Two-dimensional Ferronematics, Canonical Harmonic Maps and Minimal Connections

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Abstract

We study a variational model for ferronematics in two-dimensional domains, in the "super-dilute" regime. The free energy functional consists of a reduced Landau-de Gennes energy for the nematic order parameter, a Ginzburg-Landau type energy for the spontaneous magnetisation, and a coupling term that favours the co-alignment of the nematic director and the magnetisation. In a suitable asymptotic regime, we prove that the nematic order parameter converges to a canonical harmonic map with non-orientable point defects, while the magnetisation converges to a singular vector field, with line defects that connect the non-orientable point defects in pairs, along a minimal connection.

Keywords: Ginzburg-Landau functional, Modica-Mortola functional, canonical harmonic maps, non-orientable singularities, minimal connections.

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1 Introduction

Nematic liquid crystals (NLCs) are classical examples of mesophases or liquid crystalline phases that combine fluidity with the directionality of solids [24]. The nematic molecules are typically asymmetric in shape e.g. rod-shaped, wedge-shaped etc., and these molecules tend to align along certain locally preferred directions in space, referred to as **directors**. Consequently, NLCs have a direction-dependent response to external stimuli such as electric fields, magnetic fields, temperature and incident light. Notably, the directionality or anisotropy of NLC physical and

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mechanical responses make them the working material of choice for a range of electro-optic applications [35]. However, the magnetic susceptibility of NLCs is much weaker than their dielectric anisotropy, typically by several orders of magnitude [19]. Hence, NLCs exhibit a much stronger response to applied electric fields than their magnetic counterparts and as a result, NLC devices are mainly driven by electric fields. This naturally raises a question as to whether we can enhance the magneto-nematic coupling and induce a spontaneous magnetisation by the introduction of magnetic nanoparticles (nanoparticles with magnetic moments) in nematic media, even without external magnetic fields. If implemented successfully, these magneto-nematic systems would have a much stronger response to applied magnetic fields, compared to conventional nematic systems, rendering the possibility of magnetic-field driven NLC systems in the physical sciences and engineering.

This idea was first introduced in 1970 by Brochard and de Gennes in their pioneering work on ferronematics [19] and these composite systems of magnetic nanoparticle (MNP)-dispersed nematic media are referred to as ferronematics in the literature [20, 21, 19]. The system has two order parameters — the Landau-de Gennes (LdG) $\mathbf{Q}$-tensor order parameter to describe the nematic orientational anisotropy and the spontaneous magnetisation, $\mathbf{M}$, induced by the suspended MNPs. Brochard and de Gennes suggested that the nematic directors, denoted by $\mathbf{n}$, can be controlled by the surface-induced mechanical coupling between NLCs and MNPs. Equally, the spontaneous magnetisation, $\mathbf{M}$ profiles can be tailored by the nematic anisotropy through the MNP-NLC interactions, and this two-way coupling can stabilise exotic morphologies and defect patterns.

We work with dilute ferronematic suspensions relevant for a uniform suspension of MNPs in a nematic medium, such that the distance between pairs of MNPs is much larger than the individual MNP sizes and the volume fraction of the MNPs is small, building on the models introduced in [20, 21] and then in [15, 14]. In these dilute systems, the MNP-MNP interactions and the MNP-NLC interactions are absorbed by an empirical magneto-nematic coupling energy. These coupling energies can also be rigorously derived from homogenisation principles, as elucidated in the recent work [22]. We work with two-dimensional, simply-connected and smooth domains Ω, in a reduced LdG framework for which the $\mathbf{Q}$-tensor order parameter is a symmetric, traceless 2×2 matrix and $\mathbf{M}$ is a two-dimensional vector field. This reduced approach can be rigorously justified using Γ-Convergence techniques (see [31] since in three dimensions, the LdG $\mathbf{Q}$-tensor order parameter is a symmetric, traceless 3×3 matrix with five degrees of freedom). We use the effective re-scaled free energy for ferronematics, inspired by the experiments and results in [40] and proposed in [15, 14]. This energy has three components — a reduced LdG free energy for NLCs, a Ginzburg-Landau free energy for the magnetization and a homogenised magneto-nematic coupling term:

$$\mathscr{F}_\varepsilon(\mathbf{Q}, \mathbf{M}) := \int_\Omega \left(\frac{1}{2}|\nabla \mathbf{Q}|^2 + \frac{\xi}{2}|\nabla \mathbf{M}|^2 + \frac{1}{\varepsilon^2} f(\mathbf{Q}, \mathbf{M}) \right) \mathrm{d}x \tag{1.1}$$

In two dimensions, we have

$$f(\mathbf{Q},\, \mathbf{M}) := \frac{1}{4}(|\mathbf{Q}|^2 - 1)^2 + \frac{\xi}{4}(|\mathbf{M}|^2 - 1)^2 - c_0 \mathbf{Q}\mathbf{M} \cdot \mathbf{M}.$$

We work with a dimensionless model where ε^2 is interpreted as a material-dependent, geometry-dependent and temperature-dependent positive elastic constant, ξ is a ratio of the relative strength of the magnetic and NLC energies and c_0 is a coupling parameter. ξ is necessarily positive, positive c_0 coerces co-alignment of $\mathbf{n}$ and $\mathbf{M}$ whereas $c_0 < 0$ coerces $\mathbf{n}$ to be perpendicular to $\mathbf{M}$ [14]. We only consider positive c_0 in this paper.

For dilute suspensions, ε and ξ are necessarily small. In [14], the authors study stable critical points of this effective ferronematic free energy on square domains, with Dirichlet boundary conditions for both $\mathbf{Q}$ and $\mathbf{M}$. Their work is entirely numerical but does exhibit a plethora of exotic morphologies for different choices of ε, ξ and c_0. They demonstrate stable nematic point defects accompanied by both line defects and point defects in $\mathbf{M}$, and there is considerable freedom to manipulate the locations, multiplicity and dimensionality of defect profiles by simply tuning the values of ξ and c_0. In particular, the numerical results clearly show that line defects or jump sets are observed in stable $\mathbf{M}$-profiles for small ξ and c_0, whereas orientable point defects are stabilised in $\mathbf{M}$ for relatively large ξ and c_0. Motivated by these numerical results, we study a special limit of the effective free energy in (1.1), for which both ξ and c_0 are proportional to ε and we study the profile of the corresponding energy minimizers in the $\varepsilon \to 0$ limit, subject to Dirichlet boundary conditions for $\mathbf{Q}$ and $\mathbf{M}$. This can be interpreted as a "super-dilute" limit of the ferronematic free energy for which the magnetic energy is substantially weaker than the NLC energy, and the magneto-nematic coupling is weak. In the "super-dilute" limit, "ε" is the only model parameter and ξ, c_0 are defined by the constants of proportionality which are fixed, and hence $\varepsilon \to 0$ is the relevant asymptotic limit. Our main result shows that in this distinguished limit, the minimizing $\mathbf{Q}$-profiles are essentially canonical harmonic maps with a set of non-orientable nematic point defects, dictated by the topological degree of the Dirichlet boundary datum. This is consistent with previous powerful work in [9] in the context of the LdG theory and unsurprising since the LdG energy is the dominant energy. The minimizing $\mathbf{M}$-profiles are governed by a Modica-Mortola type of problem, quite specific to this super-dilute limit [29]. They exhibit short line defects connecting pairs of the non-orientable nematic defects, consistent with the numerical results in [14]. These line defects or jump sets in $\mathbf{M}$ are minimal connections between the nematic defects, and the location of the defects is determined by a modified renormalisation energy, which is the sum of a Ginzburg-Landau type renormalisation energy and a minimal connection energy. The modified renormalisation energy delicately captures the coupled nature of our problem, which makes it distinct and technically more complex than the usual LdG counterpart.

We complement our theoretical results with some numerical results for stable critical points of the ferronematic free energy, on square domains with topologically non-trivial Dirichlet boundary conditions for $\mathbf{Q}$ and $\mathbf{M}$. The converged numerical solutions are locally stable, and we expect multiple stable critical points for given choices of ε, ξ and c_0.The numerical results are sensitive to the choices of ε and c_0, but there is evidence that the numerically computed stable solutions do indeed converge to a canonical harmonic $\mathbf{Q}$-map and a $\mathbf{M}$-profile closely tailored by the corresponding $\mathbf{Q}$-profile. The $\mathbf{Q}$-profile has a discrete set of non-orientable nematic defects and the $\mathbf{M}$-profile exhibits line defects connecting these nematic defects, in the $\varepsilon \to 0$ limit. Whilst the practical relevance of such studies remains uncertain, it is clear that strong theoretical underpinnings are much needed for systematic scientific progress in this field, and our work is a first powerful step in an exhaustive study of ferronematic solution landscapes [46] (also see recent work in [23], [39]).

The paper is organised as follows. In Section 2, we set up our problem and state our main result, recalling the key notions of a canonical harmonic map and a minimal connection. In Section 3, we state and prove some key technical preliminary results. In Section 4, we prove the six parts of our main theorem, including convergence results for the energy-minimizing $\mathbf{Q}$ and $\mathbf{M}$-profiles in different function spaces, and the convergence of the jump set of the energy-minimizing $\mathbf{M}$ to a minimal connection between pairs of non-orientable nematic defects, in the $\varepsilon \to 0$ limit. The defect locations are captured in terms of minimizers of a modified renormalized energy, which is the sum of the Ginzburg-Landau renormalized energy and a minimal connection energy. The modified renormalized energy is derived from sharp lower and upper bounds for the energy minimizers in the $\varepsilon \to 0$ limit, in Sections 4.4.1 and 4.4.2. In Section 5, we present some numerical results and conclude with some perspectives in Section 6.

2 Statement of the main result

Let $\mathscr{S}_0^{2\times 2}$ be the set of 2×2, real, symmetric, trace-free matrices, equipped with the scalar product $\mathbf{Q}\cdot\mathbf{P} := \mathrm{tr}(\mathbf{QP}) = Q_{ij}P_{ij}$ and the induced norm $|\mathbf{Q}|^2 := \mathrm{tr}(\mathbf{Q}^2) = Q_{ij}Q_{ij}$. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded, Lipschitz, simply connected domain. The "super-dilute" limit of the ferronematic free energy is defined by

$$\xi = \varepsilon; \quad c_0 = \beta\varepsilon$$

where β, ε are *positive* parameters. For $\mathbf{Q}\colon \Omega \to \mathscr{S}_0^{2\times 2}$ and $\mathbf{M}\colon \Omega \to \mathbb{R}^2$, we define the functional

$$\mathscr{F}_\varepsilon(\mathbf{Q}, \mathbf{M}) := \int_\Omega \left(\frac{1}{2}\left|\nabla\mathbf{Q}\right|^2 + \frac{\varepsilon}{2}\left|\nabla\mathbf{M}\right|^2 + \frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}, \mathbf{M})\right) \mathrm{d}x, \tag{2.1}$$

where the potential f_ε is given by

$$f_\varepsilon(\mathbf{Q}, \mathbf{M}) := \frac{1}{4}(|\mathbf{Q}|^2 - 1)^2 + \frac{\varepsilon}{4}(|\mathbf{M}|^2 - 1)^2 - \beta\varepsilon\mathbf{QM}\cdot\mathbf{M} + \kappa_\varepsilon \tag{2.2}$$

and $\kappa_\varepsilon \in \mathbb{R}$ is a constant, uniquely determined by imposing that $\inf f_\varepsilon = 0$.

We consider minimisers of (2.1) subject to the Dirichlet boundary condition

$$\mathbf{Q} = \mathbf{Q}_{\mathrm{bd}}, \quad \mathbf{M} = \mathbf{M}_{\mathrm{bd}} \qquad \text{on } \partial\Omega, \tag{2.3}$$

We assume that $\mathbf{Q}_{\mathrm{bd}} \in C^1(\partial\Omega, \mathscr{S}_0^{2\times 2})$, $\mathbf{M}_{\mathrm{bd}} \in C^1(\partial\Omega, \mathbb{R}^2)$ are (ε-independent) maps such that

$$|\mathbf{M}_{\mathrm{bd}}| = (\sqrt{2}\beta + 1)^{1/2}, \qquad \mathbf{Q}_{\mathrm{bd}} = \sqrt{2}\left(\frac{\mathbf{M}_{\mathrm{bd}}\otimes\mathbf{M}_{\mathrm{bd}}}{\sqrt{2}\beta + 1} - \frac{\mathbf{I}}{2}\right) \tag{2.4}$$

at any point of $\partial\Omega$. Here $\mathbf{I}$ is the 2×2 identity matrix. Throughout this paper, we will denote by $(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*)$ a minimiser of the functional (2.1) subject to the boundary conditions (2.3). By routine arguments, minimisers exist and they satisfy the Euler-Lagrange system of equations

$$-\Delta\mathbf{Q}^*_\varepsilon + \frac{1}{\varepsilon^2}(|\mathbf{Q}^*_\varepsilon|^2 - 1)\mathbf{Q}^*_\varepsilon - \frac{\beta}{\varepsilon}\left(\mathbf{M}^*_\varepsilon\otimes\mathbf{M}^*_\varepsilon - \frac{|\mathbf{M}^*_\varepsilon|^2}{2}\mathbf{I}\right) = 0 \tag{2.5}$$

$$-\Delta\mathbf{M}^*_\varepsilon + \frac{1}{\varepsilon^2}(|\mathbf{M}^*_\varepsilon|^2 - 1)\mathbf{M}^*_\varepsilon - \frac{2\beta}{\varepsilon^2}\mathbf{Q}^*_\varepsilon\mathbf{M}^*_\varepsilon = 0. \tag{2.6}$$

We denote as $\mathscr{N}$ the unit circle in the space of $\mathbf{Q}$-tensors, that is,

$$\mathscr{N} := \left\{\mathbf{Q}\in\mathscr{S}_0^{2\times 2}\colon\ |\mathbf{Q}| = 1\right\} \tag{2.7}$$

Equivalently, $\mathscr{N}$ may be described as

$$\mathscr{N} = \left\{\sqrt{2}\left(\mathbf{n}\otimes\mathbf{n} - \frac{\mathbf{I}}{2}\right) : \mathbf{n}\in\mathbb{S}^1\right\} \tag{2.8}$$

As $\mathscr{S}_0^{2\times 2}$ is a real vector space of dimension 2, the set $\mathscr{N}$ is a smooth manifold, diffeomorphic to the unit circle $\mathbb{S}^1\subseteq\mathbb{C}$. A diffeomorphism is given explicitely by

$$\mathscr{N}\to\mathbb{S}^1, \qquad \mathbf{Q}\mapsto\mathbf{q} := \sqrt{2}(Q_{11},\, Q_{12}) \tag{2.9}$$

By assumption, the domain $\Omega\subseteq\mathbb{R}^2$ is bounded and convex, so its boundary $\partial\Omega$ is parametrised by a simple, closed, Lipschitz curve — in particular, $\partial\Omega$ is homeomorphic to the circle $\mathbb{S}^1$. Therefore, the boundary data $(\mathbf{Q}_{\mathrm{bd}},\, \mathbf{M}_{\mathrm{bd}})$ carries a well-defined topological degree

$$d := \deg(\mathbf{Q}_{\mathrm{bd}},\, \partial\Omega) = \deg(\mathbf{M}_{\mathrm{bd}},\, \partial\Omega)\in\mathbb{Z} \tag{2.10}$$

In principle, for a continuous map $\mathbf{Q}\colon\partial\Omega\to\mathscr{N}$, the degree may be a half-integer, that is $\deg(\mathbf{Q},\,\partial\Omega)\in\frac{1}{2}\mathbb{Z}$. However, the boundary datum $\mathbf{Q}_{\mathrm{bd}}$ is orientable, by assumption (2.4) — in fact, it is oriented by $\mathbf{M}_{\mathrm{bd}}$. This explains why d, in our case, is an integer.

The canonical harmonic map and the renormalised energy. In order to state our main result, we recall some terminology introduced by Bethuel, Brezis and Hélein [11]. Although the results in [11] are stated in terms of complex-valued maps, as opposed to $\mathbf{Q}$-tensors, they do extend to our setting, due to the change of variable (2.9). Let $a_1,\,\ldots,\,a_{2|d|}$ be distinct points in Ω (with d given by (2.10)). We say that a map $\mathbf{Q}^*\colon\Omega\to\mathscr{N}$ is a *canonical harmonic map* with singularities at $(a_1,\,\ldots,\,a_{2|d|})$ and boundary datum $\mathbf{Q}_{\mathrm{bd}}$ if the following conditions hold:

(i) $\mathbf{Q}^*$ is smooth in $\Omega\setminus\{a_1,\,\ldots,\,a_{2|d|}\}$, continuous in $\overline{\Omega}\setminus\{a_1,\,\ldots,\,a_{2|d|}\}$ and $\mathbf{Q}^* = \mathbf{Q}_{\mathrm{bd}}$ on $\partial\Omega$;

(ii) for any $\sigma > 0$ small enough and any $j\in\{1,\,\ldots,\,2\,|d|\}$, we have

$$\deg(\mathbf{Q}^*,\,\partial B_\sigma(a_j)) = \frac{\mathrm{sign}(d)}{2}$$

(iii) $\mathbf{Q}^*\in W^{1,1}(\Omega,\,\mathscr{N})$ and

$$\partial_j\left(Q^*_{11}\,\partial_j Q^*_{12} - Q^*_{12}\,\partial_j Q^*_{11}\right) = 0$$

in the sense of distributions in Ω. (Here and in what follows, we adopt Einstein's notation for the sum).

If $B \subseteq \Omega \setminus \{a_1, \ldots, a_{2|d|}\}$ is a ball that does not contain any singular point of $\mathbf{Q}^*$, then $\mathbf{Q}^*$ can written in the form

$$\mathbf{Q}^* = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta^* & \sin\theta^* \\ \sin\theta^* & -\cos\theta^* \end{pmatrix} \qquad \text{in } B, \tag{2.11}$$

where $\theta^* \colon B \to \mathbb{R}$ is a smooth function. (Equation (2.11) follows from (2.7), by classical lifting results in topology.) Then, the equation (iii) above can be written in the form

$$-\Delta\theta^* = 0 \qquad \text{in } B. \tag{2.12}$$

In other words, a canonical harmonic map can be written locally, away from its singularities, in terms of a harmonic function.

The canonical harmonic map with singularities at $(a_1, \ldots, a_{2|d|})$ and boundary datum $\mathbf{Q}_{\mathrm{bd}}$ exists and is unique, see [11, Theorem I.5, Remark I.1]. The canonical harmonic map satisfies $\mathbf{Q}^* \in W^{1,p}(\Omega, \mathscr{N})$ for any $p \in [1, 2)$, but $\mathbf{Q}^* \notin W^{1,2}(\Omega, \mathscr{N})$. Nevertheless, the limit

$$\mathbb{W}(a_1, \ldots, a_{2|d|}) := \lim_{\sigma \to 0} \left(\frac{1}{2} \int_{\Omega \setminus \bigcup_{j=1}^{2|d|} B_\sigma(a_j)} |\nabla \mathbf{Q}^*|^2 \, \mathrm{d}x - 2\pi \, |d| \, |\log \sigma| \right) \tag{2.13}$$

exists and is finite (see [11, Theorem I.8]). Following the terminology in [11], the function $\mathbb{W}$ is called the *renormalised energy.*

Minimal connections between singular points. Given distinct points $a_1, a_2, \ldots, a_{2|d|}$ in $\mathbb{R}^2$, we define a *connection* for $\{a_1, \ldots, a_{2|d|}\}$ as a finite collection of straight line segments $\{L_1, \ldots, L_{|d|}\}$ such that each a_j is an endpoint of exactly one of the segments L_k. In other words, the line segments L_j connects the points a_i in pairs. We define

$$\mathbb{L}(a_1, \ldots, a_{2|d|}) := \min \left\{ \sum_{j=1}^{|d|} \mathscr{H}^1(L_j) \colon \{L_1, \ldots, L_{|d|}\} \text{ is a connection for } \{a_1, \ldots, a_{2|d|}\} \right\} \tag{2.14}$$

Here and throughout the paper, $\mathscr{H}^1$ denotes the 1-dimensional Hausdorff measure (i.e., length). We say that a connection $\{L_1, \ldots, L_d\}$ is minimal if it is a minimiser for the right-hand side of (2.14). A notion of minimal connection, similar to (2.14), was already introduced in [17, 2]. However, the minimal connection was defined in [17] by taking the orientation into account — that is, half of the points $a_1, \ldots, a_{2|d|}$ were given positive multiplicity 1, the other half were given negative multiplicity -1, and the segments L_j were required to match points with opposite multiplicity. By constrast, here we do not distinguish between positive and negative multiplicity for the points a_i and any segment of endpoints a_i, a_k is allowed. (In the language of Geometric Measure Theory, the minimal connection was defined in [17] as the solution of a 1-dimensional Plateau problem with integer multiplicity, while (2.14) is a 1-dimensional Plateau problem modulo 2.)

The main result. We prove a convergence result for minimisers $(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*)$ of (2.1), subject to the boundary conditions (2.3)–(2.4), in the limit as $\varepsilon \to 0$. We denote by $\mathrm{SBV}(\Omega, \mathbb{R}^2)$ the space of maps $\mathbf{M} = (M_1, M_2)\colon \Omega \to \mathbb{R}^2$ whose components M_1, M_2 are special functions of bounded variation, as defined by De Giorgi and Ambrosio [25]. The distributional derivative $\mathrm{D}\mathbf{M}$ of a map $\mathbf{M} \in \mathrm{SBV}(\Omega, \mathbb{R}^2)$ can be decomposed as

$$\mathrm{D}\mathbf{M} = \nabla\mathbf{M}\,\mathscr{L}^2(\mathrm{d}x) + (\mathbf{M}^+ - \mathbf{M}^-) \otimes \nu_{\mathbf{M}}\,(\mathscr{H}^1 \llcorner \mathrm{S}_{\mathbf{M}})$$

where $\nabla\mathbf{M}\colon \Omega \to \mathbb{R}^{2\times 2}$ is the absolutely continuous component of $\mathrm{D}\mathbf{M}$, $\mathscr{L}^2(\mathrm{d}x)$ is the Lebesgue measure on $\mathbb{R}^2$, $\mathrm{S}_{\mathbf{M}}$ is the jump set of $\mathbf{M}$, $\mathbf{M}^+$, $\mathbf{M}^-$ are the traces of $\mathbf{M}$ on either side of the jump and $\nu_{\mathbf{M}}$ is the unit normal to the jump set. (See e.g. [3] for more details).

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded, Lipschitz, simply connected domain. Assume that the boundary data satisfy* (2.4)*. Let $(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*)$ be a minimiser of* (2.1) *subject to the boundary condition* (2.3)*. Then, there exists a (non-relabelled) subsequence, maps $\mathbf{Q}_*\colon \Omega \to \mathscr{N}$, $\mathbf{M}^*\colon \Omega \to \mathbb{R}^2$ and distinct points $a_1^*, \ldots, a_{2|d|}^*$ in Ω such that the following holds:*

(i) $\mathbf{Q}_\varepsilon^* \to \mathbf{Q}^*$ *strongly in $W^{1,p}(\Omega)$ for any $p < 2$;*

(ii) $\mathbf{Q}^*$ *is the canonical harmonic map with singularities $(a_1^*, \ldots, a_{2|d|}^*)$ and boundary datum $\mathbf{Q}_{\mathrm{bd}}$;*

(iii) $\mathbf{M}_\varepsilon^* \to \mathbf{M}^*$ *strongly in $L^p(\Omega)$ for any $p < +\infty$;*

(iv) $\mathbf{M}^* \in \mathrm{SBV}(\Omega, \mathbb{R}^2)$ *and it satisfies*

$$|\mathbf{M}^*| = (\sqrt{2}\beta + 1)^{1/2}, \qquad \mathbf{Q}^* = \sqrt{2}\left(\frac{\mathbf{M}^* \otimes \mathbf{M}^*}{\sqrt{2}\beta + 1} - \frac{\mathbf{I}}{2}\right)$$

at almost every point of Ω.

In addition, if the domain Ω is convex, then

(v) there exists a minimal connection $(L_1^*, \ldots, L_{|d|}^*)$ for $(a_1^*, \ldots, a_{2|d|}^*)$ such that the jump set of $\mathbf{M}^*$ coincides with $\bigcup_{j=1}^{|d|} L_j^*$ (up to sets of zero length);

(vi) $(a_1^*, \ldots, a_{2|d|}^*)$ *minimises the function*

$$\mathbb{W}_\beta(a_1, \ldots, a_{2|d|}) := \mathbb{W}(a_1, \ldots, a_{2|d|}) + \frac{2\sqrt{2}}{3}\left(\sqrt{2}\beta + 1\right)^{3/2} \mathbb{L}(a_1, \ldots, a_{2|d|})$$

among all the $(2\,|d|)$-uples $(a_1, \ldots, a_{2|d|})$ of distinct points in Ω.

Remark 2.1. Point defects and line defects connecting point defects do appear for energy minimizers in other variational models e.g. continuum models for a complex-valued map in [30] or for discrete models in [6, 5]. However, the mathematics is substantially different to our model problem for which we have two order parameters $\mathbf{Q}$ and $\mathbf{M}$, and a non-trivial coupling energy, which introduces substantive technical challenges.

3 Preliminaries

First, we state a few properties of the potential f_ε, defined in (2.2). We define

$$\kappa_* := \frac{\beta}{2\sqrt{2}}\left(\sqrt{2}\beta + 1\right). \tag{3.1}$$

Lemma 3.1. *The potential f_ε satisfies the following properties.*

(i) The constant κ_ε in (2.2), uniquely defined by imposing the condition $\inf f_\varepsilon = 0$*, satisfies*

$$\kappa_\varepsilon = \frac{1}{2}\left(\beta^2 + \sqrt{2}\beta\right)\varepsilon + \kappa_*^2\,\varepsilon^2 + \mathrm{o}(\varepsilon^2)$$

In particular, $\kappa_\varepsilon \geq 0$ for ε small enough.

(ii) If $(\mathbf{Q},\, \mathbf{M}) \in \mathscr{S}_0^{2\times 2} \times \mathbb{R}^2$ is such that

$$|\mathbf{M}| = (\sqrt{2}\beta + 1)^{1/2}, \qquad \mathbf{Q} = \sqrt{2}\left(\frac{\mathbf{M}\otimes\mathbf{M}}{\sqrt{2}\beta+1} - \frac{\mathbf{I}}{2}\right)$$

then $f_\varepsilon(\mathbf{Q},\, \mathbf{M}) = \kappa_*\,\varepsilon^2 + \mathrm{o}(\varepsilon^2)$.

(iii) If ε is sufficiently small, then

$$\frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q},\, \mathbf{M}) \geq \frac{1}{4\varepsilon^2}(|\mathbf{Q}|^2 - 1)^2 - \frac{\beta}{\sqrt{2}\varepsilon}\,|\mathbf{M}|^2\,\big||\mathbf{Q}| - 1\big|$$

and

$$\frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q},\, \mathbf{M}) \geq \frac{1}{8\varepsilon^2}(|\mathbf{Q}|^2 - 1)^2 - \beta^2\,|\mathbf{M}|^4 \tag{3.2}$$

for any $(\mathbf{Q},\, \mathbf{M}) \in \mathscr{S}_0^{2\times 2} \times \mathbb{R}^2$.

The proof of Lemma 3.1 is contained in Appendix B.

In the rest of this section, we describe an alternative expression for the functional (2.1), which will be useful in our analysis. Let $G \subseteq \Omega$ be a smooth, simply connected subdomain. Let $(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)_{\varepsilon>0}$ be any sequence in $W^{1,2}(G,\, \mathscr{S}_0^{2\times 2}) \times W^{1,2}(G,\, \mathbb{R}^2)$ (not necessarily a sequence of minimisers) that satisfies

$$\int_G \left(\frac{1}{2}\,|\nabla\mathbf{Q}_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(|\mathbf{Q}_\varepsilon|^2 - 1)^2\right)\mathrm{d}x \lesssim |\log\varepsilon| \tag{3.3}$$

$$|\mathbf{Q}_\varepsilon(x)| \geq \frac{1}{2}, \quad |\mathbf{M}_\varepsilon(x)| \leq C \qquad \text{for any } x \in G,\ \varepsilon > 0, \tag{3.4}$$

where C is some positive constant that does not depend on ε. As we have assumed that G is simply connected and that $|\mathbf{Q}_\varepsilon| \geq 1/2$ in G, we can apply lifting results [13, 12, 8] and write $\mathbf{Q}_\varepsilon$ in the form

$$\mathbf{Q}_\varepsilon = \frac{|\mathbf{Q}_\varepsilon|}{\sqrt{2}}\left(\mathbf{n}_\varepsilon\otimes\mathbf{n}_\varepsilon - \mathbf{m}_\varepsilon\otimes\mathbf{m}_\varepsilon\right) \qquad \text{in } G. \tag{3.5}$$

Here $(\mathbf{n}_\varepsilon, \mathbf{m}_\varepsilon)$ is an orthonormal set of eigenvectors for $\mathbf{Q}_\varepsilon$ with $\mathbf{n}_\varepsilon \in W^{1,2}(G, \mathbb{S}^1)$, $\mathbf{m}_\varepsilon \in W^{1,2}(G, \mathbb{S}^1)$. We define the vector field $\mathbf{u}_\varepsilon \in W^{1,2}(G, \mathbb{R}^2)$ as

$$(u_\varepsilon)_1 := \mathbf{M}_\varepsilon \cdot \mathbf{n}_\varepsilon, \qquad (u_\varepsilon)_2 := \mathbf{M}_\varepsilon \cdot \mathbf{m}_\varepsilon \tag{3.6}$$

so that $\mathbf{M}_\varepsilon = (u_\varepsilon)_1 \, \mathbf{n}_\varepsilon + (u_\varepsilon)_2 \, \mathbf{m}_\varepsilon$. Our next result expresses the energy $\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon; G)$ in terms of the variables $\mathbf{Q}_\varepsilon$ and $\mathbf{u}_\varepsilon$. We define the functions

$$g_\varepsilon(\mathbf{Q}) := \frac{1}{4\varepsilon^2}(|\mathbf{Q}|^2 - 1)^2 - \frac{2\kappa_*}{\varepsilon}(|\mathbf{Q}| - 1) + \kappa_*^2 \tag{3.7}$$

$$h(\mathbf{u}) := \frac{1}{4}(|\mathbf{u}|^2 - 1)^2 - \frac{\beta}{\sqrt{2}}(u_1^2 - u_2^2) + \frac{\beta^2 + \sqrt{2}\beta}{2} \tag{3.8}$$

for any $\mathbf{Q} \in \mathscr{S}_0^{2\times 2}$ and any $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$. We recall that κ_* is the constant defined by (3.1).

