

Sending $k \to \infty$ in (3.38) yields

$$\sup_{x\in B_{\frac{R}{2}}} c^2 = \lim_{k\to\infty}\left(\int_{B_{\frac{R}{2}}} c^{2^k}\right)^{\frac{1}{2^k}} \leq C\left(1+\|\mathbf{u}\|_{L^6}^3\right)^a \int_{B_R} c^2. \tag{3.39}$$

Then (3.22) follows from (3.39) together with the fact $c \geq 0$ and $\|c\|_{L^1} = m_2$.

We remark that for the case of boundary points, we can apply local flattening technique since the domain has smooth boundary $\partial\mathcal{O}$; while in the case when $x_0 \in \mathcal{O}$ is near the boundary, we follow similarly the ideas in [28, Section 4]. Therefore, we complete the proof of Proposition 3.1 as well as (3.4) and hence the proof of Theorem 3.1.

□

## 4. $\epsilon$-Limit for the Approximate Solutions

In this section, we shall take the $\epsilon$-limit of the approximate solutions obtained in Theorem 3.1 as $\epsilon \to 0$ for fixed $\delta \in (0,1)$, and prove the existence of solutions to the following problem:

**Theorem 4.1.** *Under the same assumptions as in Theorem 3.1, the system*

$$\begin{cases} \operatorname{div}(\rho\mathbf{u}) = 0, \\ \mathbf{u}\cdot\nabla c - \triangle c = g_1, \\ \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\left(\delta\rho^4+\rho^\gamma\right) - \operatorname{div}\left(\mathbb{S}_{ns}+\mathbb{S}_1(Q)+\mathbb{S}_2(c,Q)\right) = \rho g_2, \\ \mathbf{u}\cdot\nabla Q + Q\Omega - \Omega Q + c_* Q tr(Q^2) + \dfrac{(c-c_*)}{2}Q - b\left(Q^2 - \dfrac{1}{3}tr(Q^2)\mathbb{I}\right) - \triangle Q = g_3, \\ \mathbf{u}=0, \ \ \dfrac{\partial c}{\partial n} = 0, \ \ \dfrac{\partial Q}{\partial n} = 0, \quad \text{on } \partial\mathcal{O}, \end{cases} \tag{4.1}$$

*admits a solution $(\rho, c, \mathbf{u}, Q)$ in the sense of distributions for any $\delta \in (0,1)$, satisfying*

$$\int \rho = m_1, \ 0 \le \rho \in L^5(\mathcal{O}), \qquad \int c = m_2, \ 0 \le c \in H^2(\mathcal{O}), \tag{4.2}$$

$$\mathbf{u} \in H^1_0(\mathcal{O},\mathbb{R}^3), \qquad Q \in H^2(\mathcal{O}, S^3_0). \tag{4.3}$$

*In particular, $(4.1)_2$ and $(4.1)_4$ are satisfied almost everywhere in $\mathcal{O}$, and $(4.1)_1$ holds in the sense of renormalized solutions, namely,*

$$\operatorname{div}(b(\rho)\mathbf{u}) + (b'(\rho)\rho - b(\rho))\operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3),$$

*where $b(z) = z$, or $b \in C^1([0,\infty))$ with $b'(z) = 0$ for large $z$.*

*Proof.* We shall establish the uniform in $\epsilon$ estimates on the solutions $(\rho_\epsilon, c_\epsilon, \mathbf{u}_\epsilon, Q_\epsilon)$ obtained in Theorem 3.1 and then take the limit as $\epsilon \to 0$. We remark that the idea of the proof is in the spirit of the arguments for the steady Navier-Stokes equations; see, e.g., [28, 37, 38]. In this section, the constants $C$ and $C_1$ are generic and independent of $\epsilon$.

Firstly, it follows directly from (3.20) that, if $1 < \gamma \le 2$,

$$\begin{aligned} &\|Q_\epsilon\|^4_{L^4} + \|\nabla\mathbf{u}_\epsilon\|^2_{L^2} + \|\nabla Q_\epsilon\|^2_{L^2} + \|\triangle Q_\epsilon\|^2_{L^2} + \epsilon^2\|\nabla\rho_\epsilon\|^2_{L^2} \\ &\le \|Q_\epsilon\|^4_{L^4} + \|\nabla\mathbf{u}_\epsilon\|^2_{L^2} + \|\nabla Q_\epsilon\|^2_{L^2} + \|\triangle Q_\epsilon\|^2_{L^2} + \epsilon^2\left(\|\nabla\rho_\epsilon^{\frac{\gamma}{2}}\|^2_{L^2} + \|\nabla\rho_\epsilon^2\|^2_{L^2}\right) \\ &\le C + C\|\rho_\epsilon\|^2_{L^{\frac{6}{5}}} + C\|c_\epsilon\|^6_{L^\infty}; \end{aligned} \tag{4.4}$$

while in the case of $\gamma > 2$, we replace the artificial pressure $\delta\rho_\epsilon^4$ in (3.1) with $\delta\rho_\epsilon^4 + \delta\rho_\epsilon^2$, and repeat the deduction of (3.20) to conclude that

$$\begin{aligned} &\|Q_\epsilon\|^4_{L^4} + \|\nabla\mathbf{u}_\epsilon\|^2_{L^2} + \|\nabla Q_\epsilon\|^2_{L^2} + \|\triangle Q_\epsilon\|^2_{L^2} + \epsilon^2\left(\|\nabla\rho_\epsilon^2\|^2_{L^2} + \|\nabla\rho_\epsilon^{\frac{\gamma}{2}}\|^2_{L^2} + \|\nabla\rho_\epsilon\|^2_{L^2}\right) \\ &\le C + C\|\rho_\epsilon\|^2_{L^{\frac{6}{5}}} + C\|c_\epsilon\|^6_{L^\infty}. \end{aligned} \tag{4.5}$$

From (4.4) and (4.5) we conclude that, for all $\gamma > 1$,

$$\begin{aligned} &\|Q_\epsilon\|^4_{L^4} + \|\nabla\mathbf{u}_\epsilon\|^2_{L^2} + \|\nabla Q_\epsilon\|^2_{L^2} + \|\triangle Q_\epsilon\|^2_{L^2} + \epsilon^2\|\nabla\rho_\epsilon\|^2_{L^2} \\ &\le C\left(1 + \|\rho_\epsilon\|^2_{L^{\frac{6}{5}}} + \|c_\epsilon\|^6_{L^\infty}\right). \end{aligned} \tag{4.6}$$

It follows from (1.8), (4.6), and (3.22) that

$$\begin{aligned}\|c_\epsilon\|_{L^\infty} &\le C\left(1+\|\mathbf{u}\|_{L^6}\right)^{C_1}(1+\|g_1\|_{L^\infty})m_2\\ &\le C\left(1+\|\rho_\epsilon\|_{L^{\frac{6}{5}}}+\|c_\epsilon\|_{L^\infty}^3\right)^{C_1} m_2\\ &\le C+Cm_2\|\rho_\epsilon\|_{L^{\frac{6}{5}}}^{C_1}+Cm_2\|c_\epsilon\|_{L^\infty}^{3C_1}\\ &\le 2C+Cm_2\|\rho_\epsilon\|_{L^{\frac{6}{5}}}^{C_1},\end{aligned}\tag{4.7}$$

where the last inequality is valid as long as $m_2$ is chosen sufficiently small.

**Lemma 4.1.** *Let $(\rho_\epsilon, c_\epsilon, \mathbf{u}_\epsilon, Q_\epsilon)$ be a solution in Theorem 3.1. Then*

$$\|\rho_\epsilon^5+\rho_\epsilon^{\gamma+1}\|_{L^1}\le C,\tag{4.8}$$

*provided that $m_2$ is sufficiently small.*

*Proof.* Let $\mathcal{B}$ be the Bogovskii operator (see Lemma 3.3). Multiply $(3.1)_3$ by $\mathcal{B}(\rho_\epsilon-\rho_0)$ to obtain

$$\begin{aligned}&\int\left(\delta\rho_\epsilon^4+\rho_\epsilon^\gamma\right)\rho_\epsilon\\ &=\int\left(\delta\rho_\epsilon^4+\rho_\epsilon^\gamma\right)\rho_0-\int\rho_\epsilon g_2\cdot\mathcal{B}(\rho_\epsilon-\rho_0)\\ &\quad+\epsilon\int\rho_\epsilon\mathbf{u}_\epsilon\cdot\mathcal{B}(\rho_\epsilon-\rho_0)+\epsilon^2\int\nabla\rho_\epsilon\cdot\nabla\mathbf{u}_\epsilon\mathcal{B}(\rho_\epsilon-\rho_0)-\int\rho_\epsilon\mathbf{u}_\epsilon\otimes\mathbf{u}_\epsilon:\nabla\mathcal{B}(\rho_\epsilon-\rho_0)\\ &\quad+\int\mu(\nabla\mathbf{u}_\epsilon+(\nabla\mathbf{u}_\epsilon)^\top):\nabla\mathcal{B}(\rho_\epsilon-\rho_0)+\lambda\mathrm{div}\mathbf{u}_\epsilon\mathrm{div}\mathcal{B}(\rho_\epsilon-\rho_0)\\ &\quad+\int\left(\frac{1}{2}|\nabla Q_\epsilon|^2\mathbb{I}-\nabla Q_\epsilon\odot\nabla Q_\epsilon\right):\nabla\mathcal{B}(\rho_\epsilon-\rho_0)\\ &\quad+\frac{1}{2}\int tr(Q_\epsilon^2)\left(1+\frac{c_*}{2}tr(Q_\epsilon^2)\right)\mathrm{div}\mathcal{B}(\rho_\epsilon-\rho_0)\\ &\quad+\int(Q_\epsilon\triangle Q_\epsilon-\triangle Q_\epsilon Q_\epsilon):\nabla\mathcal{B}(\rho_\epsilon-\rho_0)+\int\sigma_* c_\epsilon^2 Q_\epsilon:\nabla\mathcal{B}(\rho_\epsilon-\rho_0)\\ &=:\sum_{i=1}^{10}K_i.\end{aligned}\tag{4.9}$$

Using $\|\rho\|_{L^1}=m_1$ and interpolation, one has

$$\begin{aligned}K_1+K_2&\le C+\frac{1}{16}\int\left(\delta\rho_\epsilon^5+\rho_\epsilon^{\gamma+1}\right)+C\|\rho_\epsilon\|_{L^{\frac{6}{5}}}\|\nabla\mathcal{B}(\rho_\epsilon-\rho_0)\|_{L^2}\\ &\le C+\frac{1}{16}\int\left(\delta\rho_\epsilon^5+\rho_\epsilon^{\gamma+1}\right)+C\|\rho_\epsilon\|_{L^{\frac{6}{5}}}\|\rho_\epsilon-\rho_0\|_{L^2}\\ &\le C+\frac{1}{8}\int\left(\delta\rho_\epsilon^5+\rho_\epsilon^{\gamma+1}\right).\end{aligned}$$

Thanks to (4.6) and (4.7),

$$\begin{aligned} K_5 &= -\int \rho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon : \nabla\mathcal{B}(\rho_\epsilon - \rho_0) \\ &\le \|\rho_\epsilon\|_{L^{\frac{12}{5}}} \|\mathbf{u}_\epsilon\|_{L^6}^2 \|\nabla\mathcal{B}(\rho_\epsilon - \rho_0)\|_{L^4} \\ &\le C\|\rho_\epsilon\|_{L^{\frac{12}{5}}} \left(1 + \|\rho_\epsilon\|_{L^{\frac{6}{5}}}^2\right) \|\nabla\mathcal{B}(\rho_\epsilon - \rho_0)\|_{L^4} \\ &\le \frac{\delta}{8}\|\rho_\epsilon\|_{L^5}^5 + C. \end{aligned}$$

In a similar way, one deduces

$$\begin{aligned} K_3 + K_4 + \sum_{i=6}^{9} K_i &\le \left(\epsilon\|\rho_\epsilon\|_{L^2} + \epsilon^2\|\nabla\rho_\epsilon\|_{L^2}\right) \|\mathbf{u}_\epsilon\|_{H^1} \|\mathcal{B}(\rho_\epsilon - \rho_0)\|_{L^\infty} \\ &\quad + C\left(\|\nabla\mathbf{u}_\epsilon\|_{L^2} + \|\nabla Q_\epsilon\|_{L^4}^2\right) \|\nabla\mathcal{B}(\rho_\epsilon - \rho_0)\|_{L^2} \\ &\quad + C\left(1 + \|Q_\epsilon\|_{L^6}^4 + \|\triangle Q_\epsilon\|_{L^2}\|Q_\epsilon\|_{L^6}\right) \|\nabla\mathcal{B}(\rho_\epsilon - \rho_0)\|_{L^3} \\ &\le C\|\rho_\epsilon\|_{L^{\frac{12}{5}}}^2 \|\mathcal{B}(\rho_\epsilon - \rho_0)\|_{W^{1,4}} \\ &\le \frac{\delta}{8}\|\rho_\epsilon\|_{L^5}^5 + C. \end{aligned}$$

