Stationary distribution and density function of a stochastic SVIR epidemic model

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Abstract We consider the long-term properties of a stochastic SVIR epidemic model with saturation incidence rates and logistic growth in this paper. We firstly derive the fitness of a unique global positive solution. Then we construct appropriate Lyapunov functions and obtain condition $R_0^s > 1$ for existence of stationary distribution, and conditions for persistence in the mean. Moreover, conditions including $R_0^e < 1$ for exponential extinction to the infected individuals are figured out. Finally, by employing Fokker-Planck equation and stochastic analysis, we derive the probability density function around the quasi-endemic equilibrium point when critical value $R_0^p > 1$ is valid. Consequently, some examples and illustrative simulations are carried out to verify the main theoretical results.

Keywords: Epidemic model, Vaccination, Persistence and extinction, Stationary distribution, Fokker-Planck equation, Probability density function

1. Introduction

Vaccines played a vital role when we controlled the spread of infectious diseases. Recently, Zhang et al. [1] studied the dynamic behaviors of SVIR epidemic model with bilinear incidence rates and vaccinations, they proposed the following model:

$$\begin{cases} \mathrm{d}S(t) = (\mu - \zeta S - \mu S - \beta SI)\mathrm{d}t + \sigma_1 S\mathrm{d}B_1(t), \\ \mathrm{d}V(t) = (\zeta S - \gamma_1 V - \beta_1 VI - \mu V)\mathrm{d}t + \sigma_2 V\mathrm{d}B_2(t), \\ \mathrm{d}I(t) = (\beta SI + \beta_1 VI - \tau I - \mu I)\mathrm{d}t + \sigma_3 I\mathrm{d}B_3(t), \end{cases} \tag{1}$$

where the bilinear incidence rate βSI described the unbounded increasing when the number of the infected raised, and infectious diseases prevailed in an infinitely increasing way for a long time in [2, 3]. In 1978, Capasso and Serio [4] improved the bilinear incidence rate and governed the saturation incidence rate in the form of $g(I)S = \frac{\beta SI}{1+aI}$ to describe the spreading of infectious diseases, where the constant a

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is called the saturated constant in [5]. When the number of the infected became very large, $g(I)$ tended to a saturation level $\frac{\beta}{a}$ in [6, 7, 8, 9, 10, 11, 12, 13, 14] and [15], which reflected the behavioral changes and the crowdedness effect of the infected. Recently, Sahu and Dhar in [6] studied an epidemic model with the saturation incidence rate, their conclusion revealed that the higher vaccination coverage rate was, the basic regeneration number declined faster. Other types of the incidence rates could be found in [16, 17, 18, 19] and the references therein.

Here, we notice that, in [1], μ is a positive constant and stands for the new recruitment rate of the susceptible, the value of which is the same with the natural death rate of a local population. While, in this paper, we do not think the constant recruitment rate μ is rational for describing the epidemic model with fast mobility of a local population, so we always assume that the mobility of a local population obeys the Logistic growth $\gamma S(1-\frac{S}{K})$ due to the fast transportation by metros, trains and airplanes within a period of time, where $\gamma = b-\mu > 0$. That is, the intrinsic rate γ equals the difference of the birth rate b and the natural death rate μ, and K is the carrying capacity of a local population. Meanwhile, in this paper, we adopt the saturation incidence rate to describe the crowdedness of the infected when infectious diseases invade a local population. Precisely, we improve the constant recruitment rate μ of model (1) and govern the Logistic growth $\gamma S\big(1-\frac{S}{K}\big)$ into the equation of the susceptible, and we assume that the density of the susceptible in the municipal cities on holidays or weekends is described by the Logistic growth, and also that the vaccinated lose their temporary immunities over time and return to the susceptible again due to immunity loss, we thus establish an SVIR epidemic model with the saturation incidence rates as follows:

$$\begin{cases} \dot{S}(t) = \gamma S\Big(1-\dfrac{S}{K}\Big) + \vartheta V - \zeta S - \dfrac{\beta SI}{1+a_1 I}, \\ \dot{V}(t) = \zeta S - \vartheta V - \dfrac{\beta VI}{1+a_2 I} - \mu V, \\ \dot{I}(t) = \dfrac{\beta SI}{1+a_1 I} + \dfrac{\beta VI}{1+a_2 I} - (\mu+\delta+\tau)I, \\ \dot{R}(t) = \tau I - \mu R. \end{cases} \tag{2}$$

Here ϑ is the rate for immunity loss of vaccines to the vaccinated; ζ is the proportion of the susceptible who take the vaccine; β is the transmission rate between the infected and the susceptible (or, the vaccinated); τ is the recovered rate of the infected; δ means the mortality rate caused by infectious diseases to the infected. Let $x(t) = S(t)+V(t)+I(t)+R(t)$, by (2), which then follows

$$\dot{x}(t) = -\mu x + bS - \frac{b-\mu}{K}S^2 - \delta I < -\mu x + bS - \frac{b-\mu}{K}S^2.$$

Here, the expression $bS - \frac{b-\mu}{K}S^2$ admits the maximum m, so we have $\dot{x}(t) < -\mu x + m$, which gives that $x(t) \to \frac{m}{\mu}$ as $t \to \infty$, which further implies that the density of a local population is always varying with the time, instead of a constant in model (1). The readers can find that the recent works in [20, 21, 22, 23, 24] also govern the Logistic growth to discuss the long-term properties of their models. In model (2), the saturation incidence rates

$$\frac{\beta I}{1+a_1 I} \to \frac{\beta}{a_1}, \quad \frac{\beta I}{1+a_2 I} \to \frac{\beta}{a_2},$$

respectively reach their boundaries when the number of the infected increases to a large amount, here the transmission rate β, the saturated constants a_1 and a_2 are positive constants. We further assume that

the saturated constants satisfy the condition $a_1 < a_2$. In other words, the probability that the vaccinated are infected by the infective is less than the probability that the susceptible are infected by the infective.