Remark 3.1. The vector fields $\mathbf{n}_\varepsilon$, $\mathbf{m}_\varepsilon$ are determined by $\mathbf{Q}_\varepsilon$ only up to their sign — Equation (3.5) still holds if we replace $\mathbf{n}_\varepsilon$ by $-\mathbf{n}_\varepsilon$ or $\mathbf{m}_\varepsilon$ by $-\mathbf{m}_\varepsilon$. Therefore, the unit vector $\mathbf{u}_\varepsilon$ is uniquely determined by $\mathbf{Q}_\varepsilon$, $\mathbf{M}_\varepsilon$ only *up to the sign of its components* $(u_\varepsilon)_1$, $(u_\varepsilon)_2$. However, the quantity $h(\mathbf{u}_\varepsilon)$ is is well-defined, irrespective of the choice of the orientations for $\mathbf{n}_\varepsilon$, $\mathbf{m}_\varepsilon$, because $h(-u_1, u_2) = h(u_1, -u_2) = h(u_1, u_2)$.

Proposition 3.2. *Let $(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)_{\varepsilon>0}$ be a sequence in $W^{1,2}(G, \mathscr{S}_0^{2\times 2}) \times W^{1,2}(G, \mathbb{R}^2)$ that satisfies* (3.3) *and* (3.4)*. Let $\mathbf{u}_\varepsilon$ be defined as in* (3.6)*. Then, we have*

$$\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon; G) = \int_G \left(\frac{1}{2}\left|\nabla \mathbf{Q}_\varepsilon\right|^2 + g_\varepsilon(\mathbf{Q}_\varepsilon)\right) \mathrm{d}x + \int_G \left(\frac{\varepsilon}{2}\left|\nabla \mathbf{u}_\varepsilon\right|^2 + \frac{1}{\varepsilon} h(\mathbf{u}_\varepsilon)\right) \mathrm{d}x + R_\varepsilon$$

where the remainder term R_ε satisfies

$$|R_\varepsilon| \lesssim \varepsilon^{1/2} \left|\log \varepsilon\right|^{1/2} \left(\int_G \left(\frac{\varepsilon}{2}\left|\nabla \mathbf{u}_\varepsilon\right|^2 + \frac{1}{\varepsilon} h(\mathbf{u}_\varepsilon)\right) \mathrm{d}x\right)^{1/2} + \mathrm{o}(1) \tag{3.9}$$

as $\varepsilon \to 0$.

In other words, the change of variables (3.6) transforms the functional into a sum of two decoupled terms, which can be studied independently, and a remainder term, which is small compared to the other ones. Before we proceed with the proof of Proposition 3.2, we state some properties of the functions g_ε, h defined in (3.7), (3.8) respectively. These properties are elementary, but will be useful later on.

Lemma 3.3. *The function $g_\varepsilon \colon \mathscr{S}_0^{2\times 2} \to \mathbb{R}$ is non-negative and satisfies*

$$g_\varepsilon(\mathbf{Q}) = \left(\frac{1}{\varepsilon}(|\mathbf{Q}| - 1) - \kappa_*\right)^2 + \frac{1}{\varepsilon^2}(|\mathbf{Q}| - 1)^2 \left(\frac{1}{4}(|\mathbf{Q}| + 1)^2 - 1\right)$$

for any $\mathbf{Q} \in \mathscr{S}_0^{2\times 2}$.

Proof. We have

$$\begin{aligned} g_\varepsilon(\mathbf{Q}) &= \frac{1}{\varepsilon^2}(|\mathbf{Q}|-1)^2 - \frac{2\kappa_*}{\varepsilon}(|\mathbf{Q}|-1) + \kappa_*^2 + \frac{1}{4\varepsilon^2}(|\mathbf{Q}|^2-1)^2 - \frac{1}{\varepsilon}(|\mathbf{Q}|-1)^2 \\ &= \left(\frac{1}{\varepsilon}(|\mathbf{Q}|-1) - \kappa_*\right)^2 + \frac{1}{\varepsilon^2}(|\mathbf{Q}|-1)^2\left(\frac{1}{4}(|\mathbf{Q}|+1)^2 - 1\right) \end{aligned}$$

If $|\mathbf{Q}| \geq 1$, then $(|\mathbf{Q}|+1)^2 \geq 4$ and hence, $g_\varepsilon(\mathbf{Q}) \geq 0$. On the other hand, if $|\mathbf{Q}| \leq 1$, then all the terms in (3.7) are non-negative. $\square$

Lemma 3.4. *The function $h\colon \mathbb{R}^2 \to \mathbb{R}$ is non-negative and its zero-set $h^{-1}(0)$ consists exactly of two points, $\mathbf{u}_\pm := (\pm(\sqrt{2}\beta+1)^{1/2},\, 0)$. Moreover, the Hessian matrix of h at both $\mathbf{u}_+$ and $\mathbf{u}_-$ is strictly positive definite.*

Proof. For any $\mathbf{u} \in \mathbb{R}^2$, we have $h(\mathbf{u}) \geq h(|\mathbf{u}|,\, 0)$ and the inequality is strict if $u_2 \neq 0$. Therefore, it suffices to study h on the line $u_2 = 0$. We have

$$h(u_1,\, 0) = \frac{1}{4}\left(u_1^2 - 1 - \sqrt{2}\beta\right)^2$$

so $h(\mathbf{u}) \geq 0$ for any $\mathbf{u} \in \mathbb{R}^2$, with equality if and only if $\mathbf{u} = (\pm(\sqrt{2}\beta+1)^{1/2},\, 0)$. Moreover,

$$\nabla^2 h(\mathbf{u}_+) = \nabla^2 h(\mathbf{u}_-) = \begin{pmatrix} 2+2\sqrt{2}\beta & 0 \\ 0 & 2\sqrt{2}\beta \end{pmatrix}$$

so the lemma follows. $\square$

Proof of Proposition 3.2. For simplicity of notation, we omit the subscript ε from all the variables.

Step 1. Let $k \in \{1,\, 2\}$. We have $\mathbf{M} = u_1\mathbf{n} + u_2\mathbf{m}$ and hence,

$$\partial_k\mathbf{M} = (\partial_k u_1)\mathbf{n} + u_1\partial_k\mathbf{n} + (\partial_k u_2)\mathbf{m} + u_2\partial_k\mathbf{m}. \tag{3.10}$$

We raise to the square both sides of (3.10). We apply the identities

$$\mathbf{n}\cdot\partial_k\mathbf{n} = \mathbf{m}\cdot\partial_k\mathbf{m} = 0, \qquad \mathbf{n}\cdot\partial_k\mathbf{m} + \mathbf{m}\cdot\partial_k\mathbf{n} = 0, \qquad \partial_k\mathbf{n}\cdot\partial_k\mathbf{m} = 0 \tag{3.11}$$

which follow by differentiating the orthonormality conditions $|\mathbf{n}|^2 = |\mathbf{m}|^2 = 1$, $\mathbf{n}\cdot\mathbf{m} = 0$. (In particular, the first identity in (3.10) implies that $\partial_k\mathbf{n}$ is parallel to $\mathbf{m}$ and $\partial_k\mathbf{m}$ is parallel to $\mathbf{n}$, so $\partial_k\mathbf{n}\cdot\partial_k\mathbf{m} = 0$.) We obtain

$$|\partial_k\mathbf{M}|^2 = |\partial_k\mathbf{u}|^2 + 2(u_1\,\partial_k u_2 - u_2\,\partial_k u_1)\mathbf{m}\cdot\partial_k\mathbf{n} + |\mathbf{u}|^2\,|\partial_k\mathbf{n}|^2 \tag{3.12}$$

We consider the potential term $f_\varepsilon(\mathbf{Q},\, \mathbf{M})$. Since $(\mathbf{n},\, \mathbf{m})$ is an orthonormal basis of $\mathbb{R}^2$, we have

$$|\mathbf{u}| = |\mathbf{M}|\,, \qquad \frac{\mathbf{Q}}{|\mathbf{Q}|}\mathbf{M}\cdot\mathbf{M} = \frac{1}{\sqrt{2}}\left(u_1^2 - u_2^2\right) \tag{3.13}$$

By substituting (3.13) into the definition (2.2) of f_ε, and recalling (3.7), (3.8), we obtain

$$
\begin{aligned}
\frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q},\, \mathbf{M}) &= \frac{1}{4\varepsilon^2}(|\mathbf{Q}|^2 - 1)^2 + \frac{1}{\varepsilon} h(\mathbf{u}) + \frac{\beta}{\sqrt{2}\,\varepsilon}(1 - |\mathbf{Q}|)\,(u_1^2 - u_2^2) \\
&\qquad + \frac{\kappa_\varepsilon}{\varepsilon^2} - \frac{1}{2\varepsilon}(\beta^2 + \sqrt{2}\beta) \\
&= g_\varepsilon(\mathbf{Q}) + \frac{1}{\varepsilon} h(\mathbf{u}) + \frac{|\mathbf{Q}| - 1}{\varepsilon}\left(2\kappa_* - \frac{\beta}{\sqrt{2}}(u_1^2 - u_2^2)\right) \\
&\qquad + \frac{\kappa_\varepsilon}{\varepsilon^2} - \frac{1}{2\varepsilon}(\beta^2 + \sqrt{2}\beta) - \kappa_*^2
\end{aligned}
\tag{3.14}
$$

Combining (3.12) with (3.14), we obtain

$$
\begin{aligned}
\mathscr{F}_\varepsilon(\mathbf{Q},\, \mathbf{M};\, G) = &\int_G \left(\frac{1}{2}\left|\nabla \mathbf{Q}\right|^2 + g_\varepsilon(\mathbf{Q})\right) \mathrm{d}x + \int_G \left(\frac{\varepsilon}{2}\left|\nabla \mathbf{u}\right|^2 + \frac{1}{\varepsilon} h(\mathbf{u})\right) \mathrm{d}x \\
&+ \varepsilon \sum_{k=1}^{2} \int_G (u_1\, \partial_k u_2 - u_2\, \partial_k u_1)\, \mathbf{m} \cdot \partial_k \mathbf{n}\, \mathrm{d}x + \frac{\varepsilon}{2}\int_G \left|\mathbf{u}\right|^2 \left|\nabla \mathbf{n}\right|^2 \, \mathrm{d}x \\
&+ \int_G \frac{|\mathbf{Q}| - 1}{\varepsilon}\left(2\kappa_* - \frac{\beta}{\sqrt{2}}(u_1^2 - u_2^2)\right) \mathrm{d}x \\
&+ \left(\frac{\kappa_\varepsilon}{\varepsilon^2} - \frac{1}{2\varepsilon}(\beta^2 + \sqrt{2}\beta) - \kappa_*^2\right) |G|
\end{aligned}
\tag{3.15}
$$

where $|G|$ denotes the area of G. We estimate separately the terms in the right-hand side of (3.15).

Step 2. In view of the identity $\mathbf{n} \otimes \mathbf{n} + \mathbf{m} \otimes \mathbf{m} = \mathbf{I}$, Equation (3.5) can be written as

$$
\frac{\mathbf{Q}}{|\mathbf{Q}|} = \sqrt{2}\left(\mathbf{n} \otimes \mathbf{n} - \frac{\mathbf{I}}{2}\right)
\tag{3.16}
$$

We differentiate both sides of (3.16) and compute the squared norm of the derivative. Recalling the assumption (3.4), after routine computations we obtain

$$
|\partial_k \mathbf{n}| = \frac{1}{2}\left|\partial_k\left(\frac{\mathbf{Q}}{|\mathbf{Q}|}\right)\right| \lesssim \frac{|\partial_k \mathbf{Q}|}{|\mathbf{Q}|} \lesssim |\partial_k \mathbf{Q}|
\tag{3.17}
$$

Thanks to (3.17), we can estimate

$$
\varepsilon \left|\sum_{k=1}^{2} \int_G (u_1\, \partial_k u_2 - u_2\, \partial_k u_1)\, \mathbf{m} \cdot \partial_k \mathbf{n}\, \mathrm{d}x\right| \lesssim \varepsilon \left\|\mathbf{u}\right\|_{L^\infty(G)} \left\|\nabla \mathbf{u}\right\|_{L^2(G)} \left\|\nabla \mathbf{Q}\right\|_{L^2(G)}
$$

By our assumptions (3.3), (3.4), the L^∞-norm of $\mathbf{u}$ is bounded and the L^2-norm of $\nabla \mathbf{Q}$ is of order $\left|\log \varepsilon\right|^{1/2}$ at most. Therefore, we obtain

$$
\varepsilon \left|\sum_{k=1}^{2} \int_G (u_1\, \partial_k u_2 - u_2\, \partial_k u_1)\, \mathbf{m} \cdot \partial_k \mathbf{n}\, \mathrm{d}x\right| \lesssim \varepsilon^{1/2} \left|\log \varepsilon\right|^{1/2} \left(\varepsilon \int_G \left|\nabla \mathbf{u}\right|^2\right)^{1/2}
\tag{3.18}
$$

Equations (3.3), (3.4) and (3.17) imply

$$\frac{\varepsilon}{2}\int_G |\mathbf{u}|^2 \, |\nabla \mathbf{n}|^2 \, \mathrm{d}x \lesssim \varepsilon \, |\log \varepsilon| \to 0 \qquad \text{as } \varepsilon \to 0. \tag{3.19}$$

Moreover, Lemma 3.1 gives

$$\left(\frac{\kappa_\varepsilon}{\varepsilon^2} - \frac{1}{2\varepsilon}(\beta^2 + \sqrt{2}\beta) - \kappa_*^2\right)|G| \to 0 \qquad \text{as } \varepsilon \to 0. \tag{3.20}$$

Step 3. By Lemma 3.4, the function h has two strict, non-degenerate minima at the points $\mathbf{u}_\pm := (\pm(\sqrt{2}\beta + 1)^{1/2}, 0)$. As a consequence, for any compact set $K \subseteq \mathbb{R}^2$ and any $\mathbf{u} \in K$, we must have

$$\begin{aligned} h(u) \geq C_K \min\left\{(\mathbf{u} - \mathbf{u}_+)^2, \, (\mathbf{u} - \mathbf{u}_-)^2\right\} &= C_K\left(|u_1| - (\sqrt{2}\beta + 1)^{1/2}\right)^2 + C_K u_2^2 \\ &= C_K \frac{\left(u_1^2 - \sqrt{2}\beta - 1\right)^2}{\left(|u_1| + (\sqrt{2}\beta + 1)^{1/2}\right)^2} + C_K u_2^2 \\ &\geq C_K\left(u_1^2 - \sqrt{2}\beta - 1\right)^2 + C_K u_2^2 \end{aligned}$$

for some constant C_K that depends only on K and β. Then, for any $\mathbf{u} \in K$ we have

$$\begin{aligned} \left|2\kappa_* - \frac{\beta}{\sqrt{2}}(u_1^2 - u_2^2)\right|^2 &\leq C_K \left|2\kappa_* - \frac{\beta}{\sqrt{2}}(u_1^2 - u_2^2)\right| \\ &\leq \frac{\beta}{\sqrt{2}}\left|\sqrt{2}\beta + 1 - u_1^2\right| + \frac{\beta}{\sqrt{2}} u_2^2 \leq C_K h(u) \end{aligned} \tag{3.21}$$

for some different constant C_K, still depending on K and β only. The assumption (3.4) guarantees that $\mathbf{u}$ takes its values in a compact subset of $\mathbb{R}^2$. Therefore, we can apply (3.21) to estimate

$$\int_G \frac{|\mathbf{Q}| - 1}{\varepsilon}\left(2\kappa_* - \frac{\beta}{\sqrt{2}}(u_1^2 - u_2^2)\right)\mathrm{d}x \lesssim \left(\frac{1}{\varepsilon^2}\int_G (|\mathbf{Q}| - 1)^2 \, \mathrm{d}x\right)^{1/2}\left(\int_G h(\mathbf{u}) \, \mathrm{d}x\right)^{1/2}$$

The elementary inequality $(x-1)^2 \leq (x^2 - 1)^2$, which applies to any $x \geq 0$, implies

$$\begin{aligned} \int_G \frac{|\mathbf{Q}| - 1}{\varepsilon}\left(2\kappa_* - \frac{\beta}{\sqrt{2}}(u_1^2 - u_2^2)\right)\mathrm{d}x &\lesssim \left(\frac{1}{\varepsilon^2}\int_G (|\mathbf{Q}|^2 - 1)^2 \, \mathrm{d}x\right)^{1/2}\left(\int_G h(\mathbf{u}) \, \mathrm{d}x\right)^{1/2} \\ &\overset{(3.3)}{\lesssim} \varepsilon^{1/2} \, |\log \varepsilon|^{1/2}\left(\frac{1}{\varepsilon}\int_G h(\mathbf{u}) \, \mathrm{d}x\right)^{1/2} \end{aligned} \tag{3.22}$$

The proposition follows by (3.15), (3.18), (3.19), (3.20) and (3.22). □

4 Proof of Theorem 2.1

4.1 Proof of Statement (i): compactness for $\mathbf{Q}^*_\varepsilon$

In this section, we prove that the $\mathbf{Q}^*_\varepsilon$-component of the minimisers converges to a limit, up to extraction of subsequences. The results in this section are largely based on the analysis in [11]. Throughout the paper, we denote by $(\mathbf{Q}^*_\varepsilon, \mathbf{M}^*_\varepsilon)$ a minimiser of the functional (2.1), subject to the boundary condition (2.3). We recall that the boundary data are of class C^1 and satisfy the assumption (2.4). Routine arguments show that minimisers exist and that they satisfy the Euler-Lagrange equations (2.5)–(2.6).

Lemma 4.1. *The maps $\mathbf{Q}^*_\varepsilon$, $\mathbf{M}^*_\varepsilon$ are smooth inside Ω and Lipschitz up to the boundary of Ω. Moreover, there exists an ε-independent constant C such that*

$$\|\mathbf{Q}^*_\varepsilon\|_{L^\infty(\Omega)} + \|\mathbf{M}^*_\varepsilon\|_{L^\infty(\Omega)} \leq C \tag{4.1}$$

$$\|\nabla\mathbf{Q}^*_\varepsilon\|_{L^\infty(\Omega)} + \|\nabla\mathbf{M}^*_\varepsilon\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon}. \tag{4.2}$$

Proof. Elliptic regularity theory implies, via a bootstrap argument, that $(\mathbf{Q}^*_\varepsilon, \mathbf{M}^*_\varepsilon)$ is smooth in the interior of Ω and continuous up to the boundary. Now, we prove (4.1). We take the scalar product of both sides of (2.5) with $\mathbf{Q}^*_\varepsilon$:

$$-\Delta\left(\frac{|\mathbf{Q}^*_\varepsilon|^2}{2}\right) + |\nabla\mathbf{Q}^*_\varepsilon|^2 + \frac{1}{\varepsilon^2}(|\mathbf{Q}^*_\varepsilon|^2 - 1)\,|\mathbf{Q}^*_\varepsilon|^2 - \frac{\beta}{\varepsilon}\mathbf{Q}^*_\varepsilon\mathbf{M}^*_\varepsilon\cdot\mathbf{M}^*_\varepsilon = 0 \tag{4.3}$$

In a similar way, by taking the scalar product of (2.6) with $\mathbf{M}^*_\varepsilon$, we obtain

$$-\Delta\left(\frac{|\mathbf{M}^*_\varepsilon|^2}{2}\right) + |\nabla\mathbf{M}^*_\varepsilon|^2 + \frac{1}{\varepsilon^2}(|\mathbf{M}^*_\varepsilon|^2 - 1)\,|\mathbf{M}^*_\varepsilon|^2 - \frac{2\beta}{\varepsilon^2}\mathbf{Q}^*_\varepsilon\mathbf{M}^*_\varepsilon\cdot\mathbf{M}^*_\varepsilon = 0. \tag{4.4}$$

By adding (4.3) and (4.4), and rearranging terms, we deduce

$$\varepsilon^2\Delta\left(\frac{|\mathbf{Q}^*_\varepsilon|^2 + |\mathbf{M}^*_\varepsilon|^2}{2}\right) \geq (|\mathbf{Q}^*_\varepsilon|^2 - 1)\,|\mathbf{Q}^*_\varepsilon|^2 + (|\mathbf{M}^*_\varepsilon|^2 - 1)\,|\mathbf{M}^*_\varepsilon|^2 - \beta(\varepsilon+2)\mathbf{Q}^*_\varepsilon\mathbf{M}^*_\varepsilon\cdot\mathbf{M}^*_\varepsilon \tag{4.5}$$

The right-hand side of (4.5) is strictly positive if $|\mathbf{Q}^*_\varepsilon|^2 + |\mathbf{M}^*_\varepsilon|^2 \geq C$, for some (sufficiently large) constant C that depends on β but not on ε. Therefore, (4.1) follows from the maximum principle. The inequality (4.2) follows by [10, Lemma A.1 and Lemma A.2]. □

Proposition 4.2. *Minimisers $(\mathbf{Q}^*_\varepsilon, \mathbf{M}^*_\varepsilon)$ of $\mathscr{F}_\varepsilon$ subject to the boundary conditions $\mathbf{Q} = \mathbf{Q}_{\mathrm{bd}}$, $\mathbf{M} = \mathbf{M}_{\mathrm{bd}}$ on $\partial\Omega$ satisfy*

$$\mathscr{F}_\varepsilon(\mathbf{Q}^*_\varepsilon, \mathbf{M}^*_\varepsilon) \leq 2\pi\,|d|\,|\log\varepsilon| + C,$$

where $d \in \mathbb{Z}$ is the degree of $\mathbf{M}^*_\varepsilon$ and C is a constant that depends only on Ω, $\mathbf{Q}_{\mathrm{bd}}$, $\mathbf{M}_{\mathrm{bd}}$ (not on ε).

Proof. We first consider the case $d = 1$. Consider balls $B_1 := B(a_1, R)$, $B_2 := B(a_2, R)$, of centres a_1, a_2 and radius $R > 0$, that are mutually disjoint. Since we have assumed that the degree of the boundary datum $\mathbf{Q}_{\mathrm{bd}}$ is $d = 1$, there exists a map $\tilde{\mathbf{Q}} \colon \Omega \setminus (B_1 \cup B_2) \to \mathscr{N}$ that is smooth (up to the boundary of $\Omega \setminus (B_1 \cup B_2)$), satisfies $\tilde{\mathbf{Q}} = \mathbf{Q}_{\mathrm{bd}}$ on $\partial\Omega$ and has degree $1/2$ on ∂B_1 and ∂B_2. We define a comparison map $\mathbf{Q}_\varepsilon$ as follows:

$$\mathbf{Q}(x) := \begin{cases} \tilde{\mathbf{Q}}(x) & \text{if } x \in \Omega \setminus (B_1 \cup B_2) \\ \mathbf{Q}^1_\varepsilon(x) & \text{if } x = a_1 + \rho e^{i\theta} \in B_1 = B(a_1, R) \\ \mathbf{Q}^2_\varepsilon(x) & \text{if } x = a_2 + \rho e^{i\theta} \in B_2 = B(a_2, R). \end{cases}$$

where $\mathbf{Q}^1_\varepsilon$, $\mathbf{Q}^2_\varepsilon$ are given as

$$\mathbf{Q}^1_\varepsilon(a_1 + \rho e^{i\theta}) := \sqrt{2} s_\varepsilon(\rho) \left(\mathbf{n}^1(\theta) \otimes \mathbf{n}^1(\theta) - \frac{\mathbf{I}}{2}\right), \qquad \mathbf{n}^1(\theta) = e^{i\theta/2}$$

$$\mathbf{Q}^2_\varepsilon(a_2 + \rho e^{i\theta}) := \sqrt{2} s_\varepsilon(\rho) \left(\mathbf{n}^2(\theta) \otimes \mathbf{n}^2(\theta) - \frac{\mathbf{I}}{2}\right), \qquad \mathbf{n}^2(\theta) = e^{i\theta/2},$$

and $s_\varepsilon(\rho)$ is the truncation at 1, $s_\varepsilon(\rho) := \min\{\frac{\rho}{\varepsilon}, 1\}$. A direct computation yields

$$\frac{1}{2} \int_\Omega |\nabla \mathbf{Q}_\varepsilon|^2 \mathrm{d}x \leq 2\pi \log\left(\frac{R}{\varepsilon}\right) + C \tag{4.6}$$

for some constant C that does not depend on ε. Indeed, since $\tilde{\mathbf{Q}}$ is regular on $\Omega \setminus (B_1 \cup B_2)$ and takes values in the manifold $\mathscr{N}$, the energy of $\mathbf{Q}_\varepsilon$ on $\Omega \setminus (B_1 \cup B_2)$ is an ε-independent constant, whereas the contribution of $\mathbf{Q}^1_\varepsilon$, $\mathbf{Q}^2_\varepsilon$ is reminiscent of the Ginzburg-Landau functional and can be computed explicitely.

Next, we construct the component $\mathbf{M}_\varepsilon$. Let Λ be the straight line segment of endpoints a_1, a_2. Thanks to Lemma A.3 in Appendix A, there exists a vector field $\tilde{\mathbf{M}}_\varepsilon \in \mathrm{SBV}(\Omega, \mathbb{R}^2)$ such that

$$|\tilde{\mathbf{M}}_\varepsilon| = (\sqrt{2}\beta + 1)^{\frac{1}{2}}, \qquad \mathbf{Q}_\varepsilon = \sqrt{2}\left(\frac{\tilde{\mathbf{M}}_\varepsilon \otimes \tilde{\mathbf{M}}_\varepsilon}{\sqrt{2}\beta + 1} - \frac{\mathbf{I}}{2}\right) \tag{4.7}$$

a.e. in Ω and, moreover, satisfies $S_{\tilde{\mathbf{M}}_\varepsilon} = \Lambda$, up to negligible sets. In particular, $\tilde{\mathbf{M}}_\varepsilon$ is smooth in a neighbourhood of $\partial\Omega$. By comparing (2.4) with (4.7), it follows that either $\tilde{\mathbf{M}}_\varepsilon = \mathbf{M}_{\mathrm{bd}}$ on $\partial\Omega$ or $\tilde{\mathbf{M}}_\varepsilon = -\mathbf{M}_{\mathrm{bd}}$ on $\partial\Omega$. Up to a change of sign, we will assume without loss of generality that $\tilde{\mathbf{M}}_\varepsilon = \mathbf{M}_{\mathrm{bd}}$ on $\partial\Omega$. In order to define our competitor $\mathbf{M}_\varepsilon$, we need to regularise $\tilde{\mathbf{M}}_\varepsilon$ near its jump set. We define

$$\mathbf{M}_\varepsilon(x) := \min\left\{\frac{\mathrm{dist}(x, \Lambda)}{\varepsilon}, 1\right\} \tilde{\mathbf{M}}_\varepsilon(x) \qquad \text{for any } x \in \Omega.$$

For ε small enough, we have $\mathbf{M}_\varepsilon = \tilde{\mathbf{M}}_\varepsilon = \mathbf{M}_{\mathrm{bd}}$ on $\partial\Omega$. The absolutely continuous part of gradient $\nabla\tilde{\mathbf{M}}_\varepsilon$ can be estimated by differentiating both sides of (4.7), by the BV-chain rule; it turns out that $|\nabla\tilde{\mathbf{M}}_\varepsilon| = c\,|\nabla\mathbf{Q}_\varepsilon|$, up to an (explicit) constant factor c that does not depend on ε. By explicit computation, we have

$$\varepsilon\,|\nabla\mathbf{M}_\varepsilon(x)|^2 \lesssim \frac{1}{\varepsilon}\chi_\varepsilon(x) + \varepsilon\,|\nabla\mathbf{Q}_\varepsilon(x)|^2 \qquad \text{for any } x\in\Omega,$$

where $\chi_\varepsilon\colon\Omega\to\mathbb{R}$ is defined as $\chi_\varepsilon(x) := 1$ if $\mathrm{dist}(x,\,\Lambda)\leq\varepsilon$ and $\chi_\varepsilon(x) := 0$ otherwise. Then, due to (4.6), we have

$$\varepsilon\int_\Omega |\nabla\mathbf{M}_\varepsilon|^2\,\mathrm{d}x \leq C\,(1+\varepsilon\,|\log\varepsilon|)\leq C \tag{4.8}$$

Finally, we compute the potential. We need to consider three different contributions. At a point $x\in\Omega\setminus(B(a_1,\varepsilon)\cup B(a_2,\varepsilon))$ such that $\mathrm{dist}(x,\,\Lambda)>\varepsilon$, we have $f_\varepsilon(\mathbf{Q}_\varepsilon(x),\,\mathbf{M}_\varepsilon(x)) = \mathrm{O}(\varepsilon^2)$ due to (4.7) and Lemma 3.1. At a point $x\in\Omega\setminus(B(a_1,\varepsilon)\cup B(a_2,\varepsilon))$ such that $\mathrm{dist}(x,\,\Lambda)<\varepsilon$, we have $|\mathbf{Q}_\varepsilon(x)| = 1$ and hence, $f_\varepsilon(\mathbf{Q}_\varepsilon(x),\,\mathbf{M}_\varepsilon(x)) = \mathrm{O}(\varepsilon)$. At a point $x\in B(a_1,\varepsilon)\cup B(a_2,\varepsilon)$, the potential $f_\varepsilon(\mathbf{Q}_\varepsilon(x),\,\mathbf{M}_\varepsilon(x))$ is bounded by a constant that does not depend on ε. Therefore, we have

$$\int_\Omega f_\varepsilon(\mathbf{Q}_\varepsilon,\,\mathbf{M}_\varepsilon)\,\mathrm{d}x \lesssim \varepsilon^2 \tag{4.9}$$

Together, (4.6), (4.8) and (4.9) imply

$$\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon^*,\,\mathbf{M}_\varepsilon^*) \leq \mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon,\,\mathbf{M}_\varepsilon)\leq 2\pi\,|\log\varepsilon| + C$$

for some constant C that does not depend on ε. The proof in case $d\neq 1$ is similar, except that in the definition of $\tilde{\mathbf{Q}}$, we need to consider $2\,|d|$ balls $B_1,\,B_2,\,\ldots\,B_{2|d|}$, each of them carrying a topological degree of $\mathrm{sign}(d)/2$. The set Λ is defined as a union of segments that connects the centres of the balls $B_1,\,B_2,\,\ldots\,B_{2|d|}$ (for instance, a minimal connection — see Appendix A). □

A compactness result for the $\mathbf{Q}_\varepsilon^*$-component of minimisers can be obtained by appealing to results in the Ginzburg-Landau theory. Given a (closed) ball $\bar{B}_\rho(a)\subseteq\Omega$ such that $|\mathbf{Q}_\varepsilon^*|\geq 1/2$ on $\partial B_\rho(a)$, the map

$$\frac{\mathbf{Q}_\varepsilon^*}{|\mathbf{Q}_\varepsilon^*|}\colon \partial B_\rho(a)\simeq\mathbb{S}^1\to\mathscr{N}\simeq\mathbb{R}\mathrm{P}^1$$

is well-defined and continuous and hence, its topological degree is well-defined as an element of $\frac{1}{2}\mathbb{Z}$. We denote the topological degree of $\mathbf{Q}_\varepsilon^*/|\mathbf{Q}_\varepsilon^*|$ on $\partial B_\rho(a)$ by $\deg(\mathbf{Q}_\varepsilon^*,\,\partial B_\rho(a_j))$. We recall that d is the degree of the boundary datum, as given in (2.10).