Finally, using (4.6) and (4.7), one deduces

$$\begin{aligned} K_{10} &= \int \sigma_* c_\epsilon^2 Q_\epsilon : \nabla\mathcal{B}(\rho_\epsilon - \rho_0) \\ &\le \|c_\epsilon\|_{L^\infty}^2 \|Q_\epsilon\|_{L^6} \|\nabla\mathcal{B}(\rho_\epsilon - \rho_0)\|_{L^{\frac{6}{5}}} \\ &\le Cm_2(1 + \|\rho_\epsilon\|_{L^{\frac{6}{5}}})^{C_1} \\ &\le C + Cm_2\|\rho_\epsilon\|_{L^5}^{C_1}. \end{aligned}$$

Substituting the last three inequalities into (4.9) and taking $m_2$ small, we get

$$\int \left(\delta\rho_\epsilon^5 + \rho_\epsilon^{\gamma+1}\right) \le C + Cm_2\|\rho_\epsilon\|_{L^5}^{C_1} \le 2C. \tag{4.10}$$

The proof of Lemma 4.1 is completed. □

With (4.8) obtained, we deduce from (4.6) and (4.7) that

$$\|c_\epsilon\|_{L^\infty} + \|Q_\epsilon\|_{L^4}^4 + \|\nabla\mathbf{u}_\epsilon\|_{L^2}^2 + \|\nabla Q_\epsilon\|_{L^2}^2 + \|\triangle Q_\epsilon\|_{L^2}^2 + \epsilon^2\|\nabla\rho_\epsilon\|_{L^2}^2 \le C. \tag{4.11}$$

Then multiply $(3.7)_2$ firstly by $c_\epsilon$ and then by $-\triangle c_\epsilon$ to deduce

$$\|\nabla c_\epsilon\|_{L^2} + \|\triangle c_\epsilon\|_{L^2} \le C. \tag{4.12}$$

As a result of (4.8), (4.11), and (4.12) we can take $\epsilon$-limit of $(\rho_\epsilon, c_\epsilon, \mathbf{u}_\epsilon, Q_\epsilon)$ subject to some subsequence so that, as $\epsilon \to 0$,

$$\rho_\epsilon \rightharpoonup \rho \text{ in } L^5 \cap L^{\gamma+1}, \tag{4.13}$$

$$(\nabla\mathbf{u}_\epsilon,\, \nabla^2 Q_\epsilon, \nabla c_\epsilon) \rightharpoonup (\nabla\mathbf{u},\, \nabla^2 Q,\, \nabla c) \quad \text{in} \quad L^2, \tag{4.14}$$

$$\mathbf{u}_\epsilon \to \mathbf{u}, \quad (Q_\epsilon,\, c_\epsilon) \to (Q,\, c) \quad \text{in} \quad W^{1,p}\ (1 \le p < 6), \tag{4.15}$$

$$\langle\nabla\mathbf{u}_\epsilon\rangle \to \nabla\mathbf{u},\ \ \langle g_3\rangle \to g_3 \quad \text{in} \quad L^2, \tag{4.16}$$

$$\epsilon\rho_\epsilon \to 0, \quad \epsilon\rho_\epsilon\mathbf{u}_\epsilon \to 0, \quad \epsilon^2\nabla\rho_\epsilon\nabla\mathbf{u}_\epsilon \to 0, \ \ \epsilon^2\nabla\rho_\epsilon \to 0 \ \text{ in } \quad L^1. \tag{4.17}$$

and moreover, it follows from (4.13) and (4.15) that

$$\rho_\epsilon^4 \rightharpoonup \overline{\rho^4} \quad \text{in } L^{\frac{5}{4}}, \quad \rho_\epsilon^\gamma \rightharpoonup \overline{\rho^\gamma} \text{ in } L^{\frac{\gamma+1}{\gamma}}, \quad \rho_\epsilon \mathbf{u}_\epsilon \rightharpoonup \rho\mathbf{u} \text{ in } L^2, \tag{4.18}$$

where and hereafter the weak limit of a function $f$ is denoted by $\overline{f}$. Therefore, with (4.13)-(4.18) in hand, we are able to pass the limit as $\epsilon \to 0$ and obtain the following equations in the weak sense:

$$\begin{cases} \mathrm{div}(\rho\mathbf{u}) = 0, \\ \mathbf{u}\cdot\nabla c - \triangle c = g_1, \\ \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\left(\delta\overline{\rho^4} + \overline{\rho^\gamma}\right) - \mathrm{div}\left(\mathbb{S}_{ns} + \mathbb{S}_1 + \mathbb{S}_2\right) = \rho g_2, \\ \mathbf{u}\cdot\nabla Q + Q\Omega - \Omega Q + c_* Q tr(Q^2) + \dfrac{(c-c_*)}{2}Q - b\left(Q^2 - \dfrac{1}{3}tr(Q^2)\mathbb{I}\right) - \triangle Q = g_3. \end{cases} \tag{4.19}$$

In addition, (4.2) and (4.3) follow from (3.2), (3.3), (4.15), and (4.29) below. The next lemma shows that $(\rho,\mathbf{u})$ is a renormalized solution to $(4.19)_1$.

**Lemma 4.2.** *Assume that $(\rho,\mathbf{u})$ is a weak solution to $(4.1)_1$, $\rho\in L^2(\mathcal{O})$ and $\mathbf{u}\in H_0^1(\mathcal{O},\mathbb{R}^3)$. If we extend $(\rho,\mathbf{u})$ by zero outside $\mathcal{O}$, we have*

$$\mathrm{div}(b(\rho)\mathbf{u}) + (b'(\rho)\rho - b(\rho))\mathrm{div}\mathbf{u} = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3), \tag{4.20}$$

*where $b(z) = z$, or $b\in C^1([0,\infty))$ with $b'(z) = 0$ for large $z$.*

*Proof.* The detailed proof is available in [38, Lemma 2.1]. □

In order to complete the proof of Theorem 4.1, we need to verify

$$\overline{\rho^4} = \rho^4, \quad \overline{\rho^\gamma} = \rho^\gamma. \tag{4.21}$$

To this end, let us define

$$C^1([0,\infty)) \ni b_n(\rho) = \begin{cases} \rho\ln(\rho + \dfrac{1}{n}), & \rho\le n; \\ (n+1)\ln(n+1+\dfrac{1}{n}), & \rho\ge n+1. \end{cases}$$

We see that $b_n(\rho)\to\rho\ln\rho$ a.e. because of the fact: $\rho\in L^1$. Select $b_n$ in (4.20) and send $n\to\infty$ to obtain

$$\mathrm{div}(\mathbf{u}\rho\ln\rho) + \rho\mathrm{div}\mathbf{u} = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3).$$

This implies

$$\int \rho\mathrm{div}\mathbf{u} = 0. \tag{4.22}$$

On the other hand, multiplying $(3.1)_1$ by $b_n'(\rho_\epsilon)$ gives

$$\begin{aligned} \int (b_n'(\rho_\epsilon)\rho_\epsilon - b_n(\rho_\epsilon))\mathrm{div}\mathbf{u}_\epsilon &= \epsilon\int\rho_0 b_n'(\rho_\epsilon) - \epsilon\int\rho_\epsilon b_n'(\rho_\epsilon) - \epsilon^4\int b_n''(\rho_\epsilon)|\nabla\rho_\epsilon|^2 \\ &\le \epsilon\int\rho_0 b_n'(\rho_\epsilon) - \epsilon\int\rho_\epsilon b_n'(\rho_\epsilon). \end{aligned} \tag{4.23}$$

Recalling (4.8) and the definition of $b_n$, one deduces that

$$\begin{aligned}
&\lim_{n\to\infty}\int \rho_0 b_n'(\rho_\epsilon)\\
&=\lim_{n\to\infty}\left(\int_{\{\rho_\epsilon\le n\}}\rho_0 b_n'(\rho_\epsilon)+\int_{\{\rho_\epsilon>n\}}\rho_0 b_n'(\rho_\epsilon)\right)\\
&\le \lim_{n\to\infty}\int_{\{\rho_\epsilon\le n\}}\rho_0\left(\ln(\rho_\epsilon+\frac{1}{n})+\frac{\rho_\epsilon}{\rho_\epsilon+\frac{1}{n}}\right)+C\lim_{n\to\infty}\text{meas}\,|\{x;\,\rho_\epsilon\ge n\}|\\
&\le \lim_{n\to\infty}\int_{\{1/2\le\rho_\epsilon\le n\}}\rho_0\ln(\rho_\epsilon+\frac{1}{n})+\lim_{n\to\infty}\int\frac{\rho_0\rho_\epsilon}{\rho_\epsilon+\frac{1}{n}}\\
&\le C.
\end{aligned}$$

Similarly,

$$\lim_{n\to\infty}\int \rho_\epsilon b_n'(\rho_\epsilon)\le C.$$

Therefore, taking sequentially $n\to\infty$ and $\epsilon\to 0$ in (4.23), using (4.22),

$$\int\overline{\rho\text{div}u}=\lim_{\epsilon\to 0}\int\rho_\epsilon\text{div}u_\epsilon\le 0=\int\rho\text{div}u. \tag{4.24}$$

Now define the following effective viscous flux:

$$\mathbb{F}_\epsilon=\delta\rho_\epsilon^4+\rho_\epsilon^\gamma-(2\mu+\lambda)\text{div}\mathbf{u}_\epsilon \quad\text{and}\quad \overline{\mathbb{F}}=\delta\overline{\rho^4}+\overline{\rho^\gamma}-(2\mu+\lambda)\text{div}\mathbf{u}. \tag{4.25}$$

**Lemma 4.3.** *Under the assumptions in Theorem 4.1, the following property holds:*

$$\lim_{\epsilon\to 0}\int\phi\rho_\epsilon\mathbb{F}_\epsilon=\int\phi\rho\overline{\mathbb{F}},\quad\forall\ \phi\in C_0^\infty(\mathcal{O}). \tag{4.26}$$

Let us continue to prove (4.21) with the aid of (4.26). The proof of Lemma 4.3 is postponed to the end of this section.

Sending $\phi\to 1$ in (4.26), using (4.24) and (4.25), we get

$$\lim_{\epsilon\to 0}\int\left(\delta\rho_\epsilon^5+\rho_\epsilon^{\gamma+1}\right)\le\int\rho\left(\delta\overline{\rho^4}+\overline{\rho^\gamma}\right). \tag{4.27}$$

According to (4.27), we have

$$\int\left(\delta\overline{\rho^5}+\overline{\rho^{\gamma+1}}\right)=\lim_{\epsilon\to 0}\int\left(\delta\rho_\epsilon^5+\rho_\epsilon^{\gamma+1}\right)\le\int\rho\left(\delta\overline{\rho^4}+\overline{\rho^\gamma}\right),$$

which implies

$$\int\delta\left(\rho\overline{\rho^4}-\overline{\rho^5}\right)\ge\int\left(\overline{\rho^{\gamma+1}}-\rho\overline{\rho^\gamma}\right)\ge 0, \tag{4.28}$$

where the last inequality is due to the convexity. Next, for given constant $\beta>0$ and $\eta\in C^\infty(\mathcal{O})$,

$$\begin{aligned}
0&\le\int\left(\rho_\epsilon^4-(\rho+\beta\eta)^4\right)\left(\rho_\epsilon-(\rho+\beta\eta)\right)\\
&=\int\left(\rho_\epsilon^5-\rho_\epsilon^4\rho-\rho_\epsilon^4\beta\eta-(\rho+\beta\eta)^4\rho_\epsilon+(\rho+\beta\eta)^5\right).
\end{aligned}$$

By (4.28), sending $\epsilon \to 0$ yields

$$0 \le \int \left(\overline{\rho^5} - \rho\overline{\rho^4} - \overline{\rho^4}\beta\eta + (\rho+\beta\eta)^4\beta\eta\right) \le \int \left(-\overline{\rho^4} + (\rho+\beta\eta)^4\right)\beta\eta.$$

Replacing $-\beta$ with $\beta$ in the argument above, and then sending $\beta \to 0$, we get

$$\int \left(\rho^4 - \overline{\rho^4}\right)\eta = 0,$$

which implies $\overline{\rho^4} = \rho^4$, and thus $\rho_\epsilon \to \rho$ a.e. in $\mathcal{O}$ due to the arbitrariness of $\eta$, and hence for all $s \in [1,5)$, from (4.13),

$$\rho_\epsilon \to \rho \quad \text{in} \quad L^s. \tag{4.29}$$

As a result of (4.29) and (4.13), we obtain (4.21) and thus complete the proof of Theorem 4.1. □

It remains to prove Lemma 4.3.