In the real world, many tiny and independent random fluctuations, such as small changes in temperature, humidity, wind and the like, usually affect the population size. So, model (2) can be improved into a stochastic epidemic model with the fluctuation circumstances by introducing Gaussian white noises $\xi(t) = \mathrm{d}B(t)/\mathrm{d}t$, here $\mathrm{d}B(t) = B(t+\Delta t) - B(t)$ is a Wiener increment with zero mean and Δt variance. Therefore, the epidemic models with fluctuations describe the real circumstances in the appropriate ways when modelling infectious diseases. Motivated by the recent contributions in [25, 26, 27, 28], we assume that the environmental noises are proportional to the variables S, V, I, R in this paper. Moreover, we notice that, the first three equations of model (2) are independent of the recovered, so we leave the fourth equation of model (2), and consider a stochastic epidemic model (3) with the saturation incidence rates and the Logistic growth as follows:

$$\begin{cases} \mathrm{d}S(t) = \left[\gamma S\Big(1 - \dfrac{S}{K}\Big) + \vartheta V - \zeta S - \dfrac{\beta SI}{1+a_1 I}\right]\mathrm{d}t + \sigma_1 S\mathrm{d}B_1(t), \\ \mathrm{d}V(t) = \left[\zeta S - \vartheta V - \dfrac{\beta VI}{1+a_2 I} - \mu V\right]\mathrm{d}t + \sigma_2 V\mathrm{d}B_2(t), \\ \mathrm{d}I(t) = \left[\dfrac{\beta SI}{1+a_1 I} + \dfrac{\beta VI}{1+a_2 I} - (\mu+\delta+\tau)I\right]\mathrm{d}t + \sigma_3 I\mathrm{d}B_3(t), \end{cases} \tag{3}$$

where $B_1(t), B_2(t)$ and $B_3(t)$ are three independent standard Brownian motions (or Wiener processes), σ_1, σ_2 and σ_3 respectively are the intensities of the white noises; $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ is a complete probability space with its filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$.

Next, we start to show the existence and uniqueness of a global positive solution of model (3). Then the sufficient conditions for the persistence of model (3) are given in Section 2. Further, the sufficient conditions of the existence of an ergodic stationary distribution to model (3) is obtained in Section 3. We derive the sufficient conditions for the extinction of model (3) in Section 4. By means of the developed approaches in solving the general three-dimensional Fokker-Planck equation, the exact expression of the probability density function for the stationary distribution is presented in Section 5.

2. Fitness and persistence

We firstly concern the existence and uniqueness of a global positive solution to model (3) before we investigate other long-term properties, further we concern the persistence in the mean of the density of the infected to model (3) in this section.

2.1 Existence and uniqueness of a global solution

By the similar discussions in [8, 9, 10, 11], we derive the following Theorem 2.1.

Theorem 2.1. *For any initial value $(S(0), V(0), I(0)) \in \mathbb{R}^3_+$, model (3) admits a unique solution $(S(t), V(t), I(t)) \in \mathbb{R}^3_+$ for $t \geqslant 0$, and the solution will remain in $\mathbb{R}^3_+$ with probability one.*

Proof. It is easy to verify that the coefficients of model (3) satisfy the local Lipschitz condition. Therefore, model (3) admits a unique local solution $(S(t), V(t), I(t))$ on the interval $[0, \tau_e)$, where τ_e is the explosion time. Next, we will prove that the assertion $\tau_e = \infty$ holds almost surely. In other words, the solution $(S(t), V(t), I(t))$ does not explode within a finite time. Let $m_0 > 1$ be a sufficiently large

number which can ensure each component of $(S(t), V(t), I(t))$ all lying within $[\frac{1}{m_0}, m_0]$. For any integer $m \geqslant m_0$, we define the stopping time

$$\tau_m = \inf\Big\{t \in [0, \tau_e) : \min\{S(t), V(t), I(t)\} \leqslant \frac{1}{m} \text{ or } \max\{S(t), V(t), I(t)\} \geqslant m\Big\},$$

where $\inf \emptyset = \infty$. Obviously, τ_m is increasing as $m \to \infty$. We denote $\lim\limits_{m\to\infty} \tau_m = \tau_\infty$. The assertion $\tau_\infty \leqslant \tau_e$ is valid by definition of the stopping time. We claim that the assertion $\tau_\infty = \infty$ is valid almost surely. If the assertion is not valid, then there exist a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_m \leqslant T\} \geqslant \varepsilon$ for each integer $m \geqslant m_0$. We define a C^2-function $V : \mathbb{R}^3_+ \to \mathbb{R}_+$ as follows:

$$V_1(S, V, I) = S - 1 - \ln S + V - 1 - \ln V + I - 1 - \ln I,$$

by the nonnegativity of V and the generalized Itô's formula, we get

$$\mathrm{d}V_1(S, V, I) = \mathcal{L}V_1(S, V, I)\mathrm{d}t + (S-1)\sigma_1\mathrm{d}B_1(t) + (V-1)\sigma_2\mathrm{d}B_2(t) + (I-1)\sigma_3\mathrm{d}B_3(t),$$

where

$$\begin{aligned}
\mathcal{L}V_1(S, V, I) = {} & \Big(1 - \frac{1}{S}\Big)\Big[\gamma S\Big(1 - \frac{S}{K}\Big) + \vartheta V - \zeta S - \frac{\beta SI}{1 + a_1 I}\Big] \\
& + \Big(1 - \frac{1}{V}\Big)\Big(\zeta S - \vartheta V - \frac{\beta VI}{1 + a_2 I} - \mu V\Big) \\
& + \Big(1 - \frac{1}{I}\Big)\Big[\frac{\beta SI}{1 + a_1 I} + \frac{\beta VI}{1 + a_2 I} - (\mu + \delta + \tau)I\Big] \\
& + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2),
\end{aligned}$$