Lemma 4.3. *There exist distinct points $a_1^*,\,\ldots,\,a_{2|d|}^*$ in Ω, distinct points $b_1^*,\,\ldots,\,b_K^*$ in $\overline{\Omega}$ and a (non-relabelled) subsequence such that the following statement holds. For any $\sigma>0$ sufficiently small there exists $\varepsilon_0(\sigma)>0$ such that, if $0<\varepsilon\leq\varepsilon_0(\sigma)$, then*

$$\frac{1}{2}\leq|\mathbf{Q}_\varepsilon^*(x)|\leq\frac{3}{2} \qquad \textit{for any } x\notin\bigcup_{j=1}^{2|d|}B_\sigma(a_j^*)\cup\bigcup_{k=1}^{K}B_\sigma(b_k^*) \tag{4.10}$$

$$\deg(\mathbf{Q}_\varepsilon^*,\,\partial B_\sigma(a_j^*)) = \frac{1}{2}\,\mathrm{sign}(d), \qquad \deg(\mathbf{Q}_\varepsilon^*,\,\partial(B_\sigma(b_k^*)\cap\Omega)) = 0 \tag{4.11}$$

for any $j \in \{1, \ldots, 2|d|\}$, any $k \in \{1, \ldots, K\}$. Moreover, for any σ sufficiently small and any $0 < \varepsilon \leq \varepsilon_0(\sigma)$, there holds

$$\mathscr{F}_\varepsilon\left(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon;\, \Omega \setminus \cup_{j=1}^{2|d|} B_\sigma(a_j^*)\right) \leq 2\pi\,|d|\,|\log\sigma| + C \tag{4.12}$$

where C is a positive constant C that does not depend on ε, σ. Finally, there exists a limit map $\mathbf{Q}^*\colon \Omega \to \mathscr{N}$ such that

$$\mathbf{Q}_\varepsilon^* \rightharpoonup \mathbf{Q}^* \qquad \textit{weakly in } W^{1,p}(\Omega) \textit{ for any } p < 2 \textit{ and in } W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{a_1^*, \ldots, a_{2|d|}^*\}). \tag{4.13}$$

Proof. The analysis of the $\mathbf{Q}_\varepsilon^*$-component can be recast in the classical Ginzburg-Landau setting, by menas of a change of variables. We define $\mathbf{q}_\varepsilon^*\colon \Omega \to \mathbb{R}^2$ as

$$\mathbf{q}_\varepsilon^* := \sqrt{2}((Q_\varepsilon^*)_{11},\, (Q_\varepsilon^*)_{12}) \tag{4.14}$$

Since $\mathbf{Q}_\varepsilon^*$ is symmetric and trace-free, we have $|\mathbf{q}_\varepsilon^*| = |\mathbf{Q}_\varepsilon^*|$ and $|\nabla\mathbf{q}_\varepsilon^*| = |\nabla\mathbf{Q}_\varepsilon^*|$. With the help of Lemma 3.1, we deduce

$$E_\varepsilon(\mathbf{q}_\varepsilon^*) := \int_\Omega \left(\frac{1}{2}\,|\nabla\mathbf{q}_\varepsilon^*|^2 + \frac{1}{8\varepsilon^2}(|\mathbf{q}_\varepsilon^*|^2 - 1)^2\right) \mathrm{d}x \overset{(3.2)}{\leq} \mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*) + \beta^2 \int_\Omega |\mathbf{M}_\varepsilon^*|^4 \,\mathrm{d}x$$

The terms at the right-hand side can be bounded by Proposition 4.2 and Lemma 4.1, respectively. We obtain

$$E_\varepsilon(\mathbf{q}_\varepsilon^*) \leq 2\pi\,|d|\,|\log\varepsilon| + C, \tag{4.15}$$

where C is an ε-independent constant. Moreover, due to the boundary condition (2.3) and (2.4), $\mathbf{q}_\varepsilon^*$ restricted to the boundary $\partial\Omega$ coincides with an ε-independent map of class C^1. More precisely, if we identify vectors in $\mathbb{R}^2$ with complex numbers so that $\mathbf{M}_{\mathrm{bd}}$ is identified with a complex number, $\mathbf{M}_{\mathrm{bd}} = \mathbf{M}_{\mathrm{bd}1} + \mathrm{i}\mathbf{M}_{\mathrm{bd}2}$, then a routine computation shows

$$\mathbf{q}_\varepsilon^* = \frac{\mathbf{M}_{\mathrm{bd}}^2}{\sqrt{2}\beta + 1} \qquad \text{on } \partial\Omega$$

(the square is taken in the sense of complex numbers). In particular, $|\mathbf{q}_\varepsilon^*| = 1$ on $\partial\Omega$ and

$$\deg(\mathbf{q}_\varepsilon^*,\, \partial\Omega) = 2\deg(\mathbf{M}_{\mathrm{bd}},\, \partial\Omega) \overset{(2.10)}{=} 2d. \tag{4.16}$$

Now, (4.10), (4.11), (4.12) follow from classical results in the Ginzburg-Landau literature (see e.g [36, Theorem 2.4], [37, Proposition 1.1], [33, Theorems 1.2 and 1.3], [42, Theorem 1]). Moreover, the arguments in [45, Theorem 1.1] prove that, for any $p \in (1, 2)$, there exists a constant C_p such that

$$\int_\Omega |\nabla\mathbf{Q}_\varepsilon^*|^p \,\mathrm{d}x \leq C_p. \tag{4.17}$$

for any ε sufficiently small. Then, (4.13) follows from (4.12) and (4.17), by means of a compactness argument. □

Remark 4.1. Another well-known estimate in the Ginzburg-Landau literature [26, Theorem 1.1] is that

$$\frac{1}{4\varepsilon^2}\int_\Omega (|\mathbf{Q}_\varepsilon^*|^2 - 1)^2 \, \mathrm{d}x \leq C$$

for some ε-independent constant C. This inequality will be useful in some of our arguments below.

In order to complete the proof of Statement (i) in Theorem 2.1, it only remains to show that the convergence $\mathbf{Q}_\varepsilon^* \to \mathbf{Q}^*$ is not only weak, but also strong in $W^{1,p}(\Omega)$. The proof of this fact relies on an auxiliary lemma. We consider the function $g_\varepsilon \colon \mathscr{S}_0^{2\times 2} \to \mathbb{R}$ defined in (3.7).

Lemma 4.4. *Let $B = B_r(x_0) \subseteq \Omega$ be an open ball. Suppose that $\mathbf{Q}_\varepsilon^* \rightharpoonup \mathbf{Q}^*$ weakly in $W^{1,2}(\partial B)$ and that*

$$\int_{\partial B} \left(\frac{1}{2}\,|\nabla \mathbf{Q}_\varepsilon^*|^2 + g_\varepsilon(\mathbf{Q}_\varepsilon^*)\right) \mathrm{d}\mathscr{H}^1 \leq C \tag{4.18}$$

for some constant C that may depend on the radius r, but not on ε. Then, there exists a map $\mathbf{Q}_\varepsilon \in W^{1,2}(B, \mathscr{S}_0^{2\times 2})$ such that

$$\mathbf{Q}_\varepsilon = \mathbf{Q}_\varepsilon^* \quad \text{on } \partial B, \qquad |\mathbf{Q}_\varepsilon| \geq \frac{1}{2} \quad \text{in } B \tag{4.19}$$

$$\int_B \left(\frac{1}{2}\,|\nabla \mathbf{Q}_\varepsilon|^2 + g_\varepsilon(\mathbf{Q}_\varepsilon)\right) \mathrm{d}x \to \frac{1}{2}\int_B |\nabla \mathbf{Q}^*|^2 \, \mathrm{d}x \tag{4.20}$$

The proof of Lemma 4.4 is given in Appendix C.

Proposition 4.5. *As $\varepsilon \to 0$, we have*

$$\mathbf{Q}_\varepsilon^* \to \mathbf{Q}^* \qquad \text{strongly in } W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{a_1^*, \ldots, a_{2|d|}^*, b_1, \ldots, b_K\}) \tag{4.21}$$

$$\mathbf{Q}_\varepsilon^* \to \mathbf{Q}^* \qquad \text{strongly in } W^{1,p}(\Omega) \quad \text{for any } p \in [1, 2) \tag{4.22}$$

Proof. Let $B := B_R(x_0) \subset\subset \Omega \setminus \{a_1^*, \ldots, a_{2|d|}^*, b_1, \ldots, b_K\}$ be an open ball. We have $|\mathbf{Q}_\varepsilon^*| \geq 1/2$ in B, so we can apply the change of variables described in Section 3. We consider the vector field $\mathbf{u}_\varepsilon^* \colon B \to \mathbb{R}^2$ defined as in (3.6) — that is, we write

$$\mathbf{Q}_\varepsilon^* = \frac{|\mathbf{Q}_\varepsilon^*|}{\sqrt{2}}\,(\mathbf{n}_\varepsilon^* \otimes \mathbf{n}_\varepsilon^* - \mathbf{m}_\varepsilon^* \otimes \mathbf{m}_\varepsilon^*) \qquad \text{in } B,$$

where $(\mathbf{n}_\varepsilon^*, \mathbf{m}_\varepsilon^*)$ is an orthonormal basis of eigevectors for $\mathbf{Q}_\varepsilon$, and we define $(u_\varepsilon^*)_1 := \mathbf{M}_\varepsilon^* \cdot \mathbf{n}_\varepsilon^*$, $(u_\varepsilon^*)_2 := \mathbf{M}_\varepsilon^* \cdot \mathbf{m}_\varepsilon^*$. By Proposition 3.2, we have

$$\begin{aligned}\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*; B) = &\int_B \left(\frac{1}{2}\,|\nabla \mathbf{Q}_\varepsilon^*|^2 + g_\varepsilon(\mathbf{Q}_\varepsilon^*)\right) \mathrm{d}x \\ &+ \int_B \left(\frac{\varepsilon}{2}\,|\nabla \mathbf{u}_\varepsilon^*|^2 + \frac{1}{\varepsilon} h(\mathbf{u}_\varepsilon^*)\right) \mathrm{d}x + \mathrm{o}_{\varepsilon\to 0}(1)\end{aligned} \tag{4.23}$$

where the functions g_ε and h are defined in (3.7) and (3.8), respectively. (The remainder term R_ε, given by Proposition 3.2, tends to zero as $\varepsilon \to 0$, due to (3.9) and the energy bound (4.12)).

By Lemma 4.3, we know that $\mathscr{F}_\varepsilon(\mathbf{Q}^*_\varepsilon, \mathbf{M}^*_\varepsilon; B) \leq C$ for some constant C that depends on the ball B, but not on ε. By Fubini theorem, and possibly up to extraction of a subsequence, we find a radius $r \in (R/2, R)$ such that

$$\int_{\partial B_r(x_0)} \left(\frac{1}{2}\left|\nabla \mathbf{Q}^*_\varepsilon\right|^2 + g_\varepsilon(\mathbf{Q}^*_\varepsilon)\right) \mathrm{d}\mathscr{H}^1 + \frac{1}{2}\int_{\partial B_r(x_0)} \left|\nabla \mathbf{Q}^*\right|^2 \, \mathrm{d}\mathscr{H}^1 \leq \frac{C}{R} \tag{4.24}$$

with C that does not depend on ε. Moreover, without loss of generality we can assume that $\mathbf{Q}^*_\varepsilon \rightharpoonup \mathbf{Q}^*$ weakly in $W^{1,2}(\partial B_r(x_0))$. Let $B' := B_r(x_0)$. By Lemma 4.4, there exists a map $\mathbf{Q}_\varepsilon \in W^{1,2}(B', \mathscr{S}_0^{2\times 2})$ such that

$$\mathbf{Q}_\varepsilon = \mathbf{Q}^*_\varepsilon \quad \text{on } \partial B', \qquad |\mathbf{Q}_\varepsilon| \geq \frac{1}{2} \quad \text{in } B' \tag{4.25}$$

$$\int_{B'} \left(\frac{1}{2}\left|\nabla \mathbf{Q}_\varepsilon\right|^2 + g_\varepsilon(\mathbf{Q}_\varepsilon)\right) \mathrm{d}x \to \frac{1}{2}\int_{B'} \left|\nabla \mathbf{Q}^*\right|^2 \, \mathrm{d}x \tag{4.26}$$

Thanks to (4.25), we can write

$$\mathbf{Q}_\varepsilon = \frac{|\mathbf{Q}_\varepsilon|}{\sqrt{2}} \left(\mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon - \mathbf{m}_\varepsilon \otimes \mathbf{m}_\varepsilon\right) \qquad \text{in } B',$$

where $(\mathbf{n}_\varepsilon, \mathbf{m}_\varepsilon)$ is an orthonormal basis of eigevectors for $\mathbf{Q}_\varepsilon$. We define

$$\mathbf{M}_\varepsilon := (u^*_\varepsilon)_1 \, \mathbf{n}_\varepsilon + (u^*_\varepsilon) \, \mathbf{m}_\varepsilon \qquad \text{in } B'.$$

The pair $(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)$ is an admissible competitor for $(\mathbf{Q}^*_\varepsilon, \mathbf{M}^*_\varepsilon)$: $\mathbf{Q}_\varepsilon = \mathbf{Q}^*_\varepsilon$ on $\partial B'$ by construction and, if the orientation of $\mathbf{n}_\varepsilon$ and $\mathbf{m}_\varepsilon$ is chosen suitably, then $\mathbf{M}_\varepsilon = \mathbf{M}^*_\varepsilon$ on $\partial B'$. By minimality of $(\mathbf{Q}^*_\varepsilon, \mathbf{M}^*_\varepsilon)$, we have $\mathscr{F}_\varepsilon(\mathbf{Q}^*, \mathbf{M}^*_\varepsilon; B') \leq \mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon; B')$ By applying Proposition 3.2, we deduce

$$\begin{aligned}\int_{B'} \left(\frac{1}{2}\left|\nabla \mathbf{Q}^*_\varepsilon\right|^2 + g_\varepsilon(\mathbf{Q}^*_\varepsilon)\right) \mathrm{d}x &\leq \int_{B'} \left(\frac{1}{2}\left|\nabla \mathbf{Q}_\varepsilon\right|^2 + g_\varepsilon(\mathbf{Q}_\varepsilon)\right) \mathrm{d}x + \mathrm{o}_{\varepsilon\to 0}(1) \\ &\overset{(4.26)}{=} \frac{1}{2}\int_{B'} \left|\nabla \mathbf{Q}^*\right|^2 \, \mathrm{d}x + \mathrm{o}_{\varepsilon\to 0}(1)\end{aligned} \tag{4.27}$$

As we know already that $\mathbf{Q}^*_\varepsilon \rightharpoonup \mathbf{Q}^*$ weakly in $W^{1,2}(B')$ (by Lemma 4.3), from (4.27) we deduce that $\mathbf{Q}^*_\varepsilon \to \mathbf{Q}^*$ strongly in $W^{1,2}(B')$ and (4.21) follows.

We turn to the proof of (4.22). Let p, q be such that $1 \leq p < q < 2$. Let $\sigma > 0$ be a small parameter, and let

$$U_\sigma := \bigcup_{j=1}^{2|d|} B_\sigma(a^*_j) \cup \bigcup_{k=1}^{K} B_\sigma(b_k)$$

By the Hölder inequality, we obtain

$$\|\nabla \mathbf{Q}^*_\varepsilon - \nabla \mathbf{Q}^*\|_{L^p(\Omega)} \lesssim |U_\sigma|^{1/p-1/q} \left(\|\nabla \mathbf{Q}^*_\varepsilon\|_{L^q(U_\sigma)} + \|\nabla \mathbf{Q}^*\|_{L^q(U_\sigma)}\right) + \|\nabla \mathbf{Q}^*_\varepsilon - \nabla \mathbf{Q}^*\|_{L^2(\Omega\setminus U_\sigma)}$$

Thanks to Lemma 4.3 and (4.21), we deduce

$$\limsup_{\varepsilon\to 0} \|\nabla \mathbf{Q}^*_\varepsilon - \nabla \mathbf{Q}^*\|_{L^p(\Omega)} \lesssim \sigma^{2/p-2/q}$$

and, as σ may be taken arbitrarily small, (4.22) follows. □

Remark 4.2. As a byproduct of the estimate (4.27), we deduce that $g_\varepsilon(\mathbf{Q}_\varepsilon^*) \to 0$ strongly in $L^1_{\mathrm{loc}}(\Omega \setminus \{a_1, \ldots, a_{2|d|}, b_1, \ldots, b_K\})$.

We state an additional convergence property for $\mathbf{Q}_\varepsilon^*$, which will be useful later on. We recall that the vector product of two vectors $\mathbf{u} \in \mathbb{R}^2$, $\mathbf{v} \in \mathbb{R}^2$ can be identified with a scalar, $\mathbf{u} \times \mathbf{v} := u_1 v_2 - u_2 v_1$. In a similar way, we define the vector product of two matrices $\mathbf{Q} \in \mathscr{S}_0^{2\times 2}$, $\mathbf{P} \in \mathscr{S}_0^{2\times 2}$ as

$$\mathbf{Q} \times \mathbf{P} := Q_{11}P_{12} - Q_{12}P_{11} + Q_{21}P_{22} - Q_{22}P_{21} = 2\,(Q_{11}P_{12} - Q_{12}P_{11}) \tag{4.28}$$

Now, for any $\mathbf{Q} \in (L^\infty \cap W^{1,1})(\Omega, \mathscr{S}_0^{2\times 2})$, we define the vector field $j(\mathbf{Q})\colon \Omega \to \mathbb{R}^2$ as

$$j(\mathbf{Q}) := \frac{1}{2}\,(\mathbf{Q} \times \partial_1 \mathbf{Q},\, \mathbf{Q} \times \partial_2 \mathbf{Q}) \tag{4.29}$$

For any $\mathbf{Q} \in (L^\infty \cap W^{1,1})(\Omega, \mathscr{S}_0^{2\times 2})$, the vector field $j(\mathbf{Q})$ is integrable. Therefore, it makes sense to define

$$J(\mathbf{Q}) := \partial_1(j(\mathbf{Q}))_2 - \partial_2(j(\mathbf{Q}))_1 \tag{4.30}$$

if we take the derivatives in the sense of distributions. If $\mathbf{Q}$ is smooth, then $J(\mathbf{Q})$ is the Jacobian determinant of $\mathbf{q} := (\sqrt{2}Q_{11}, \sqrt{2}Q_{12})$:

$$J(\mathbf{Q}) = 2\,\partial_1 Q_{11}\,\partial_2 Q_{12} - 2\,\partial_2 Q_{11}\,\partial_1 Q_{12} = \det \nabla \mathbf{q} \tag{4.31}$$

More generally, for any $\mathbf{Q} \in (L^\infty \cap W^{1,1})(\Omega, \mathbb{R}^2)$, $J(\mathbf{Q})$ coincides with the distributional Jacobian of $\mathbf{q}$ (see e.g. [34] and the references therein).

Lemma 4.6. *We have*

$$J(\mathbf{Q}_\varepsilon^*) \to J(\mathbf{Q}^*) = \pi \,\mathrm{sign}(d) \sum_{j=1}^{2|d|} \delta_{a_j^*} \qquad \text{in } W^{-1,1}(\Omega)$$

as $\varepsilon \to 0$.

Proof. Let $\mathbf{q}^* := (\sqrt{2}Q_{11}^*, \sqrt{2}Q_{12}^*)$. By Lemma 4.3, the vector field $\mathbf{q}^*$ belongs to $W^{1,1}(\Omega, \mathbb{S}^1)$ (globally in Ω) and to $W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{a_1^*, \ldots, a_{2|d|}^*\}, \mathbb{S}^1)$. At each point a_j^*, $\mathbf{q}^*$ has a singularity of degree $2\deg(\mathbf{Q}^*, \partial B_\sigma(a_j^*)) = \mathrm{sign}(d)$, due to (4.11). By reasoning e.g. as in [34, Example 3.1], we obtain

$$J(\mathbf{Q}^*) = \pi\,\mathrm{sign}(d) \sum_{j=1}^{2|d|} \delta_{a_j^*} \tag{4.32}$$

It remains to show that $J(\mathbf{Q}_\varepsilon^*) \to J(\mathbf{Q}^*)$ in $W^{-1,1}(\Omega)$. Let $p \in [1, 2)$ and $q \in (2, +\infty]$ be such that $1/p + 1/q = 1$. By, e.g., [18, Theorem 1], we have

$$\|J(\mathbf{Q}_\varepsilon^*) - J(\mathbf{Q}^*)\|_{W^{-1,1}(\Omega)} \leq C\,\|\mathbf{Q}_\varepsilon^* - \mathbf{Q}^*\|_{L^q(\Omega)} \left(\|\nabla \mathbf{Q}_\varepsilon^*\|_{L^p(\Omega)} + \|\nabla \mathbf{Q}^*\|_{L^p(\Omega)}\right) \tag{4.33}$$

for some constant C that depends only on Ω. The sequence $\mathbf{Q}_\varepsilon^*$ is bounded in $W^{1,p}(\Omega)$, by Lemma 4.3. By compact Sobolev embedding, we have $\mathbf{Q}_\varepsilon^* \to \mathbf{Q}^*$ pointwise a.e., up to extraction of a subsequence. As $\mathbf{Q}_\varepsilon^*$ is also bounded in $L^\infty(\Omega)$, by Lemma 4.1, we deduce that $\mathbf{Q}_\varepsilon^* \to \mathbf{Q}^*$ strongly in $L^q(\Omega)$ (via Lebesgue's dominated convergence theorem). Then, (4.33) implies that $J(\mathbf{Q}_\varepsilon^*) \to J(\mathbf{Q}^*)$ in $W^{-1,1}(\Omega)$ and the lemma follows. □

4.2 Proof of Statement (ii): $\mathbf{Q}^*$ is a canonical harmonic map

Next, we show that $\mathbf{Q}^*$ is the canonical harmonic map with singularities at $(a_1^*, \ldots, a_{2|d|}^*)$ and boundary datum $\mathbf{Q}_{\mathrm{bd}}$, as defined in Section 2. The proof relies on an auxiliary lemma.

Lemma 4.7. *The minimisers* $(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*)$ *satisfy*

$$-\partial_j \left(\mathbf{Q}_\varepsilon^* \times \partial_j \mathbf{Q}_\varepsilon^*\right) = \frac{\varepsilon}{2}\, \partial_j \left(\mathbf{M}_\varepsilon^* \times \partial_j \mathbf{M}_\varepsilon^*\right) \qquad \text{in } \Omega.$$

Proof. For ease of notation, we drop the subscript ε and the superscript * from all the variables. We consider the Euler-Lagrange equation for $\mathbf{Q}$, Equation (2.5), and take the vector product with $\mathbf{Q}$:

$$-\mathbf{Q} \times \Delta \mathbf{Q} - \frac{\beta}{\varepsilon} \mathbf{Q} \times \left(\mathbf{M} \otimes \mathbf{M} - \frac{|\mathbf{M}|^2}{2} \mathbf{I}\right) = 0 \tag{4.34}$$

We have

$$\mathbf{Q} \times \Delta \mathbf{Q} = \partial_j \left(\mathbf{Q} \times \partial_j \mathbf{Q}\right) - \partial_j \mathbf{Q} \times \partial_j \mathbf{Q} = \partial_j \left(\mathbf{Q} \times \partial_j \mathbf{Q}\right)$$

and

$$\mathbf{Q} \times \left(\mathbf{M} \otimes \mathbf{M} - \frac{|\mathbf{M}|^2}{2} \mathbf{I}\right) = 2 Q_{11} M_1 M_2 - Q_{12} M_1^2 + Q_{12} M_2^2 = \mathbf{Q}\mathbf{M} \times \mathbf{M}$$

so Equation (4.34) rewrites as

$$-\partial_j \left(\mathbf{Q} \times \partial_j \mathbf{Q}\right) = \frac{\beta}{\varepsilon} \mathbf{Q}\mathbf{M} \times \mathbf{M} \tag{4.35}$$

Now, we consider the Euler-Lagrange equation for $\mathbf{M}$, Equation (2.5), and take the vector product with $\mathbf{M}$:

$$-\mathbf{M} \times \Delta \mathbf{M} - \frac{2\beta}{\varepsilon^2} \mathbf{M} \times \mathbf{Q}\mathbf{M} = 0. \tag{4.36}$$

As before, we have $\mathbf{M} \times \Delta \mathbf{M} = \partial_j (\mathbf{M} \times \partial_j \mathbf{M})$, so (4.36) can be written as

$$\partial_j (\mathbf{M} \times \partial_j \mathbf{M}) = \frac{2\beta}{\varepsilon^2} \mathbf{Q}\mathbf{M} \times \mathbf{M} \tag{4.37}$$

The lemma follows from (4.35) and (4.37). □

Proposition 4.8. $\mathbf{Q}^*$ *is the canonical harmonic map with singularities at* $(a_1^*, \ldots, a_{2|d|}^*)$ *and boundary datum* $\mathbf{Q}_{\mathrm{bd}}$.

Proof. First, we show that $\mathbf{Q}^*$ satisfies

$$\partial_j \left(\mathbf{Q}^* \times \partial_j \mathbf{Q}^*\right) = 0 \tag{4.38}$$

in the sense of distributions in Ω. To this end, we pass to the limit in both sides of Lemma 4.7. Let $p \in (1, 2)$. By Lemma 4.3, we have $\mathbf{Q}_\varepsilon^* \rightharpoonup \mathbf{Q}^*$ weakly in $W^{1,p}(\Omega)$ and, up to extraction of subsequences, pointwise a.e. As $\mathbf{Q}_\varepsilon^*$ is bounded in $L^\infty(\Omega)$ by Lemma 4.1, Lebesgue's dominated convergence theorem implies that $\mathbf{Q}_\varepsilon^* \to \mathbf{Q}^*$ strongly in $L^q(\Omega)$ for any $q < +\infty$. As a consequence, we have

$$\partial_j \left(\mathbf{Q}_\varepsilon^* \times \partial_j \mathbf{Q}_\varepsilon^*\right) \rightharpoonup^* \partial_j \left(\mathbf{Q}^* \times \partial_j \mathbf{Q}^*\right) \qquad \text{as distributions in } \Omega \text{ as } \varepsilon \to 0. \tag{4.39}$$

On the other hand, Proposition 4.2 implies

$$\|\nabla \mathbf{M}_\varepsilon^*\|_{L^2(\Omega)}^2 \leq \frac{1}{\varepsilon} \mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*) \lesssim \frac{|\log \varepsilon|}{\varepsilon}$$

As $\mathbf{M}_\varepsilon^*$ is bounded in $L^\infty(\Omega)$ by Lemma 4.1, we deduce

$$\varepsilon \|\mathbf{M}_\varepsilon^* \times \nabla \mathbf{M}_\varepsilon^*\|_{L^2(\Omega)} \leq \varepsilon \|\mathbf{M}_\varepsilon^*\|_{L^\infty(\Omega)} \|\nabla \mathbf{M}_\varepsilon^*\|_{L^2(\Omega)} \lesssim \varepsilon^{1/2} |\log \varepsilon|^{1/2} \to 0$$

as $\varepsilon \to 0$. Therefore,

$$\varepsilon\, \partial_j \left(\mathbf{M}_\varepsilon^* \times \partial_j \mathbf{M}_\varepsilon^*\right) \to 0 \qquad \text{in } W^{-1,2}(\Omega) \text{ as } \varepsilon \to 0. \tag{4.40}$$

Combining (4.39) and (4.40) with Lemma 4.7, we obtain (4.38).

To prove that $\mathbf{Q}^*$ is canonical harmonic, it only remains to check that $\mathbf{Q}^*$ is smooth in $\Omega \setminus \{a_1^*, \ldots, a_{2|d|}^*\}$ and continuous in $\overline{\Omega} \setminus \{a_1^*, \ldots, a_{2|d|}^*\}$. Both these properties follow from (4.38). Indeed, let $G \subseteq \overline{\Omega} \setminus \{a_1^*, \ldots, a_{2|d|}^*\}$ be a simply connected domain. As $\mathbf{Q}^* \in W^{1,2}(G, \mathscr{N})$, we can apply lifting results (see e.g. [12, Theorem 1]) and write

$$\mathbf{Q}^* = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta^* & \sin\theta^* \\ \sin\theta^* & -\cos\theta^* \end{pmatrix}$$

for some scalar function $\theta^* \in W^{1,2}(G)$. Equation (4.38) may be written in terms of θ^* as

$$\Delta\theta^* = 0 \qquad \text{as distributions in } G.$$

Therefore, θ^* is smooth in G and so is $\mathbf{Q}^*$. In case G touches the boundary of Ω, θ^* is continuous up to $\partial\Omega$ and hence $\mathbf{Q}^*$ is. □

4.3 Proof of Statements (iii) and (iv): compactness for $\mathbf{M}_\varepsilon^*$

In this section, we prove a compactness result for the component $\mathbf{M}_\varepsilon^*$ of a sequence of minimisers. The proof relies on the change of variables we introduced in Section 3.