*Proof of Lemma 4.3.* Let $\triangle^{-1}(h) = K * h$ be the convolution of $h$ with the fundamental solution $K$ of Laplacian in $\mathbb{R}^3$. For convenience, we write $(3.1)_3$ equivalently as

$$\begin{aligned}
&\epsilon\rho_\epsilon\mathbf{u}_\epsilon^i + \partial_j(\rho_\epsilon\mathbf{u}_\epsilon^j\mathbf{u}_\epsilon^i) + \partial_i\mathbb{F}_\epsilon + \epsilon^2\nabla\rho_\epsilon\cdot\nabla\mathbf{u}_\epsilon^i \\
&= \rho_\epsilon g_2^i + \mu\triangle\mathbf{u}_\epsilon^i \\
&\quad - \partial_j\left(\partial_jQ_\epsilon\partial_iQ_\epsilon\right) + \frac{1}{2}\partial_i|\nabla Q_\epsilon|^2 + \frac{1}{2}\partial_i\left(tr(Q_\epsilon^2)(1+\frac{c_*}{2}tr(Q_\epsilon^2))\right) \\
&\quad + \partial_j\left(Q_\epsilon^{ik}\triangle Q_\epsilon^{kj} - \triangle Q_\epsilon^{ik}Q_\epsilon^{kj} + \sigma_*c_\epsilon^2Q_\epsilon^{ij}\right), \quad i=1,2,3,
\end{aligned} \tag{4.30}$$

where the Einstein summation is used on $k, j$, and $\mathbb{F}_\epsilon$ is taken from (4.25).

Making zero extension of $\rho_\epsilon$ to the whole space $\mathbb{R}^3$, multiplying (4.30) by $\phi\partial_i\triangle^{-1}(\rho_\epsilon)$ with $\phi \in C_0^\infty(\mathcal{O})$, we deduce

$$\begin{aligned}
&\int \phi\rho_\epsilon\mathbb{F}_\epsilon \\
&= -\int \partial_i\triangle^{-1}(\rho_\epsilon)\partial_i\phi\left(\delta\rho_\epsilon^4 + \rho_\epsilon^\gamma - (\mu+\lambda)\mathrm{div}\mathbf{u}_\epsilon\right) - \int \rho_\epsilon g_2^i\phi\partial_i\triangle^{-1}(\rho_\epsilon) \\
&\quad + \mu\int\left(\partial_j\mathbf{u}_\epsilon^i\partial_i\triangle^{-1}(\rho_\epsilon)\partial_j\phi - \mathbf{u}_\epsilon^i\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\partial_j\phi + \rho_\epsilon\mathbf{u}_\epsilon\cdot\nabla\phi\right) \\
&\quad - \int \rho_\epsilon\mathbf{u}_\epsilon^j\mathbf{u}_\epsilon^i\partial_j\phi\partial_i\triangle^{-1}(\rho_\epsilon) - \int \rho_\epsilon\mathbf{u}_\epsilon^j\mathbf{u}_\epsilon^i\phi\partial_j\partial_i\triangle^{-1}(\rho_\epsilon) \\
&\quad + \int\left(\frac{1}{2}\int|\nabla Q_\epsilon|^2 + \frac{1}{2}tr(Q_\epsilon^2)(1+\frac{c_*}{2}tr(Q_\epsilon^2))\right)\left(\partial_i\phi\partial_i\triangle^{-1}(\rho_\epsilon) + \phi\rho_\epsilon\right) \\
&\quad - \int \partial_jQ_\epsilon\partial_iQ_\epsilon\left(\partial_j\phi\partial_i\triangle^{-1}(\rho_\epsilon) + \phi\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\right) \\
&\quad + \int\left(Q_\epsilon^{ik}\triangle Q_\epsilon^{kj} - \triangle Q_\epsilon^{ik}Q_\epsilon^{kj} + \sigma_*c_\epsilon^2Q_\epsilon^{ij}\right)\left(\partial_j\phi\partial_i\triangle^{-1}(\rho_\epsilon) + \phi\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\right) \\
&\quad + \epsilon\int\rho_\epsilon\mathbf{u}_\epsilon^i\phi\partial_i\triangle^{-1}(\rho_\epsilon) + \epsilon^2\int\nabla\rho_\epsilon\cdot\nabla\mathbf{u}_\epsilon^i\phi\partial_i\triangle^{-1}(\rho_\epsilon),
\end{aligned} \tag{4.31}$$

where the second line on the right-hand side is due to

$$\begin{aligned}&\int \partial_j \mathbf{u}_\epsilon^i \left(\partial_i \triangle^{-1}(\rho_\epsilon)\partial_j\phi + \partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\phi\right)\\&= \int \left(\partial_j \mathbf{u}_\epsilon^i \partial_i \triangle^{-1}(\rho_\epsilon)\partial_j\phi - \mathbf{u}_\epsilon^i\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\partial_j\phi + \rho_\epsilon\mathbf{u}_\epsilon\cdot\nabla\phi\right) + \int \rho_\epsilon \mathrm{div}\mathbf{u}_\epsilon\phi.\end{aligned}$$

Making use of

$$\epsilon(\rho_\epsilon - \rho_0) + \mathrm{div}(\rho_\epsilon\mathbf{u}_\epsilon) = \epsilon^2\mathrm{div}(\mathbf{1}_\mathcal{O}\nabla\rho_\epsilon) \quad \text{in} \quad \mathbb{R}^3,$$

we write the third line on the right-hand side of (4.31) as

$$\begin{aligned}&-\int \rho_\epsilon \mathbf{u}_\epsilon^j\mathbf{u}_\epsilon^i\partial_j\phi\partial_i\triangle^{-1}(\rho_\epsilon) - \int \rho_\epsilon\mathbf{u}_\epsilon^j\mathbf{u}_\epsilon^i\phi\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\\&= -\int \rho_\epsilon \mathbf{u}_\epsilon^j\mathbf{u}_\epsilon^i\partial_j\phi\partial_i\triangle^{-1}(\rho_\epsilon) + \int \mathbf{u}_\epsilon^i\phi\left[\rho_\epsilon\partial_i\partial_j\triangle^{-1}(\rho_\epsilon\mathbf{u}_\epsilon^j) - \rho_\epsilon\mathbf{u}_\epsilon^j\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\right]\\&\quad - \int \rho_\epsilon\mathbf{u}_\epsilon^i\phi\partial_i\partial_j\triangle^{-1}(\rho_\epsilon\mathbf{u}_\epsilon^j) \qquad (4.32)\\&= -\int \rho_\epsilon \mathbf{u}_\epsilon^j\mathbf{u}_\epsilon^i\partial_j\phi\partial_i\triangle^{-1}(\rho_\epsilon) + \int \mathbf{u}_\epsilon^i\phi\left[\rho_\epsilon\partial_i\partial_j\triangle^{-1}(\rho_\epsilon\mathbf{u}_\epsilon^j) - \rho_\epsilon\mathbf{u}_\epsilon^j\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\right]\\&\quad - \epsilon^2\int \rho_\epsilon\mathbf{u}_\epsilon^i\phi\partial_i\triangle^{-1}(\mathrm{div}(\mathbf{1}_\mathcal{O}\nabla\rho_\epsilon)) + \epsilon\int\rho_\epsilon\mathbf{u}_\epsilon^i\phi\partial_i\triangle^{-1}(\rho_\epsilon-\rho_0).\end{aligned}$$

Substituting (4.32) into (4.31) gives us

$$\begin{aligned}&\int \phi\rho_\epsilon\mathbb{F}_\epsilon\\&= -\int \partial_i\triangle^{-1}(\rho_\epsilon)\partial_i\phi\left(\delta\rho_\epsilon^4 + \rho_\epsilon^\gamma - (\mu+\lambda)\mathrm{div}\mathbf{u}_\epsilon\right) - \int \rho_\epsilon g_2^i\phi\partial_i\triangle^{-1}(\rho_\epsilon)\\&\quad + \mu\int\left(\partial_j\mathbf{u}_\epsilon^i\partial_i\triangle^{-1}(\rho_\epsilon)\partial_j\phi - \mathbf{u}_\epsilon^i\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\partial_j\phi + \rho_\epsilon\mathbf{u}_\epsilon\cdot\nabla\phi\right)\\&\quad - \int \rho_\epsilon \mathbf{u}_\epsilon^j\mathbf{u}_\epsilon^i\partial_j\phi\partial_i\triangle^{-1}(\rho_\epsilon) + \int \mathbf{u}_\epsilon^i\phi\left[\rho_\epsilon\partial_i\partial_j\triangle^{-1}(\rho_\epsilon\mathbf{u}_\epsilon^j) - \rho_\epsilon\mathbf{u}_\epsilon^j\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\right]\\&\quad + \int\left(\frac{1}{2}\int|\nabla Q_\epsilon|^2 + \frac{1}{2}tr(Q_\epsilon^2)(1+\frac{c_*}{2}tr(Q_\epsilon^2))\right)\left(\partial_i\phi\partial_i\triangle^{-1}(\rho_\epsilon) + \phi\rho_\epsilon\right)\\&\quad - \int \partial_jQ_\epsilon\partial_iQ_\epsilon\left(\partial_j\phi\partial_i\triangle^{-1}(\rho_\epsilon) + \phi\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\right) \qquad (4.33)\\&\quad + \int\left(Q_\epsilon^{ik}\triangle Q_\epsilon^{kj} - \triangle Q_\epsilon^{ik}Q_\epsilon^{kj}\right)\left(\partial_j\phi\partial_i\triangle^{-1}(\rho_\epsilon) + \phi\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\right)\\&\quad + \int \sigma_* c_\epsilon^2 Q_\epsilon^{ij}\left(\partial_j\phi\partial_i\triangle^{-1}(\rho_\epsilon) + \phi\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\right)\\&\quad - \epsilon^2\int \rho_\epsilon\mathbf{u}_\epsilon^i\phi\partial_i\triangle^{-1}(\mathrm{div}(\mathbf{1}_\mathcal{O}\nabla\rho_\epsilon)) - \nabla\rho_\epsilon\cdot\nabla\mathbf{u}_\epsilon^i\phi\partial_i\triangle^{-1}(\rho_\epsilon)\\&\quad + \epsilon\int\rho_\epsilon\mathbf{u}_\epsilon^i\phi\partial_i\triangle^{-1}(2\rho_\epsilon - \rho_0)\\&=: \sum_{n=1}^{11} T_n^\epsilon,\end{aligned}$$

where $T_n^\epsilon$ denotes the $n^{th}$ integral on the right hand side of (4.33).

On the other hand, if we multiply $(4.19)_2$ by $\phi\partial_i\triangle^{-1}(\rho)$, we obtain

$$
\begin{aligned}
\int \phi\rho\mathbb{F} = &-\int \partial_i\triangle^{-1}(\rho)\partial_i\phi\left(\delta\overline{\rho^4}+\overline{\rho^\gamma}-(\mu+\lambda)\mathrm{div}\mathbf{u}\right)-\int \rho g_2^i\phi\partial_i\triangle^{-1}(\rho)\\
&+\mu\int\left(\partial_j\mathbf{u}^i\partial_i\triangle^{-1}(\rho)\partial_j\phi-\mathbf{u}^i\partial_j\partial_i\triangle^{-1}(\rho)\partial_j\phi+\rho\mathbf{u}\cdot\nabla\phi\right)\\
&-\int\rho\mathbf{u}^j\mathbf{u}^i\partial_j\phi\partial_i\triangle^{-1}(\rho)+\int\mathbf{u}^i\phi\left[\rho\partial_i\partial_j\triangle^{-1}(\rho\mathbf{u}^j)-\rho\mathbf{u}^j\partial_j\partial_i\triangle^{-1}(\rho)\right]\\
&+\int\left(\frac{1}{2}\int|\nabla Q|^2+\frac{1}{2}tr(Q^2)(1+\frac{c_*}{2}tr(Q^2))\right)\left(\partial_i\phi\partial_i\triangle^{-1}(\rho)+\phi\rho\right)\\
&-\int\partial_jQ\partial_iQ\left(\partial_j\phi\partial_i\triangle^{-1}(\rho)+\phi\partial_j\partial_i\triangle^{-1}(\rho)\right)\\
&+\int\left(Q^{ik}\triangle Q^{kj}-\triangle Q^{ik}Q^{kj}\right)\left(\partial_j\phi\partial_i\triangle^{-1}(\rho)+\phi\partial_j\partial_i\triangle^{-1}(\rho)\right)\\
&+\int\sigma_*c^2Q^{ij}\left(\partial_j\phi\partial_i\triangle^{-1}(\rho)+\phi\partial_j\partial_i\triangle^{-1}(\rho)\right)\\
=:&\sum_{n=1}^{9}T_n.
\end{aligned}
\tag{4.34}
$$

In terms of (4.33) and (4.34), to prove (4.26) it suffices to check

$$
\lim_{\epsilon\to0}T_n^\epsilon=T_n\ (n=1,2,\cdots,9)\quad\text{and}\quad\lim_{\epsilon\to0}T_n^\epsilon=0\ (n=10,11).
$$