after the proper simplification, which implies that

$$\begin{aligned}
\mathcal{L}V_1(S, V, I) &\leqslant \gamma S\Big(1 - \frac{S}{K}\Big) - \mu V - (\mu + \delta + \tau)I + \frac{\gamma S}{K} + \zeta + \frac{\beta I}{1 + a_1 I} + \vartheta \\
&\quad + \frac{\beta I}{1 + a_2 I} + 2\mu + \delta + \tau + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \\
&\leqslant \max_{S\in\mathbb{R}_+}\Big\{\gamma S\Big(1 - \frac{\gamma S}{K}\Big) + \frac{\gamma S}{K}\Big\} + \zeta + \vartheta + 2\mu + \delta + \tau \\
&\quad + \beta\Big(\frac{1}{a_1} + \frac{1}{a_2}\Big) + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) := G > 0.
\end{aligned}$$

It then follows that

$$\mathrm{d}V_1(S, V, I) \leqslant G\mathrm{d}t + (S-1)\sigma_1\mathrm{d}B_1(t) + (V-1)\sigma_2\mathrm{d}B_2(t) + (I-1)\sigma_3\mathrm{d}B_3(t).$$

For any $t \in [0, T]$ and $m \geqslant m_0$, integrating from 0 to $\tau_m \wedge t$ and taking expectation, which gives

$$\begin{aligned}
\mathbb{E}V_1(S(\tau_m \wedge t), V(\tau_m \wedge t), I(\tau_m \wedge t)) &\leqslant V_1(S(0), V(0), I(0)) + \mathbb{E}\int_0^{\tau_m\wedge t} G\mathrm{d}t \\
&\leqslant V_1(S(0), V(0), I(0)) + GT < \infty.
\end{aligned}$$

We set $\Omega_m = \{\tau_m \leqslant T\}$ for $m \geqslant m_0$, so $\mathbb{P}(\Omega_m) \geqslant \varepsilon$ holds. And each component of $(S(\tau_m \wedge t), V(\tau_m \wedge t), I(\tau_m \wedge t))$ equals either m or $\frac{1}{m}$ for all $\omega \in \Omega_m$. Hence

$$\infty > V_1(S(0), V(0), I(0)) + GT \geqslant \varepsilon \min\Big\{m - 1 - \ln m, \frac{1}{m} - 1 + \ln m\Big\},$$

letting $m \to \infty$, there arises a contradiction as follows:

$$\infty > V_1(S(0), V(0), I(0)) + GT \geqslant \infty.$$

The proof is complete.

2.2. Persistence in the mean

By the results from [29, 30], we derive the following Lemma 2.1, we omit the proof herewith.

Lemma 2.1. *For any initial value $(S(0), V(0), I(0)) \in \mathbb{R}^3_+$, model (3) has a unique positive solution $(S(t), V(t), I(t)) \in \mathbb{R}^3_+$, the solution has the following properties*

$$\lim_{t\to\infty} \frac{S(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{V(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{I(t)}{t} = 0,$$

and

$$\lim_{t\to\infty} \frac{\ln S(t)}{t} \leqslant 0, \quad \lim_{t\to\infty} \frac{\ln V(t)}{t} \leqslant 0, \quad \lim_{t\to\infty} \frac{\ln I(t)}{t} \leqslant 0.$$

If $\mu > 0.5(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)$, then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S(s)\mathrm{d}B_1(s) = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t V(s)\mathrm{d}B_2(s) = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t I(s)\mathrm{d}B_3(s) = 0 \quad \text{a.s..}$$

By similar approaches in Theorem 3.1 of [31], Theorem 4.2 of [10] and Theorem 4.1 of [32], we next provide the sufficient conditions of the persistence in the mean for the infected to model (3). Let

$$R_0^s = \frac{n_1}{n_2 n_3\big(1 + \frac{a_2\gamma K}{4(\mu+\delta+\tau)}\big)}, \quad A = \beta\bigg(\frac{3n_1}{n_2 n_3(1 - \frac{\sigma_1^2}{2\gamma} - \frac{\zeta}{\gamma})\gamma} + c_1\bigg), \tag{4}$$

where

$$n_1 = \beta\zeta K\Big(1 - \frac{\sigma_1^2}{2\gamma} - \frac{\zeta}{\gamma}\Big)^3, \quad n_2 = \vartheta + \mu + \frac{\sigma_2^2}{2}, \quad n_3 = \mu + \delta + \tau + \frac{\sigma_3^2}{2}.$$

Theorem 2.2. *If the following conditions hold*

$$R_0^s > 1, \quad \mu > 0.5(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2), \quad \sigma_1^2 < 2(\gamma - \zeta), \tag{5}$$

then density of the infected to model (3) is persistent in the mean

$$\liminf_{t\to\infty} A\langle I\rangle_t \geqslant \Big(1 + \frac{a_2\gamma K}{4(\mu+\delta+\tau)}\Big)(R_0^s - 1) > 0 \quad \text{a.s..} \tag{6}$$

In other words, when $R_0^s > 1$ is valid, the lower boundary of the infected exists and infectious diseases will prevail for a long run.