We recall that in Lemma 4.3, we found a finite number of points $a_1^*, \ldots, a_{2|d|}^*, b_1^*, \ldots, b_K^*$ such that $|\mathbf{Q}_\varepsilon^*|$ is uniformly bounded away from zero, except for some small balls of radius σ around these points. Let

$$G \subset\subset \Omega \setminus \{a_1^*, \ldots, a_{2|d|}^*, b_1^*, \ldots, b_K^*\}$$

be a smooth, simply connected domain. The sequence of minimisers $(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*)$ satisfies the assumptions (3.3)–(3.4), thanks to Lemma 4.1, Proposition 4.2 and Lemma 4.3. Therefore, we are in position to apply the results from Section 3. We define the vector field $\mathbf{u}_\varepsilon^*\colon G \to \mathbb{R}^2$ as in (3.6) — that is, we write

$$\mathbf{Q}_\varepsilon^* = \frac{|\mathbf{Q}_\varepsilon^*|}{\sqrt{2}} \left(\mathbf{n}_\varepsilon^* \otimes \mathbf{n}_\varepsilon^* - \mathbf{m}_\varepsilon^* \otimes \mathbf{m}_\varepsilon^*\right) \qquad \text{in } G, \tag{4.41}$$

where $(\mathbf{n}_\varepsilon^*, \mathbf{m}_\varepsilon^*)$ is an orthonormal set of eigenvectors for $\mathbf{Q}^*$ with $\mathbf{n}_\varepsilon^* \in W^{1,2}(G, \mathbb{S}^1)$, $\mathbf{m}_\varepsilon^* \in W^{1,2}(G, \mathbb{S}^1)$, and we define

$$(u_\varepsilon^*)_1 := \mathbf{M}_\varepsilon^* \cdot \mathbf{n}_\varepsilon^*, \qquad (u_\varepsilon^*)_2 := \mathbf{M}_\varepsilon^* \cdot \mathbf{m}_\varepsilon^* \tag{4.42}$$

The next lemma is key to prove compactness of the sequence $\mathbf{u}_\varepsilon^*$ and, hence, of $\mathbf{M}_\varepsilon^*$.

Lemma 4.9. *Let h be the function defined by (3.8). For any simply connected domain $G \subset\subset \Omega \setminus \{a_1^*, \ldots, a_{2|d|}^*, b_1^*, \ldots, b_K^*\}$, there holds*

$$\int_G \left(\frac{\varepsilon}{2} |\nabla \mathbf{u}_\varepsilon^*|^2 + \frac{1}{\varepsilon} h(\mathbf{u}_\varepsilon^*)\right) \mathrm{d}x \leq C,$$

where C is a positive constant that depends only on Ω, β and the boundary datum (in particular, it is independent of ε, G).

Proof. By classical lower bounds in the Ginzburg-Landau theory, such as [33, Theorem 1.1] or [42, Theorem 2], we have

$$\int_\Omega \left(\frac{1}{2} |\nabla \mathbf{Q}_\varepsilon^*|^2 + \frac{1}{4\varepsilon^2}(|\mathbf{Q}_\varepsilon^*|^2 - 1)^2\right) \mathrm{d}x \geq 2\pi\, |d|\, |\log \varepsilon| - C, \tag{4.43}$$

for some constant C that depends only on Ω and the boundary datum $\mathbf{Q}_{\mathrm{bd}}$. The results in [33, 42] extend to our setting due to change of variables $\mathbf{Q}_\varepsilon^* \mapsto \mathbf{q}_\varepsilon^*$, given by (4.14). The coefficient $2\pi\,|d|$ in the right-hand side of (4.43) depends on this change of variables, which transforms the boundary condition of degree d for $\mathbf{Q}_\varepsilon^*$ into a boundary condition of degree $2d$ for $\mathbf{q}_\varepsilon^*$ — see (4.16).

From (4.43) and Remark 4.1, we deduce

$$\frac{1}{2} \int_\Omega |\nabla \mathbf{Q}_\varepsilon^*|^2 \, \mathrm{d}x \geq 2\pi\, |d|\, |\log \varepsilon| - C \tag{4.44}$$

and then, by Proposition 4.2,

$$\int_\Omega \left(\frac{\varepsilon}{2} |\nabla \mathbf{M}_\varepsilon^*|^2 + \frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*)\right) \mathrm{d}x \leq C, \tag{4.45}$$

for some constant C that depends only on the domain and the boundary data.

Now, we apply Proposition 3.2:

$$\begin{aligned} \mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*; G) \geq &\int_G \left(\frac{1}{2} |\nabla \mathbf{Q}_\varepsilon^*|^2 + g_\varepsilon(\mathbf{Q}_\varepsilon^*)\right) \mathrm{d}x \\ &+ \frac{1}{2} \int_G \left(\frac{\varepsilon}{2} |\nabla \mathbf{u}_\varepsilon^*|^2 + \frac{1}{\varepsilon} h(\mathbf{u}_\varepsilon^*)\right) \mathrm{d}x + \mathrm{o}(1) \end{aligned} \tag{4.46}$$

We have used (3.9) and the elementary inequality $ab \leq a^2/2 + b^2/2$ to estimate the remainder term R_ε. From (4.46), we obtain

$$\begin{aligned} &\int_G \left(\frac{\varepsilon}{2}\left|\nabla \mathbf{u}_\varepsilon^*\right|^2 + \frac{1}{\varepsilon}h(\mathbf{u}_\varepsilon^*)\right) \mathrm{d}x + 2\int_G g_\varepsilon(\mathbf{Q}_\varepsilon^*)\,\mathrm{d}x \\ &\qquad \leq 2\int_G \left(\frac{\varepsilon}{2}\left|\nabla \mathbf{M}_\varepsilon^*\right|^2 + \frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}_\varepsilon^*,\, \mathbf{M}_\varepsilon^*)\right) \mathrm{d}x + \mathrm{o}(1) \overset{(4.45)}{\leq} C \end{aligned} \tag{4.47}$$

Lemma 3.3 gives $g_\varepsilon \geq 0$, so the lemma follows. □

Proposition 4.10. *There exist a map* $\mathbf{M}^* \in \mathrm{SBV}(\Omega,\, \mathbb{R}^2)$ *and a (non-relabelled) subsequence such that* $\mathbf{M}_\varepsilon^* \to \mathbf{M}^*$ *a.e. and strongly in* $L^p(\Omega,\, \mathbb{R}^2)$ *for any* $p < +\infty$*, as* $\varepsilon \to 0$*. Moreover,* $\mathscr{H}^1(\mathrm{S}_{\mathbf{M}^*}) < +\infty$ *and* $\mathbf{M}^*$ *satisfies*

$$|\mathbf{M}^*| = (\sqrt{2}\beta + 1)^{1/2}, \tag{4.48}$$

$$\mathbf{Q}^* = \sqrt{2}\left(\frac{\mathbf{M}^* \otimes \mathbf{M}^*}{\sqrt{2}\beta + 1} - \frac{\mathbf{I}}{2}\right) \tag{4.49}$$

a.e. on Ω*.*

Proof. Let $G \subset\subset \Omega \setminus \{a_1^*,\, \ldots,\, a_{2|d|}^*,\, b_1^*,\, \ldots,\, b_K^*\}$. By Proposition 4.5, we have $\mathbf{Q}_\varepsilon^* \to \mathbf{Q}^*$ strongly in $W^{1,2}(G)$ and, up to extraction of a subsequence, pointwise a.e. in G. By differentiating the identity (4.41), we obtain that

$$|\nabla \mathbf{n}_\varepsilon^*|^2 = |\nabla \mathbf{m}_\varepsilon^*|^2 \lesssim \left|\nabla\left(\frac{\mathbf{Q}_\varepsilon^*}{|\mathbf{Q}_\varepsilon^*|}\right)\right|^2 \lesssim |\nabla \mathbf{Q}_\varepsilon^*|^2$$

(the last inequality follows because $|\mathbf{Q}_\varepsilon^*| \geq 1/2$ in G, by Lemma 4.3). In particular, $\mathbf{n}_\varepsilon^*$, $\mathbf{m}_\varepsilon^*$ are bounded in $W^{1,2}(G)$. Therefore, there exists vector fields $\mathbf{n}^* \in W^{1,2}(G,\, \mathbb{S}^1)$, $\mathbf{m}^* \in W^{1,2}(G,\, \mathbb{S}^1)$ such that, up to extraction of a subsequence, there holds

$$\mathbf{n}_\varepsilon^* \rightharpoonup \mathbf{n}^*, \quad \mathbf{m}_\varepsilon^* \rightharpoonup \mathbf{m}^* \qquad \text{weakly in } W^{1,2}(G) \text{ and pointwise a.e. in } G. \tag{4.50}$$

By passing to the limit pointwise a.e. in (4.41), we obtain

$$\mathbf{Q}^* = \frac{1}{\sqrt{2}}\left(\mathbf{n}^* \otimes \mathbf{n}^* - \mathbf{m}^* \otimes \mathbf{m}^*\right) \qquad \text{in } G, \tag{4.51}$$

hence $(\mathbf{n}^*,\, \mathbf{m}^*)$ is an orthonormal set of eigenvectors for $\mathbf{Q}^*$. In fact, $\mathbf{n}^*$, $\mathbf{m}^*$ must be smooth, because $\mathbf{Q}^*$ is smooth (by Proposition 4.8).

Lemma 4.9, combined with compactness results for the vectorial Modica-Mortola functional (see e.g. [7] or [28, Theorems 3.1 and 4.1]), implies that there exists a (non-relabelled) subsequence and a map $\mathbf{u}^* \in \mathrm{BV}(G,\, \mathbb{R}^2)$ such that

$$\mathbf{u}_\varepsilon^* \to \mathbf{u}^* \quad \text{strongly in } L^1(G) \text{ and a.e. in } G, \qquad h(\mathbf{u}^*) = 0 \quad \text{a.e. in } G \tag{4.52}$$

and

$$\mathscr{H}^1(\mathrm{S}_{\mathbf{u}^*} \cap G) \lesssim \liminf_{\varepsilon\to 0} \int_G \left(\frac{\varepsilon}{2} \left|\nabla \mathbf{u}^*_\varepsilon\right|^2 + \frac{1}{\varepsilon} h(\mathbf{u}^*_\varepsilon)\right) \mathrm{d}x \leq C \tag{4.53}$$

for some constant C that does not depend on G. As $h(\mathbf{u}^*) = 0$ a.e., necessarily $\mathbf{u}^*$ must take the form

$$\mathbf{u}^*(x) = \left(\tau(x)\,(\sqrt{2}\beta + 1)^{1/2},\, 0\right) \qquad \text{for a.e. } x \in G,$$

where $\tau(x) \in \{1, -1\}$ is a sign (see Lemma 3.4). Since $\mathbf{u}^*$ takes values in a finite set, the distributional derivative $\mathrm{D}\mathbf{u}^*$ must be concentrated on $\mathrm{S}_{\mathbf{u}^*}$, so $\mathbf{u}^* \in \mathrm{SBV}(G, \mathbb{R}^2)$.

We define

$$\mathbf{M}^* := (u^*)_1\, \mathbf{n}^* + (u^*)_2\, \mathbf{m}^* = \tau\, (\sqrt{2}\beta + 1)^{1/2}\, \mathbf{n}^* \qquad \text{in } G. \tag{4.54}$$

The vector field $\mathbf{M}^*$ is well-defined and does not depend on the choice of the orientation for $\mathbf{n}^*_\varepsilon$, $\mathbf{m}^*_\varepsilon$ (so long as the orientation is chosen consistently as $\varepsilon \to 0$, in such a way that (4.50) is satisfied). Indeed, if we replace $\mathbf{n}^*_\varepsilon$ by $-\mathbf{n}^*_\varepsilon$, then also $(\mathbf{u}^*_\varepsilon)_1$ will change its sign and the product at the right-hand side of (4.54) will remain unaffected. Therefore, by letting G vary in $\Omega \setminus \{a^*_1, \ldots, a^*_{2|d|}, b^*_1, \ldots, b^*_K\}$, we can define $\mathbf{M}^*$ almost everywhere in Ω. An explicit computation, based on (4.51) and (4.54), shows that $\mathbf{M}^*$ satisfies (4.48) and (4.49). Moreover, due to (4.50) and (4.52), we have $\mathbf{M}^*_\varepsilon \to \mathbf{M}^*$ a.e. in G. As the sequence $\mathbf{M}_\varepsilon$ is uniformly bounded in $L^\infty(\Omega)$ (by Lemma 4.1), Lebesgue's dominated convergence theorem implies that $\mathbf{M}^*_\varepsilon \to \mathbf{M}^*$ in $L^p(\Omega)$ for any $p < +\infty$.

As we have seen, $\mathbf{u}^* \in \mathrm{SBV}(G, \mathbb{R}^2)$ for any $G \subset\subset \Omega \setminus \{a^*_1, \ldots, a^*_{2|d|}, b^*_1, \ldots, b^*_K\}$. Therefore, by applying the BV-chain rule (see e.g. [3, Theorem 3.96]) to (4.56), and letting G vary, we obtain

$$\mathbf{M}^* \in \mathrm{SBV}_{\mathrm{loc}}(\Omega \setminus \{a^*_1, \ldots, a^*_{2|d|}, b^*_1, \ldots, b^*_K\}; \mathbb{R}^2) \tag{4.55}$$

Moreover, we claim that

$$\mathbf{M}^* \in \mathrm{SBV}(\Omega \setminus \{a^*_1, \ldots, a^*_{2|d|}, b^*_1, \ldots, b^*_K\}; \mathbb{R}^2), \qquad \mathscr{H}^1(\mathrm{S}_{\mathbf{M}^*}) < +\infty \tag{4.56}$$

Indeed, the absolutely continuous part $\nabla\mathbf{M}^*$ of the distributional derivative $\mathrm{D}\mathbf{M}^*$ can be bounded by differentiating (4.49): the BV-chain rule implies

$$\left|\nabla\mathbf{M}^*\right| = \frac{\sqrt{2}\beta + 1}{2} \left|\nabla\mathbf{Q}^*\right|,$$

and hence,

$$\|\nabla\mathbf{M}^*\|_{L^1(\Omega)} \leq \frac{\sqrt{2}\beta + 1}{2} \|\nabla\mathbf{Q}^*\|_{L^1(\Omega)} < +\infty$$

due to Lemma 4.3. The total variation of the jump part of $\mathrm{D}\mathbf{M}^*$ is uniformly bounded, too, because of (4.53) (the constant at the right-hand side of (4.53) does not depend on G, so we may take the limit as $G \searrow \Omega$). Then, (4.56) follows.

In order to complete the proof, it only remains to show that $\mathbf{M} \in \mathrm{SBV}(\Omega, \mathbb{R}^2)$. Let $\varphi \in C^\infty_{\mathrm{c}}(\Omega)$ be a test function, and let $\sigma > 0$ be fixed. We define

$$U_\sigma := \bigcup_{i=1}^{2|d|} B_\sigma(a_i) \cup \bigcup_{k=1}^{K} B_\sigma(b_k)$$

We choose a smooth cut-off function ψ_σ such that $0 \leq \psi_\sigma \leq 1$ in Ω, $\psi_\sigma = 0$ in $\Omega \setminus U_\sigma$, $\psi_\sigma = 1$ in a neighbourhood of each point $a_1, \ldots, a_{2|d|}$, $b_1, \ldots, b_K$, and $\|\nabla\psi_\sigma\|_{L^\infty(\Omega)} \leq C\sigma$ for some constant C that does not depend on σ. Then, for $j \in \{1, 2\}$, we have

$$\int_\Omega \mathbf{M}^* \, \partial_j\varphi = \int_\Omega \mathbf{M}^* \, \partial_j \left(\varphi(1 - \psi_\sigma)\right) + \int_\Omega \mathbf{M}^* \left(\psi_\sigma \, \partial_j\varphi + \varphi \, \partial_j\psi_\sigma\right)$$

We bound the first term in the right-hand side by applying (4.56). To estimate the second term, we observe that the integrand is bounded and supported in U_σ. Therefore, we obtain

$$\begin{aligned} \int_\Omega \mathbf{M}^* \, \partial_j\varphi &\lesssim \|\varphi\|_{L^\infty(\Omega)} + \|\mathbf{M}^*\|_{L^\infty(\Omega)} \|\nabla\varphi\|_{L^\infty(\Omega)} |U_\sigma| \\ &\qquad + \|\mathbf{M}^*\|_{L^\infty(\Omega)} \|\varphi\|_{L^\infty(\Omega)} \|\nabla\psi_\sigma\|_{L^\infty(\Omega)} |U_\sigma| \\ &\lesssim \|\varphi\|_{L^\infty(\Omega)} + \|\mathbf{M}^*\|_{L^\infty(\Omega)} \left(\sigma^2 \|\nabla\varphi\|_{L^\infty(\Omega)} + \sigma \|\varphi\|_{L^\infty(\Omega)}\right) \end{aligned} \tag{4.57}$$

By taking the limit as $\sigma \to 0$, we deduce that $\mathbf{M}^* \in \mathrm{BV}(\Omega, \mathbb{R}^2)$. In fact, we must have $\mathbf{M}^* \in \mathrm{SBV}(\Omega, \mathbb{R}^2)$, because the Cantor part of $\mathrm{D}\mathbf{M}^*$ cannot be supported on a finite number of points, $a_1, \ldots, a_{2|d|}$, $b_1, \ldots, b_K$. This completes the proof. $\square$

4.4 Proof of Statements (v) and (vi): sharp energy estimates

In this section, we complete the proof of Theorem 2.1, by describing the structure of the jump set of $\mathbf{M}^*$ and characterising the optimal position of the defects of $\mathbf{Q}^*$ (in case the domain Ω is convex). As a byproduct of our arguments, we will also show a refined energy estimate for the minimisers $(\mathbf{Q}^*_\varepsilon, \mathbf{M}^*_\varepsilon)$, i.e. Proposition 4.11 below.

First, we set some notations. We let

$$c_\beta := \frac{2\sqrt{2}}{3} \left(\sqrt{2}\beta + 1\right)^{3/2} \tag{4.58}$$

For any $(2\,|d|)$-uple of distinct points $a_1, \ldots, a_{2|d|}$ in Ω, we define

$$\mathbb{W}_\beta(a_1, \ldots, a_{2|d|}) := \mathbb{W}(a_1, \ldots, a_{2|d|}) + c_\beta \, \mathbb{L}(a_1, \ldots, a_{2|d|}) \tag{4.59}$$

where $\mathbb{W}$, $\mathbb{L}$ are, respectively, the Ginzburg-Landau renormalised energy (defined in (2.13)) and the length of a minimal connection (defined in (2.14)). We also recall the definition of the Ginzburg-Landau *core energy*, which was introduced in [11]. Let $B_1 \subseteq \mathbb{R}^2$ be the unit disk. For any $\varepsilon > 0$, let

$$\gamma(\varepsilon) := \inf\left\{\int_{B_1} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(|u|^2 - 1)^2\right) \mathrm{d}x \colon u \in W^{1,2}(B_1, \mathbb{C}),\ u(x) = x \text{ for } x \in \partial B_1\right\}$$

It can be proved (see [11, Lemma III.3]) that the function $\varepsilon \mapsto \gamma(\varepsilon) - \pi\,|\log\varepsilon|$ is finite in (0, 1) and non-decreasing. Therefore, the limit

$$\gamma_* := \lim_{\varepsilon\to 0}\left(\gamma(\varepsilon) - \pi\,|\log\varepsilon|\right) > 0 \tag{4.60}$$

exists and is finite. The number γ_* is the so-called core energy. In this section, we will prove the following result:

Proposition 4.11. *If the domain* $\Omega \subseteq \mathbb{R}^2$ *is* convex*, then*

$$\mathscr{F}_\varepsilon(\mathbf{Q}^*_\varepsilon,\, \mathbf{M}^*_\varepsilon) = 2\pi\,|d|\,|\log\varepsilon| + \mathbb{W}_\beta(a^*_1,\, \ldots,\, a^*_{2|d|}) + 2\,|d|\,\gamma_* + \mathrm{o}(1) \tag{4.61}$$

as $\varepsilon \to 0$.

We will prove the lower and upper inequality in (4.61) separately. From now on, we alwasy assume that the domain Ω is *convex.*

4.4.1 Sharp lower bounds for the energy of minimisers

The aim of this section is to prove a sharp lower bound for $\mathscr{F}_\varepsilon(\mathbf{Q}^*_\varepsilon,\, \mathbf{M}^*_\varepsilon)$. We know from previous results (Lemma 4.3, Proposition 4.10), that, up to extraction of a subsequence, we have $\mathbf{Q}^*_\varepsilon \to \mathbf{Q}^*$, $\mathbf{M}^*_\varepsilon \to \mathbf{M}^*$ a.e., where

$$\mathbf{Q}^* \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{a^*_1,\, \ldots,\, a^*_{2|d|}\},\, \mathscr{N}), \qquad \mathbf{M}^* \in \mathrm{SBV}(\Omega,\, \mathbb{R}^2)$$

Due to Remark 4.1, we may further assume that

$$\frac{|\mathbf{Q}^*_\varepsilon| - 1}{\varepsilon} \rightharpoonup \xi_* \qquad \text{weakly in } L^2(\Omega). \tag{4.62}$$

Proposition 4.12. *There holds*

$$\begin{aligned}\liminf_{\varepsilon\to 0}&\left(\mathscr{F}_\varepsilon(\mathbf{Q}^*_\varepsilon,\, \mathbf{M}^*_\varepsilon) - 2\pi\,|d|\,|\log\varepsilon|\,\right)\\ &\geq \mathbb{W}(a^*_1,\, \ldots,\, a^*_{2|d|}) + c_\beta\,\mathscr{H}^1(\mathrm{S}_{\mathbf{M}^*}) + \int_\Omega (\xi_* - \kappa_*)^2\,\mathrm{d}x + 2\,|d|\,\gamma_*\end{aligned}$$

where the constants c_β*,* κ_* *are given, respectively, by* (4.58) *and* (3.1)*.*

The length of the jump set $\mathrm{S}_{\mathbf{M}^*}$ can be further bounded from below, in terms of the singular points $a^*_1,\, \ldots,\, a^*_{2|d|}$. We recall from Section 2 that a connection for $a^*_1,\, \ldots,\, a^*_{2|d|}$ is a finite collection of straight line segments $L_1,\, \ldots,\, L_{|d|}$ that connects the points a^*_j in pairs, and that $\mathbb{L}(a^*_1,\, \ldots,\, a^*_{2|d|})$ is the minimal length of a connection for the points a^*_j (see (2.14)). Given two sets A, B, we denote by $A\Delta B$ their symmetric difference, i.e. $A\Delta B := (A \setminus B) \cup (B \setminus A)$.

Proposition 4.13. *We have*

$$\mathscr{H}^1(\mathrm{S}_{\mathbf{M}^*}) \geq \mathbb{L}(a^*_1,\, \ldots,\, a^*_{2d}) \tag{4.63}$$

The equality in (4.63) *holds if and only if there exists a minimal connection* $\{L^*_1,\, \ldots,\, L^*_{|d|}\}$ *for* $\{a^*_1, \ldots, a^*_{2|d|}\}$ *such that*

$$\mathscr{H}^1\left(\mathrm{S}_{\mathbf{M}^*}\,\Delta\,\bigcup_{j=1}^{d} L^*_j\right) = 0$$

We will give the proof of Proposition 4.13 in Appendix A. Here, instead, we focus on the proof of Proposition 4.12.

Lemma 4.14. *Let $h\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined in* (3.8)*, and let* $\mathbf{u}_\pm := (\pm(\sqrt{2}\beta+1)^{1/2},\, 0)$*. Then, there holds*

$$\inf\left\{\int_0^1 \sqrt{2h(\mathbf{u}(t))}\,\left|\mathbf{u}'(t)\right| \mathrm{d}t\colon \mathbf{u}\in W^{1,1}([0,\,1],\,\mathbb{R}^2),\ \mathbf{u}(0)=\mathbf{u}_-,\ \mathbf{u}(1)=\mathbf{u}_+\right\} = c_\beta \tag{4.64}$$

with c_β given by (4.58)*.*

Proof. Let $\mathbf{u}\in W^{1,1}([0,\,1],\,\mathbb{R}^2)$ be such that $\mathbf{u}(0)=\mathbf{u}_-$, $\mathbf{u}(1)=\mathbf{u}_+$. We define $\tilde{\mathbf{u}}\colon [0,\,1]\to\mathbb{R}^2$ as $\tilde{\mathbf{u}}(t) := (\left|\mathbf{u}(t)\right|,\, 0)$. We have $h(\tilde{\mathbf{u}}(t)) \le h(\mathbf{u}(t))$ for any t and

$$\left|\tilde{\mathbf{u}}'(t)\right| = \left|\mathbf{u}'(t)\cdot\frac{\mathbf{u}(t)}{\left|\mathbf{u}(t)\right|}\right| \le \left|\mathbf{u}'(t)\right|$$

for a.e. $t\in[0,\,1]$ such that $\mathbf{u}(t)\neq 0$. On the other hand, Stampacchia's lemma implies that $\mathbf{u}'=0$ a.e. on the set $\mathbf{u}^{-1}(0)$ and similarly, $\tilde{\mathbf{u}}'=0$ a.e. on $\tilde{\mathbf{u}}^{-1}(0)$. Therefore, we have

$$\int_0^1 \sqrt{h(\tilde{\mathbf{u}}(t))}\,\left|\tilde{\mathbf{u}}'(t)\right| \mathrm{d}t \le \int_0^1 \sqrt{h(\mathbf{u}(t))}\,\left|\mathbf{u}'(t)\right| \mathrm{d}t.$$

As a consequence, in the left-hand side of (4.64) we can minimise under the additional constraint that $u_2\equiv 0$, without loss of generality. In other words, we have shown that

$$\begin{aligned} I &:= \inf\left\{\int_0^1 \sqrt{2h(\mathbf{u}(t))}\,\left|\mathbf{u}'(t)\right| \mathrm{d}t\colon \mathbf{u}\in W^{1,1}([0,\,1],\,\mathbb{R}^2),\ \mathbf{u}(0)=\mathbf{u}_-,\ \mathbf{u}(1)=\mathbf{u}_+\right\} \\ &= \inf\left\{\int_0^1 \sqrt{2h(u_1(t),\,0)}\,\left|u_1'(t)\right| \mathrm{d}t\colon u_1\in W^{1,1}(0,\,1),\ u_1(0)=-\lambda,\ u_1(1)=\lambda\right\} \end{aligned}$$

where $\lambda := (\sqrt{2}\beta+1)^{1/2}$. Equation (3.8) implies, by an explicit computation,

$$\sqrt{2h(u_1,\,0)} = \frac{1}{\sqrt{2}}\left|\lambda^2 - u_1^2\right|$$

By making the change of variable $y = u_1(t)$, we deduce

$$\begin{aligned} I &\ge \inf\left\{\int_0^1 \sqrt{2h(u_1(t),\,0)}\,u_1'(t)\,\mathrm{d}t\colon u_1\in W^{1,1}(0,\,1),\ u_1(0)=-\lambda,\ u_1(1)=\lambda\right\} \\ &= \int_{-\lambda}^{\lambda}\sqrt{2h(y,\,0)}\,\mathrm{d}y = \frac{1}{\sqrt{2}}\int_{-\lambda}^{\lambda}\left(\lambda^2-y^2\right)\mathrm{d}y = \frac{2\sqrt{2}}{3}\lambda^3 \end{aligned} \tag{4.65}$$

We take as a competitor in (4.64) the map $\mathbf{v}(t) := (-\lambda + 2t\lambda,\, 0)$. By similar computations, we obtain

$$I \le \int_0^1 \sqrt{2h(\mathbf{v}(t))}\,\left|\mathbf{v}'(t)\right| \mathrm{d}t = \frac{2\sqrt{2}}{3}\lambda^3 \tag{4.66}$$

and the lemma follows. □

Lemma 4.15. *Let $G \subset\subset \Omega \setminus \{a_1^*, \ldots, a_{2|d|}^*, b_1^*, \ldots, b_K^*\}$ be a simply connected domain. Then,*

$$\liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*; G) \geq \frac{1}{2}\int_G |\nabla \mathbf{Q}^*|^2 \,\mathrm{d}x + \int_G (\xi_* - \kappa_*)^2 \,\mathrm{d}x + c_\beta \mathscr{H}^1(\mathrm{S}_{\mathbf{M}^*} \cap G)$$

Proof. We make a change of variable, as introduced in Section 3. Let $\mathbf{u}_\varepsilon^*$ be the vector field defined in (3.6). By Proposition 3.2, we have

$$\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*; G) = \int_G \left(\frac{1}{2}|\nabla \mathbf{Q}_\varepsilon^*|^2 + g_\varepsilon(\mathbf{Q}_\varepsilon^*)\right)\mathrm{d}x + \int_G \left(\frac{\varepsilon}{2}|\nabla \mathbf{u}_\varepsilon^*|^2 + \frac{1}{\varepsilon}h(\mathbf{u}_\varepsilon^*)\right)\mathrm{d}x + R_\varepsilon \tag{4.67}$$

and the remainder term R_ε satisfies

$$|R_\varepsilon| \lesssim \varepsilon^{1/2}|\log\varepsilon|^{1/2}\left(\int_G \left(\frac{\varepsilon}{2}|\nabla \mathbf{u}_\varepsilon^*|^2 + \frac{1}{\varepsilon}h(\mathbf{u}_\varepsilon^*)\right)\mathrm{d}x\right)^{1/2} + \mathrm{o}(1) \qquad \text{as } \varepsilon \to 0. \tag{4.68}$$

Lemma 4.9 implies that $R_\varepsilon \to 0$ as $\varepsilon \to 0$. We estimate separately the other terms in the right-hand side of (4.67). The weak convergence $\mathbf{Q}_\varepsilon^* \rightharpoonup \mathbf{Q}^*$ in $W^{1,2}(G)$ implies

$$\liminf_{\varepsilon\to 0} \frac{1}{2}\int_G |\nabla \mathbf{Q}_\varepsilon^*|^2 \,\mathrm{d}x \geq \frac{1}{2}\int_G |\nabla \mathbf{Q}^*|^2 \,\mathrm{d}x \tag{4.69}$$

We claim that

$$\liminf_{\varepsilon\to 0} \int_G g_\varepsilon(\mathbf{Q}_\varepsilon^*)\,\mathrm{d}x \geq \int_G (\xi_* - \kappa_*)^2 \,\mathrm{d}x \tag{4.70}$$

Indeed, Lemma 3.3 gives

$$g_\varepsilon(\mathbf{Q}_\varepsilon^*) = (\xi_\varepsilon - \kappa_*)^2 + \xi_\varepsilon^2\,\zeta_\varepsilon \tag{4.71}$$

where

$$\xi_\varepsilon := \frac{1}{\varepsilon}(|\mathbf{Q}_\varepsilon^*| - 1), \qquad \zeta_\varepsilon := \frac{1}{4}(|\mathbf{Q}_\varepsilon^*| + 1)^2 - 1 \geq -1$$

Let $\delta > 0$ be a small parameter. By Lemma 4.3, we have $|\mathbf{Q}_\varepsilon^*| \to |\mathbf{Q}^*| = 1$ a.e. in Ω and hence, $\zeta_\varepsilon \to 0$ a.e. in G. Therefore, by the Severini-Egoroff theorem, there exists a Borel set $\tilde{G} \subseteq G$ such that $|G \setminus \tilde{G}| \leq \delta$ and $\zeta_\varepsilon \to 0$ uniformly in $\tilde{G}$ as $\varepsilon \to 0$. Now, we have

$$\int_G g_\varepsilon(\mathbf{Q}_\varepsilon^*)\,\mathrm{d}x \overset{(4.71)}{\geq} \int_{\tilde{G}} (\xi_\varepsilon - \kappa_*)^2\,\mathrm{d}x + \int_{\tilde{G}} \xi_\varepsilon^2\,\zeta_\varepsilon\,\mathrm{d}x + \int_{G\setminus\tilde{G}} (-2\xi_\varepsilon\,\kappa_* + \kappa_*^2)\,\mathrm{d}x$$

The integral of $\xi_\varepsilon^2\,\zeta_*$ on $\tilde{G}$ tends to zero, because ξ_ε is bounded in $L^2(G)$ (by Remark 4.1) and $\zeta_\varepsilon \to 0$ uniformly in $\tilde{G}$. As $\xi_\varepsilon \rightharpoonup \xi_*$ weakly in $L^2(G)$ (see (4.62)), we deduce

$$\liminf_{\varepsilon\to 0} \int_G g_\varepsilon(\mathbf{Q}_\varepsilon^*)\,\mathrm{d}x \geq \int_{\tilde{G}} (\xi_* - \kappa_*)^2\,\mathrm{d}x + \int_{G\setminus\tilde{G}} (-2\xi_*\,\kappa_* + \kappa_*^2)\,\mathrm{d}x$$

The area of $G \setminus \tilde{G}$ may be taken arbitrarily small, so (4.70) follows.