In fact, by the Mikhlin multiplier theory (cf. [43]), and the Rellich-Kondrachov compactness theorem (cf. [15]), one has

$$
\partial_j\partial_i\triangle^{-1}(h_\epsilon)\rightharpoonup\partial_j\partial_i\triangle^{-1}(h)\quad\text{in }L^p,\quad\partial_i\triangle^{-1}(h_\epsilon)\to\partial_i\triangle^{-1}(h)\quad\text{in }L^q,\tag{4.35}
$$

where $q<(1/p-1/3)^{-1}$ if $p<3$ and $q\le\infty$ if $p>3$. By (4.35), as well as (4.13) and (4.15), we have

$$
\begin{aligned}
T_8^\epsilon=&\int\left(Q_\epsilon^{ik}\triangle Q_\epsilon^{kj}-\triangle Q_\epsilon^{ik}Q_\epsilon^{kj}\right)\left(\partial_j\phi\partial_i\triangle^{-1}(\rho_\epsilon)+\phi\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\right)\\
=&\int\left(\nabla Q_\epsilon^{ik}\nabla Q_\epsilon^{kj}-\nabla Q_\epsilon^{ik}\nabla Q_\epsilon^{kj}\right)\left(\partial_j\phi\partial_i\triangle^{-1}(\rho_\epsilon)+\phi\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\right)\\
&+\int\left(Q_\epsilon^{ik}\nabla Q_\epsilon^{kj}-\nabla Q_\epsilon^{ik}Q_\epsilon^{kj}\right)\nabla\left(\partial_j\phi\partial_i\triangle^{-1}(\rho_\epsilon)+\phi\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\right)\\
=&\int\left(\nabla Q_\epsilon^{ik}\nabla Q_\epsilon^{kj}-\nabla Q_\epsilon^{ik}\nabla Q_\epsilon^{kj}\right)\left(\partial_i\phi\partial_i\triangle^{-1}(\rho_\epsilon)+\phi\rho_\epsilon\right)\\
\to&\int\left(\nabla Q^{ik}\nabla Q^{kj}-\nabla Q^{ik}\nabla Q^{kj}\right)\left(\partial_i\phi\partial_i\triangle^{-1}(\rho)+\phi\rho\right)\\
=&\int\left(Q\triangle Q-\triangle QQ\right)\left(\partial_i\phi\partial_i\triangle^{-1}(\rho)+\phi\rho\right)\\
=&T_8,
\end{aligned}
\tag{4.36}
$$

where the third equality is valid after summing up $i, j = 1, 2, 3$, due to the fact that the matrix $Q$ is symmetric and the following computation:

$$
\begin{aligned}
&\int \left(Q_\epsilon^{ik}\nabla Q_\epsilon^{kj} - \nabla Q_\epsilon^{ik} Q_\epsilon^{kj}\right) \nabla\left(\partial_j\phi\partial_i\triangle^{-1}(\rho_\epsilon) + \phi\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\right)\\
&\quad + \int \left(Q_\epsilon^{jk}\nabla Q_\epsilon^{ki} - \nabla Q_\epsilon^{jk} Q_\epsilon^{ki}\right) \nabla\left(\partial_i\phi\partial_j\triangle^{-1}(\rho_\epsilon) + \phi\partial_i\partial_j\triangle^{-1}(\rho_\epsilon)\right)\\
&= \int \left(Q_\epsilon^{ik}\nabla Q_\epsilon^{kj} - \nabla Q_\epsilon^{jk} Q_\epsilon^{ki}\right) \nabla\left(\partial_j\phi\partial_i\triangle^{-1}(\rho_\epsilon) + \phi\partial_j\partial_i\triangle^{-1}(\rho_\epsilon)\right)\\
&\quad + \int \left(Q_\epsilon^{jk}\nabla Q_\epsilon^{ki} - \nabla Q_\epsilon^{ik} Q_\epsilon^{kj}\right) \nabla\left(\partial_i\phi\partial_j\triangle^{-1}(\rho_\epsilon) + \phi\partial_i\partial_j\triangle^{-1}(\rho_\epsilon)\right)\\
&= 0.
\end{aligned}
$$

Next, utilizing (4.8), (4.13)-(4.18), and (4.35) again, we deduce that

$$\lim_{\epsilon\to 0} T_n^\epsilon = T_n, \quad \text{for } i = 1, 2, 3, 4, 6, 7, 9,$$

and

$$\lim_{\epsilon\to 0} T_n^\epsilon = T_n, \quad \text{for } i = 10, 11.$$

In order to justify

$$\lim_{\epsilon\to 0} T_5^\epsilon = T_5, \tag{4.37}$$

we present the following Lemma (cf. [18]):

**Lemma 4.4** (div-curl). *Let $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}$ and $1 \le r, r_1, r_2 < \infty$. Suppose that*

$$v_\epsilon \rightharpoonup v \text{ in } L^{r_1} \quad \text{and} \quad w_\epsilon \rightharpoonup w \text{ in } L^{r_2}.$$

*Then,*

$$v_\epsilon\partial_i\partial_j\triangle^{-1}(w_\epsilon) - w_\epsilon\partial_i\partial_j\triangle^{-1}(v_\epsilon) \rightharpoonup v\partial_i\partial_j\triangle^{-1}(w) - w\partial_i\partial_j\triangle^{-1}(v) \quad \text{in } L^r,\ \ (i, j = 1, 2, 3).$$

Taking $v_\epsilon = \rho_\epsilon \mathbf{u}_\epsilon^j$ and $w_\epsilon = \rho_\epsilon$, we obtain (4.37) by Lemma 4.4. The proof of Lemma 4.3 is thus completed.

## 5. Vanishing Artificial Pressure

In this section, we will complete the proof of Theorem 1.1 by taking the limit as $\delta \to 0$ in the solutions $(\rho_\delta, c_\delta, \mathbf{u}_\delta, Q_\delta)$ obtained in Theorem 4.1.

### 5.1. Refined estimates on energy function.

We first derive the refined estimates on $(\rho_\delta, c_\delta, \mathbf{u}_\delta, Q_\delta)$ uniform in $\delta$, which helps us relax the restriction on $\gamma$.

**Proposition 5.1.** *Let $(\rho_\delta, c_\delta, \mathbf{u}_\delta, Q_\delta)$ be the solution obtained in Theorem 4.1. Then, under the assumptions in Theorem 1.1, the following inequality holds for all $s \in (1, \frac{3}{2})$,*

$$\|\delta\rho_\delta^4 + \rho_\delta^\gamma\|_{L^s} + \|Q_\delta\|_{L^6} + \|\mathbf{u}_\delta\|_{H_0^1} + \|\nabla Q_\delta\|_{H^1} + \|c_\delta\|_{L^\infty} \le C, \tag{5.1}$$

*where, and in what follows, the constant $C$ is independent of $\delta$.*

The proof of Proposition 5.1 borrows some ideas developed in [19, 28, 34, 40]. We present the details below through several lemmas.

**Lemma 5.1.** *Let $(\rho_\delta, c_\delta, \mathbf{u}_\delta, Q_\delta)$ be the solution obtained in Theorem 4.1. Then there are constants $C$ and $C_1$ independent of $\delta$ such that,*

$$\begin{aligned}
&\|c\|_{L^\infty}^6 + \int \left(|\nabla \mathbf{u}_\delta|^2 + |\triangle Q_\delta|^2 + |\nabla Q_\delta|^2 + |Q_\delta|^6\right) \\
&\le C\left(1 + \|\rho_\delta \mathbf{u}_\delta\|_{L^1} + m_2 \|\rho_\delta \mathbf{u}_\delta\|_{L^1}^{3C_1}\right),
\end{aligned} \tag{5.2}$$

*provided that $m_2$ is sufficiently small.*

*Proof.* Using the same computation as that in (3.12)-(3.15), we multiply $(4.1)_3$ by $\mathbf{u}_\delta$ and $(4.1)_4$ by $-\triangle Q_\delta + Q_\delta + c_* Q_\delta tr(Q_\delta^2)$ to deduce

$$\begin{aligned}
&\mu \int |\nabla \mathbf{u}_\delta|^2 + (\lambda + \mu) \int |\mathrm{div} \mathbf{u}_\delta|^2 + \int |\triangle Q_\delta|^2 + \int |\nabla Q_\delta|^2 + c_* \int \left(|Q_\delta|^4 + c_* |Q_\delta|^6\right) \\
&\le \int \rho_\delta g_2 \cdot \mathbf{u}_\delta + \int b\left(Q_\delta^2 - \frac{1}{3} tr(Q_\delta^2)\mathbb{I}\right) : \left(-\triangle Q_\delta + Q_\delta + c_* Q_\delta tr(Q_\delta^2)\right) \\
&\quad + \int g_3 : \left(-\triangle Q_\delta + Q_\delta + c_* Q_\delta tr(Q_\delta^2)\right) \\
&\quad - \sigma_* \int c_\delta^2 Q_\delta : \nabla \mathbf{u}_\delta + \int \frac{(c_\delta - c_*)}{2} Q_\delta : (\triangle Q_\delta - Q_\delta - c_* Q_\delta tr(Q_\delta^2)).
\end{aligned}$$

From (1.8) it follows that

$$\begin{aligned}
&\left| \int b\left(Q_\delta^2 - \frac{1}{3} tr(Q_\delta^2)\mathbb{I}\right) : \left(-\triangle Q_\delta + Q_\delta + c_* Q_\delta tr(Q_\delta^2)\right)\right| \\
&\quad + \left|\int g_3 : \left(-\triangle Q_\delta + Q_\delta + c_* Q_\delta tr(Q_\delta^2)\right)\right| \\
&\le C(1 + \|Q_\delta\|_{L^4}^2 + \|g_3\|_{L^2}) \left(\|\triangle Q_\delta\|_{L^2} + \|Q_\delta\|_{L^6}^3 + 1\right) \\
&\le C + \frac{c_*^2}{4} \int |Q_\delta|^6 + \frac{1}{4} \int |\triangle Q_\delta|^2,
\end{aligned}$$

and

$$\begin{aligned}
&\left| -\sigma_* \int c_\delta^2 Q_\delta : \nabla \mathbf{u}_\delta + \int \frac{(c_\delta - c_*)}{2} Q_\delta : (\triangle Q_\delta - Q_\delta - c_* Q_\delta tr(Q_\delta^2))\right| \\
&\le C \|c_\delta\|_{L^\infty}^2 \|Q_\delta\|_{L^2} \|\nabla \mathbf{u}_\delta\|_{L^2} + C(1 + \|c_\delta\|_{L^\infty}) \left(\|Q_\delta\|_{L^2} \|\triangle Q_\delta\|_{L^2} + \|Q_\delta\|_{L^6}^4 + 1\right) \\
&\le C + C\|c_\delta\|_{L^\infty}^6 + \frac{c_*^2}{4} \int |Q_\delta|^6 + \frac{\mu}{2} \int |\nabla \mathbf{u}_\delta|^2 + \frac{1}{4} \int |\triangle Q_\delta|^2.
\end{aligned}$$

The last three inequalities provide us

$$\int |\nabla \mathbf{u}_\delta|^2 + \int |\triangle Q_\delta|^2 + \int |\nabla Q_\delta|^2 + \int |Q_\delta|^4 \le C + C\|c_\delta\|_{L^\infty}^6 + \|\rho_\delta \mathbf{u}_\delta\|_{L^1}. \tag{5.3}$$

With the aid of (5.3) and (3.22), we choose $m_2$ sufficiently small such that

$$\begin{aligned}
\|c_\delta\|_{L^\infty} &\le C(1 + \|\mathbf{u}_\delta\|_{L^6})^{C_1} (1 + \|g_1\|_{L^\infty}) m_2 \\
&\le C(1 + \|c_\delta\|_{L^\infty}^3 + \|\rho_\delta \mathbf{u}_\delta\|_{L^1}^{\frac{1}{2}})^{C_1} m_2 \\
&\le 2C + C m_2 \|\rho_\delta \mathbf{u}_\delta\|_{L^1}^{\frac{C_1}{2}},
\end{aligned}$$

which together with (5.3) lead to the desired estimate (5.2). □

**Lemma 5.2.** *Let $(\rho_\delta, c_\delta, \mathbf{u}_\delta, Q_\delta)$ be the solution obtained in Theorem 4.1. Then, for any $s \in (1, \frac{3}{2})$, the following inequality holds true*

$$\|\delta\rho_\delta^4 + \rho_\delta^\gamma\|_{L^s} \le C\left(1 + \|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^s} + m_2\|\rho_\delta\mathbf{u}_\delta\|_{L^1}^{3C_1}\right), \tag{5.4}$$

*provided that $m_2$ is sufficiently small.*

*Proof.* As in Lemma 3.3, we introduce the Bogovskii operator

$$\mathcal{B} := \mathcal{B}(h - (h)_\mathcal{O}) \quad \text{with} \quad (h)_\mathcal{O} = |\mathcal{O}|^{-1}\int h.$$