Proof. We construct a non-negative C^2-function

$$V_2 = \frac{a_2}{\mu+\delta+\tau}(S + V + I) - c_1 \ln V - c_2 \ln I, \tag{7}$$

where c_1 and c_2 are positive constants determined later. Itô's formula implies that

$$\mathrm{d}V_2 = \mathcal{L}V_2\mathrm{d}t + \frac{a_2 S\sigma_1}{\mu+\delta+\tau}\mathrm{d}B_1(t) + \Big(\frac{a_2 V}{\mu+\delta+\tau} - c_1\Big)\sigma_2\mathrm{d}B_2(t) + \Big(\frac{a_2 I}{\mu+\delta+\tau} - c_2\Big)\sigma_3\mathrm{d}B_3(t), \tag{8}$$

where

$$\begin{aligned}\mathcal{L}V_2 =& \frac{a_2}{\mu+\delta+\tau}\Big[\gamma S\Big(1-\frac{S}{K}\Big)-\mu V-(\mu+\delta+\tau)I\Big]-c_1\frac{1}{V}\Big(\zeta S-\vartheta V-\frac{\beta VI}{1+a_2I}-\mu V\Big)\\&+\frac{1}{2}(c_1\sigma_2^2+c_2\sigma_3^2)-c_2\frac{1}{I}\Big[\frac{\beta SI}{1+a_1I}+\frac{\beta VI}{1+a_2I}-(\mu+\delta+\tau)I\Big].\end{aligned}\tag{9}$$

Noticing that

$$\max_{S\in\mathbb{R}_+}\Big\{\gamma S\Big(1-\frac{S}{K}\Big)\Big\}=\frac{\gamma K}{4},$$

which then follows that

$$\begin{aligned}\mathcal{L}V_2 <& \frac{a_2\gamma K}{4(\mu+\delta+\tau)}-a_2I-\frac{c_1\zeta S}{V}+\Big(\vartheta+\mu+\frac{\sigma_2^2}{2}\Big)c_1\\&+\frac{c_1\beta I}{1+a_2I}-\frac{c_2\beta S}{1+a_1I}-\frac{c_2\beta V}{1+a_2I}+c_2\Big(\mu+\delta+\tau+\frac{\sigma_3^2}{2}\Big)\\<&-\frac{c_1\zeta S}{V}-\frac{c_2\beta V}{1+a_2I}-(1+a_2I)+1\\&+c_1\Big(\vartheta+\mu+\frac{\sigma_2^2}{2}\Big)+c_2\Big(\mu+\delta+\tau+\frac{\sigma_3^2}{2}\Big)+\frac{a_2\gamma K}{4(\mu+\delta+\tau)}+c_1\beta I\\<&-3\sqrt[3]{c_1c_2\beta\zeta S}+c_1\Big(\vartheta+\mu+\frac{\sigma_2^2}{2}\Big)+c_2\Big(\mu+\delta+\tau+\frac{\sigma_3^2}{2}\Big)+\frac{a_2\gamma K}{4(\mu+\delta+\tau)}+c_1\beta I+1.\end{aligned}\tag{10}$$

Again, we define

$$V_3=\frac{2(S+V)}{3\gamma K}-\frac{\ln S}{\gamma},\tag{11}$$

we obtain by Itô's formula that

$$\mathrm{d}V_3=\mathcal{L}V_3\mathrm{d}t+\Big(\frac{2S}{3\gamma K}-\frac{1}{\gamma}\Big)\sigma_1\mathrm{d}B_1(t)+\frac{2V}{3\gamma K}\sigma_2\mathrm{d}B_2(t),\tag{12}$$

Lemma 4.2 in [33] and the proper simplification implies

$$\begin{aligned}\mathcal{L}V_3 =& \Big(\frac{2}{3\gamma K}-\frac{1}{\gamma S}\Big)\Big[\gamma S\Big(1-\frac{S}{K}\Big)+\vartheta V-\zeta S-\frac{\beta SI}{1+a_1I}\Big]\\&+\frac{2}{3\gamma K}\Big(\zeta S-\vartheta V-\frac{\beta VI}{1+a_2I}-\mu V\Big)+\frac{\sigma_1^2}{2\gamma}\\<&\frac{2S}{3K}\Big(1-\frac{S}{K}\Big)+\frac{S}{K}-1+\frac{\zeta}{\gamma}+\frac{\beta I}{\gamma}+\frac{\sigma_1^2}{2\gamma}\\<&\sqrt[3]{\frac{S}{K}}+\frac{\beta I}{\gamma}-\Big(1-\frac{\sigma_1^2}{2\gamma}-\frac{\zeta}{\gamma}\Big).\end{aligned}\tag{13}$$

We thus define

$$V_4=V_2+3\sqrt[3]{c_1c_2\beta\zeta K}V_3,\tag{14}$$

combining (10) with (13), we can get

$$\begin{aligned}\mathcal{L}V_4 =& \mathcal{L}V_2+3\sqrt[3]{c_1c_2\beta\zeta K}\mathcal{L}V_3\\<&1+\frac{a_2\gamma K}{4(\mu+\delta+\tau)}+c_1\Big(\vartheta+\mu+\frac{\sigma_2^2}{2}\Big)\\&+c_2\Big(\mu+\delta+\tau+\frac{\sigma_3^2}{2}\Big)-3\sqrt[3]{c_1c_2\beta\zeta K}\Big(1-\frac{\sigma_1^2}{2}-\frac{\zeta}{\gamma}\Big)+\Big(\frac{3}{\gamma}\sqrt[3]{c_1c_2\beta\zeta K}+c_1\Big)\beta I.\end{aligned}\tag{15}$$

By (4), we choose

$$c_1 = \frac{n_1}{n_2^2 n_3}, \quad c_2 = \frac{n_1}{n_2 n_3^2},$$

then

$$\mathcal{L}V_4 < -\Big(1 + \frac{a_2\gamma K}{4(\mu+\delta+\tau)}\Big)(R_0^s - 1) + AI := -\lambda + AI. \tag{16}$$