Finally, for the term in $\mathbf{u}_\varepsilon^*$, we apply classical Γ-convergence results for the vectorial Modica-Mortola functional (see e.g. [7, 28]), as well as Lemma 4.14:

$$\liminf_{\varepsilon\to 0} \int_G \left(\frac{\varepsilon}{2}\left|\nabla \mathbf{u}_\varepsilon\right|^2 + \frac{1}{\varepsilon}h(\mathbf{u}_\varepsilon)\right) \mathrm{d}x \geq c_\beta\, \mathscr{H}^1(\mathrm{S}_{\mathbf{u}^*} \cap G) \tag{4.72}$$

where $\mathbf{u}^*$ is the $L^1(G)$-limit of the sequence $\mathbf{u}_\varepsilon^*$, as in (4.52). By (4.54), we have $\mathrm{S}_{\mathbf{u}^*} = \mathrm{S}_{\mathbf{M}^*}$ and hence,

$$\liminf_{\varepsilon\to 0} \int_G \left(\frac{\varepsilon}{2}\left|\nabla \mathbf{u}_\varepsilon\right|^2 + \frac{1}{\varepsilon}h(\mathbf{u}_\varepsilon)\right) \mathrm{d}x \geq c_\beta\, \mathscr{H}^1(\mathrm{S}_{\mathbf{M}^*} \cap G) \tag{4.73}$$

Combining (4.67), (4.68), (4.69), (4.70) and (4.73), the lemma follows. □

Lemma 4.16. *For any $\sigma > 0$ sufficiently small and any $j \in \{1, \ldots, 2\,|d|\}$, we have*

$$\liminf_{\varepsilon\to 0} \Big(\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon^*,\, \mathbf{M}_\varepsilon^*;\, B_\sigma(a_j^*)) - \pi\,|\log\varepsilon|\Big) \geq \pi\log\sigma + \gamma_* - C\sigma,$$

where γ_* is the constant given by (4.60) *and C is a constant that does not depend on ε, σ.*

Proof. Take σ is so small that the ball $B_\sigma(a_j^*)$ does not contain any other singular point a_k^*, with $k \neq j$. We consider the function $J(\mathbf{Q}_\varepsilon^*)$ defined in (4.30). By Lemma 4.6, we have

$$J(\mathbf{Q}_\varepsilon^*) \to \pi\,\mathrm{sign}(d)\,\delta_{a_j^*} \qquad \text{in } W^{-1,1}(B_\sigma(a_j^*)).$$

Then, we can apply pre-existing Γ-convergence results for the Ginzburg-Landau functional — for instance, [1, Theorem 5.3]. We obtain a (sharp) lower bound for the Ginzburg-Landau energy of $\mathbf{Q}_\varepsilon^*$:

$$\liminf_{\varepsilon\to 0}\left(\int_{B_\sigma(a_j^*)} \left(\frac{1}{2}\left|\nabla \mathbf{Q}_\varepsilon^*\right|^2 + \frac{1}{4\varepsilon^2}(|\mathbf{Q}_\varepsilon^*|^2 - 1)^2\right)\mathrm{d}x - \pi\,|\log\varepsilon|\right) \geq \pi\log\sigma + \gamma_* \tag{4.74}$$

On the other hand, Lemma 3.1 gives

$$\frac{1}{\varepsilon^2}\int_{B_\sigma(a_j^*)} f_\varepsilon(\mathbf{Q}_\varepsilon^*,\, \mathbf{M}_\varepsilon^*)\,\mathrm{d}x \geq \frac{1}{4\varepsilon^2}\int_{B_\sigma(a_j^*)} (|\mathbf{Q}_\varepsilon^*|^2 - 1)^2\,\mathrm{d}x - \frac{\beta}{\sqrt{2}\varepsilon}\int_{B_\sigma(a_j^*)} |\mathbf{M}_\varepsilon^*|^2\,\big||\mathbf{Q}_\varepsilon^*| - 1\big|\,\mathrm{d}x$$

As $\mathbf{M}_\varepsilon^*$ is uniformly bounded in $L^\infty(\Omega)$ (by Lemma 4.1), we obtain via the Hölder inequality

$$\frac{1}{\varepsilon^2}\int_{B_\sigma(a_j^*)} f_\varepsilon(\mathbf{Q}_\varepsilon^*,\, \mathbf{M}_\varepsilon^*)\,\mathrm{d}x \geq \frac{1}{4\varepsilon^2}\int_{B_\sigma(a_j^*)} (|\mathbf{Q}_\varepsilon^*|^2 - 1)^2\,\mathrm{d}x - C\sigma\left(\frac{1}{\varepsilon^2}\int_\Omega (|\mathbf{Q}_\varepsilon^*| - 1)^2\,\mathrm{d}x\right)^{1/2}$$

The constant C here depends only on β. Finally, Remark 4.1 implies

$$\frac{1}{\varepsilon^2}\int_\Omega (|\mathbf{Q}_\varepsilon^*| - 1)^2\,\mathrm{d}x \leq \frac{1}{\varepsilon^2}\int_\Omega (|\mathbf{Q}_\varepsilon^*|^2 - 1)^2\,\mathrm{d}x \leq C$$

and hence,

$$\frac{1}{\varepsilon^2}\int_{B_\sigma(a_j^*)} f_\varepsilon(\mathbf{Q}_\varepsilon^*,\, \mathbf{M}_\varepsilon^*)\,\mathrm{d}x \geq \frac{1}{4\varepsilon^2}\int_{B_\sigma(a_j^*)} (|\mathbf{Q}_\varepsilon^*|^2 - 1)^2\,\mathrm{d}x - C\sigma \tag{4.75}$$

Combining (4.74) with (4.75), the lemma follows. □

We can now complete the proof of Proposition 4.12.

Proof of Proposition 4.12. Let $\sigma > 0$ be small enough that the balls $B_\sigma(a_j^*)$, $B_\sigma(b_k^*)$ are pairwise disjoint. We define

$$\Omega_\sigma := \Omega \setminus \left(\bigcup_{j=1}^{2|d|} \bar{B}_\sigma(a_j^*) \cup \bigcup_{k=1}^{K} \bar{B}_\sigma(b_k^*) \right)$$

We construct open sets $G_1, \ldots, G_N$ with the following properties:

(i) the sets G_i are pairwise disjoint;

(ii) their closures, $\overline{G}_i$, cover all of Ω_σ;

(iii) each G_j is simply connected;

(iv) $\mathscr{H}^1(\mathrm{S}_{\mathbf{M}^*} \cap \partial G_i \cap \Omega_\sigma) = 0$ for any j.

For instance, we can partition Ω_σ by considering a grid, consisting of finitely many vertical and horizontal lines. Since $\mathscr{H}^1(\mathrm{S}_{\mathbf{M}^*}) < +\infty$ by Proposition 4.10, we have

$$\mathscr{H}^1\left(\mathrm{S}_{\mathbf{M}^*} \cap (\{c\} \times \mathbb{R})\right) = \mathscr{H}^1\left(\mathrm{S}_{\mathbf{M}^*} \cap (\mathbb{R} \times \{d\})\right) = 0 \tag{4.76}$$

for all but countably many values of $c \in \mathbb{R}$, $d \in \mathbb{R}$. We choose numbers

$$c_0 < c_1 < \ldots < c_{N_1}, \qquad d_0 < d_1 < \ldots < d_{N_2}$$

that satisfy (4.76), in such a way that $\overline{\Omega} \subseteq (c_0, c_{N_1}) \times (d_0, d_{N_2})$. For a suitable choice of c_h, d_ℓ, we can make sure that no ball $B_\sigma(a_j^*)$ or $B_\sigma(b_k^*)$ is entirely contained in a rectangle of the form $(c_h, c_{h+1}) \times (d_\ell, d_{\ell+1})$, and that any rectangle $(c_h, c_{h+1}) \times (d_\ell, d_{\ell+1})$ intersects at most one of the balls. Then, the sets

$$G_{h,\ell} := ((c_h, c_{h+1}) \times (d_\ell, d_{\ell+1})) \cap \Omega_\sigma$$

are all simply connected and satisfy the properties (i)–(iv) above. We relabel the $G_{h,\ell}$'s as G_i.

We apply Lemma 4.15 on each G_i, then sum over all the indices i. We obtain

$$\begin{aligned} \mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*; \Omega_\sigma) \geq \frac{1}{2} \int_{\Omega_\sigma} |\nabla \mathbf{Q}^*|^2 \, \mathrm{d}x + \int_{\Omega_\sigma} (\xi_* - \kappa_*)^2 \, \mathrm{d}x \\ + c_\beta \, \mathscr{H}^1(\mathrm{S}_{\mathbf{M}^*} \cap \Omega_\sigma) + \mathrm{o}_{\varepsilon\to 0}(1) \end{aligned} \tag{4.77}$$

On the other hand, Lemma 4.16 implies

$$\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*; B_\sigma(a_j^*)) - \pi \, |\log \varepsilon| \geq -\pi \, |\log \sigma| + \gamma_* - C\sigma + \mathrm{o}_{\varepsilon\to 0}(1) \tag{4.78}$$

for any $j \in \{1, \ldots, 2\,|d|\}$. Combining (4.77) with (4.78), we obtain

$$\begin{aligned} \mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*) - 2\pi \, |d| \, |\log \varepsilon| \geq \frac{1}{2} \int_{\Omega_\sigma} |\nabla \mathbf{Q}^*|^2 \, \mathrm{d}x - 2\pi \, |d| \, |\log \sigma| + 2\,|d| \, \gamma_* \\ + c_\beta \, \mathscr{H}^1(\mathrm{S}_{\mathbf{M}^*} \cap \Omega_\sigma) + \int_{\Omega_\sigma} (\xi_* - \kappa_*)^2 \, \mathrm{d}x + \mathrm{o}_{\varepsilon\to 0}(1) + \mathrm{o}_{\sigma\to 0}(1) \end{aligned} \tag{4.79}$$

By Proposition 4.8, $\mathbf{Q}^*$ is the canonical harmonic map with singularities at $(a_1^*, \ldots, a_{2|d|}^*)$ and boundary datum $\mathbf{Q}_{\mathrm{bd}}$. Then, we can write the right-hand side of (4.79) in terms of the renormalised energy, $\mathbb{W}$, defined in (2.13). First, we observe that

$$\frac{1}{2}\int_{\bigcup_{k=1}^K B_\sigma(b_k^*)} |\nabla \mathbf{Q}^*|^2 \,\mathrm{d}x \to 0 \qquad \text{as } \sigma \to 0 \tag{4.80}$$

because $\mathbf{Q}^* \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{a_1^*, \ldots, a_{2|d|}^*\}, \mathscr{S}_0^{2\times 2})$. Then, from (4.79), (4.80) and (2.13) we deduce

$$\begin{aligned} \mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*) - 2\pi\,|d|\,|\log\varepsilon| &\geq \mathbb{W}(a_1^*, \ldots, a_{2|d|}^*) + 2\,|d|\,\gamma_* \\ &+ c_\beta\,\mathscr{H}^1(\mathrm{S}_{\mathbf{M}^*} \cap \Omega_\sigma) + \int_{\Omega_\sigma} (\xi_* - \kappa_*)^2\,\mathrm{d}x + \mathrm{o}_{\varepsilon\to 0}(1) + \mathrm{o}_{\sigma\to 0}(1) \end{aligned} \tag{4.81}$$

Now we pass to the limit in both sides of (4.81), first as $\varepsilon \to 0$, then as $\sigma \to 0$. The proposition follows. □

4.4.2 Sharp upper bounds

In this section we will prove an upper bound for the energy of minimizers, namely the following:

Proposition 4.17. *Let $a_1, \ldots, a_{2|d|}$ be distinct points in Ω. Then, there exist maps $\mathbf{Q}_\varepsilon \in W^{1,2}(\Omega, \mathscr{S}_0^{2\times 2})$, $\mathbf{M}_\varepsilon \in W^{1,2}(\Omega, \mathbb{R}^2)$ that satisfy the boundary condition* (2.3) *and*

$$\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) \leq 2\pi\,|d|\,|\log\varepsilon| + \mathbb{W}_\beta(a_1, \cdots, a_{2|d|}) + 2\,|d|\,\gamma_* + \mathrm{o}_{\varepsilon\to 0}(1) \tag{4.82}$$

where $\mathbb{W}_\beta$ and γ_* are as in (4.59), (4.60) *respectively.*

The proof of Proposition 4.17 is based on a rather explicit construction. For the component $\mathbf{Q}_\varepsilon$, we follow classical arguments from the Ginzburg-Landau literature (see e.g. [11, 1]), with minor modifications. For the component $\mathbf{M}_\varepsilon$, we first construct a vector field $\tilde{\mathbf{M}}_\varepsilon\colon \Omega \to \mathbb{R}^2$ of constant norm, such that $\tilde{\mathbf{M}}_\varepsilon(x)$ is an eigenvector of $\mathbf{Q}_\varepsilon(x)$ at each point $x \in \Omega$. As $\mathbf{Q}_\varepsilon$ has non-orientable singularities at the points a_j, there is no smooth vector field $\tilde{\mathbf{M}}_\varepsilon$ with this property. However, we can construct a BV-vector field $\tilde{\mathbf{M}}_\varepsilon$, which jumps along finitely many line segments that join the points a_j along a minimal connection (see Appendix A). Then, we define $\mathbf{M}_\varepsilon$ by regularising $\tilde{\mathbf{M}}_\varepsilon$ in a small neighbourhood of the jump set. The regularisation procedure is reminiscent of the optimal profile problem for the Modica-Mortola functional [41].

Proof of Proposition 4.17. We follow the argument of [1], Theorem 5.3. Let $\sigma > 0$ be such that $B_\sigma(a_i)$ are disjoints and contained in Ω and set $\Omega_\sigma := \Omega \setminus \bigcup_{i=1}^{2|d|} B_\sigma(a_i)$. First, we minimize the functional

$$(\mathbf{Q}, \mathbf{R}_1, \ldots, \mathbf{R}_{2|d|}) \mapsto \frac{1}{2}\int_{\Omega_\sigma} |\nabla \mathbf{Q}|^2\,\mathrm{d}x \tag{4.83}$$

over all maps $\mathbf{Q} \in H^1(\Omega_\sigma, \mathscr{N})$ and all rotation matrices $\mathbf{R}_i \in \mathrm{SO}(2)$ such that $\mathbf{Q} = \mathbf{Q}_{\mathrm{bd}}$ on $\partial\Omega$ and

$$\mathbf{Q}(x) = \sqrt{2}\left(\frac{(\mathbf{R}_i(x - a_i)) \otimes (\mathbf{R}_i(x - a_i))}{\sigma^2} - \frac{\mathbf{I}}{2}\right) \quad \text{for } x \in \partial B_\sigma(a_i).$$

We denote by $m(\sigma)$ the minimum value and by $\tilde{\mathbf{P}}_1$, $\tilde{\mathbf{R}}_i$ the minimisers of this functional. Next, we minimise the Ginzburg-Landau energy, on a ball B_σ of radius σ centered at the origin,

$$GL_\varepsilon(\mathbf{Q}, B_\sigma) := \int_{B_\sigma} \Big(\frac{1}{2}|\nabla \mathbf{Q}|^2 + \frac{1}{4\varepsilon^2}(1 - |\mathbf{Q}|^2)^2\Big)\, \mathrm{d}x \tag{4.84}$$

among all the maps $\mathbf{Q} \in H^1(B_\sigma, \mathscr{S}_0^{2\times 2})$ such that

$$\mathbf{Q}(x) = \sqrt{2}\left(\frac{x\otimes x}{\sigma^2} - \frac{\mathbf{I}}{2}\right) \quad \text{for } x \in \partial B_\sigma.$$

We denote by $\gamma(\varepsilon, \sigma)$ the minimum value and by $\tilde{\mathbf{P}}_2$ the minimiser of this functional.

We define a map $\tilde{\mathbf{Q}}_\varepsilon \in H^1(\Omega, \mathscr{S}_0^{2\times 2})$ as

$$\tilde{\mathbf{Q}}_\varepsilon(x) := \begin{cases} \tilde{\mathbf{P}}_1(x) & \text{if } x \in \Omega_\sigma \\ \tilde{\mathbf{R}}_i \tilde{\mathbf{P}}_2(x - a_i)\tilde{\mathbf{R}}_i^\mathsf{T} & \text{if } x \in B_\sigma(a_i) \end{cases}$$

This map satisfies $\tilde{\mathbf{Q}}_\varepsilon = \mathbf{Q}_{\mathrm{bd}}$ on $\partial\Omega$, $|\tilde{\mathbf{Q}}_\varepsilon| \leq 1$ in Ω, $|\tilde{\mathbf{Q}}_\varepsilon| = 1$ in Ω_σ. Moreover, thanks to [11, Theorem I.9 and Section III.1], we have

$$\begin{aligned}
&\int_\Omega \Big(\frac{1}{2}|\nabla \tilde{\mathbf{Q}}_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1 - |\tilde{\mathbf{Q}}_\varepsilon|^2)^2\Big)\, \mathrm{d}x \\
&\qquad = m(\sigma) + 2\,|d|\,\gamma(\sigma, \varepsilon) \\
&\qquad = m(\sigma) + 2\,|d|\,\gamma_* - 2\,|d|\,\pi \log\frac{\varepsilon}{\sigma} + \mathrm{o}_{\frac{\varepsilon}{\sigma}\to 0}(1) \\
&\qquad = \mathbb{W}(a_1, \cdots, a_{2|d|}) - 2\pi\,|d| \log\sigma + 2\,|d|\,\gamma_* - 2\,|d|\,\pi\log\frac{\varepsilon}{\sigma} + \mathrm{o}_{\sigma\to 0}(1) + \mathrm{o}_{\frac{\varepsilon}{\sigma}\to 0}(1) \\
&\qquad = \mathbb{W}(a_1, \cdots, a_{2|d|}) + 2\,|d|\,\gamma_* + 2\,|d|\,\pi\log|\varepsilon| + \mathrm{o}_{\sigma\to 0}(1) + \mathrm{o}_{\frac{\varepsilon}{\sigma}\to 0}(1).
\end{aligned}$$

We will choose $\sigma = \sigma_\varepsilon$ in such a way that

$$\sigma_\varepsilon \to 0, \quad \frac{\varepsilon}{\sigma_\varepsilon} \to 0.$$

Define $\mathbf{Q}_\varepsilon := (1 + \varepsilon\kappa_*)\tilde{\mathbf{Q}}_\varepsilon$ on Ω. Therefore, $|\mathbf{Q}_\varepsilon| = 1 + \varepsilon\kappa_*$ on $\Omega_{\sigma_\varepsilon} = \Omega \setminus \bigcup_{j=1}^{2|d|} B_{\sigma_\varepsilon}(a_j)$. Moreover, we have

$$\left|\int_\Omega |\nabla \tilde{\mathbf{Q}}_\varepsilon|^2 - |\nabla \mathbf{Q}_\varepsilon|^2\right| \lesssim \kappa_*\varepsilon \int_\Omega |\nabla\tilde{\mathbf{Q}}_\varepsilon|^2 \lesssim \varepsilon\,|\log\varepsilon| \tag{4.85}$$

On the other hand, for the Ginzburg-Landau potential we have

$$\begin{aligned}
&\frac{1}{4\varepsilon^2}\left|\int_{\bigcup_{j=1}^{2|d|} B_{\sigma_\varepsilon}(a_j)} \Big(1 - |\tilde{\mathbf{Q}}_\varepsilon|^2\Big)^2 - \int_{\bigcup_{j=1}^{2|d|} B_{\sigma_\varepsilon}(a_j)} \Big(1 - |\mathbf{Q}_\varepsilon|^2\Big)^2\right| \\
&= \frac{1}{4\varepsilon^2}\left|\int_{\bigcup_{j=1}^{2|d|} B_{\sigma_\varepsilon}(a_j)} \Big(2(1+\varepsilon\kappa_*)^2 - 2\Big)|\tilde{\mathbf{Q}}_\varepsilon|^2 - \Big((1+\varepsilon\kappa_*)^4 - 1\Big)|\tilde{\mathbf{Q}}_\varepsilon|^4\right| = \mathrm{O}(\varepsilon)
\end{aligned}$$

since by construction $|\tilde{\mathbf{Q}}_\varepsilon| \leq 1$. In conclusion, we have:

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla\mathbf{Q}_\varepsilon|^2\,\mathrm{d}x + \frac{1}{4\varepsilon^2}\int_{\bigcup_{j=1}^{2|d|} B_{\sigma_\varepsilon}(a_j)} (1-|\mathbf{Q}_\varepsilon|^2)^2\,\mathrm{d}x \\ = \mathbb{W}(a_1,\cdots,a_{2|d|}) + 2\,|d|\,\gamma_* + 2\,|d|\,\pi\,|\log\varepsilon| + \mathrm{o}_{\varepsilon\to0}(1).\end{aligned} \tag{4.86}$$

We will estimate the contribution of the potential on $\Omega_{\sigma_\varepsilon}$ later on.

We construct the component $\mathbf{M}_\varepsilon$. Using the results of Appendix A, we find a minimal connection $L_1,\cdots,L_{|d|}$ for $a_1,\cdots,a_{2|d|}$ with L_i pairwise disjoint (see Lemma A.2). By reasoning as in Lemma A.3, we define a lifting $\tilde{\mathbf{M}}_\varepsilon \in \mathrm{SBV}(\Omega_{\sigma_\varepsilon},\mathbb{R}^2)$ of $\tilde{\mathbf{Q}}_\varepsilon$ — that is, a vector field $\tilde{\mathbf{M}}_\varepsilon\colon\Omega_{\sigma_\varepsilon}\to\mathbb{R}^2$ such that $|\tilde{\mathbf{M}}_\varepsilon| = (\sqrt{2}\beta+1)^{\frac{1}{2}}$ and

$$\tilde{\mathbf{Q}}_\varepsilon = \sqrt{2}\left(\frac{\tilde{\mathbf{M}}_\varepsilon\otimes\tilde{\mathbf{M}}_\varepsilon}{\sqrt{2}\beta+1} - \frac{\mathbf{I}}{2}\right) \tag{4.87}$$

— which, in addition, satisfies $S_{\tilde{\mathbf{M}}_\varepsilon} = (\bigcup_{i=1}^{|d|} L_i)\cap\Omega_{\sigma_\varepsilon}$, up to negligible sets. By the same arguments as in the proof of Proposition 4.2, we can assume with no loss of generality that $\tilde{\mathbf{M}}_\varepsilon = \mathbf{M}_{\mathrm{bd}}$ on $\partial\Omega$. In order to define our competitor $\mathbf{M}_\varepsilon$, we need to regularise $\tilde{M}_\varepsilon$ near its jump set. We will do this by considering a Modica-Mortola optimal profile problem. Define $u\colon[0,\infty]\to\mathbb{R}$ as a minimiser for the following variational problem:

$$\min\left\{\int_0^{+\infty}\Big(\frac{1}{2}u'^2+\frac{1}{2}H^2(u)\Big)\mathrm{d}t\colon\, u\colon[0,\,+\infty)\to\mathbb{R},\ u(0)=0,\ u(+\infty)=(\sqrt{2}\beta+1)^{\frac{1}{2}}\right\} \tag{4.88}$$

where $H(u) := \sqrt{2h(u,0)} = \frac{1}{\sqrt{2}}|\sqrt{2}\beta+1-u^2|$. A minimiser for (4.88) exists, by the direct method of the calculus of variations. The Euler-Lagrange equation for (4.88) reads as:

$$-u'u'' + \frac{1}{2}(H^2)'(u) = 0$$

This can be rewritten as

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u'^2}{2}\right) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(H^2(u)) = 0$$

that is

$$-\,u'^2 + H^2(u) = \text{constant} = 0 \tag{4.89}$$

due to the conditions at infinity. We can compute the integral in (4.88):

$$\int_0^{+\infty}\Big(\frac{1}{2}u'^2+\frac{1}{2}H(u)\Big)\mathrm{d}t \overset{(4.89)}{=} \int_0^{+\infty} u'H(u)\,\mathrm{d}t = \int_0^{(\sqrt{2}\beta+1)^{1/2}} H(u)\,\mathrm{d}u = \frac{1}{2}c_\beta \tag{4.90}$$

where c_β is given by (4.58) (see Lemma 4.14).