Then, for any $h \in L^{\frac{s}{s-1}}$ with $s \in (1, \frac{3}{2})$, Lemma 3.3 implies

$$\|\mathcal{B}\|_{L^\infty} + \|\nabla\mathcal{B}\|_{L^2} + \|\nabla\mathcal{B}\|_{L^{\frac{s}{s-1}}} \le C\|h\|_{L^{\frac{s}{s-1}}}. \tag{5.5}$$

Multiplying $(4.1)_3$ by $\mathcal{B}$ gives

$$\begin{aligned}
&\int \left(\delta\rho_\delta^4 + \rho_\delta^\gamma\right) h \\
&= (h)_\mathcal{O}\int\left(\delta\rho_\delta^4 + \rho_\delta^\gamma\right) - \int \rho g_2 \cdot \mathcal{B}(h - (h)_\mathcal{O}) + \int \mathbb{S}_{ns} : \nabla\mathcal{B} \\
&\quad - \int \rho_\delta\mathbf{u}_\delta \otimes \mathbf{u} : \nabla\mathcal{B} + \int \mathbb{S}_1 : \nabla\mathcal{B} + \int \mathbb{S}_2 : \nabla\mathcal{B} \\
&\le C\|h\|_{L^{\frac{s}{s-1}}}\|\delta\rho_\delta^4 + \rho_\delta^\gamma\|_{L^1} + C\|\mathcal{B}\|_{L^\infty} + C\|\nabla\mathbf{u}_\delta\|_{L^2}\|\nabla\mathcal{B}\|_{L^2} \\
&\quad + C\left(\|\rho_\delta|\mathbf{u}_\delta|^2 + c^2Q_\delta + |Q_\delta|^2 + |Q_\delta|^4 + |\nabla Q_\delta|^2 + Q_\delta\triangle Q_\delta\|_{L^s})\|\nabla\mathcal{B}\|_{L^{\frac{s}{s-1}}}\right) \\
&\le C\|h\|_{L^{\frac{s}{s-1}}}\left(1 + \|\delta\rho_\delta^4 + \rho_\delta^\gamma\|_{L^1} + \|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^s} + \|\nabla\mathbf{u}_\delta\|_{L^2}\right) \\
&\quad + C\|h\|_{L^{\frac{s}{s-1}}}\left(\|c^2Q_\delta + |Q_\delta|^2 + |Q_\delta|^4 + |\nabla Q_\delta|^2 + Q_\delta\triangle Q_\delta\|_{L^s}\right),
\end{aligned} \tag{5.6}$$

where, for the last inequalities, we have used (1.4), (1.5), (1.8), (5.5) and $\|\rho_\delta\|_{L^1} = m_1$. Due to the arbitrariness of $h \in L^{\frac{s}{s-1}}$, it yields from (5.6) that

$$\begin{aligned}
\|\delta\rho_\delta^4 + \rho_\delta^\gamma\|_{L^s} &\le C\left(1 + \|\delta\rho_\delta^4 + \rho_\delta^\gamma\|_{L^1} + \|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^s} + \|\nabla\mathbf{u}_\delta\|_{L^2}\right) \\
&\quad + C\|c^2Q_\delta + |Q_\delta|^2 + |Q_\delta|^4 + |\nabla Q_\delta|^2 + Q_\delta\triangle Q_\delta\|_{L^s} \\
&\le \frac{1}{2}\|\delta\rho_\delta^4 + \rho_\delta^\gamma\|_{L^s} + \left(1 + \|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^s} + \|\nabla\mathbf{u}_\delta\|_{L^2}\right) \\
&\quad + C\|c^2Q_\delta + |Q_\delta|^2 + |Q_\delta|^4 + |\nabla Q_\delta|^2 + Q_\delta\triangle Q_\delta\|_{L^s}.
\end{aligned} \tag{5.7}$$

Since $s \in (1, \frac{3}{2})$, one has

$$\begin{aligned}
&\|c_\delta^2 Q_\delta + |Q_\delta|^2 + |Q_\delta|^4 + |\nabla Q_\delta|^2 + Q_\delta\triangle Q_\delta\|_{L^s} \\
&\le C\left(1 + \|c_\delta\|_{L^\infty}^3 + \|Q_\delta\|_{L^6}^4 + \|\nabla Q_\delta\|_{L^2}^2 + \|\triangle Q_\delta\|_{L^2}^2\right) \\
&\le C\left(1 + \|c_\delta\|_{L^\infty}^3 + \|Q_\delta\|_{L^6}^6 + \|\nabla Q_\delta\|_{L^2}^2 + \|\triangle Q_\delta\|_{L^2}^2\right).
\end{aligned}$$

Therefore, substituting it into (5.7) and utilizing (5.2) we obtain (5.4). □

Next, we shall deduce a weighted estimate on both the pressure and kinetic energy.

**Lemma 5.3.** *Let $(\rho_\delta, c_\delta, \mathbf{u}_\delta, Q_\delta)$ be the solution obtained in Theorem 4.1. Then, for any $\alpha \in (0,1)$ and $s \in (1, \frac{3}{2})$, the following inequality holds true*

$$\begin{aligned} &\sup_{x^* \in \overline{\mathcal{O}}} \int \frac{\left(\delta\rho_\delta^4 + \rho_\delta^\gamma + \rho_\delta |\mathbf{u}_\delta|^2\right)(x)}{|x - x^*|^\alpha} dx \\ &\le C\left(1 + \|\rho_\delta |\mathbf{u}_\delta|^2\|_{L^s} + m_2 \|\rho_\delta \mathbf{u}_\delta\|_{L^1}^{3C_1}\right), \end{aligned} \tag{5.8}$$

*provided that $m_2$ is sufficiently small.*

*Proof.* We adopt some ideas in [19, 28, 34], and divide the process into two cases.

*Case 1: The boundary point case $x^* \in \partial\mathcal{O}$.*

As in [45, Exercise 1.15] we introduce a function $\phi(x) \in C^2(\overline{\mathcal{O}})$ that behaviors like the distance when $x \in \mathcal{O}$ is near the boundary and is extended smoothly to the whole domain $\mathcal{O}$, and moreover,

$$\begin{cases} \phi(x) > 0 \text{ in } \mathcal{O} \text{ and } \phi(x) = 0 \text{ on } \partial\mathcal{O}, \\ |\phi(x)| \ge k_1 \text{ if } x \in \mathcal{O} \text{ and } \operatorname{dist}(x, \partial\mathcal{O}) \ge k_2, \\ \nabla\phi = \dfrac{x - \tilde{x}}{\phi(x)} = \dfrac{x - \tilde{x}}{|x - \tilde{x}|} \text{ if } x \in \mathcal{O} \text{ and } \operatorname{dist}(x, \partial\mathcal{O}) = |x - \tilde{x}| \le k_2, \end{cases} \tag{5.9}$$

where the positive constants $k_1$ and $k_2$ are given. Following [19], we define

$$\xi(x) = \frac{\phi(x)\nabla\phi(x)}{\left(\phi(x) + |x - x^*|^{\frac{2}{2-\alpha}}\right)^\alpha} \quad \text{with} \quad x,\, x^* \in \overline{\mathcal{O}}. \tag{5.10}$$

It follows from (5.9) that, for all points $x$ satisfying $\operatorname{dist}(x, \partial\mathcal{O}) \le k_2$,

$$\phi < \phi + |x - x^*|^{\frac{2}{2-\alpha}} \le C|x - x^*|, \tag{5.11}$$

owing to $\frac{2}{2-\alpha} > 1$. By (5.10), a careful computation gives

$$\begin{aligned} \partial_j \xi^i = &\frac{\phi \partial_j \partial_i \phi}{\left(\phi + |x - x^*|^{\frac{2}{2-\alpha}}\right)^\alpha} + \frac{\partial_j \phi \partial_i \phi}{\left(\phi + |x - x^*|^{\frac{2}{2-\alpha}}\right)^\alpha} \\ &- \alpha \frac{\phi \partial_i \phi \partial_j \phi}{\left(\phi + |x - x^*|^{\frac{2}{2-\alpha}}\right)^{\alpha+1}} - \alpha \frac{\phi \partial_i \phi \partial_j |x - x^*|^{\frac{2}{2-\alpha}}}{\left(\phi + |x - x^*|^{\frac{2}{2-\alpha}}\right)^{\alpha+1}}, \quad i, j = 1, 2, 3. \end{aligned} \tag{5.12}$$

Thus, $|\nabla\xi| \in L^q$ for all $q \in [2, \frac{3}{\alpha})$. In view of (5.10)-(5.12), one has

$$\begin{aligned} C + \frac{C}{|x - x^*|^\alpha} \ge \operatorname{div}\xi &\ge -C + \frac{(1-\alpha)}{2} \frac{|\nabla\phi|^2}{\left(\phi + |x - x^*|^{\frac{2}{2-\alpha}}\right)^\alpha} \\ &\ge -C + \frac{C}{|x - x^*|^\alpha}. \end{aligned} \tag{5.13}$$

In addition, by (5.9),

$$\partial_j \partial_i \phi = \frac{\partial_i (x - \tilde{x})^j}{\phi} - \frac{\partial_j \phi \partial_i \phi}{\phi}. \tag{5.14}$$

Hence, if we multiply $(4.1)_3$ by $\xi$, we find

$$\begin{aligned}&\int\left(\delta\rho_\delta^4+\rho_\delta^\gamma\right)\mathrm{div}\xi+\int\rho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta:\nabla\xi\\&=\int\left(\mathbb{S}_{ns}(\nabla\mathbf{u}_\delta)+\mathbb{S}_1(Q_\delta)+\mathbb{S}_2(c_\delta,Q_\delta)\right):\nabla\xi-\int\rho_\delta g_2\cdot\xi.\end{aligned}\tag{5.15}$$

Making use of (1.2), (1.4), (1.5), (1.8), (5.13), and the fact $\xi\in W_0^{1,3}$, one deduces

$$\begin{aligned}&\left|\int\left(\mathbb{S}_{ns}(\nabla\mathbf{u}_\delta)+\mathbb{S}_1(Q_\delta)+\mathbb{S}_2(c_\delta,Q_\delta)\right):\nabla\xi-\int\rho_\delta g_2\cdot\xi\right|\\&\le C(\alpha)\left(1+\|\nabla\mathbf{u}_\delta\|_{L^2}+\||\nabla Q_\delta|^2+|Q_\delta|^4+|Q_\delta||\triangle Q_\delta|+c_\delta^2Q_\delta\|_{L^{\frac32}}\right)\\&\le C(\alpha)\left(1+\|\nabla\mathbf{u}_\delta\|_{L^2}+\|\triangle Q_\delta\|_{L^2}^2+\|\nabla Q_\delta\|_{L^2}^2+\|Q_\delta\|_{L^6}^4+\|c_\delta\|_{L^\infty}^3\right),\end{aligned}\tag{5.16}$$

and

$$\int\left(\delta\rho_\delta^4+\rho_\delta^\gamma\right)\mathrm{div}\xi\ \ge -C\int\left(\delta\rho_\delta^4+\rho_\delta^\gamma\right)+C\int_{\mathcal{O}\cap B_{k_2}(x^*)}\frac{\left(\delta\rho_\delta^4+\rho_\delta^\gamma\right)}{|x-x^*|^\alpha}.\tag{5.17}$$

By (5.14),

$$\int\frac{\phi\rho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta\partial_j\partial_i\phi}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha}}\right)^\alpha}=\int\frac{\rho_\delta|\mathbf{u}_\delta|^2}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha}}\right)^\alpha}-\int\frac{\rho_\delta|\mathbf{u}_\delta\cdot\nabla\phi|^2}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha}}\right)^\alpha},$$

which along with (5.9), (5.11), (5.14) and the Schwarz inequality imply

$$\begin{aligned}&\int\rho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta:\nabla\xi\\&=\int\frac{\rho_\delta|\mathbf{u}_\delta|^2}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha}}\right)^\alpha}-\alpha\int\frac{\phi\rho_\delta(\mathbf{u}_\delta\cdot\nabla\phi)^2}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha}}\right)^{\alpha+1}}\\&\quad-\alpha\int\frac{\phi\rho_\delta(\mathbf{u}_\delta\cdot\nabla|x-x^*|^{\frac{2}{2-\alpha}})(\mathbf{u}_\delta\cdot\nabla\phi)}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha}}\right)^{\alpha+1}}\\&\ge(1-\alpha)\int\frac{\rho_\delta|\mathbf{u}_\delta|^2}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha}}\right)^\alpha}-\alpha\int\frac{\phi\rho_\delta(\mathbf{u}_\delta\cdot\nabla|x-x^*|^{\frac{2}{2-\alpha}})(\mathbf{u}_\delta\cdot\nabla\phi)}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha}}\right)^{\alpha+1}}\\&\ge\frac{(1-\alpha)}{2}\int\frac{\rho_\delta|\mathbf{u}_\delta|^2}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha}}\right)^\alpha}-C\int\frac{\phi^2\rho_\delta|\mathbf{u}_\delta|^2|x-x^*|^{\frac{2\alpha}{2-\alpha}}}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha}}\right)^{\alpha+2}}\\&\ge C\int_{\mathcal{O}\cap B_{k_2}(x^*)}\frac{\rho_\delta|\mathbf{u}_\delta|^2}{|x-x^*|^\alpha}-C\|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^1}.\end{aligned}\tag{5.18}$$

Therefore, the inequalities (5.15)-(5.18) yield that, for some $C$ independent of $x^*$,

$$\begin{aligned}
&\int_{\mathcal{O}\cap B_{k_2}(x^*)} \frac{\left(\delta\rho_\delta^4+\rho_\delta^\gamma+\rho_\delta|\mathbf{u}_\delta|^2\right)(x)}{|x-x^*|^\alpha}dx \\
&\le C\left(\|\delta\rho_\delta^4+\rho_\delta^\gamma\|_{L^1}+\|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^1}\right) \\
&\quad + C\left(\|\nabla\mathbf{u}_\delta\|_{L^2}+\|\triangle Q_\delta\|_{L^2}^2+\|\nabla Q_\delta\|_{L^2}^2+\|Q_\delta\|_{L^6}^4+\|c_\delta\|_{L^\infty}^3\right) \\
&\le C\left(1+\|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^s}+m_2\|\rho_\delta\mathbf{u}_\delta\|_{L^1}^{3C_1}\right),
\end{aligned} \tag{5.19}$$

where the last inequality follows from (5.4) and (5.2).