Further, it follows that

$$\begin{aligned} \mathrm{d}V_4 < \mathcal{L}V_4\mathrm{d}t + \Big[\frac{a_2 S}{\mu+\delta+\tau} + 3\sqrt[3]{c_1c_2\beta\zeta K}\Big(\frac{2S}{3\gamma K} - \frac{1}{\gamma}\Big)\Big]\sigma_1\mathrm{d}B_1(t) \\ + \Big(\frac{a_2 V}{\mu+\delta+\tau} + \sqrt[3]{c_1c_2\beta\zeta K}\frac{2V}{\gamma K} - c_1\Big)\sigma_2\mathrm{d}B_2(t) + \Big(\frac{a_2 I}{\mu+\delta+\tau} - c_2\Big)\sigma_3\mathrm{d}B_3(t). \end{aligned} \tag{17}$$

We integrate both sides of (17) from 0 to t and divided by t, and we get

$$\frac{1}{t}[V_4(t) - V_4(0)] < A\langle I\rangle_t - \lambda + \frac{\varphi_1(t)}{t}, \tag{18}$$

with

$$\begin{aligned} \varphi_1(t) = \int_0^t \Big[\frac{a_2 S(s)}{\mu+\delta+\tau} + 3\sqrt[3]{c_1c_2\beta\zeta K}\Big(\frac{2S(s)}{3\gamma K} - \frac{1}{\gamma}\Big)\Big]\sigma_1\mathrm{d}B_1(s) \\ + \int_0^t \Big(\frac{a_2 V(s)}{\mu+\delta+\tau} + \sqrt[3]{c_1c_2\beta\zeta K}\frac{2V(s)}{\gamma K} - c_1\Big)\sigma_2\mathrm{d}B_2(s) + \int_0^t \Big(\frac{a_2 I(s)}{\mu+\delta+\tau} - c_2\Big)\sigma_3\mathrm{d}B_3(s), \end{aligned} \tag{19}$$

Lemma 2.1 and the strong law of large numbers in [34] give

$$\limsup_{t\to\infty} \frac{V_4(t)}{t} = 0, \quad \limsup_{t\to\infty} \frac{\varphi_1(t)}{t} = 0,$$

it then by (18) follows

$$\liminf_{t\to\infty} A\langle I\rangle_t > \lambda > 0. \tag{20}$$

So, the infected admit the lower boundary as $R_0^s > 1$ holds, which implies that infectious diseases thus prevail for a long time.

3. Stationary distribution

In this section, we investigate the long-term property for the solution of model (3) by constructing several Lyapunov functions and using Hasminskii's theory in [35], the aim is to prove the solution of model (3) declines outside some compact set. In other words, there exists an ergodic stationary distribution within the compact set for model (3), which implies that the solution of model (3) is stable around the endemic equilibrium point, instead of exploding to the infinity. Precisely, the solution of model (3) provides some fluctuations, and the densities of the susceptible, the vaccinated and the infected are kind of stable in a long run.

Lemma 3.1. [35] *The Markov process $x(t)$ has a unique ergodic stationary distribution $\nu(\cdot)$, if there exists a bounded domain $D_0 \subset \mathbb{R}_+^n$ with a regular boundary Γ and has the following two conditions:*

(H1) *there exists a positive number η such that $\sum_{i,j=1}^n a_{ij}(x)\zeta_i\zeta_j \geqslant \eta|\zeta|^2, x \in D_0, \zeta \in \mathbb{R}_+^n$;*

(H2) *there exists a non-negative C^2-function V such that $\mathcal{L}V$ is negative for any $\mathbb{R}^n_+\backslash D_0$, then for all $x \in \mathbb{R}^n_+$, it follows*

$$\mathbb{P}\Big\{\lim_{T\to\infty}\frac{1}{T}\int_0^T g(x(t),t)\mathrm{d}t = \int_{\mathbb{R}^n_+} g(x)\nu(\mathrm{d}x)\Big\} = 1,$$

where $g(\cdot)$ is an integral function with respect to the measure $\nu(\cdot)$.

Stationary distributions for a stochastic SEIR model in Theorem 3.3 of [11] and also for a stochastic SIR model in Theorem 5.1 of [32] are investigated. We thus derive our main results in Theorem 3.1 as follows.

Theorem 3.1. *If $R_0^s > 1$, then model (3) admits a unique stationary distribution $\nu(\cdot)$, which has the ergodic property.*

Proof. According to Lemma 3.1, the solution $(S(t), V(t), I(t))$ of model (3) is Markov process, because the infected (I) in a local population contact with the susceptible (S) or the vaccinated (V) randomly, and the infected do not have any memories regarding the contacting histories. In other words, the contact between the infected (I) and the susceptible (S) or the vaccinated (V) are memoryless. Moreover, the future state only depends on the present state, that is, the future state is independent of the past state. We usually describe the SVI epidemic model by using the stochastic differential equations, which are Markov processes for each equation of model (3).

We thus find a non-negative C^2-function $\tilde{V}$ and a bounded set D_ε to satisfy conditions (H1) and (H2). Therefore, the proof of Theorem 3.1 is split into two steps.

Step 1. Construct a bounded set D_ε to make condition (H1) of Lemma 3.1 valid in this set. Firstly, we define

$$D_\varepsilon = \Big\{(S,V,I) \in \mathbb{R}^3_+, \varepsilon \leqslant S \leqslant \frac{1}{\varepsilon}, \varepsilon^2 \leqslant V \leqslant \frac{1}{\varepsilon^2}, \varepsilon \leqslant I \leqslant \frac{1}{\varepsilon}\Big\}, \tag{21}$$

here $\varepsilon > 0$ is a sufficiently small constant, the diffusion matrix of model (3) is as follows

$$\tilde{A} = \mathrm{diag}\{\sigma_1^2 S^2, \sigma_2^2 V^2, \sigma_3^2 I^2\} = (a_{ij})_{3\times 3},$$

for any $(S,V,I) \in D_\varepsilon, \zeta = (\zeta_1,\zeta_2,\zeta_3) \in \mathbb{R}^3_+$, we have

$$\sum_{i,j=1}^n a_{ij}\zeta_i\zeta_j = (\zeta_1,\zeta_2,\zeta_3)\tilde{A}(\zeta_1,\zeta_2,\zeta_3)^T = (\sigma_1 S)^2\zeta_1^2 + (\sigma_2 V)^2\zeta_2^2 + (\sigma_3 I)^2\zeta_3^2 \geqslant \eta\|\zeta\|^2,$$

where

$$\eta = \min_{(S,V,I)\in D_\varepsilon}\{\sigma_1^2 S^2, \sigma_2^2 V^2, \sigma_3^2 I^2\} > 0,$$

which means that condition (H1) in Lemma 3.1 holds.