We define the competitor $\mathbf{M}_\varepsilon$ in $\Omega_{\sigma_\varepsilon}$ by a suitable regularisation of $\tilde{\mathbf{M}}_\varepsilon$ in a neighbourhood of each singular line segment L_j. To simplify the notation, we focus on L_1 and we assume without loss of generality, up to rotations and translations, that $L_1 = [0,\, a] \times \{0\}$ for some $a > 0$. We assume that ε is small enough, so that $\sigma_\varepsilon \ll \frac{a}{4}$. Let $A_\varepsilon := [0,a]\times[-\sigma_\varepsilon,\sigma_\varepsilon]\setminus\Big(B_{\sigma_\varepsilon}(0,0)\bigcup B_{\sigma_\varepsilon}(a,0)\Big)$. We define

$$\mathbf{M}_\varepsilon(x) := \begin{cases} \dfrac{u\left(\frac{|x_2|}{\varepsilon}\right)}{u\left(\frac{\sigma_\varepsilon}{\varepsilon}\right)}\tilde{\mathbf{M}}_\varepsilon(x), & \text{in } A^1_\varepsilon \text{ (and similarly in each } A^j_\varepsilon) \\ \tilde{\mathbf{M}}_\varepsilon(x), & \text{on } \Omega_{\sigma_\varepsilon} \setminus \bigcup_{j=1}^{2|d|} A^j_\varepsilon. \end{cases} \tag{4.91}$$

For ε small enough, we have $\mathbf{M}_\varepsilon = \tilde{\mathbf{M}}_\varepsilon = \mathbf{M}_{\mathrm{bd}}$ on $\partial\Omega$. In $\Omega_{\sigma_\varepsilon} \setminus \bigcup_{j=1}^{2|d|} A^j_\varepsilon$, we have $|\nabla\mathbf{M}_\varepsilon|^2 \lesssim |\nabla\tilde{\mathbf{M}}_\varepsilon|^2$. The latter can be estimated by differentiating both sides of (4.87), by the BV-chain rule; this gives $\|\nabla\tilde{\mathbf{M}}_\varepsilon\|^2_{L^2(\Omega_{\sigma_\varepsilon})} \lesssim \|\nabla\tilde{\mathbf{Q}}_\varepsilon\|^2_{L^2(\Omega_{\sigma_\varepsilon})} \lesssim |\log\varepsilon|$. Let

$$\eta_\varepsilon := \frac{\sqrt{2}\beta + 1}{u\left(\frac{\sigma_\varepsilon}{\varepsilon}\right)^2}$$

We observe that $\eta_\varepsilon \to 1$ as $\varepsilon \to 0$, due to the condition at infinity in (4.88). We have in A^1_ε:

$$\frac{\varepsilon}{2}|\nabla\mathbf{M}_\varepsilon|^2 = \frac{\eta_\varepsilon}{2}\left|u'\left(\frac{|x_2|}{\varepsilon}\right)\right|^2 + \mathrm{O}(\varepsilon|\nabla\tilde{\mathbf{M}}_\varepsilon|^2)$$

and therefore:

$$\begin{aligned} \frac{\varepsilon}{2}\int_{A^1_\varepsilon}|\nabla\mathbf{M}_\varepsilon|^2 \mathrm{d}x &\leq \mathrm{O}(\varepsilon|\log\varepsilon|) + \frac{\eta_\varepsilon}{2}\int_{A^1_\varepsilon}\left|u'\left(\frac{|x_2|}{\varepsilon}\right)\right|^2 \mathrm{d}x + \mathrm{O}(\varepsilon\|\nabla\tilde{\mathbf{M}}_\varepsilon\|^2_{L^2(\Omega_{\sigma_\varepsilon})}) \\ &= \mathrm{O}(\varepsilon|\log\varepsilon|) + \frac{\eta_\varepsilon}{2}\mathscr{H}^1(L_1)\cdot\int_{-\sigma_\varepsilon}^{\sigma_\varepsilon}\left|u'\left(\frac{|x_2|}{\varepsilon}\right)\right|^2 \mathrm{d}x_2 \\ &= \mathrm{O}(\varepsilon|\log\varepsilon|) + \eta_\varepsilon\,\mathscr{H}^1(L_1)\cdot\int_0^{\frac{\sigma_\varepsilon}{\varepsilon}}|u'(t)|^2\,\mathrm{d}t. \end{aligned}$$

By repeating this argument on each A^j_ε, we deduce

$$\frac{\varepsilon}{2}\int_{\Omega_{\sigma_\varepsilon}}|\nabla\mathbf{M}_\varepsilon|^2 \mathrm{d}x \leq \mathrm{O}(\varepsilon|\log\varepsilon|) + \eta_\varepsilon\,\mathbb{L}(a_1,\,\ldots,\,a_{2|d|})\cdot\int_0^{\frac{\sigma_\varepsilon}{\varepsilon}}|u'(t)|^2\,\mathrm{d}t. \tag{4.92}$$

Next, we estimate the potential term. On $\Omega_{\sigma_\varepsilon} \setminus \bigcup_{j=1}^{|d|} A^j_\varepsilon$, we have $|\mathbf{Q}_\varepsilon| = 1 + \kappa_*\varepsilon$ and $|\mathbf{M}_\varepsilon| = (\sqrt{2}\beta + 1)^{\frac{1}{2}}$. The identity (4.87) can be written as

$$\frac{\mathbf{Q}_\varepsilon}{1+\kappa_*\varepsilon} = \sqrt{2}\left(\frac{\mathbf{M}_\varepsilon\otimes\mathbf{M}_\varepsilon}{\sqrt{2}\beta+1} - \frac{\mathbf{I}}{2}\right)$$

which implies

$$\mathbf{Q}_\varepsilon\mathbf{M}_\varepsilon\cdot\mathbf{M}_\varepsilon = \sqrt{2}(1+\kappa_*\varepsilon)\left(\frac{|\mathbf{M}_\varepsilon|^4}{\sqrt{2}\beta+1} - \frac{1}{2}|\mathbf{M}_\varepsilon|^2\right) = \frac{\sqrt{2}}{2}(1+\kappa_*\varepsilon)(\sqrt{2}\beta+1).$$

In conclusion, at each point of $\Omega_{\sigma_\varepsilon} \setminus \bigcup_{j=1}^{|d|} A_\varepsilon^j$ we have

$$
\begin{aligned}
f_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) &= \frac{1}{4}(2\kappa_*\varepsilon + \kappa_*^2\varepsilon^2)^2 + \frac{\varepsilon\beta^2}{2} - \frac{\beta\varepsilon}{\sqrt{2}}(1+\kappa_*\varepsilon)(\sqrt{2}\beta + 1) + \kappa_\varepsilon \\
&= \mathrm{o}_{\varepsilon\to 0}(\varepsilon^2)
\end{aligned} \tag{4.93}
$$

by taking Lemma 3.1 into account. Therefore, the total contribution from the potential on $\Omega_{\sigma_\varepsilon} \setminus \bigcup_{j=1}^{|d|} A_\varepsilon^j$ is negligible. Let us compute the potential on A_ε^j. Considering for simplicity the case $j = 1$, again we have $|\mathbf{Q}_\varepsilon| = 1 + \kappa_*\varepsilon$, but

$$
|\mathbf{M}_\varepsilon(x)| = \eta_\varepsilon^{1/2}\, u\left(\frac{|x_2|}{\varepsilon}\right)
$$

Then, (4.87) can be written as

$$
\mathbf{Q}_\varepsilon = \sqrt{2}(1+\kappa_*\varepsilon)\left(\frac{\mathbf{M}_\varepsilon \otimes \mathbf{M}_\varepsilon}{|\mathbf{M}_\varepsilon|^2} - \frac{\mathbf{I}}{2}\right)
$$

which implies

$$
\mathbf{Q}_\varepsilon \mathbf{M}_\varepsilon \cdot \mathbf{M}_\varepsilon = \frac{\sqrt{2}}{2}(1+\kappa_*\varepsilon)\,\eta_\varepsilon\, u^2\left(\frac{|x_2|}{\varepsilon}\right)
$$

At a generic point $x \in A_\varepsilon^1$, we have (writing $v_\varepsilon := \eta_\varepsilon^{1/2} u(|x_2|/\varepsilon)$ for simplicity)

$$
\begin{aligned}
f_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) &= \frac{\kappa_*\varepsilon^2}{4}(2+\kappa_*\varepsilon)^2 + \frac{\varepsilon}{4}(1-v_\varepsilon^2) - \beta\varepsilon\frac{\sqrt{2}}{2}(1+\kappa_*\varepsilon)v_\varepsilon^2 + \frac{1}{2}(\beta^2 + \sqrt{2}\beta)\varepsilon + \kappa_*^2\varepsilon^2 + \mathrm{o}(\varepsilon^2) \\
&= 2\kappa_*^2\varepsilon^2 + \frac{\varepsilon}{4}(1-v_\varepsilon^2)^2 - \frac{\beta\varepsilon}{\sqrt{2}}v_\varepsilon^2 - \frac{\kappa_*\beta\varepsilon^2}{\sqrt{2}}v_\varepsilon^2 + \frac{1}{2}(\beta^2+\sqrt{2}\beta)\varepsilon + \mathrm{o}(\varepsilon^2) \\
&= \mathrm{O}(\varepsilon^2) + \varepsilon\left(h(v_\varepsilon, 0) - \frac{\beta^2+\sqrt{2}\beta}{2}\right) + \frac{1}{2}(\beta^2+\sqrt{2}\beta)\varepsilon \\
&= \mathrm{O}(\varepsilon^2) + \varepsilon h(v_\varepsilon, 0) \\
&= \mathrm{O}(\varepsilon^2) + \frac{\varepsilon}{2}H^2(v_\varepsilon).
\end{aligned}
$$

By repeating this argument on each A_ε^j, and taking the integral over A_ε^j, we obtain

$$
\begin{aligned}
\frac{1}{\varepsilon^2}\int_{\bigcup_{j=1}^{|d|} A_\varepsilon^j} f_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)\,\mathrm{d}x &= \int_{\bigcup_{j=1}^{|d|} A_\varepsilon^j} \mathrm{O}(1) + \frac{1}{2\varepsilon}H^2\left(\eta_\varepsilon^{1/2}\, u\left(\frac{|x_2|}{\varepsilon}\right)\right)\mathrm{d}x \\
&= \mathrm{O}(\sigma_\varepsilon) + \frac{1}{2\varepsilon}\sum_{j=1}^{|d|}\int_{A_\varepsilon^j} H^2\left(\eta_\varepsilon^{1/2}\, u\left(\frac{|x_2|}{\varepsilon}\right)\right)\mathrm{d}x \\
&= \mathrm{o}_{\varepsilon\to 0}(1) + \mathbb{L}(a_1, \cdots, a_{2|d|}) \cdot \int_0^{\frac{\sigma_\varepsilon}{\varepsilon}} H^2(\eta_\varepsilon^{1/2}\, u(t))\,\mathrm{d}t.
\end{aligned} \tag{4.94}
$$

By combining (4.86), (4.92), (4.93) and (4.94), keeping in mind that $\eta_\varepsilon \to 1$, $\sigma_\varepsilon/\varepsilon \to +\infty$ as $\varepsilon \to 0$, and applying Lebesgue's dominated convergence theorem, we obtain

$$
\begin{aligned}
\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon; \Omega \setminus \bigcup_{j=1}^{2|d|} B_{\sigma_\varepsilon}(a_j)) &\leq \mathrm{o}_{\varepsilon\to 0}(1) + \mathbb{L}(a_1, \cdots, a_{2d}) \int_0^{+\infty} \left(u'^2(t) + H^2(u(t))\right) \mathrm{d}t \\
&\qquad + 2\pi\,|d|\,|\log\varepsilon| + \mathbb{W}(a_1, \cdots, a_{2|d|}) + 2\,|d|\,\gamma_* \\
&\overset{(4.90)}{=} 2\pi\,|d|\,|\log\varepsilon| + \mathbb{W}_\beta(a_1, \cdots, a_{2|d|}) + 2\,|d|\,\gamma_* + \mathrm{o}_{\varepsilon\to 0}(1).
\end{aligned} \tag{4.95}
$$

It only remains to define $\mathbf{M}_\varepsilon$ in each ball $B_{\sigma_\varepsilon}(a_j)$. For each j, there exists $\rho = \rho(j) \in (\sigma_\varepsilon, 2\sigma_\varepsilon)$ such that

$$
\int_{\partial B_\rho(a_j)} |\nabla \mathbf{M}_\varepsilon|^2\,\mathrm{d}\mathscr{H}^1 \leq \frac{1}{\sigma_\varepsilon} \int_{B_{2\sigma_\varepsilon}(a_j)\setminus B_{\sigma_\varepsilon}(a_j)} |\nabla \mathbf{M}_\varepsilon|^2\,\mathrm{d}x = \mathrm{O}\left(\frac{|\log\varepsilon|}{\sigma_\varepsilon}\right) + \mathrm{O}(\frac{1}{\varepsilon})
$$

Define $\mathbf{M}_\varepsilon$ on $B_\rho(a_j)$ as

$$
\mathbf{M}_\varepsilon(x) := \frac{|x - a_j|}{\rho} \mathbf{M}_\varepsilon \left(\frac{\rho(x - a_j)}{|x - a_j|}\right) \tag{4.96}
$$

The vector field $\mathbf{M}_\varepsilon$ was already defined in $B_\rho(a_j) \setminus B_{\sigma_\varepsilon}(a_j)$, but we disregard its previous values and re-define it according to (4.96). We have

$$
\begin{aligned}
\varepsilon \int_{B_\rho(a_j)} |\nabla \mathbf{M}_\varepsilon|^2\,\mathrm{d}x &\leq \sigma_\varepsilon \int_{\partial B_\rho(a_j)} \varepsilon |\nabla \mathbf{M}_\varepsilon|^2\,\mathrm{d}\mathscr{H}^1 + \varepsilon \int_{B_\rho(a_j)} \mathrm{O}\left(\frac{1}{\rho^2}\right) \mathrm{d}x \\
&\leq \mathrm{O}(\varepsilon|\log\varepsilon|) + \mathrm{O}(\sigma_\varepsilon) + \mathrm{O}(\varepsilon) \to 0
\end{aligned} \tag{4.97}
$$

and

$$
\frac{1}{\varepsilon^2} \int_{B_\rho(a_j)} \left(f_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) - \frac{1}{4}(1 - |\mathbf{Q}_\varepsilon|^2)^2\right) \mathrm{d}x = \mathrm{O}(\frac{\sigma_\varepsilon^2}{\varepsilon}). \tag{4.98}
$$

If we choose $\varepsilon \ll \sigma_\varepsilon \ll \varepsilon^{\frac{1}{2}}$, then the total contribution of $\mathbf{M}_\varepsilon$ to the energy on each ball $B_\rho(a_j)$ tends to zero as $\varepsilon \to 0$. □

We can now complete the proof of our main result, Theorem 2.1.

Conclusion of the proof of Theorem 2.1, proof of Proposition 4.11. From Proposition 4.12 and Proposition 4.17, we deduce

$$
\begin{aligned}
\mathbb{W}(a_1^*, \ldots, a_{2|d|}^*) + c_\beta\,\mathscr{H}^1(\mathrm{S}_{\mathbf{M}^*}) &+ \int_\Omega (\xi_* - \kappa_*)^2\,\mathrm{d}x + 2\,|d|\,\gamma_* \\
&\leq \liminf_{\varepsilon\to 0} \left(\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*) - 2\pi\,|d|\,|\log\varepsilon|\right) \\
&\leq \limsup_{\varepsilon\to 0} \left(\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon^*, \mathbf{M}_\varepsilon^*) - 2\pi\,|d|\,|\log\varepsilon|\right) \\
&\leq \mathbb{W}(a_1, \ldots, a_{2|d|}) + c_\beta\,\mathbb{L}(a_1, \ldots, a_{2|d|}) + 2\,|d|\,\gamma_*
\end{aligned} \tag{4.99}
$$

for any $(2\,|d|)$-uple of distinct points $a_1, \ldots, a_{2|d|}$ in Ω. In particular, choosing $a_j = a_j^*$, we obtain

$$\mathscr{H}^1(\mathrm{S}_{\mathbf{M}^*}) = \mathbb{L}(a_1^*, \ldots, a_{2|d|}^*), \qquad \xi_* = \kappa_* \tag{4.100}$$

and Proposition (4.11) follows. Moreover, Proposition 4.13 and (4.100) imply that the jump set $\mathrm{S}_{\mathbf{M}^*}$ coincides (up to negligible sets) with $\cup_{j=1}^{|d|} L_j$, where $(L_1, \ldots, L_{|d|})$ is a minimal connection for $(a_1, \ldots, a_{2|d|})$. Finally, from (4.99) and (4.100) we deduce

$$\mathbb{W}_\beta(a_1^*, \ldots, a_{2|d|}^*) \leq \mathbb{W}_\beta(a_1, \ldots, a_{2|d|}) \tag{4.101}$$

for any $(2\,|d|)$-uple of distinct points $a_1, \ldots, a_{2|d|}$ in Ω — that is, $(a_1^*, \ldots, a_{2|d|}^*)$ minimises $\mathbb{W}_\beta$. □

5 Numerics

In this section, we provide some numerical results for stable critical points of the ferronematic free energy, on square domains with topologically non-trivial Dirichlet boundary conditions for $\mathbf{Q}$ and $\mathbf{M}$.

Instead of solving the Euler-Lagrange system directly, we solve an L^2-gradient flow associated with the effective re-scaled free energy for ferronematics (2.1), given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}_\varepsilon(\mathbf{Q},\mathbf{M}) = -\int_\Omega (\eta_1|\partial_t\mathbf{Q}|^2 + \eta_2|\partial_t\mathbf{M}|^2)\mathrm{d}\mathbf{x}. \tag{5.1}$$

Here $\eta_1 > 0$ and $\eta_2 > 0$ are arbitrary friction coefficients. Due to limited physical data, we do not comment on physically relevant values of ε, β and the friction coefficients. The system of L^2-gradient flow equations for Q_{11}, Q_{12} and the components, M_1, M_2 of the magnetisation vector, can be written as

$$\begin{cases} 2\eta_1\,\partial_t Q_{11} = 2\Delta Q_{11} - \frac{1}{\varepsilon^2}(4Q_{11}(Q_{11}^2 + Q_{12}^2 - 1/2) - \beta\varepsilon(M_1^2 - M_2^2)) \\ 2\eta_1\,\partial_t Q_{12} = 2\Delta Q_{12} - \frac{1}{\varepsilon^2}(4Q_{12}(Q_{11}^2 + Q_{12}^2 - 1/2) - 2\beta\varepsilon M_1 M_2) \\ \eta_2\,\partial_t M_1 = \varepsilon\Delta M_1 - \frac{1}{\varepsilon^2}(\varepsilon(M_1^2 + M_2^2 - 1)M_1 - \beta\varepsilon(2Q_{11}M_1 + 2Q_{12}M_2)) \\ \eta_2\,\partial_t M_2 = -\varepsilon\Delta M_2 - \frac{1}{\varepsilon^2}(\varepsilon(M_1^2 + M_2^2 - 1)M_2 - \beta\varepsilon(-2Q_{11}M_2 + 2Q_{12}M_1)), \end{cases} \tag{5.2}$$

In the following simulations, we take $\eta_1 = 1$ and $\eta_2 = \varepsilon$ and do not offer rigorous justifications for these choices, except as numerical experiments to validate out theoretical results. We impose the continuous degree $+k$ boundary condition

$$\mathbf{M}_b = (\sqrt{2}\beta + 1)^{1/2}(\cos k\theta, \sin k\theta), \quad \mathbf{Q}_b = \sqrt{2}\begin{pmatrix} \frac{1}{2}\cos 2k\theta & \frac{1}{2}\sin 2k\theta \\ \frac{1}{2}\sin 2k\theta & \frac{1}{2}\cos 2k\theta \end{pmatrix}, \tag{5.3}$$

where

$$\theta(x, y) = \mathrm{atan2}\,(y - 0.5,\ x - 0.5) - \pi/2, \quad (x, y) \in \partial\Omega. \tag{5.4}$$

and $\text{atan2}(y, x)$ is the 2-argument arctangent that computes the principal value of the argument function applied to the complex number $x + iy$. So $-\pi \leq \text{atan2}(y, x) \leq \pi$. For example, if $x > 0$, then $\text{atan2}(y, x) = \arctan\left(\frac{y}{x}\right)$. The initial condition is prescribed to be

$$\mathbf{M}_0 = (\sqrt{2}\beta + 1)^{1/2}(\cos k\theta, \sin k\theta), \quad \mathbf{Q}_0 = \sqrt{2}\begin{pmatrix} \frac{1}{2}\cos 2k\theta & \frac{1}{2}\sin 2k\theta \\ \frac{1}{2}\sin 2k\theta & \frac{1}{2}\cos 2k\theta, \end{pmatrix} \tag{5.5}$$

where

$$\theta(x, y) = \text{atan2}\,(y - 0.5,\, x - 0.5) - \pi/2, \quad (x, y) \in (0, 1)^2. \tag{5.6}$$

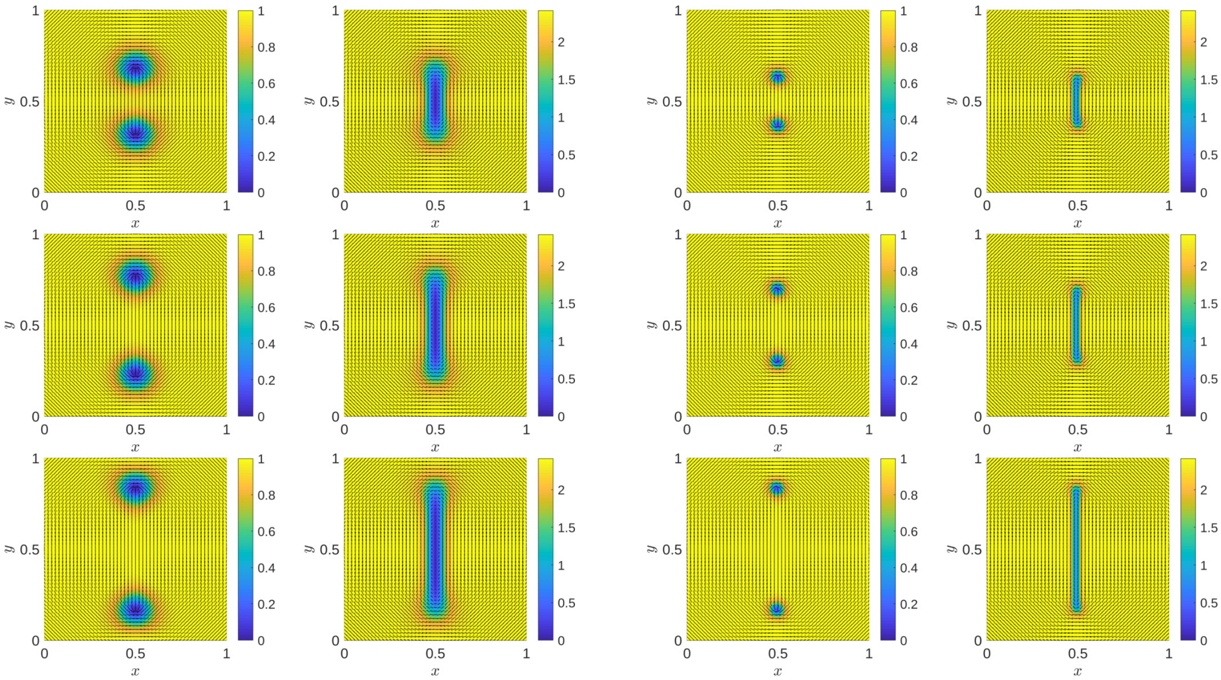
In Figure 1, we plot the converged numerical solution of the gradient flow equations for $k = 1$. We cannot conclusively argue that the converged solution is an energy minimizer but it is locally stable, the converged $\mathbf{Q}$-profile has two non-orientable defects and the corresponding $\mathbf{M}$-profile has a jump set composed of a straight line connecting the nematic defect pair, consistent with our theoretical results. We consider two different values of ε and it is clear that the $\mathbf{Q}$-defects and the jump set in $\mathbf{M}$ become more localised as ε becomes smaller, as expected from the theoretical-results. We have also investigated the effects of β on the converged solutions — the defects become closer as β increases. We do not have a clear explanation for this except that the cost of the minimal connection between the nematic defects increases as β increases, and hence the shorter connections require the defects to be closer to each other (at least in a pairwise sense).

In Figure 2, we plot the converged numerical solution of the gradient flow equations for $k = 2$. Again, the converged solution is locally stable, the $\mathbf{Q}$-profile has four non-orientable defects,the $\mathbf{M}$-profile has two distinct jump sets connecting two pairs of non-orientable nematic defects, and the jump sets are indeed approximately straight lines as predicted by our analysis. Smaller values of ε correspond to the sharp interface limit, with more localised defects and larger values of β push the defects closer together, in qualitative agreement with our theoretical results.

We expect multiple local energy minimizers or even energy minimizers with different jump sets in $\mathbf{M}$ i.e. different choices of the minimal connection of equal length, or different choices of the connections between the nematic defect pairs. For example, it is conceivable that a locally stable $\mathbf{M}$-profile also connects the nematic defects by means of straight lines, but this connection is not minimal. There may be non energy-minimising critical points with orientable point defects in $\mathbf{M}$ tailored by the non-orientable nematic defects. Similarly, there may be non energy-minimising critical points with non-orientable and orientable nematic defects, whose locations are not minimisers but critical points of the modified renormalised energy in Theorem 2.1. We defer these interesting questions to future work.

6 Conclusions

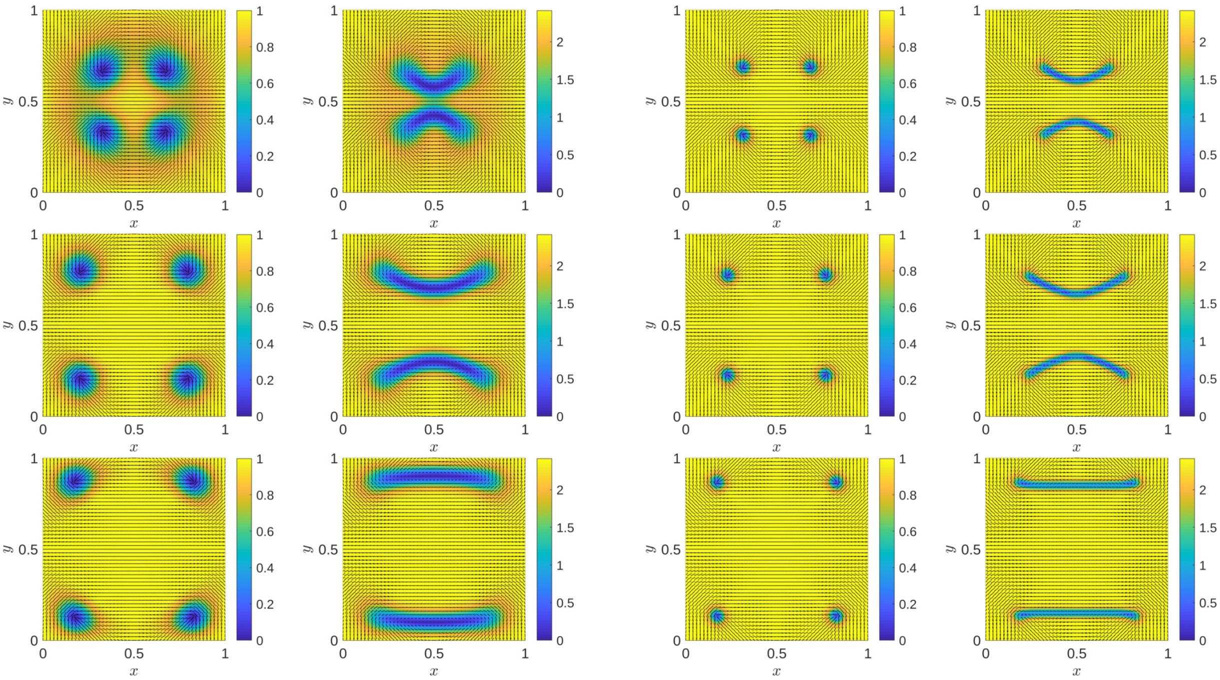
We study a simplified model for ferronematics in two-dimensional domains, with Dirichlet boundary conditions, building on previous work in [14]. The model is only valid for dilute ferronematic suspensions and we do not expect quantitative agreement with experiments. Further, the experimentally relevant choices for the boundary conditions for $\mathbf{M}$ are not well established and our methods can be adapted to other choices of boundary conditions e.g. Neumann conditions for



(a) $\beta = 1, \varepsilon = 0.05$ **(b)** $\beta = 1, \varepsilon = 0.02$

Figure 1: Numerical results for the gradient flows (5.2) with (a) $\beta = 1, \varepsilon = 0.05$ at $t = 0.05$, 0.1 and 1 and (b) $\beta = 1, \varepsilon = 0.02$ at $t = 0.02$, 0.05 and 1 (Continuous degree $+1$ boundary condition, $h = 1/50$, $\tau = 1/1000$). In each sub-figure, the nematic configuration is shown in the left panel, where the black bars represent nematic field $\mathbf{n}$ (the eigenvector of $\mathbf{Q}$ with the largest positive eigenvalue) and the color represents $\operatorname{tr}\mathbf{Q}^2 = 2(Q_{11}^2 + Q_{12}^2)$; the $\mathbf{M}$-profile is shown in the right panel, where the black bars represent magnetic field $\mathbf{M}$ and the color bar represents $|\mathbf{M}|^2 = M_1^2 + M_2^2$.

the magnetisation vector. Similarly, it is not clear if topologically non-trivial Dirichlet conditions can be imposed on the nematic directors, for physically relevant experimental scenarios. Having said that, our model problem is a fascinating mathematical problem because of the tremendous complexity of ferronematic solution landscapes, the multiplicity of the energy minimizers and non energy-minimizing critical points, and the multitude of admissible coupled defect profiles for the nematic and magnetic profiles. There are several forward research directions, some of which could facilitate experimental observations of the theoretically predicted morphologies in this manuscript. For example, one could study the experimentally relevant generalisation of our model problem with Dirichlet conditions for $\mathbf{Q}$ and Neumann conditions for $\mathbf{M}$, or study different asymptotic limits of the ferronematic free energy in (1.1), a prime candidate being the $\varepsilon \to 0$ limit for fixed ξ and c_0 (independent of ε). This limit, although relevant for dilute suspensions, would significantly change the vacuum manifold $\mathscr{N}$ in the $\varepsilon \to 0$ limit. In fact, we expect to observe stable point defects in the energy-minimizing $\mathbf{M}$-profiles for this limit, where ξ and c_0 are independent of ε, as $\varepsilon \to 0$. Further, there is the interesting question of how this ferronematic



(a) $\beta = 1, \varepsilon = 0.05$ **(b)** $\beta = 1, \varepsilon = 0.02$

Figure 2: Numerical results for the gradient flows (5.2) with (a) $\beta = 1, \varepsilon = 0.05$ at $t = 0.02$, 0.05 and 1 and (b) $\beta = 1, \varepsilon = 0.02$ at $t = 0.02$, 0.05 and 1 (Continuous degree +2 boundary condition, $h = 1/50$, $\tau = 1/1000$). In each sub-figure, the nematic configuration is shown in the left panel, where the black bars represent nematic field $\mathbf{n}$ and the color represents $\operatorname{tr} \mathbf{Q}^2 = 2(Q_{11}^2 + Q_{12}^2)$; the magnetic configurations is shown in the right panel, where the black bars represent magnetic field $\mathbf{M}$ and the color represents $|\mathbf{M}|^2 = M_1^2 + M_2^2$.

model can be generalised to non-dilute suspensions or to propose a catalogue of magneto-nematic coupling energies for different kinds of MNP-MNP interactions and MNP-NLC interactions. The physics of ferronematics is complex, and it is challenging to translate the physics to tractable mathematical problems with multiple order parameters, and we hope that our work is solid progress in this direction with bright interdisciplinary prospects.

Taxonomy: GC, BS and AM conceived the project based on a model developed by AM and her ex-collaborators. GC and BS led the analysis, followed by AM. YW performed the numerical simulations, as advised by AM and GC. All authors contributed to the scientific writing.

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A Lifting of a map with non-orientable singularities

The aim of this section is to prove Proposition 4.13. We reformulate the problem in a slightly more general setting.

Let $a \in \mathbb{R}^2$, and let $\mathbf{Q} \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{a\}, \mathscr{N})$. By Fubini theorem and Sobolev embedding, the restriction of $\mathbf{Q}$ on the circle $\partial B_\rho(a)$ is well-defined and continuous for a.e. $\rho > 0$. Therefore, it makes sense to define the topological degree of $\mathbf{Q}$ on $\partial B_\rho(a)$ as an half-integer, $\deg(\mathbf{Q}, a) \in \frac{1}{2}\mathbb{Z}$. As the notation suggests, the degree is independent of the choice of ρ: for a.e. $0 < \rho_1 < \rho_2$, the degrees of $\mathbf{Q}$ on $\partial B_{\rho_1}(a)$ and $\partial B_{\rho_2}(a)$ are the same. If $\mathbf{Q}$ is smooth, this is a consequence of the homotopy lifting property; for more general $\mathbf{Q} \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{a\}, \mathscr{N})$, this follows from an approximation argument (based on [43, Proposition p. 267]). We will say that a is a non-orientable singularity of $\mathbf{Q}$ if $\deg(\mathbf{Q}, a) \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.