*Case 2: The interior point case $x^*\in\mathcal{O}$.*

We set $\mathrm{dist}(x^*,\,\partial\mathcal{O})=3r>0$. Define the smooth cut-off function

$$\chi(x)=1 \text{ if } x\in B_r(x^*),\quad \chi(x)=0 \text{ if } x\notin B_{2r}(x^*),\quad |\nabla\chi(x)|\le 2r^{-1}. \tag{5.20}$$

Multiplying $(4.1)_3$ by $\frac{x-x^*}{|x-x^*|^\alpha}\chi^2$ yields

$$\begin{aligned}
&\int\left(\delta\rho_\delta^4+\rho_\delta^\gamma\right)\frac{3-\alpha}{|x-x^*|^\alpha}\chi^2+\int\rho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta:\nabla\left(\frac{x-x^*}{|x-x^*|^\alpha}\chi^2\right) \\
&=\int\left(\mathbb{S}_{ns}(\nabla\mathbf{u}_\delta)+\mathbb{S}_1(Q_\delta)+\mathbb{S}_2(c_\delta,Q_\delta)\right):\nabla\left(\frac{x-x^*}{|x-x^*|^\alpha}\chi^2\right)-\int\rho_\delta g_2\cdot\frac{x-x^*}{|x-x^*|^\alpha}\chi^2 \\
&\quad -2\int\left(\delta\rho_\delta^4+\rho_\delta^\gamma\right)\chi\frac{\nabla\chi\cdot(x-x^*)}{|x-x^*|^\alpha}.
\end{aligned} \tag{5.21}$$

A simple calculation shows

$$\begin{aligned}
&\partial_i\left(\frac{x^j-(x^*)^j}{|x-x^*|^\alpha}\chi^2\right) \\
&=\frac{\partial_i(x^j-(x^*)^j)}{|x-x^*|^\alpha}\chi^2-\alpha\frac{(x^j-(x^*)^j)(x^i-(x^*)^i)}{|x-x^*|^{\alpha+2}}\chi^2+2\chi\frac{x^j-(x^*)^j}{|x-x^*|^\alpha}\partial_i\chi,
\end{aligned} \tag{5.22}$$

thus, for some constant $C$ independent of $x^*$ and $r$,

$$\begin{aligned}
&\int\rho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta:\nabla\left(\frac{x-x^*}{|x-x^*|^\alpha}\chi^2\right) \\
&\ge(1-\alpha)\int\frac{\rho_\delta|\mathbf{u}_\delta|^2}{|x-x^*|^\alpha}\chi^2+2\int\frac{\chi\rho_\delta(\mathbf{u}_\delta\cdot\nabla\chi)(\mathbf{u}_\delta\cdot(x-x^*))}{|x-x^*|^\alpha} \\
&\ge\frac{1-\alpha}{2}\int\frac{\rho_\delta|\mathbf{u}_\delta|^2}{|x-x^*|^\alpha}\chi^2-C\int_{B_{2r}(x^*)\backslash B_r(x^*)}\frac{\rho_\delta|\mathbf{u}_\delta|^2}{|x-x^*|^\alpha},
\end{aligned}$$

where we have used

$$|\nabla\chi||x-x^*|\le 4,\quad \forall\ x\in B_{2r}(x^*)\backslash B_r(x^*). \tag{5.23}$$

Observe from (5.20) and (5.22) that $\nabla\left(\frac{x-x^*}{|x-x^*|^\alpha}\chi^2\right)\in L^q$ for all $q\in[1,\frac{3}{\alpha})$. By the similar argument to (5.16), one has, for some constant $C$ independent of $r$,

$$\begin{aligned}&\left|\int\left(\mathbb{S}_{ns}(\nabla\mathbf{u}_\delta)+\mathbb{S}_1(Q_\delta)+\mathbb{S}_2(c_\delta,Q_\delta)\right):\nabla\left(\frac{x-x^*}{|x-x^*|^\alpha}\chi^2\right)\right|\\&\quad+\left|\int\rho_\delta g_2\cdot\frac{x-x^*}{|x-x^*|^\alpha}\chi^2\right|\\&\le C\left(1+\|\nabla\mathbf{u}_\delta\|_{L^2}+\|\triangle Q_\delta\|_{L^2}^2+\|\nabla Q_\delta\|_{L^2}^2+\|Q_\delta\|_{L^6}^4+\|c_\delta\|_{L^\infty}^3\right)\end{aligned}$$

and

$$\left|-2\int\left(\delta\rho_\delta^4+\rho_\delta^\gamma\right)\chi\frac{\nabla\chi\cdot(x-x^*)}{|x-x^*|^\alpha}\right|\le C\int_{B_{2r}(x^*)\setminus B_r(x^*)}\frac{\left(\delta\rho_\delta^4+\rho_\delta^\gamma\right)}{|x-x^*|^\alpha},$$

due to (5.23). Therefore, taking the above inequalities into accounts, utilizing (5.2), we deduce from (5.21) that

$$\begin{aligned}&\int_{B_r(x^*)}\frac{\left(\delta\rho_\delta^4+\rho_\delta^\gamma+\rho_\delta|\mathbf{u}_\delta|^2\right)(x)}{|x-x^*|^\alpha}dx\\&\le C\left(1+\|\nabla\mathbf{u}_\delta\|_{L^2}+\|\triangle Q_\delta\|_{L^2}^2+\|\nabla Q_\delta\|_{L^2}^2+\|Q_\delta\|_{L^6}^4+\|c_\delta\|_{L^\infty}^3\right)\\&\quad+C\int_{B_{2r}(x^*)\setminus B_r(x^*)}\frac{\left(\delta\rho_\delta^4+\rho_\delta^\gamma+\rho_\delta|\mathbf{u}_\delta|^2\right)(x)}{|x-x^*|^\alpha}dx\\&\le C\left(1+\|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^s}+m_2\|\rho_\delta\mathbf{u}_\delta\|_{L^1}^{3C_1}\right)\\&\quad+C\int_{B_{2r}(x^*)\setminus B_r(x^*)}\frac{\left(\delta\rho_\delta^4+\rho_\delta^\gamma+\rho_\delta|\mathbf{u}_\delta|^2\right)(x)}{|x-x^*|^\alpha}dx.\end{aligned}\tag{5.24}$$

It remains to estimate the last term appeared in (5.24). To this end, we adopt the ideas in [28] and discuss two cases: (1) $x^*$ is far away from the boundary; (2) $x^*$ is close to the boundary.

(1) Assume $\operatorname{dist}(x^*,\partial\mathcal{O})=3r\ge\frac{k_2}{2}$ with $k_2$ given in (5.9). Then

$$\begin{aligned}&\int_{B_{2r}(x^*)\setminus B_r(x^*)}\frac{\left(\delta\rho_\delta^4+\rho_\delta^\gamma+\rho_\delta|\mathbf{u}_\delta|^2\right)(x)}{|x-x^*|^\alpha}dx\\&\le k_2^{-\alpha}\int_{B_{2r}(x^*)\setminus B_r(x^*)}\left(\delta\rho_\delta^4+\rho_\delta^\gamma+\rho_\delta|\mathbf{u}_\delta|^2\right)\\&\le C\left(1+\|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^s}+m_2\|\rho_\delta\mathbf{u}_\delta\|_{L^1}^{3C_1}\right),\end{aligned}\tag{5.25}$$

where the last inequality follows from (5.4).

(2) Assume that $x^*\in\mathcal{O}$ is close to the boundary. By (5.9), we have

$$|x^*-\tilde{x}^*|=dist(x^*,\partial\mathcal{O})=3r<\frac{k_2}{2}\quad\text{with}\quad\tilde{x}^*\in\partial\mathcal{O},$$

and hence,

$$4|x-x^*|\ge|x-\tilde{x}^*|,\quad\forall\ x\notin B_r(x^*).\tag{5.26}$$

Making use of (5.26) and (5.19), we get

$$\begin{aligned} &C\int_{B_{2r}(x^*)\setminus B_r(x^*)}\frac{\left(\delta\rho_\delta^4+\rho_\delta^\gamma+\rho_\delta|\mathbf{u}_\delta|^2\right)(x)}{|x-x^*|^\alpha}dx \\ &\le C\int_{\mathcal{O}\cap B_{k_2}(\tilde{x}^*)}\frac{\left(\delta\rho_\delta^4+\rho_\delta^\gamma+\rho_\delta|\mathbf{u}_\delta|^2\right)(x)}{|x-\tilde{x}^*|^\alpha}dx \\ &\le C\left(1+\|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^s}+m_2\|\rho_\delta\mathbf{u}_\delta\|_{L^1}^{3C_1}\right). \end{aligned} \tag{5.27}$$

In summary, substituting (5.25) and (5.27) back into (5.24) yields

$$\begin{aligned} &\int_{B_r(x^*)}\frac{\left(\delta\rho_\delta^4+\rho_\delta^\gamma+\rho_\delta|\mathbf{u}_\delta|^2\right)(x)}{|x-x^*|^\alpha}dx \\ &\le C\left(1+\|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^s}+m_2\|\rho_\delta\mathbf{u}_\delta\|_{L^1}^{3C_1}\right), \end{aligned} \tag{5.28}$$

where the constant $C$ is independent of $x^*$. The combination of (5.19) with (5.28) yoelds the desired estimate (5.8). □

**Lemma 5.4.** *Assume that $\mathbf{u}\in H_0^1(\mathcal{O},\mathbb{R}^3)$ and $f(x)\ge 0$ a.e. in $\mathcal{O}$. Then there is a constant $C$ depending only on $|\mathcal{O}|$ such that*

$$\int_{\mathcal{O}}|\mathbf{u}|^2 f dx\le C\|\nabla\mathbf{u}\|_{H_0^1(\mathcal{O})}^2\int_{\mathcal{O}}\frac{f(x)}{|x-x^*|}dx, \tag{5.29}$$

*as long as the right-hand side quantity is finite.*

*Proof.* The proof is based on the Green representation and integration by parts; see, e.g., [40, Lemma 4]. □

**Lemma 5.5.** *Let $\theta=\frac{\gamma-1}{2\gamma}\in(0,\,\frac12)$. Then,*

$$\int\rho_\delta|\mathbf{u}_\delta|^{2(2-\theta)}\le C. \tag{5.30}$$

*Proof.* Denote by

$$\mathbb{A}=\int\rho_\delta|\mathbf{u}_\delta|^{2(2-\theta)}.$$

Noting $\|\rho_\delta\|_{L^1}=m_1$, we have

$$\|\rho_\delta\mathbf{u}_\delta\|_{L^1}\le\|\rho_\delta|\mathbf{u}_\delta|^{2(2-\theta)}\|_{L^1}^{\frac{1}{2(2-\theta)}}\|\rho_\delta\|_{L^1}^{\frac{3-2\theta}{2(2-\theta)}}\le C\mathbb{A}^{\frac{1}{2(2-\theta)}}, \tag{5.31}$$

and

$$\|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^{\frac32}}\le\|\rho_\delta|\mathbf{u}_\delta|^{2(2-\theta)}\|_{L^1}^{\frac{1}{2-\theta}}\|\rho_\delta\|_{L^1}^{\frac{(1-\theta)}{2-\theta}}\le C\mathbb{A}^{\frac{1}{2-\theta}}. \tag{5.32}$$