Step 2. Find a C^2-Lyapunov function $\tilde{V}$ such that $\mathcal{L}\tilde{V} \leqslant -1$ for any $(S,V,I) \in \mathbb{R}^3_+\backslash D_\varepsilon$. We define

$$\tilde{V} = M(V_4 + V_5) + V_6 + V_7, \tag{22}$$

where

$$V_5 = \frac{A}{\mu+\delta+\tau}(V+I), \quad V_6 = \frac{1}{\theta+2}(S+I+V)^{\theta+2}, \quad V_7 = -\ln V,$$

where $\theta > 0$ is a sufficiently small constant satisfying

$$\theta < \frac{\mu - 0.5(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)}{\mu + 0.5(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)},$$

and $M > 0$ is a sufficiently large constant satisfying

$$-M\lambda + B + \theta + \mu + \frac{\sigma_2^2}{2} + \frac{\beta}{a_2} \leqslant -2. \tag{23}$$

Obviously, $\tilde{V}(S, V, I)$ is a continuous function and takes minimum at the point $(\bar{S}, \bar{V}, \bar{I})$, so we define a non-negative C^2-function $W : \mathbb{R}^3_+ \to \mathbb{R}$ as follows:

$$W = M(V_4 + V_5) + V_6 + V_7 - \tilde{V}(\bar{S}, \bar{V}, \bar{I}). \tag{24}$$

Applying Itô's formula to V_5, together with inequality (16), we get

$$\begin{aligned} \mathcal{L}V_5 &= \frac{A}{\mu + \delta + \tau}\Big[\zeta S - (\vartheta + \mu)V + \frac{\beta SI}{1 + a_1 I} - (\mu + \delta + \tau)I\Big] \\ &< \frac{A\zeta S}{\mu + \delta + \tau} + \frac{A\beta SI}{(1 + a_1 I)(\mu + \delta + \tau)} - AI, \end{aligned} \tag{25}$$

then

$$\mathcal{L}(V_4 + V_5) < -\lambda + \frac{A\zeta S}{\mu + \delta + \tau} + \frac{A\beta SI}{(1 + a_1 I)(\mu + \delta + \tau)}. \tag{26}$$

Similarly, one obtains that

$$\begin{aligned} \mathcal{L}V_6 &= (S + V + I)^{\theta+1}\Big[\gamma S\Big(1 - \frac{S}{K}\Big) - \mu V - (\mu + \delta + \tau)I\Big] \\ &\quad + \frac{\theta + 1}{2}(S + V + I)^{\theta}(\sigma_1^2 S^2 + \sigma_2^2 V^2 + \sigma_3^2 I^2) \\ &< \gamma S(S + V + I)^{\theta+1} - \frac{\gamma}{K}S^{\theta+3} - \mu V^{\theta+2} - (\mu + \delta + \tau)I^{\theta+2} \\ &\quad + \frac{\theta + 1}{2}(S + V + I)^{\theta+2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2) \\ &= \gamma S(S + V + I)^{\theta+1} - \frac{\gamma}{2K}S^{\theta+3} - \frac{\gamma}{2K}S^{\theta+3} - \mu(1 - \theta)V^{\theta+2} \\ &\quad - \mu\theta V^{\theta+2} - (\mu + \delta + \tau)(1 - \theta)I^{\theta+2} - \theta(\mu + \delta + \tau)I^{\theta+2} \\ &\quad + \frac{\theta + 1}{2}(S + V + I)^{\theta+2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2), \end{aligned} \tag{27}$$

letting

$$\begin{aligned} B = \max_{(S,V,I)\in\mathbb{R}^3_+} \Big\{ &-\frac{\gamma}{2K}S^{\theta+3} - \mu(1 - \theta)V^{\theta+2} - (\mu + \delta + \tau)(1 - \theta)I^{\theta+2} \\ &+ \frac{\theta + 1}{2}(S + V + I)^{\theta+2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2) + \gamma S(S + V + I)^{\theta+1}\Big\}, \end{aligned}$$

then

$$\mathcal{L}V_6 < -\frac{\gamma}{2K}S^{\theta+3} - \mu\theta V^{\theta+2} - \theta(\mu + \delta + \tau)I^{\theta+2} + B, \tag{28}$$

$$\mathcal{L}V_7 = -\frac{\zeta S}{V} + \frac{\beta I}{1 + a_2 I} + \vartheta + \mu + \frac{\sigma_2^2}{2} \leqslant -\frac{\zeta S}{V} + \vartheta + \mu + \frac{\beta}{a_2} + \frac{\sigma_2^2}{2}. \tag{29}$$