Given an open set $\Omega \subseteq \mathbb{R}^2$, a map $\mathbf{Q}\colon \Omega \to \mathscr{N}$ and a unit vector field $\mathbf{M}\colon \Omega \to \mathbb{S}^1$, we say that $\mathbf{M}$ is a lifting for $\mathbf{Q}$ if

$$\mathbf{Q}(x) = \sqrt{2}\left(\mathbf{M}(x) \otimes \mathbf{M}(x) - \frac{\mathbf{I}}{2}\right) \qquad \text{for a.e. } x \in \Omega. \tag{A.1}$$

Any map $\mathbf{Q} \in \mathrm{BV}(\Omega, \mathscr{N})$ admits a lifting $\mathbf{M} \in \mathrm{BV}(\Omega, \mathbb{S}^1)$ (see e.g. [32]). The vector field $\mathbf{M}^*$ given by Theorem 2.1 is not a lifting of $\mathbf{Q}^*$, according to the definition above, because $|\mathbf{M}^*| \neq 1$. However, $|\mathbf{M}^*|$ is still a positive constant (see Proposition 4.10), so we can construct a lifting of unit-norm simply by rescaling.

We focus on properties of the lifting for $\mathbf{Q}$-tensors of a particular form, namely, we assume that $\mathbf{Q}$ has an even number of non orientable singularities at distinct points $a_1, \ldots, a_{2d}$. We recall that a *connection* for $\{a_1, \ldots, a_{2d}\}$ as a finite collection of straight line segments $\{L_1, \ldots, L_d\}$, with endpoints in $\{a_1, \ldots, a_{2d}\}$, such that each a_i is an endpoint of one of the segments L_j. We recall that

$$\mathbb{L}(a_1, \ldots, a_{2d}) := \min\left\{\sum_{i=1}^d \mathscr{H}^1(L_i)\colon \{L_1, \ldots, L_d\} \text{ is a connection for } \{a_1, \ldots, a_{2d}\}\right\}. \tag{A.2}$$

A minimal connection for $\{a_1, \ldots, a_{2d}\}$ is a connection that attains the minimum in the right-hand side of (A.2). Given two sets A, B, we denote their symmetric difference as $A \Delta B := (A \setminus B) \cup (B \setminus A)$.

Proposition A.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded, convex domain, let $d \geq 1$ be an integer, and let $a_1, \ldots, a_{2d}$ be distinct points in Ω. Let $\mathbf{Q} \in W^{1,1}(\Omega, \mathscr{N}) \cap W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{a_1, \ldots, a_{2d}\}, \mathscr{N})$ be* *a map with a non-orientable singularity at each* a_j*. If* $\mathbf{M} \in \mathrm{SBV}(\Omega, \mathbb{S}^1)$ *is a lifting for* $\mathbf{Q}$ *such that* $\mathrm{S}_{\mathbf{M}} \subset\subset \Omega$*, then*

$$\mathscr{H}^1(\mathrm{S}_{\mathbf{M}}) \geq \mathbb{L}(a_1, \ldots, a_{2d})$$

The equality holds if and only if there exists a minimal connection $\{L_1, \ldots, L_d\}$ *for* $\{a_1, \ldots, a_d\}$ *such that* $\mathscr{H}^1(\mathrm{S}_{\mathbf{M}} \Delta \cup_{j=1}^d L_j) = 0$*.*

Proposition 4.13 is an immediate consequence of Proposition A.1. The proof of Proposition A.1 is based on classical results in Geometric Measure Theory, but we provide it in full detail for the reader's convenience. Before we prove Proposition A.1, we state a few preliminary results.

Lemma A.2. *If* $\{L_1, \ldots, L_d\}$ *is a minimal connection for* $\{a_1, \ldots, a_{2d}\}$*, then the* L_j*'s are pairwise disjoint.*

Proof. Suppose, towards a contradiction, that $\{L_1, \ldots, L_d\}$ is a minimal connection with $L_1 \cap L_2 \neq \emptyset$. The intersection $L_1 \cap L_2$ must be either a non-degenerate sub-segment of both L_1 and L_2 or a point. If $L_1 \cap L_2$ is non-degenerate, then $(L_1 \cup L_2) \setminus (L_1 \cap L_2)$ can be written as the disjoint union of two straight line segments, K_1 and K_2, and

$$\mathscr{H}^1(K_1) + \mathscr{H}^1(K_2) = \mathscr{H}^1((L_1 \cup L_2) \setminus (L_1 \cap L_2)) < \mathscr{H}^1(L_1) + \mathscr{H}^1(L_2)$$

This contradicts the minimality of $\{L_1, \ldots, L_d\}$. Now, suppose that $L_1 \cap L_2$ is a point. By the pigeon-hole principle, $L_1 \cap L_2$ cannot be an endpoint for either L_1 or L_2. Say, for instance, that L_1 is the segment of endpoints a_1, a_2, while L_2 is the segment of endpoints a_3, a_4. Let H_1, H_2 be the segments of endpoints (a_1, a_3), (a_2, a_4) respectively. Then, by the triangular inequality,

$$\mathscr{H}^1(H_1) + \mathscr{H}^1(H_2) < \mathscr{H}^1(L_1) + \mathscr{H}^1(L_2),$$

which contradicts again the minimality of $\{L_1, \ldots, L_d\}$. □

Lemma A.3. *Let* $\Omega \subseteq \mathbb{R}^2$ *be a bounded, convex domain and let* $a_1, \ldots, a_{2d}$ *be distinct points in* Ω*. Let* $\mathbf{Q} \in W^{1,1}(\Omega, \mathscr{N}) \cap W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{a_1, \ldots, a_{2d}\}, \mathscr{N})$ *be a map with a non-orientable singularity at each* a_j*. If* $\{L_1, \ldots, L_d\}$ *is a minimal connection for* $\{a_1, \ldots, a_{2d}\}$*, then there exists a lifting* $\mathbf{M}^* \in \mathrm{SBV}(\Omega, \mathbb{S}^1)$ *such that* $\mathscr{H}^1(\mathrm{S}_{\mathbf{M}^*} \Delta \cup_{j=1}^d L_j) = 0$*.*

Proof. For any $\rho > 0$ and $j \in \{1, \ldots, d\}$, we define

$$U_{j,\rho} := \Big\{x \in \mathbb{R}^2 \colon \operatorname{dist}(x, L_j) < \rho\Big\}.$$

and

$$\Omega_\rho := \Omega \setminus \bigcup_{j=1}^d U_{j,\rho}.$$

Since Ω is convex, $L_j \subseteq \Omega$ for any j and hence, $U_{j,\rho} \subseteq \Omega$ for any j and ρ small enough. Each $U_{j\,\rho}$ is a simply connected domain with piecewise smooth boundary. Moreover, for ρ fixed and small, the sets $U_{j,\rho}$ are pairwise disjoint, because the L_j's are pairwise disjoint (Lemma A.2). The trace of $\mathbf{Q}$ on $\partial U_{j,\rho}$ is orientable, because $\partial U_{j,\rho}$ contains exactly two non-orientable singularities of $\mathbf{Q}$. Then, for any $\rho > 0$ small enough, $\mathbf{Q}_{|\Omega_\rho}$ has a lifting $\mathbf{M}^*_\rho \in W^{1,2}(\Omega_\rho, \mathbb{S}^1)$ [8, Proposition 7]. In fact, the lifting is unique up to the choice of the sign [8, Proposition 2]; in particular, if $0 < \rho_1 < \rho_2$ then we have either $\mathbf{M}^*_{\rho_2} = \mathbf{M}^*_{\rho_1}$ a.e. in Ω_{ρ_2} or $\mathbf{M}^*_{\rho_2} = -\mathbf{M}^*_{\rho_1}$ a.e. in Ω_{ρ_2}. As a consequence, for any sequence $\rho_k \searrow 0$, we can choose liftings $\mathbf{M}^*_{\rho_k} \in W^{1,2}(\Omega_{\rho_k}, \mathbb{S}^1)$ of $\mathbf{Q}^*_{|\Omega_{\rho_k}}$ in such a way that $\mathbf{M}^*_{\rho_{k+1}} = \mathbf{M}^*_{\rho_k}$ a.e. in Ω_{ρ_k}. By glueing the $\mathbf{M}^*_{\rho_k}$'s, we obtain a lifting

$$\mathbf{M}^* \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus \cup_j L_j, \mathbb{S}^1)$$

of $\mathbf{Q}$. By differentiating the identity (A.1), we obtain $\sqrt{2}\,|\nabla \mathbf{M}^*| = |\nabla \mathbf{Q}|$ a.e. and, since $\nabla \mathbf{Q} \in L^1(\Omega, \mathbb{R}^2 \otimes \mathbb{R}^{2\times 2})$ by assumption, we deduce that so $\mathbf{M}^* \in W^{1,1}(\Omega \setminus \cup_j L_j, \mathbb{S}^1)$. The set $\cup_j L_j$ has finite length and $\mathbf{M}^*$ is bounded, so we also have $\mathbf{M}^* \in \mathrm{SBV}(\Omega, \mathbb{S}^1)$ (see [3, Proposition 4.4]).

By construction, we have $\mathrm{S}_{\mathbf{M}^*} \subseteq \cup_j L_j$. Therefore, it only remains to prove that $\mathrm{S}_{\mathbf{M}^*}$ contains $\mathscr{H}^1$-almost all of $\cup_j L_j$. Consider, for instance, the segment L_1; up to a rotation and traslation, we can assume that $L_1 = [0, b] \times \{0\}$ for some $b > 0$. Given a small parameter $\rho > 0$ and $t \in (0, b)$, we define $K_{\rho,t} := (-\rho, t) \times (-\rho, \rho)$. Fubini theorem implies that, for a.e. ρ and t, $\mathbf{Q}$ restricted to $\partial K_{\rho,t}$ belongs to $W^{1,2}(\partial K_{\rho,t}, \mathscr{N})$ and hence, by Sobolev embedding, is continuous. Since the segments L_j are pairwise disjoint by Lemma A.2, for ρ small enough there is exactly one non-orientable singularity of $\mathbf{Q}$ inside $K_{\rho,t}$. Therefore, $\mathbf{Q}$ is non-orientable on $\partial K_{\rho,t}$ for a.e. $t \in (0, b)$ and a.e. $\rho > 0$ small enough; in particular, there is no continuous lifting of $\mathbf{Q}$ on $\partial K_{\rho,t}$. Since $\mathbf{M}^*$ is continuous on $\partial K_{\rho,t} \setminus L_1$ for a.e. ρ and t, we conclude that $\mathrm{S}_{\mathbf{M}^*}$ contains $\mathscr{H}^1$-almost all of L_1. □

Given a countably 1-rectifiable set $\Sigma \subseteq \mathbb{R}^2$ and a $\mathscr{H}^1$-measurable unit vector field $\boldsymbol{\tau}\colon \Sigma \to \mathbb{S}^1$, we say that $\boldsymbol{\tau}$ is an orientation for Σ if $\boldsymbol{\tau}(x)$ spans the (approximate) tangent line of Σ at x, for $\mathscr{H}^1$-a.e. $x \in \Sigma$. In case Σ is the jump set of an SBV-map $\mathbf{M}$, $\boldsymbol{\tau}\colon \mathrm{S}_{\mathbf{M}} \to \mathbb{S}^1$ is an orientation for $\mathrm{S}_{\mathbf{M}}$ if and only if $\boldsymbol{\tau}(x) \cdot \boldsymbol{\nu}_{\mathbf{M}}(x) = 0$ for $\mathscr{H}^1$-a.e. $x \in \mathrm{S}_{\mathbf{M}}$.

Lemma A.4. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded, convex domain and let $a_1, \ldots, a_{2d}$ be distinct points in Ω. Let $\mathbf{Q} \in W^{1,1}(\Omega, \mathscr{N}) \cap W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{a_1, \ldots, a_{2d}\}, \mathscr{N})$ be a map with a non-orientable singularity at each a_j. Let $\{L_1, \ldots, L_d\}$ be a minimal connection for $\{a_1, \ldots, a_{2d}\}$. Up to relabelling, we assume that L_j is the segment of endpoints a_{2j-1}, a_{2j}, for any $j \in \{1, \ldots, d\}$. Let $\mathbf{M} \in \mathrm{SBV}(\Omega, \mathbb{S}^1)$ be a lifting for $\mathbf{Q}$ such that $\mathrm{S}_{\mathbf{M}} \subset\subset \Omega$. Then, there exist $\mathscr{H}^1$-measurable sets $T_j \subseteq L_j$ and an orientation $\boldsymbol{\tau}_{\mathbf{M}}$ for $\mathrm{S}_{\mathbf{M}}$ such that, for any $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^2)$, there holds*

$$\int_{\mathrm{S}_{\mathbf{M}}} \nabla\varphi \cdot \boldsymbol{\tau}_{\mathbf{M}} \,\mathrm{d}\mathscr{H}^1 = \sum_{j=1}^d \left(\varphi(a_{2j-1}) - \varphi(a_{2j})\right) - 2\sum_{j=1}^d \int_{T_j} \nabla\varphi \cdot \frac{a_{2j-1} - a_{2j}}{|a_{2j-1} - a_{2j}|} \,\mathrm{d}\mathscr{H}^1$$

Proof. Let $\mathbf{M}^* \in \mathrm{SBV}(\Omega, \mathbb{S}^1)$ be the lifting of $\mathbf{Q}$ given by Lemma A.3. By construction, $\mathrm{S}_{\mathbf{M}^*}$ coincides with $\cup_j L_j \subset\subset \Omega$ up to $\mathscr{H}^1$-negligible sets. Since we have assumed that $\mathrm{S}_{\mathbf{M}} \subset\subset \Omega$, there exists a neighbourhood $U \subseteq \overline{\Omega}$ of $\partial\Omega$ in $\overline{\Omega}$ such that $\mathbf{M} \in W^{1,1}(U, \mathbb{S}^1)$, $\mathbf{M}^* \in W^{1,1}(U, \mathbb{S}^1)$. A map that belongs to $W^{1,1}(U, \mathscr{N})$ has at most two different liftings in $W^{1,1}(U, \mathbb{S}^1)$, which differ only for the sign [8, Proposition 2]. Therefore, since both $\mathbf{M}$ and $\mathbf{M}^*$ are liftings of $\mathbf{Q}$ in U, we have that either $\mathbf{M} = \mathbf{M}^*$ a.e. in U or $\mathbf{M} = -\mathbf{M}^*$ a.e. in U. Changing the sign of $\mathbf{M}^*$ if necessary, we can assume that $\mathbf{M} = -\mathbf{M}^*$ a.e. in U. Then, the set

$$A := \{x \in \Omega \colon \mathbf{M}(x) \cdot \mathbf{M}^*(x) = 1\}$$

is compactly contained in Ω.

The Leibnitz rule for BV-functions (see e.g. [3, Example 3.97]) implies that $\mathbf{M} \cdot \mathbf{M}^* \in \mathrm{SBV}(\Omega;\, \{-1,\, 1\})$ As a consequence, A has finite perimeter in Ω (see e.g. [3, Theorem 3.40]); since $A \subset\subset \Omega$, A has also finite perimeter in $\mathbb{R}^2$. By the Gauss-Green formula (see e.g. [3, Theorem 3.36, Eq. (3.47)]), for any $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^2)$ we have

$$0 = \int_A \operatorname{curl} \nabla\varphi = \int_{\partial^* A} \nabla\varphi \cdot \boldsymbol{\tau}_A \, \mathrm{d}\mathscr{H}^1, \tag{A.3}$$

where $\partial^* A$ is the reduced boundary of A and $\boldsymbol{\tau}_A$ is an orientation for $\partial^* A$. Up to $\mathscr{H}^1$-negligible sets, $\partial^* A$ coincides with $\mathrm{S}_{\mathbf{M}\cdot\mathbf{M}^*}$ (see e.g. [3, Example 3.68 and Theorem 3.61]). By the Leibnitz rule for BV-functions, $\mathrm{S}_{\mathbf{M}\cdot\mathbf{M}^*}$ coincides with $\mathrm{S}_{\mathbf{M}} \Delta \mathrm{S}_{\mathbf{M}^*}$ up to $\mathscr{H}^1$-negligible sets, so

$$\mathscr{H}^1 \left(\partial^* A \, \Delta \, (\mathrm{S}_{\mathbf{M}} \Delta \cup_{j=1}^d L_j)\right) = 0 \tag{A.4}$$

For any $j \in \{1, \ldots, d\}$, let $\boldsymbol{\tau}_j := (a_{2j-1} - a_{2j})/|a_{2j-1} - a_{2j}|$. We define an orientation $\boldsymbol{\tau}_{\mathbf{M}}$ for $\mathrm{S}_{\mathbf{M}}$ as $\boldsymbol{\tau}_{\mathbf{M}} := \boldsymbol{\tau}_A$ on $\mathrm{S}_{\mathbf{M}} \setminus (\cup_j L_j)$ (observing that, by (A.4), $\mathscr{H}^1$-almost all of $\mathrm{S}_{\mathbf{M}} \setminus (\cup_j L_j)$ is contained in $\partial^* A$) and $\boldsymbol{\tau}_{\mathbf{M}} := \boldsymbol{\tau}_j$ on $\mathrm{S}_{\mathbf{M}} \cap L_j$, for any j. Then, (A.3) and (A.4) imply

$$\int_{\mathrm{S}_{\mathbf{M}}} \nabla\varphi \cdot \boldsymbol{\tau}_{\mathbf{M}} \, \mathrm{d}\mathscr{H}^1 - \sum_{j=1}^d \int_{L_j} \nabla\varphi \cdot \boldsymbol{\tau}_j \, \mathrm{d}\mathscr{H}^1 + \sum_{j=1}^d \int_{L_j \setminus \mathrm{S}_{\mathbf{M}}} (1 + \boldsymbol{\tau}_A \cdot \boldsymbol{\tau}_j) \nabla\varphi \cdot \boldsymbol{\tau}_j \, \mathrm{d}\mathscr{H}^1 = 0, \tag{A.5}$$

On $\mathscr{H}^1$-almost all of $L_j \setminus \mathrm{S}_{\mathbf{M}}$, both $\boldsymbol{\tau}_j$ and $\boldsymbol{\tau}_A$ are tangent to L_j. Therefore, for $\mathscr{H}^1$-a.e. $x \in L_j \setminus \mathrm{S}_{\mathbf{M}}$ we have $\boldsymbol{\tau}_A(x) \cdot \boldsymbol{\tau}_j(x) \in \{-1,\, 1\}$. If we define $T_j := \{x \in L_j \setminus \mathrm{S}_{\mathbf{M}} \colon \boldsymbol{\tau}_A(x) \cdot \boldsymbol{\tau}_j(x) = 1\}$, then the lemma follows from (A.5). □

Lemma A.4 can be reformulated in terms of currents. We recall a few basic definitions in the theory of currents, because they will be useful to complete the proof of Proposition A.1. Actually, we will only work with currents of dimension 0 or 1. We refer to, e.g., [27, 44] for more details.

A 0-dimensional current, or 0-current, in $\mathbb{R}^2$ is just a distribution on $\mathbb{R}^2$, i.e. an element of the topological dual of $C^\infty_{\mathrm{c}}(\mathbb{R}^2)$ (where $C^\infty_{\mathrm{c}}(\mathbb{R}^2)$ is given a suitable topology). A 1-dimensional current, or 1-current, in $\mathbb{R}^2$ is an element of the topological dual of $C^\infty_{\mathrm{c}}(\mathbb{R}^2;\, (\mathbb{R}^2)')$, where $(\mathbb{R}^2)'$ denotes the dual of $\mathbb{R}^2$ and $C^\infty_{\mathrm{c}}(\mathbb{R}^2;\, (\mathbb{R}^2)')$ is given a suitable topology, in much the same way as $C^\infty_{\mathrm{c}}(\mathbb{R}^2)$. In other words, a 1-dimensional current is an $\mathbb{R}^2$-valued distribution. The boundary of a 1-current T is the 0-current ∂T defined by

$$\langle \partial T,\, \varphi \rangle := \langle T,\, \mathrm{d}\varphi \rangle \qquad \text{for any } \varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^2).$$

The mass of a 1-current T is defined as

$$\mathbb{M}(T) := \sup\left\{ \langle T,\, \omega \rangle \colon \omega \in C^\infty_{\mathrm{c}}(\mathbb{R}^2;\, (\mathbb{R}^2)'),\ |\omega(x)| \le 1 \quad \text{for any } x \in \mathbb{R}^2 \right\};$$

the mass of a 0-current is defined analogously.

We single out a particular subset of currents, called integer-multiplicity rectifiable currents or rectifiable currents for short. A rectifiable 0-current is a current of the form

$$T = \sum_{k=1}^{p} n_k \, \delta_{b_k}, \tag{A.6}$$

where $k \in \mathbb{N}$, $n_k \in \mathbb{Z}$ and $b_k \in \mathbb{R}^2$. A rectifibiable 0-current has finite mass: for the current T given by (A.6), we have $\mathbb{M}(T) = \sum_{k=1}^{p} |n_k|$. A 1-current is called rectifiable if there exist a countably 1-rectifiable set $\Sigma \subseteq \mathbb{R}^2$ with $\mathscr{H}^1(\Sigma) < +\infty$, an orientation $\boldsymbol{\tau}\colon \Sigma \to \mathbb{S}^1$ for Σ and an integer-valued, $\mathscr{H}^1$-integrable function $\theta\colon \Sigma \to \mathbb{Z}$ such that

$$\langle T, \, \omega \rangle = \int_\Sigma \theta(x) \langle \boldsymbol{\tau}(x), \, \omega(x) \rangle \, \mathrm{d}\mathscr{H}^1(x) \qquad \text{for any } \omega \in C^\infty_{\mathrm{c}}(\mathbb{R}^2; \, (\mathbb{R}^2)'). \tag{A.7}$$

The current T defined by (A.7) is called the rectifiable 1-current carried by Σ, with multiplicity θ and orientation $\boldsymbol{\tau}$; it satisfies

$$\mathbb{M}(T) = \int_\Sigma |\theta(x)| \, \mathrm{d}\mathscr{H}^1(x) < +\infty.$$

The set of rectifiable 0-currents, respectively rectifiable 1-currents, is denoted by $\mathscr{R}_0(\mathbb{R}^2)$, respectively $\mathscr{R}_1(\mathbb{R}^2)$.

Given a Lipschitz, injective map $\mathbf{f}\colon [0, \, 1] \to \mathbb{R}^2$, we denote by $\mathbf{f}_{\#} I$ the rectifiable 1-current carried by $\mathbf{f}([0, \, 1])$, with unit multiplicity and orientation given by $\mathbf{f}'$. The mass of $\mathbf{f}_{\#} I$ is the length of the curve parametrised by $\mathbf{f}$ and $\partial(\mathbf{f}_{\#} I) = \delta_{\mathbf{f}(1)} - \delta_{\mathbf{f}(0)}$; in particular, $\partial(\mathbf{f}_{\#} I) = 0$ if $\mathbf{f}(1) = \mathbf{f}(0)$. The assumption that $\mathbf{f}$ is injective can be relaxed; for instance, if the curve parametrised by $\mathbf{f}$ has only a finite number of self-intersections, then $\mathbf{f}_{\#} I$ is still well-defined and the properties above remain valid.

We take a bounded, convex domain $\Omega \subseteq \mathbb{R}^2$, distinct points $a_1, \, \ldots a_{2d}$ and a map $\mathbf{Q} \in W^{1,1}(\Omega, \, \mathscr{N}) \cap W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{a_1, \, \ldots, \, a_{2d}\}, \, \mathscr{N})$ with a non-orientable singularity at each a_i. Let $\mathbf{M} \in \mathrm{SBV}(\Omega, \, \mathbb{S}^1)$ be a lifting of $\mathbf{Q}$ such that $\mathrm{S}_{\mathbf{M}} \subset\subset \Omega$. By Federer-Vol'pert theorem (see e.g. [3, Theorem 3.78]), the set $\mathrm{S}_{\mathbf{M}}$ is countably 1-rectifiable. We claim that $\mathscr{H}^1(\mathrm{S}_{\mathbf{M}}) < +\infty$. Indeed, since $\mathbf{Q}$ has no jump set, by the BV-chain rule (see e.g. [3, Theorem 3.96]) we deduce that $\mathbf{M}^+(x) = -\mathbf{M}^-(x)$ at $\mathscr{H}^1$-a.e. point $x \in \mathrm{S}_{\mathbf{M}}$. This implies

$$\mathscr{H}^1(\mathrm{S}_{\mathbf{M}}) \leq \frac{1}{2} \int_{\mathrm{S}_{\mathbf{M}}} |\mathbf{M}^+ - \mathbf{M}^-| \, \mathrm{d}\mathscr{H}^1 \lesssim |\mathrm{D}\mathbf{M}| \, (\Omega) < +\infty,$$

as claimed. In particular, there is a well-defined, rectifiable 1-current carried by $\mathrm{S}_{\mathbf{M}}$, with unit multiplicity and orientation $\boldsymbol{\tau}_{\mathbf{M}}$ given by Lemma A.4; we denote it by $[\![\mathrm{S}_{\mathbf{M}}]\!]$. Lemma A.4 provides information on the boundary of $[\![\mathrm{S}_{\mathbf{M}}]\!]$. More precisely, Lemma A.4 implies

$$\partial [\![\mathrm{S}_{\mathbf{M}}]\!] = \sum_{i=1}^{2d} \delta_{a_i} + 2 \, \partial Q \tag{A.8}$$

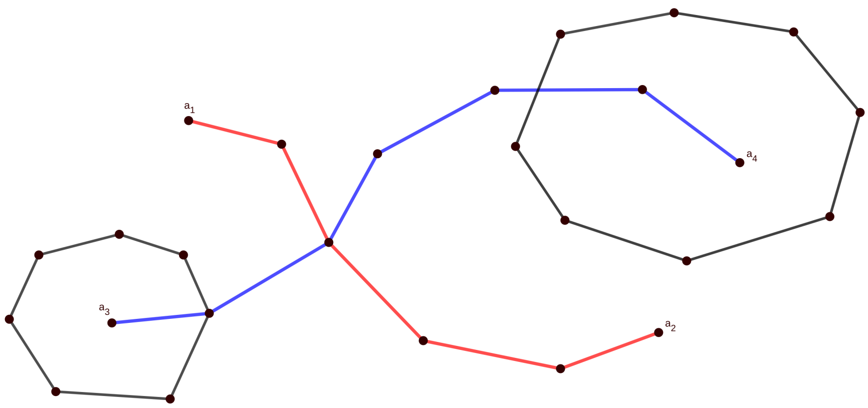

Figure 3: A decomposition of the graph $\mathscr{G}$, as defined in the proof of Lemma A.5, into edge-disjoint trails $\mathscr{E}_1$ (in red) and $\mathscr{E}_2$ (in blue). In addition to the edges of $\mathscr{G}$, there may be other cycles, carried by the curves $\mathbf{g}_j([0,\,1])$ with $j \geq q+1$; they are shown in black.

where Q is a rectifiable 1-chain, defined as

$$\langle Q,\,\psi\rangle := \sum_{j=1}^{d} \int_{T_j} \left\langle \psi(x), \frac{a_{2j-1}-a_{2j}}{|a_{2j-1}-a_{2j}|} \right\rangle \mathrm{d}\mathscr{H}^1(x) \tag{A.9}$$

for any $\psi \in C^\infty_{\mathrm{c}}(\mathbb{R}^2,\,(\mathbb{R}^2)')$. The T_j's are 1-rectifiable sets that depend only on $\mathbf{M}$, not on ψ, as given by Lemma A.4.

Lemma A.5. *Let Ω, $\mathbf{Q}$ be as above. Let $\mathbf{M} \in \mathrm{SBV}(\Omega,\,\mathbb{S}^1)$ be a lifting of $\mathbf{Q}$ with $\mathrm{S}_{\mathbf{M}} \subset\subset \Omega$. Then, there exist countably may Lipschitz functions $\mathbf{f}_j\colon [0,\,1] \to \mathbb{R}^2$, with finitely many self-intersections, a rectifiable 1-current $R \in \mathscr{R}_1(\mathbb{R}^2)$ and a permutation σ of the indices $\{1,\,\ldots,\,2d\}$ such that the following properties hold:*

$$[\![\mathrm{S}_{\mathbf{M}}]\!] = \sum_{j\geq 1} \mathbf{f}_{j,\#} I + 2R \tag{A.10}$$

$$\mathbb{M}([\![\mathrm{S}_{\mathbf{M}}]\!]) = \sum_{j\geq 1} \mathbb{M}\left(\mathbf{f}_{j,\#} I\right) \tag{A.11}$$

$$\partial(\mathbf{f}_{j,\#} I) = \delta_{\sigma(2j)} - \delta_{\sigma(2j-1)} \quad \textit{if } j \in \{1,\,\ldots,\,d\}, \qquad \partial(\mathbf{f}_{j,\#} I) = 0 \quad \textit{otherwise.} \tag{A.12}$$

Proof. By applying, e.g., [47, Theorem 6.3] or [4, Corollary 4.2], we find rectifiable 1-currents T, $R \in \mathscr{R}_1(\mathbb{R}^2)$ such that $\mathbb{M}(T) = \mathbb{M}([\![\mathrm{S}_{\mathbf{M}}]\!]) = \mathscr{H}^1(\mathrm{S}_{\mathbf{M}})$, $\partial T \in \mathscr{R}_0(\mathbb{R}^2)$ and

$$T = [\![\mathrm{S}_{\mathbf{M}}]\!] + 2R. \tag{A.13}$$

By taking the boundary of both sides of (A.13), and applying (A.8), we obtain

$$\partial T = \sum_{i=1}^{2d} \delta_{a_i} + 2P \tag{A.14}$$

with $P := \partial(R+Q)$ (and Q as in (A.9)). The current $2P = \partial T - \sum_{i=1}^{2d} \delta_{a_i}$ is rectifiable, so $\mathbb{M}(P) < +\infty$. Moreover, P isthe boundary of a rectifiable 1-current. Then, Federer's closure theorem [27, 4.2.16] implies that P itself is rectifiable. As a consequence, we can re-write (A.14) as

$$\partial T = \sum_{i=1}^{2d} \delta_{a_i} + 2 \sum_{k=1}^{p} n_k \, \delta_{b_k}, \tag{A.15}$$

for some integers n_k and some distinct points $b_k \in \mathbb{R}^2$. By applying [27, 4.2.25], we find countably many Lipschitz, injective maps $\mathbf{g}_j \colon [0,\, 1] \to \mathbb{R}$ such that

$$T = \sum_{j\geq 1} \mathbf{g}_{j,\#} I, \qquad \sum_{j \geq 1} \left(\mathbb{M}(\mathbf{g}_{j,\#} I) + \mathbb{M}(\partial(\mathbf{g}_{j,\#} I))\right) = \mathbb{M}(T) + \mathbb{M}(\partial T) < +\infty. \tag{A.16}$$

For any j, we have either $\partial(\mathbf{g}_{j,\#} I) = 0$ (if $\mathbf{g}_{j,\#}(1) = \mathbf{g}_{j,\#}(0)$) or $\mathbb{M}(\partial(\mathbf{g}_{j,\#} I)) = 2$ (otherwise). Therefore, by (A.16), there are only finitely many indices j such that $\mathbf{g}_{j,\#}(1) \neq \mathbf{g}_{j,\#}(0)$. Up to a relabelling of the $\mathbf{g}_j$'s, we assume that there is an integer q such that $\mathbf{g}_{j,\#}(1) \neq \mathbf{g}_{j,\#}(0)$ if and only if $j \leq q$.