Thanks to (5.31), it follows from (5.2) that

$$\begin{aligned} &\|c_\delta\|_{L^\infty}^6+\int\left(|\nabla\mathbf{u}_\delta|^2+|\triangle Q_\delta|^2+|\nabla Q_\delta|^2+|Q_\delta|^6\right) \\ &\le C\left(1+\mathbb{A}^{\frac{1}{2(2-\theta)}}+m_2\mathbb{A}^{\frac{3C_1}{2(2-\theta)}}\right). \end{aligned} \tag{5.33}$$

A direct calculation shows

$$\frac{\rho_\delta|\mathbf{u}_\delta|^{2(1-\theta)}}{|x-x^*|} = \left(\frac{\rho_\delta|\mathbf{u}_\delta|^2}{|x-x^*|^\alpha}\right)^{1-\theta}\left(\frac{\rho_\delta^\gamma}{|x-x^*|^\alpha}\right)^{\frac{\theta}{\gamma}}\left(\frac{1}{|x-x^*|^{\alpha+\frac{\gamma(1-\alpha)}{(\gamma-1)\theta}}}\right)^{\frac{(\gamma-1)\theta}{\gamma}}. \tag{5.34}$$

Noting that $\theta = \frac{\gamma-1}{2\gamma}$, we have $\alpha + \frac{\gamma(1-\alpha)}{(\gamma-1)\theta} \in (0,3)$ if $\alpha \in (\frac{2\gamma-1}{\gamma^2}, 1)$, thus

$$\int \frac{1}{|x-x^*|^{\alpha+\frac{\gamma(1-\alpha)}{(\gamma-1)\theta}}}dx \le C.$$

Therefore, by (5.8), (5.31), (5.32), we integrate (5.34) to obtain

$$\begin{aligned}\int \frac{\rho_\delta|\mathbf{u}_\delta|^{2(1-\theta)}(x)}{|x-x^*|}dx &\le C + \int \frac{\rho_\delta|\mathbf{u}_\delta|^2(x)}{|x-x^*|^\alpha}dx + \int \frac{\rho_\delta^\gamma(x)}{|x-x^*|^\alpha}dx \\ &\le C + C\int \frac{\left(\delta\rho_\delta^4 + \rho_\delta^\gamma + \rho_\delta|\mathbf{u}_\delta|^2\right)(x)}{|x-x^*|^\alpha}dx \\ &\le C\left(1 + \mathbb{A}^{\frac{1}{(2-\theta)}} + m_2\mathbb{A}^{\frac{3C_1}{2(2-\theta)}}\right).\end{aligned} \tag{5.35}$$

From the definition of $\mathbb{A}$, (5.33) and Lemma 5.4, we obtain

$$\begin{aligned}\mathbb{A} &\le \|\nabla u\|_{L^2}^2 \sup_{x^*\in\overline{\mathcal{O}}}\int \frac{\rho_\delta|\mathbf{u}_\delta|^{2(1-\theta)}(x)}{|x-x^*|}dx \\ &\le C\left(1 + \mathbb{A}^{\frac{1}{2(2-\theta)}} + m_2\mathbb{A}^{\frac{3C_1}{2(2-\theta)}}\right)\sup_{x^*\in\overline{\mathcal{O}}}\int \frac{\rho_\delta|\mathbf{u}_\delta|^{2(1-\theta)}(x)}{|x-x^*|}dx,\end{aligned}$$

which together with (5.35) implies

$$\mathbb{A} \le 1 + C\mathbb{A}^{\frac{3}{2(2-\theta)}} + m_2\mathbb{A}^{\frac{3C_1}{(2-\theta)}}.$$

Since $\theta \in (0, \frac{1}{2})$, we choose $m_2 \le 1$ sufficiently small to conclude (5.30). The proof of Lemma 5.7 is completed. □

Finally, Proposition 5.1 is a direct consequence of Lemmas 5.1-5.5.

5.2. **Vanishing artificial pressure.** Now we take the limit as $\delta \to 0$ in the spirit of [10,37, 38]. Thanks to (5.1), the following estimate follows similarly to (3.27):

$$\|\nabla c_\delta\|_{L^2} + \|\triangle c_\delta\|_{L^2} \le C. \tag{5.36}$$

With (5.1) and (5.36) in hand, we are allowed to take the following limits as $\delta \to 0$, subject to a subsequence,

$$(\nabla\mathbf{u}_\delta,\, \nabla^2 c_\delta,\, \nabla^2 Q_\delta) \rightharpoonup (\nabla u,\, \nabla^2 c,\, \nabla^2 Q) \text{ in } L^2, \tag{5.37}$$

$$\mathbf{u}_\delta \to \mathbf{u} \text{ in } L^{p_1}, \quad (c_\delta,\, Q_\delta) \to (c,\, Q) \text{ in } W^{1,p_1} \quad (1 \le p_1 < 6), \tag{5.38}$$

$$\delta\rho_\delta^4 \to 0 \text{ in } \mathcal{D}', \quad \rho_\delta \rightharpoonup \rho \text{ in } L^{\gamma s}, \quad \text{for all } s \in (1, \frac{3}{2}). \tag{5.39}$$

As $\gamma > 1$, we can choose $s \in (1, \frac{3}{2})$ such that $\gamma s > \frac{3}{2}$. Then, from (5.38)-(5.39) one has

$$\rho_\delta\mathbf{u}_\delta \rightharpoonup \rho\mathbf{u}, \quad \text{in } L^{p_2} \quad \text{for some} \quad p_2 > \frac{6}{5}, \tag{5.40}$$

and from (5.38)-(5.40), for some $p_3 > 1$,

$$\begin{cases} \rho_\delta \mathbf{u}_\delta^i \mathbf{u}_\delta^j \rightharpoonup \rho \mathbf{u}^i \mathbf{u}^j, \\ Q_\delta^{ik} \triangle Q_\delta^{kj} \rightharpoonup Q^{ik} \triangle Q^{kj}, \\ Q_\delta^{ik}(\partial_k \mathbf{u}_\delta^j - \partial_j \mathbf{u}_\delta^k) \rightharpoonup Q^{ik}(\partial_k \mathbf{u}^j - \partial_j \mathbf{u}^k), \end{cases} \tag{5.41}$$

in $L^{p_3}$. Using (5.37)-(5.41), we take $\delta$-limit in (4.1) and obtain the equations in the sense of distributions:

$$\begin{cases} \operatorname{div}(\rho \mathbf{u}) = 0, \\ \mathbf{u} \cdot \nabla c - \triangle c = g_1, \\ \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \overline{\rho^\gamma} - \operatorname{div}\left(\mathbb{S}_{ns}(\nabla \mathbf{u}) + \mathbb{S}_1(Q) + \mathbb{S}_2(c, Q)\right) = \rho g_2, \\ \mathbf{u} \cdot \nabla Q + Q\Omega - \Omega Q + c_* Q tr(Q^2) + \dfrac{(c - c_*)}{2} Q - b\left(Q^2 - \dfrac{1}{3} tr(Q^2)\mathbb{I}\right) - \triangle Q = g_3, \\ \mathbf{u} = 0, \ \dfrac{\partial c}{\partial n} = 0, \ \dfrac{\partial Q}{\partial n} = 0, \quad \text{on } \partial \mathcal{O}. \end{cases} \tag{5.42}$$

Additionally, (1.7) follows from (3.2)-(3.3), (5.38), (5.39) and (5.50) below.

Next, we define an increasing and concave function $T_k(z) \in C_0^1([0, \infty))$, satisfying

$$T_k(z) = z \ \text{ if } z \le k, \quad T_k(z) = k + 1 \ \text{ if } z \ge k + 1. \tag{5.43}$$

Clearly, for any $1 \le p \le \infty$,

$$T_k(\rho_\delta) \rightharpoonup \overline{T_k(\rho)} \quad \text{in} \quad L^p. \tag{5.44}$$

**Lemma 5.6.** *Let $(\rho_\delta, \mathbf{u}_\delta, c_\delta, Q_\delta)$ be the solution obtained in Theorem 4.1. Then,*

$$\lim_{\delta \to 0} \int T_k(\rho_\delta)\left(\rho_\delta^\gamma - (2\mu + \lambda)\operatorname{div}\mathbf{u}_\delta\right) = \int \overline{T_k(\rho)}\left(\overline{\rho^\gamma} - (2\mu + \lambda)\operatorname{div}\mathbf{u}\right), \tag{5.45}$$

*where $T_k$ is defined in* (5.43)*.*

*Proof.* With the help of (5.37)-(5.41), we may slightly modify the argument of Lemma 4.3 to complete the proof of Lemma 5.6. The detail is omitted here. □

Since $T_k$ is concave, one has

$$\left(\rho_\delta^\gamma - \rho^\gamma\right)\left(T_k(\rho_\delta) - T_k(\rho)\right) \ge \left(T_k(\rho_\delta) - T_k(\rho)\right)^{\gamma+1}.$$

Then, from (5.45), (5.39), $\overline{\rho^\gamma} \ge \rho^\gamma$, and $T_k(\rho) \le \overline{T_k(\rho)}$, we obtain

$$\begin{aligned} &(2\mu + \lambda)\lim_{\delta \to 0} \int \left(T_k(\rho_\delta)\operatorname{div}\mathbf{u}_\delta - \overline{T_k(\rho)}\operatorname{div}\mathbf{u}\right) \\ &= \lim_{\delta \to 0} \int \left(T_k(\rho_\delta)\rho_\delta^\gamma - \overline{T_k(\rho)}\ \overline{\rho^\gamma}\right) \\ &= \lim_{\delta \to 0} \int \left(\rho_\delta^\gamma - \rho^\gamma\right)\left(T_k(\rho_\delta) - T_k(\rho)\right) + \int \left(\overline{\rho^\gamma} - \rho^\gamma\right)\left(T_k(\rho) - \overline{T_k(\rho)}\right) \\ &\ge \lim_{\delta \to 0} \int \left(\rho_\delta^\gamma - \rho^\gamma\right)\left(T_k(\rho_\delta) - T_k(\rho)\right) \\ &\ge \lim_{\delta \to 0} \int \left(T_k(\rho_\delta) - T_k(\rho)\right)^{\gamma+1}. \end{aligned} \tag{5.46}$$

Noticing that $\mathrm{div}\mathbf{u}_\delta \in L^2$ is bounded uniformly in $\delta$, and

$$\lim_{\delta\to 0}\int (T_k(\rho_\delta))^2 \geq \int \left(\overline{T_k(\rho)}\right)^2,$$

one has

$$\begin{aligned}
&2C\lim_{\delta\to 0}\|T_k(\rho_\delta)-T_k(\rho)\|_{L^2}\\
&\geq C\lim_{\delta\to 0}\left(\|T_k(\rho_\delta)-T_k(\rho)\|_{L^2}+\|T_k(\rho)-\overline{T_k(\rho)}\|_{L^2}\right)\\
&\geq (2\mu+\lambda)\lim_{\delta\to 0}\int\left(T_k(\rho_\delta)-T_k(\rho)+T_k(\rho)-\overline{T_k(\rho)}\right)\mathrm{div}\mathbf{u}_\delta\\
&=(2\mu+\lambda)\lim_{\delta\to 0}\int\left(T_k(\rho_\delta)-\overline{T_k(\rho)}\right)\mathrm{div}\mathbf{u}_\delta\\
&=(2\mu+\lambda)\lim_{\delta\to 0}\int\left(T_k(\rho_\delta)\mathrm{div}\mathbf{u}_\delta-\overline{T_k(\rho)}\mathrm{div}\mathbf{u}\right).
\end{aligned}\tag{5.47}$$

In terms of (5.46) and (5.47), it holds that

$$\lim_{\delta\to 0}\|T_k(\rho_\delta)-T_k(\rho)\|_{L^{\gamma+1}}\leq C,\tag{5.48}$$

where the constant $C$ is independent of $k$ and $\delta$.

We remark that (5.48) measures oscillation of the density, which helps us prove that $(5.42)_1$ holds in the sense of renormalized solutions as in [37].