We combine inequalities (26), (28) and (29), and derive that

$$\begin{aligned}\mathcal{L}W < M\Big[-\lambda+\frac{A\zeta S}{\mu+\delta+\tau}+\frac{A\beta SI}{(1+a_1I)(\mu+\delta+\tau)}\Big]-\frac{\gamma}{2K}S^{\theta+3}-\mu\theta V^{\theta+2}\\ -\theta(\mu+\delta+\tau)I^{\theta+2}+B-\frac{\zeta S}{V}+\vartheta+\mu+\frac{\beta}{a_2}+\frac{\sigma_2^2}{2},\end{aligned} \tag{30}$$

where ε is a sufficiently small constant in the set D_ε, and ε satisfies the following conditions

$$\frac{AM\beta}{\mu+\delta+\tau}\varepsilon\leqslant\min\Big\{\frac{a_1\beta}{a_1\zeta+\beta},\frac{\gamma(\theta+3)}{2K},\frac{(\theta+3)(1-C)}{\theta+2}\Big\}, \tag{31}$$

$$E+1\leqslant\min\Big\{\frac{\zeta}{\varepsilon},\frac{\gamma}{4K\varepsilon^{\theta+3}},\frac{\theta(\mu+\delta+\tau)}{\varepsilon^{\theta+2}},\frac{\mu\theta}{\varepsilon^{2\theta+4}}\Big\}, \tag{32}$$

where

$$C=\sup_{S\in\mathbb{R}_+}\left\{\frac{AM\zeta S}{\mu+\delta+\tau}+\frac{AM\beta\varepsilon S^{\theta+3}}{(\mu+\delta+\tau)(\theta+3)}-\frac{\gamma}{2K}S^{\theta+3}\right\},$$

$$E=\sup_{(S,V,I)\in\mathbb{R}^3_+}\left\{\frac{AM\zeta S}{\mu+\delta+\tau}+\frac{AM\beta S}{a_1(\mu+\delta+\tau)}-\frac{\gamma}{4K}S^{\theta+3}+B+\mu+\vartheta+\frac{\sigma_2^2}{2}+\frac{\beta}{a_2}\right\}.$$

We next separate $\mathbb{R}^3_+\backslash D_\varepsilon$ into six parts to prove the assertion $\mathcal{L}W\leqslant-1$ in $\mathbb{R}^3_+\backslash D_\varepsilon$ as follows:

$$\begin{aligned}&D_1=\Big\{(S,V,I)\in\mathbb{R}^3_+,0<S<\varepsilon\Big\},&&D_2=\Big\{(S,V,I)\in\mathbb{R}^3_+,0<I<\varepsilon\Big\},\\&D_3=\Big\{(S,V,I)\in\mathbb{R}^3_+,S>\varepsilon,I>\varepsilon,0<V<\varepsilon^2\Big\},&&D_4=\Big\{(S,V,I)\in\mathbb{R}^3_+,S>\frac{1}{\varepsilon}\Big\},\\&D_5=\Big\{(S,V,I)\in\mathbb{R}^3_+,I>\frac{1}{\varepsilon}\Big\},&&D_6=\Big\{(S,V,I)\in\mathbb{R}^3_+,V>\frac{1}{\varepsilon^2}\Big\},\end{aligned}$$

and $D_\varepsilon^c=D_1\cup D_2\cup D_3\cup D_4\cup D_5\cup D_6$.

Case 1. When $(S,V,I)\in D_1$, by (23), (30), (31), one can derive

$$\mathcal{L}W<-2+\frac{AM\zeta\varepsilon}{\mu+\delta+\tau}+\frac{AM\beta\varepsilon}{a_1(\mu+\delta+\tau)}\quad\leqslant-1. \tag{33}$$

Case 2. When $(S,V,I)\in D_2$, we can obtain an inequality

$$\frac{SI}{1+a_1I}\leqslant SI\leqslant\varepsilon S\leqslant\varepsilon\frac{\theta+2+S^{\theta+3}}{\theta+3}. \tag{34}$$

It follows from (23), (30), (31) that

$$\begin{aligned}\mathcal{L}W&<-2+\frac{AM\zeta S}{\mu+\delta+\tau}+\frac{AM\beta\varepsilon(\theta+2)}{(\mu+\delta+\tau)(\theta+3)}+\frac{AM\beta\varepsilon S^{\theta+3}}{(\mu+\delta+\tau)(\theta+3)}-\frac{\gamma}{2K}S^{\theta+3}\\&\leqslant-2+C+\frac{AM\beta\varepsilon(\theta+2)}{(\mu+\delta+\tau)(\theta+3)}\\&\leqslant-2+1=-1.\end{aligned} \tag{35}$$

Case 3. When $(S,V,I)\in D_3$, according to the inequalities (30) and (32), we obtain

$$\begin{aligned}\mathcal{L}W&<-M\lambda+\frac{AM\zeta S}{\mu+\delta+\tau}+\frac{AM\beta S}{a_1(\mu+\delta+\tau)}-\frac{\gamma}{4K}S^{\theta+3}-\frac{\gamma}{4K}S^{\theta+3}-\mu\theta V^{\theta+2}\\&\quad-\theta(\mu+\delta+\tau)I^{\theta+2}+B-\frac{\zeta S}{V}+\Big(\vartheta+\mu+\frac{\beta}{a_2}+\frac{\sigma_2^2}{2}\Big)\\&\leqslant-\frac{\zeta}{\varepsilon}+E\leqslant-1.\end{aligned} \tag{36}$$

Case 4. For any $(S,V,I)\in D_4$, it follows from (30) and (32) that

$$\mathcal{L}W<-\frac{\gamma}{4K}S^{\theta+3}+E<-\frac{\gamma}{4K\varepsilon^{\theta+3}}+E\leqslant -1. \tag{37}$$

Case 5. When $(S,V,I)\in D_5$, by (30) and (32), we get

$$\mathcal{L}W<-(\mu+\delta+\tau)\theta I^{\theta+2}+E\leqslant-\frac{(\mu+\delta+\tau)\theta}{\varepsilon^{\theta+2}}+E\leqslant -1. \tag{38}$$

Case 6. When $(S,V,I)\in D_6$, by (30) and (32), we derive

$$\mathcal{L}W<-\mu\theta V^{\theta+2}+E\leqslant-\frac{\mu\theta}{\varepsilon^{2\theta+4}}+E\leqslant -1. \tag{39}$$

Hence, model (3) admits a unique ergodic stationary distribution $\nu(\cdot)$.