Now, the problem reduces to a combinatorial, or graph-theoretical, one. We consider the finite (multi-)graph $\mathscr{G}$ whose edges are the curves parametrised by $\mathbf{g}_1, \ldots, \mathbf{g}_q$, and whose vertices are the endpoints of such curves. There can be two or more edges that join the same pair of vertices. However, we can disregard the orientation of the edges: changing the orientation of the curve parametrised by $\mathbf{g}_j$ corresponds to passing from the current $\mathbf{g}_{j,\#} I$ to the current $-\mathbf{g}_{j,\#} I$; the difference $\mathbf{g}_{j,\#} I - (-\mathbf{g}_{j,\#} I) = 2\mathbf{g}_{j,\#} I$ can be absorbed into the term $2R$ that appears in (A.10).

We would like to partition the set of edges of $\mathscr{G}$ into d disjoint subsets $\mathscr{E}_1, \ldots \mathscr{E}_d$, where each $\mathscr{E}_j$ is a trail (i.e., a sequence of distinct edges such that each edge is adjacent to the next one) and, for a suitable permutation σ of $\{1, \ldots, 2d\}$, the trail $\mathscr{E}_j$ connects $a_{\sigma(2j-1)}$ with $a_{\sigma(2j)}$. If we do so, then we can define $\mathbf{f}_j \colon [0,\, 1] \to \mathbb{R}^2$ for $j \in \{1, \ldots, d\}$ as a Lipschitz map that parameterises the trail $\mathscr{E}_j$, with suitable orientations of each edge; for $j \geq d+1$, we define $\mathbf{f}_j := \mathbf{g}_{q+j-d}$. With this choice of $\mathbf{f}_j$, the lemma follows. It is possible to find $\mathscr{E}_1, \ldots \mathscr{E}_d$ as required because the graph $\mathscr{G}$ has the following property: any a_i is an endpoint of an *odd* number of edges of $\mathscr{G}$; conversely, any vertex of $\mathscr{G}$ other than the a_i's is an endpoint of an *even* number of edges of $\mathscr{G}$. This property follows from (A.15). Then, we can construct $\mathscr{E}_1, \ldots \mathscr{E}_d$ by reasoning along the lines of, e.g., [16, Theorem 12]. □

We can now conclude the proof of Proposition A.1.

Proof of Proposition A.1. We consider the decomposition of $[\![\mathrm{S}_{\mathbf{M}}]\!]$ given by Lemma A.5. Thanks to (A.12), for any $j \in \{1, \ldots, d\}$ the curve parametrised by $\mathbf{f}_j$ joins $a_{\sigma(2j-1)}$ with $a_{\sigma(2j)}$. Then,

$$\mathscr{H}^1(\mathrm{S}_{\mathbf{M}}) = \mathbb{M}([\![\mathrm{S}_{\mathbf{M}}]\!]) \geq \sum_{j=1}^{d} \mathbb{M}(\mathbf{f}_{j,\#} I) \geq \sum_{j=1}^{d} \left|a_{\sigma(2j)} - a_{\sigma(2j-1)}\right| \geq \mathbb{L}(a_1, \ldots, a_{2d}).$$

The equality can only be attained if there are exactly d maps $\mathbf{f}_j$ and each of them parametrises a straight line segment. □

B Properties of f_ε

The aim of this section is to prove Lemma 3.1. We first of all, we characterise the zero-set of the potential f_ε, in terms of the (unique) solution to an algebraic system depending on ε and β.

Lemma B.1. *For any $\varepsilon > 0$, the algebraic system*

$$\begin{cases} X(X - 1 - \beta^2\varepsilon)^2 = \dfrac{\beta^2\varepsilon^2}{2} \\ X > 1 + \beta^2\varepsilon \end{cases} \tag{B.1}$$

admits a unique solution X_ε, which satisfies

$$X_\varepsilon = 1 + \frac{1}{\sqrt{2}}\left(\sqrt{2}\beta + 1\right)\beta\varepsilon - \frac{1}{4}\left(\sqrt{2}\beta + 1\right)\beta^2\varepsilon^2 + \mathrm{o}(\varepsilon^2) \qquad \text{as } \varepsilon \to 0.$$

Proof. The function $P(X) := X(X - 1 - \beta^2\varepsilon)^2$ is continuous and strictly increasing in the interval $[1 + \beta^2\varepsilon, +\infty)$, because $P'(X) = (X - 1 - \beta^2\varepsilon)(3X - 1 - \beta^2\varepsilon) > 0$ for $X > 1 + \beta^2\varepsilon$. Moreover, $P(1 + \beta^2\varepsilon) = 0$ and $P(X) \to +\infty$ as $X \to +\infty$. Therefore, the system (B.1) admits a unique solution. Let $Y_\varepsilon > 0$ be such that

$$X_\varepsilon = 1 + \beta^2\varepsilon + \beta\varepsilon\, Y_\varepsilon.$$

Then, (B.1) can be rewritten as

$$Y_\varepsilon^2 = \frac{1}{2 + 2\beta^2\varepsilon + 2\beta\varepsilon\, Y_\varepsilon}, \tag{B.2}$$

which implies $Y_\varepsilon \to 1/\sqrt{2}$ as $\varepsilon \to 0$. Using (B.2) again, we obtain

$$Y_\varepsilon = \frac{1}{\left(2 + 2\beta^2\varepsilon + \sqrt{2}\beta\varepsilon + \mathrm{o}(\varepsilon)\right)^{1/2}} = \frac{1}{\sqrt{2}} - \frac{1}{4}\left(\sqrt{2}\beta + 1\right)\beta\varepsilon + \mathrm{o}(\varepsilon)$$

as $\varepsilon \to 0$, and the lemma follows. □

For any $\varepsilon > 0$, we define

$$s_\varepsilon := X_\varepsilon^{1/2}, \qquad \lambda_\varepsilon := \left(\frac{X_\varepsilon - 1}{X_\varepsilon - 1 - \beta^2\varepsilon}\right)^{1/2}. \tag{B.3}$$

Lemma B.1 implies, via routine algebraic manipulations, that

$$s_\varepsilon = 1 + \frac{1}{2\sqrt{2}}\left(\sqrt{2}\beta + 1\right)\beta\varepsilon + \mathrm{o}(\varepsilon), \qquad \lambda_\varepsilon^2 = \sqrt{2}\beta + 1 + \frac{1}{2}\left(\sqrt{2}\beta + 1\right)\beta^2\varepsilon + \mathrm{o}(\varepsilon) \tag{B.4}$$

as $\varepsilon \to 0$.

Lemma B.2. *A pair $(\mathbf{Q}, \mathbf{M}) \in \mathscr{S}_0^{2\times 2} \times \mathbb{R}^2$ satisfies $f_\varepsilon(\mathbf{Q}, \mathbf{M}) = 0$ if and only if*

$$|\mathbf{M}| = \lambda_\varepsilon, \qquad \mathbf{Q} = \sqrt{2}\, s_\varepsilon \left(\frac{\mathbf{M} \otimes \mathbf{M}}{\lambda_\varepsilon^2} - \frac{\mathbf{I}}{2}\right).$$

Proof. By imposing that the gradient of f_ε is equal to zero, we obtain the system

$$(|\mathbf{Q}|^2 - 1)\mathbf{Q} = \beta\varepsilon\left(\mathbf{M}\otimes\mathbf{M} - \frac{|\mathbf{M}|^2}{2}\mathbf{I}\right) \tag{B.5}$$

$$(|\mathbf{M}|^2 - 1)\mathbf{M} = 2\beta\,\mathbf{Q}\mathbf{M}. \tag{B.6}$$

Suppose first that $\mathbf{M} = 0$. Then, Equation (B.5) implies that either $\mathbf{Q} = 0$ or $|\mathbf{Q}| = 1$. The pair $\mathbf{Q} = 0$, $\mathbf{M} = 0$ is not a minimiser for f_ε, because $\nabla^2_{\mathbf{Q}} f_\varepsilon(0, 0) = -\mathbf{I} < 0$. If $|\mathbf{Q}| = 1$, $\mathbf{M} = 0$, then $\nabla^2_{\mathbf{M}} f_\varepsilon(\mathbf{Q}, 0) = -\varepsilon(\mathbf{I} + 2\beta\mathbf{Q})$. Since $\mathbf{Q}$ is non-zero, symmetric and trace-free, there exists $\mathbf{n}\in\mathbb{S}^1$ such that $\mathbf{Q}\mathbf{n}\cdot\mathbf{n} > 0$. Then, $\nabla^2_{\mathbf{M}} f_\varepsilon(\mathbf{Q}, 0)\mathbf{n}\cdot\mathbf{n} < 0$, so the pair $(\mathbf{Q}, \mathbf{M} = 0)$ is not a minimiser of f_ε. It remains to consider the case $\mathbf{M}\neq 0$. In this case, we have $\mathbf{Q}\neq 0$ and $|\mathbf{Q}|\neq 1$, due to (B.5). Solving (B.5) for $\mathbf{Q}$, and then substituting in (B.6), we obtain

$$|\mathbf{M}|^2 - 1 = \frac{\beta^2\varepsilon\,|\mathbf{M}|^2}{|\mathbf{Q}|^2 - 1}$$

and hence, solving for $|\mathbf{M}|^2$,

$$|\mathbf{M}|^2 = \frac{|\mathbf{Q}|^2 - 1}{|\mathbf{Q}|^2 - 1 - \beta^2\varepsilon}. \tag{B.7}$$

By taking the squared norm of both sides of (B.5), we obtain

$$(|\mathbf{Q}|^2 - 1)^2\,|\mathbf{Q}|^2 = \frac{\beta^2\varepsilon^2}{2}\,|\mathbf{M}|^4$$

and hence, using (B.7),

$$|\mathbf{Q}|^2 = \frac{\beta^2\varepsilon^2}{2(|\mathbf{Q}|^2 - 1 - \beta^2\varepsilon)^2} \tag{B.8}$$

We either have $|\mathbf{Q}|^2 < 1$ or $|\mathbf{Q}|^2 > 1 + \beta^2\varepsilon$, because of (B.7). On the other hand, by imposing that the second derivative of f_ε with respect to $\mathbf{Q}$ is non-negative, we obtain $|\mathbf{Q}|^2 \geq 1$. Therefore, we conclude that $|\mathbf{Q}|^2 = X_\varepsilon$ is the unique solution to the system (B.1) and, taking (B.7) into account, the proposition follows. □

We can now prove Lemma 3.1. For convenience, we recall the statement here.

Lemma B.3. *The potential f_ε satisfies the following properties.*

(i) The constant κ_ε in (2.2)*, uniquely defined by imposing the condition* $\inf f_\varepsilon = 0$*, satisfies*

$$\kappa_\varepsilon = \frac{1}{2}\left(\beta^2 + \sqrt{2}\beta\right)\varepsilon + \kappa_*^2\,\varepsilon^2 + \mathrm{o}(\varepsilon^2) \tag{B.9}$$

In particular, $\kappa_\varepsilon \geq 0$ for ε small enough.

(ii) If $(\mathbf{Q}, \mathbf{M}) \in \mathscr{S}_0^{2\times 2} \times \mathbb{R}^2$ *is such that*

$$|\mathbf{M}| = (\sqrt{2}\beta + 1)^{1/2}, \qquad \mathbf{Q} = \sqrt{2}\left(\frac{\mathbf{M}\otimes\mathbf{M}}{\sqrt{2}\beta+1} - \frac{\mathbf{I}}{2}\right) \tag{B.10}$$

then $f_\varepsilon(\mathbf{Q}, \mathbf{M}) = \kappa_*\,\varepsilon^2 + \mathrm{o}(\varepsilon^2)$.

(iii) If ε *is sufficiently small, then*

$$\frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}, \mathbf{M}) \geq \frac{1}{4\varepsilon^2}(|\mathbf{Q}|^2 - 1)^2 - \frac{\beta}{\sqrt{2}\varepsilon}\,|\mathbf{M}|^2\, ||\mathbf{Q}| - 1| \tag{B.11}$$

$$\frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}, \mathbf{M}) \geq \frac{1}{8\varepsilon^2}(|\mathbf{Q}|^2 - 1)^2 - \beta^2\,|\mathbf{M}|^4 \tag{B.12}$$

for any $(\mathbf{Q}, \mathbf{M}) \in \mathscr{S}_0^{2\times 2} \times \mathbb{R}^2$.

Proof of Statement (i). Let $(\mathbf{Q}_*^*, \mathbf{M}^*) \in \mathscr{S}_0^{2\times 2} \times \mathbb{R}^2$ be a minimiser for f_ε, i.e. $f_\varepsilon(\mathbf{Q}_*^*, \mathbf{M}^*) = 0$. By Lemma B.2, we have

$$\begin{aligned} \kappa_\varepsilon &= -\frac{1}{4}(|\mathbf{Q}_*^*|^2 - 1)^2 - \frac{\varepsilon}{4}(|\mathbf{M}^*|^2 - 1)^2 + \beta\varepsilon\,\mathbf{Q}_*^*\mathbf{M}^*\cdot\mathbf{M}^* \\ &= -\frac{1}{4}(s_\varepsilon^2 - 1)^2 - \frac{\varepsilon}{4}(\lambda_\varepsilon^2 - 1)^2 + \frac{\beta\varepsilon}{\sqrt{2}}\, s_\varepsilon\lambda_\varepsilon^2 \end{aligned}$$

We expand s_ε, λ_ε in terms of ε, as given by (B.4). Equation (B.9) then follows by standard algebraic manipulations.

Proof of Statement (ii). The assumption (B.10) implies

$$|\mathbf{Q}| = 1, \qquad \mathbf{Q}\mathbf{M}\cdot\mathbf{M} = \sqrt{2}\left(\frac{|\mathbf{M}|^4}{\sqrt{2}\beta+1} - \frac{1}{2}|\mathbf{M}|^2\right) = \frac{\sqrt{2}}{2}\left(\sqrt{2}\beta + 1\right) = \beta + \frac{\sqrt{2}}{2}$$

Therefore,

$$f_\varepsilon(\mathbf{Q}, \mathbf{M}) = \frac{\varepsilon\,\beta^2}{2} - \beta\varepsilon\left(\beta + \frac{\sqrt{2}}{2}\right) + \kappa_\varepsilon = -\frac{\varepsilon}{2}\left(\beta^2 + \sqrt{2}\beta\right) + \kappa_\varepsilon \overset{\text{(B.9)}}{=} \kappa_*\,\varepsilon^2 + \mathrm{o}(\varepsilon^2)$$

Proof of Statement (iii). When $\mathbf{Q} = 0$, we have $f_\varepsilon(0, \mathbf{M}) \geq 1/4 + \kappa_\varepsilon$ and $\kappa_\varepsilon > 0$ is positive for ε small enough, due to (B.9). Then, (B.11) follows. When $\mathbf{Q} \neq 0$, it is convenient to make the change of variables we have introduced in Section 3. We write

$$\mathbf{Q} = \frac{|\mathbf{Q}|}{\sqrt{2}}\,(\mathbf{n}\otimes\mathbf{n} - \mathbf{m}\otimes\mathbf{m})$$

where $(\mathbf{n}, \mathbf{m})$ is an orthonormal basis of eigenvalues for $\mathbf{Q}$. We define $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ as $u_1 := \mathbf{M}\cdot\mathbf{n}$, $u_2 := \mathbf{M}\cdot\mathbf{m}$. The potential f_ε can be expressed in terms of $\mathbf{Q}$, $\mathbf{u}$ as (see Equation (3.14)),

$$\begin{aligned} \frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}, \mathbf{M}) = \frac{1}{4\varepsilon^2}(|\mathbf{Q}|^2 - 1)^2 + \frac{1}{\varepsilon}h(\mathbf{u}) + \frac{\beta}{\sqrt{2}\,\varepsilon}(1 - |\mathbf{Q}|)\,(u_1^2 - u_2^2) \\ + \frac{\kappa_\varepsilon}{\varepsilon^2} - \frac{1}{2\varepsilon}(\beta^2 + \sqrt{2}\beta) \end{aligned}$$

where h is defined in (3.8). By Lemma (3.4), we know that $h \geq 0$. Moreover, Equation (B.9) implies

$$\frac{\kappa_\varepsilon}{\varepsilon^2} - \frac{1}{2\varepsilon}(\beta^2 + \sqrt{2}\beta) = \kappa_*^2 + \mathrm{o}(1) \geq 0$$

for ε small enough. Then, (B.11) follows. Equation (B.12) follows from (B.11), as

$$\frac{\beta}{\sqrt{2}\varepsilon} \left|\mathbf{M}\right|^2 \left|\left|\mathbf{Q}\right| - 1\right| \leq \beta^2 \left|\mathbf{M}\right|^4 + \frac{1}{8\varepsilon^2}(\left|\mathbf{Q}\right| - 1)^2 \leq \beta^2 \left|\mathbf{M}\right|^4 + \frac{1}{8\varepsilon^2}(\left|\mathbf{Q}\right|^2 - 1)^2$$

□

C Proof of Lemma 4.4

The aim of this section is to prove Lemma C.1, which we recall here for the convenience of the reader. We recall that $g_\varepsilon \colon \mathscr{S}_0^{2\times 2} \to \mathbb{R}$ is the function defined in (3.7).

Lemma C.1. *Let $B = B_r(x_0) \subseteq \Omega$ be an open ball. Suppose that $\mathbf{Q}_\varepsilon^* \rightharpoonup \mathbf{Q}^*$ weakly in $W^{1,2}(\partial B)$ and that*

$$\int_{\partial B} \left(\frac{1}{2} \left|\nabla \mathbf{Q}_\varepsilon^*\right|^2 + g_\varepsilon(\mathbf{Q}_\varepsilon^*)\right) \mathrm{d}\mathscr{H}^1 \leq C \tag{C.1}$$

for some constant C that may depend on the radius r, but not on ε. Then, there exists a map $\mathbf{Q}_\varepsilon \in W^{1,2}(B, \mathscr{S}_0^{2\times 2})$ such that

$$\mathbf{Q}_\varepsilon = \mathbf{Q}_\varepsilon^* \quad \text{on } \partial B, \qquad \left|\mathbf{Q}_\varepsilon\right| \geq \frac{1}{2} \quad \text{in } B \tag{C.2}$$

$$\int_B \left(\frac{1}{2} \left|\nabla \mathbf{Q}_\varepsilon\right|^2 + g_\varepsilon(\mathbf{Q}_\varepsilon)\right) \mathrm{d}x \to \frac{1}{2} \int_B \left|\nabla \mathbf{Q}^*\right|^2 \mathrm{d}x \tag{C.3}$$

Lemma C.1 is inspired by interpolation results in the literature on harmonic maps (see e.g. [38, Lemma 1]). As we work in a two-dimensional domain, we can simplify some points of the proof in [38]. On the other hand, we need to estimate the contributions from the term $g_\varepsilon(\mathbf{Q}_\varepsilon)$, which is not present in [38].

Proof of Lemma C.1. Without loss of generality, we can assume that $x_0 = 0$. By assumption, we have $\mathbf{Q}_\varepsilon^* \rightharpoonup \mathbf{Q}^*$ weakly in $W^{1,2}(\partial B)$ and hence, by Sobolev embedding, uniformly on ∂B. In particular, $\left|\mathbf{Q}_\varepsilon^*\right| \to 1$ uniformly on ∂B. Let $\lambda_\varepsilon > 0$ be a small number, to be chosen later on. We consider the decomposition $B = A_\varepsilon^1 \cup A_\varepsilon^2 \cup A_\varepsilon^3$, where

$$A_\varepsilon^1 := B_r \setminus \bar{B}_{r-\lambda_\varepsilon r}, \qquad A_\varepsilon^2 := \bar{B}_{r-\lambda_\varepsilon r} \setminus \bar{B}_{r-2\lambda_\varepsilon r}, \qquad A_\varepsilon^3 := \bar{B}_{r-2\lambda_\varepsilon r}$$

We define the map $\mathbf{Q}_\varepsilon$ using polar coordinates (ρ, θ), as follows. If $x = \rho e^{i\theta} \in A_\varepsilon^1$, we define

$$\mathbf{Q}_\varepsilon(x) := t_\varepsilon(\rho)\, \mathbf{Q}_\varepsilon^*(re^{i\theta}) + (1 + \kappa_*\varepsilon)(1 - t_\varepsilon(\rho))\, \frac{\mathbf{Q}_\varepsilon^*(re^{i\theta})}{\left|\mathbf{Q}_\varepsilon^*(re^{i\theta})\right|}$$

where $t_\varepsilon\colon \mathbb{R}\to\mathbb{R}$ is an affine function such that $t_\varepsilon(r)=1$, $t_\varepsilon(r-\lambda_\varepsilon r)=0$. If $x=\rho e^{i\theta}\in A^2_\varepsilon$, we define

$$\mathbf{Q}_\varepsilon(x) := (1+\kappa_*\varepsilon)\,\frac{s_\varepsilon(\rho)\,\mathbf{Q}^*_\varepsilon(re^{i\theta}) + (1-s_\varepsilon(\rho))\,\mathbf{Q}^*(re^{i\theta})}{|s_\varepsilon(\rho)\,\mathbf{Q}^*_\varepsilon(re^{i\theta}) + (1-s_\varepsilon(\rho))\,\mathbf{Q}^*(re^{i\theta})|}$$

where $s_\varepsilon\colon \mathbb{R}\to\mathbb{R}$ is an affine function such that $s_\varepsilon(r-\lambda_\varepsilon r)=1$, $s_\varepsilon(r-2\lambda_\varepsilon r)=0$. Finally, if $x\in A^3_\varepsilon$, we define

$$\mathbf{Q}_\varepsilon(x) := (1+\kappa_*\varepsilon)\,\mathbf{Q}^*\left(\frac{x}{1-2\lambda_\varepsilon}\right)$$

The map $\mathbf{Q}_\varepsilon$ is well-defined in B, beacuse $|\mathbf{Q}_\varepsilon|\to 1$ uniformly on ∂B. Moreover, we have $|\mathbf{Q}_\varepsilon|\geq 1/2$ for ε small enough, $\mathbf{Q}_\varepsilon\in W^{1,2}(B,\,\mathscr{S}_0^{2\times 2})$ (at the interfaces between A^1_ε, A^2_ε, A^3_ε, the traces of $\mathbf{Q}_\varepsilon$ on either side of the interface match), and $\mathbf{Q}_\varepsilon=\mathbf{Q}^*_\varepsilon$ on ∂B.

It only remains to prove (C.3). First, we estimate the integral of $g_\varepsilon(\mathbf{Q}_\varepsilon)$. On $A^2_\varepsilon\cup A^3_\varepsilon$, we have $|\mathbf{Q}_\varepsilon|=1+\kappa_*\varepsilon$ and hence, substituting in (3.7),

$$g_\varepsilon(\mathbf{Q}_\varepsilon)=\kappa_*^2\left(\frac{1}{4}(2+\kappa_*\varepsilon)^2-1\right)=\kappa_*^2\left(\kappa_*\varepsilon+\kappa_*^2\varepsilon^2\right)=\mathrm{O}(\varepsilon) \tag{C.4}$$

We consider the annulus A^1_ε. By Lemma 3.3, we have

$$g_\varepsilon(\mathbf{Q}_\varepsilon)\leq\left(\frac{1}{\varepsilon}(|\mathbf{Q}_\varepsilon|-1)-\kappa_*\right)^2+\frac{C}{\varepsilon^2}(|\mathbf{Q}_\varepsilon|-1)^2$$

For $x\in A^1_\varepsilon$, we have $|\mathbf{Q}_\varepsilon(x)|=t_\varepsilon\,|\mathbf{Q}^*_\varepsilon(rx/\,|x|)|+(1-t_\varepsilon)(1+\kappa_*\varepsilon)$, with $t_\varepsilon=t_\varepsilon(\rho)\in[0,\,1]$. As a consequence,

$$\int_{A^1_\varepsilon} g_\varepsilon(\mathbf{Q}_\varepsilon)\,\mathrm{d}x\lesssim\lambda_\varepsilon\int_{\partial B}\left(\frac{1}{\varepsilon}(|\mathbf{Q}^*_\varepsilon|-1)-\kappa_*\right)^2\mathrm{d}\mathscr{H}^1+\frac{\lambda_\varepsilon}{\varepsilon^2}\int_{\partial B}(|\mathbf{Q}^*_\varepsilon|-1)^2\,\mathrm{d}\mathscr{H}^1+\lambda_\varepsilon\kappa_*^2 \tag{C.5}$$

On the other hand, as $|\mathbf{Q}^*_\varepsilon|\to 1$ uniformly on ∂B, from Lemma 3.3 we deduce that

$$g_\varepsilon(\mathbf{Q}^*_\varepsilon)\geq\left(\frac{1}{\varepsilon}(|\mathbf{Q}^*_\varepsilon|-1)-\kappa_*\right)^2-\frac{3}{4\varepsilon^2}(|\mathbf{Q}^*_\varepsilon|-1)^2\geq\frac{1}{8\varepsilon^2}(|\mathbf{Q}^*_\varepsilon|-1)^2-7\kappa_*^2 \tag{C.6}$$

at any point of ∂B, for ε small enough. Combining (C.5) and (C.6), we obtain

$$\int_{A^1_\varepsilon} g_\varepsilon(\mathbf{Q}_\varepsilon)\,\mathrm{d}x\lesssim\lambda_\varepsilon\int_{\partial B}g_\varepsilon(\mathbf{Q}^*_\varepsilon)\,\mathrm{d}\mathscr{H}^1+\lambda_\varepsilon\kappa_*^2\overset{\text{(C.1)}}{\lesssim}\lambda_\varepsilon \tag{C.7}$$

If we choose λ_ε in such a way that $\lambda_\varepsilon\to 0$ as $\varepsilon\to 0$, then (C.4) and (C.7) imply

$$\int_B g_\varepsilon(\mathbf{Q}_\varepsilon)\,\mathrm{d}x\to 0\qquad\text{as }\varepsilon\to 0. \tag{C.8}$$

Finally, we estimate the gradient term. An explicit computation shows that

$$\begin{aligned}\int_{A^1_\varepsilon\cup A^2_\varepsilon}|\nabla\mathbf{Q}_\varepsilon|^2\,\mathrm{d}x&\lesssim\lambda_\varepsilon\int_{\partial B}\left(|\nabla\mathbf{Q}^*_\varepsilon|^2+|\nabla\mathbf{Q}^*|^2+\frac{1}{\lambda_\varepsilon^2}\,|\mathbf{Q}^*_\varepsilon-\mathbf{Q}^*|^2+\frac{1}{\lambda_\varepsilon^2}\,(|\mathbf{Q}^*_\varepsilon|-1-\kappa_*\varepsilon)^2\right)\mathrm{d}\mathscr{H}^1\\&\overset{\text{(C.6)}}{\lesssim}\lambda_\varepsilon\int_{\partial B}\left(|\nabla\mathbf{Q}^*_\varepsilon|^2+|\nabla\mathbf{Q}^*|^2+\frac{1}{\lambda_\varepsilon^2}\,|\mathbf{Q}^*_\varepsilon-\mathbf{Q}^*|^2+\frac{\varepsilon^2}{\lambda_\varepsilon^2}g_\varepsilon(\mathbf{Q}^*_\varepsilon)+\frac{\varepsilon^2\kappa_*}{\lambda_\varepsilon^2}\right)\mathrm{d}\mathscr{H}^1\end{aligned}$$

By the assumption (C.1), we deduce

$$\int_{A^1_\varepsilon \cup A^2_\varepsilon} |\nabla \mathbf{Q}_\varepsilon|^2 \, \mathrm{d}x \overset{(C.6)}{\lesssim} \lambda_\varepsilon + \frac{\varepsilon^2}{\lambda_\varepsilon} + \frac{1}{\lambda_\varepsilon} \int_{\partial B} |\mathbf{Q}^*_\varepsilon - \mathbf{Q}^*|^2 \, \mathrm{d}\mathscr{H}^1 \tag{C.9}$$

We take

$$\lambda_\varepsilon := \varepsilon + \left(\int_{\partial B} |\mathbf{Q}^*_\varepsilon - \mathbf{Q}^*|^2 \, \mathrm{d}\mathscr{H}^1 \right)^{1/2} \tag{C.10}$$

By assumption, we have $\mathbf{Q}^*_\varepsilon \rightharpoonup \mathbf{Q}^*$ weakly in $W^{1,2}(\partial B)$, hence strongly in $L^2(\partial B)$. Therefore, $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$. Moreover, (C.9) and (C.10) imply

$$\int_{A^1_\varepsilon \cup A^2_\varepsilon} |\nabla \mathbf{Q}_\varepsilon|^2 \, \mathrm{d}x \to 0 \qquad \text{as } \varepsilon \to 0, \tag{C.11}$$

On the other hand, we have

$$\int_{A^3_\varepsilon} |\nabla \mathbf{Q}_\varepsilon|^2 \, \mathrm{d}x = \int_B |\nabla \mathbf{Q}^*|^2 \, \mathrm{d}x \tag{C.12}$$

for any ε. Therefore, (C.3) follows from (C.8), (C.11) and (C.12). □

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