**Lemma 5.7.** [37] *For the solution $(\rho,\mathbf{u})$,*

$$\mathrm{div}(b(\rho)\mathbf{u})+(b'(\rho)\rho-b(\rho))\mathrm{div}\mathbf{u}=0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3),\tag{5.49}$$

*where $b(z)=z$, or $b\in C^1([0,\infty))$ with $b'(z)=0$ for large $z$.*

*Proof.* Thanks to Lemma 4.2, we see that $(\rho_\delta,\mathbf{u}_\delta)$ is a renormalized solution. If we multiply the equation (4.20) satisfied by $(\rho_\delta,\mathbf{u}_\delta)$ by $T_k(\rho_\delta)$, and use (5.43), (5.37)-(5.41), (5.48), we conclude (5.49) by taking $\delta\to 0$ and then $k\to\infty$. The detailed proof may be found in [37]. □

In order to complete the proof of Theorem 1.1 we only need to verify

$$\overline{\rho^\gamma}=\rho^\gamma.\tag{5.50}$$

To this end, it suffices to prove the strong convergence of $\rho_\delta$ in $L^1$ by (5.39). The idea is to compare the limit of the renormalized solution $(\rho_\delta,\mathbf{u}_\delta)$ with $(\rho,\mathbf{u})$. In more detail, we introduce

$$L_k=\begin{cases} z\ln z, & z\leq k;\\ z\ln k+z\displaystyle\int_k^z\frac{T_k(s)}{s^2}ds, & z\geq k.\end{cases}$$

A direct computation shows that

$$C([0,\infty))\cap C^1((0,\infty))\ni b_k(z)=L_k(z)-\left(\ln k+\int_k^{k+1}\frac{T_k(s)}{s^2}+1\right)z,$$

and moreover, $b'(z)=0$ if $z\geq k+1$ and $b_k'(z)z-b_k(z)=T_k(z)$. In view of Lemma 4.2 and Lemma 5.7, one has

$$\begin{aligned}
0&=\mathrm{div}(b_k(\rho)\mathbf{u})+T_k(\rho)\mathrm{div}\mathbf{u}\\
&=\mathrm{div}(L_k(\rho)\mathbf{u})+T_k(\rho)\mathrm{div}\mathbf{u}, \quad \text{in } \mathcal{D}(\mathbb{R}^3),
\end{aligned}$$

and

$$0 = \mathrm{div}(L_k(\rho_\delta)\mathbf{u}_\delta) + T_k(\rho_\delta)\mathrm{div}\mathbf{u}_\delta, \quad \text{in } \mathcal{D}(\mathbb{R}^3).$$

Integration of the difference of above two equations leads to

$$\int (T_k(\rho)\mathrm{div}\mathbf{u} - T_k(\rho_\delta)\mathrm{div}\mathbf{u}_\delta) = 0,$$

which along with (5.46) and the fact $\mathrm{div}\mathbf{u} \in L^2$ implies

$$\begin{aligned} C\|T_k(\rho) - \overline{T_k(\rho)}\|_{L^2} &\geq (2\mu+\lambda)\int \left(T_k(\rho) - \overline{T_k(\rho)}\right)\mathrm{div}\mathbf{u} \\ &= (2\mu+\lambda)\int \left(T_k(\rho_\delta)\mathrm{div}\mathbf{u}_\delta - \overline{T_k(\rho)}\mathrm{div}\mathbf{u}\right) \\ &\geq \int (T_k(\rho_\delta) - T_k(\rho))^{\gamma+1}. \end{aligned} \tag{5.51}$$

Recalling Proposition 5.1, we have $\|\rho_\delta\|_{L^\gamma} \leq C$. Thus, the limit

$$\begin{aligned} \lim_{k\to\infty} \|T_k(\rho_\delta) - \rho_\delta\|_{L^1} &= \|T_k(\rho_\delta) - \rho_\delta\|_{L^1(\{\rho_\delta \geq k\})} \\ &\leq 2\lim_{k\to\infty} \|\rho_\delta\|_{L^\gamma(\{\rho_\delta\geq k\})} \\ &\leq C\lim_{k\to\infty} k^{1-\gamma}\|\rho_\delta\|^\gamma_{L^\gamma(\{\rho_\delta\geq k\})} = 0 \end{aligned} \tag{5.52}$$

is uniform in $\delta$. In a similar way,

$$\lim_{k\to\infty} \|T_k(\rho) - \rho\|_{L^1} = 0. \tag{5.53}$$

Making use of (5.54)-(5.53), and

$$\begin{aligned} \|T_k(\rho) - T_k(\rho_\delta)\|_{L^1} &\leq C\|T_k(\rho) - \rho + \rho_\delta - \overline{T_k(\rho)}\|_{L^1} \\ &\leq C\left(\|T_k(\rho) - \rho\|_{L^1} + \lim_{\delta\to 0}\|T_k(\rho_\delta) - \rho_\delta\|_{L^1}\right), \end{aligned} \tag{5.54}$$

we conclude

$$\begin{aligned} &\lim_{\delta\to 0}\|\rho_\delta - \rho\|_{L^1} \\ &\leq \lim_{k\to\infty}\lim_{\delta\to 0}(\|\rho_\delta - T_k(\rho_\delta)\|_{L^1} + \|T_k(\rho_\delta) - T_k(\rho)\|_{L^1} + \|T_k(\rho) - \rho\|_{L^1}) \\ &= 0. \end{aligned}$$

The proof of Theorem 1.1 is completed.

## Acknowledgements

The research of Z. Liang was supported by the fundamental research funds for central universities (JBK 2202045). A. Majumdar (AM) acknowledges support from the University of Strathclyde New Professors Fund and a University of Strathclyde Global Engagement Grant. AM is also supported by a Leverhulme International Academic Fellowship. AM gratefully acknowledges hospitality from the University of Oxford and IIT Bombay, through her OCIAM Visiting Fellowship and Visiting Professorship at IIT Bombay. The research of D. Wang was partially supported by the National Science Foundation under grant DMS-1907519.

## References

[1] R. Adams, Sobolev spaces, New York: Academic Press, 1975.
[2] A. Ahmadi; M. C. Marchetti; T. B. Liverpool, Hydrodynamics of isotropic and liquid crystalline active polymer solutions, Phys. Rev. E 74 (2006), 061913.
[3] L. Antanovskii, A phase field model of capillarity, Phys. Fluids A 7 (1995), 747-753.
[4] D. Anderson; G. McFadden; A. Wheeler, Diffuse-interface methods in fluid mechanics. Annual review of fluid mechanics, Annu. Rev. Fluid Mech. Annual Reviews 30 (1998), 139-165.
[5] J. M. Ball; A. Majumdar, Nematic liquid crystals: from Maier-Saupe to a continuum theory, Molecular Crystals and Liquid Crystals, 525 (2010), 1-11.
[6] G. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press, Cambridge, 1999.
[7] A. N. Beris; B. J. Edwards, Thermodynamics of Flowing Systems with Internal Microstructure, Oxford University Press: New York, 1994.
[8] M. L. Blow; S. P. Thampi; J. M. Yeomans, Biphasic, lyotropic, active nematics, Phys. Rev. Lett. 113 (2014), 248303.
[9] G. Chen; A. Majumdar; D. Wang; R. Zhang, Global existence and regularity of solutions for the active liquid crystal, J. Diff. Equa., 50(4), (2017), 202-239.
[10] G. Chen; A. Majumdar; D. Wang; R. Zhang, Global weak solutions for the compressible active liquid crystal system, SIAM J. Math. Anal., 263, (2018), 3632-3675.
[11] R.-M. Chen; Z Liang; D. Wang; R. Xu, Energy equality in compressible fluids with physical boundaries, SIAM J. Math. Anal., 52 (2020), 1363-1385.
[12] H. Davis; L. Scriven, Stress and structure in fluid interfaces, Adv. Chem. Phys. 49 (1982), 357-454.
[13] W. Dreyer; J. Giesselmann; C. Kraus, A compressible mixture model with phase transition, Phys. D. 273-274 (2014), 1-13.
[14] D. Edwards; H. Brenner; D. Wasan, Interfacial Transport Process and Rheology, Butterworths/Heinemann, London, 1991.
[15] L. Evans, Partial differential equations. Second edition, Graduate Studies in Mathematics 19. American Mathematical Society, Providence, RI, 2010.
[16] E. Feireisl, Dynamics of viscous compressible fluids. Oxford University Press (2004).
[17] E. Feireisl; A. Novotný; H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations, J. Math. Fluid Mech. **3** (2001), 358-392.
[18] E. Feireisl, A. Novotný, Singular Limits in Thermodynamics of Viscous Fluids. Advances in Mathematical Fluid Mechanics, Birkhäuser, Basel, 2009.
[19] J. Frehse; M. Steinhauer; W. Weigant, The Dirichlet problem for steady viscous compressible flow in three dimensions, J. Math. Pures Appl. 97 (2012), 85-97.
[20] P. De Gennes; J. Prost, The physics of Liquid crystals, The Clarendon Press, Oxford University Press, New York, 1995.
[21] D. Gilbarg; N. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd edition, Grundlehren Math. Wiss., vol. 224, Springer-Verlag, Berlin, Heidelberg, New York, 1983.
[22] L. Giomi; L. Mahadevan; B. Chakraborty; M. Hagan, Banding, excitability and chaos in active nematic suspensions, Nonlinearity, 25 (2012), 2245.
[23] D. Henao; A. Majumdar; A. Pisante, Uniaxial versus biaxial character of nematic equilibria in three dimensions, Calc. Var. Partial Differential Equations 56 (2017), no. 2, Paper No. 55.
[24] S. Jiang; P. Zhang, Global spherically symmetry solutions of the compressible isentropic Navier-Stokes equations, Comm. Math. Phys. 215 (2001), 85-97.
[25] S. Jiang; C. Zhou, On the existence theory to the three-dimensional steady compressible Navier-Stokes equations, Ann. Inst. H. Poincareé Anal. Non Linéaire 28 (2011) 485-498.
[26] J. Kierfeld; K. Frentzel; P. Kraikivski; R. Lipowsky, Active dynamics of filaments in motility assays, Eur. Phys. J. Special Topics, 157 (2008), 123.
[27] W. Lian; R. Zhang, Global weak solutions to the active hydrodynamics of liquid crystals., J. Differential Equations 268(8) (2020), 4194-4221.
[28] Z. Liang; D. Wang, Stationary Cahn-hilliard-Navier-Stokes equations for the diffuse interface model of compressible flows, Math. Mod. Meth. Appl. Sci. 30 (2020), 2445-2486.
[29] Z. Liang; D. Wang, Stationary weak solutions for compressible Cahn-Hilliard/Navier-Stokes equations, J. Nonlinear Sci. (2022) 32:41.

[30] F. Lin; C. Wang, Recent developments of analysis for hydrodynamic flow of nematic liquid crystals, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 372 (2014), no. 2029, 20130361, 18 pp.

[31] P. Lions, Mathematical topics in fluid mechanics. Vol. 2. Compressible models. Oxford Lecture Series in Mathematics and its Applications, 10. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.

[32] T. B. Liverpool; M. C. Marchetti, Rheology of Active Filament Solutions, Phys. Rev. Lett. 97 (2006), 268101.

[33] M. C. Marchetti; J. F. Joanny; S. Ramaswamy, T. B. Liverpool; J. Prost; M. Rao; R. Aditi Simha, Hydrodynamics of soft active matter, Rev. Mod. Phys. 85 (2013), 1143.

[34] P. B. Mucha; M. Pokorný; E. Zatorska, Existence of stationary weak solutions for compressible heat conducting flows. Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, 2595-2662, Springer, Cham, 2018.

[35] W. F. Paxton; K. C. Kistler; C. C. Olmeda; A. Sen; S. K. St. Angelo; Y. Cao; T. E. Mallouk; P. E. Lammert; V. H. Crespi, Catalytic nanomotors: autonomous movement of striped nanorods, J. Amer. Chem. Soc. 126 (41) (2014), 13424–13431.

[36] Z. Qiu; Y. Wang, Martingale solution for stochastic active liquid crystal system, Discrete Contin. Dyn. Syst. 41 (2021), no. 5, 2227–2268.

[37] A. Novotný; I. Straškraba, Introduction to the Mathematical Theory of Compressible Flow. Oxford Lecture Series in Mathematics and its Applications, 27. Oxford University Press, Oxford, 2004.

[38] S. Novo; A. Novotný, On the existence of weak solutions to the steady compressible Navier-Stokes equations when the density is not square integrable, J. Math. Fluid Mech. 42 3 (2002), 531-550.

[39] P. Plotnikov; J. Sokolowski, Concentrations of solutions to time-discretized compressible Navier-Stokes equations, Comm. Math. Phys. 258(3) (2005), 567-608.

[40] P. Plotnikov; W. Weigant, Steady 3D viscous compressible flows with adiabatic exponent $\gamma \in (1, \infty)$, J. Math. Pures Appl. 104,(2015) 58-82.

[41] M. Ravnik; J. M. Yeomans, Confined active nematic flow in cylindrical capillaries, Phys. Rev. Lett. 110 (2013), 026001.

[42] D. Saintillan; M. Shelley, Instabilities and pattern formation in active particle suspensions: Kinetic theory and continuum simulations, Phys. Rev. Lett. 100 (2008), 178103.

[43] E. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, New Jersey, 1970.

[44] X. Xu, Recent analytic development of the dynamic Q-tensor theory for nematic liquid crystals, Electron. Res. Arch. 30 (2022), no. 6, 2220-2246.

[45] W. P. Ziemer, Weakly Differentiable Functions. Springer, New York, 1989.

School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China
*Email address*: zhilei0592@gmail.com

Department of Mathematics and Statistics, University of Strathclyde, G1 1XQ, UK.
*Email address*: apala.majumdar@strath.ac.uk

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA
*Email address*: dwang@math.pitt.edu

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA
*Email address*: yiw119@pitt.edu