4. Extinction

There are several techniques to investigate the extinction in [7, 9, 10, 11, 20, 21, 30, 31, 32, 33, 36]. Extinction in the epidemiology usually means the elimination of infectious diseases over a long period of time. In this section, we adopt the approaches in Theorem 1 of [9] and Theorem 3.1 of [10, 32], by constructing moderate Lyapunov functions, together with the generalized Itô's formula and the strong law of large numbers, we obtain the critical value for the extinction of infectious diseases to model (3), which implies that infectious diseases eventually disappear in a local population.

Theorem 4.1. *If the following conditions hold*

$$R_0^e=\frac{\beta\gamma K(\mu+\theta+\zeta)}{4\big(\mu+\delta+\tau+\frac{\sigma_3^2}{2}\big)\zeta\mu}<1,\quad 2\mu>\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2, \tag{40}$$

then the solution $(S(t),V(t),I(t))$ *of model* (3) *has*

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t}<\Big(\mu+\delta+\tau+\frac{\sigma_3^2}{2}\Big)(R_0^e-1)<0, \tag{41}$$

which means that infectious diseases to model (3) *will exponentially go to extinction.*

Proof. Integrating the first equation of model (3) from 0 to t, and then divided by t gives

$$\begin{aligned}\frac{1}{t}[S(t)-S(0)]&=\Big\langle\gamma S\Big(1-\frac{S}{K}\Big)\Big\rangle_t+\vartheta\langle V\rangle_t-\zeta\langle S\rangle_t-\Big\langle\frac{\beta SI}{1+a_1I}\Big\rangle_t+\frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}B_1(s)\\&<\frac{\gamma K}{4}+\vartheta\langle V\rangle_t-\zeta\langle S\rangle_t+\varphi_2(t),\end{aligned} \tag{42}$$

according to Lemma 2.1, we obtain

$$\limsup_{t\to\infty}\varphi_2(t)=\limsup_{t\to\infty}\left\{\frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}B_1(s)-\frac{1}{t}[S(t)-S(0)]\right\}=0.$$

Taking superior limit on both sides of (42), we have

$$\limsup_{t\to\infty}\langle S\rangle_t<\frac{\gamma K}{4\zeta}+\frac{\vartheta}{\zeta}\limsup_{t\to\infty}\langle V\rangle_t. \tag{43}$$

Similarly, all three equations of model (3) imply

$$\begin{aligned}&\frac{1}{t}[S(t)-S(0)]+\frac{1}{t}[V(t)-V(0)]+\frac{1}{t}[I(t)-I(0)]\\&=\Big\langle\gamma S\Big(1-\frac{S}{K}\Big)\Big\rangle_t-\mu\langle V\rangle_t-(\mu+\delta+\tau)\langle I\rangle_t\\&\quad+\frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}B_1(s)+\frac{\sigma_2}{t}\int_0^t V(s)\mathrm{d}B_2(s)+\frac{\sigma_3}{t}\int_0^t I(s)\mathrm{d}B_3(s),\end{aligned}\tag{44}$$

which gives that

$$\mu\langle V\rangle_t<\frac{\gamma K}{4}+\varphi_3(t),\tag{45}$$

with

$$\begin{aligned}\varphi_3(t)=\Big\{&\frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}B_1(s)+\frac{\sigma_2}{t}\int_0^t V(s)\mathrm{d}B_2(s)+\frac{\sigma_3}{t}\int_0^t I(s)\mathrm{d}B_3(s)\\&-\frac{1}{t}[S(t)-S(0)]-\frac{1}{t}[V(t)-V(0)]-\frac{1}{t}[I(t)-I(0)]\Big\}.\end{aligned}$$

Lemma 2.1 implies $\limsup_{t\to\infty}\varphi_3(t)=0$, taking superior limit on both sides of (45) gives

$$\limsup_{t\to\infty}\langle V\rangle_t<\frac{\gamma K}{4\mu}.\tag{46}$$

By the same discussion, for $\ln I(t)$, generalized Itô's formula implies

$$\begin{aligned}\frac{1}{t}[\ln I(t)-\ln I(0)]&=\Big\langle\frac{\beta S}{1+a_1I}\Big\rangle_t+\Big\langle\frac{\beta V}{1+a_2I}\Big\rangle_t-\frac{\sigma_3^2}{2}+\frac{\sigma_3B_3(t)}{t}-(\mu+\delta+\tau)\\&<\beta\langle S\rangle_t+\beta\langle V\rangle_t-\Big(\mu+\delta+\tau+\frac{\sigma_3^2}{2}\Big)+\frac{\sigma_3B_3(t)}{t},\end{aligned}\tag{47}$$

the strong law of large numbers in [34] yields

$$\lim_{t\to\infty}\frac{B_3(t)}{t}=0,$$

taking superior limit on both sides of (47), by (40), one derives

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln I(t)}{t}&<\beta\limsup_{t\to\infty}\langle S\rangle_t+\beta\limsup_{t\to\infty}\langle V\rangle_t-\Big(\mu+\delta+\tau+\frac{\sigma_3^2}{2}\Big)\\&<\frac{\beta\gamma K}{4}\Big(\frac{1}{\zeta}+\frac{\theta}{\zeta\mu}+\frac{1}{\mu}\Big)-\Big(\mu+\delta+\tau+\frac{\sigma_3^2}{2}\Big)\\&=\Big(\mu+\delta+\tau+\frac{\sigma_3^2}{2}\Big)(R_0^e-1)<0.\end{aligned}\tag{48}$$

So, the number of the infected declines to zero with an exponential rate in a long run.

5. Probability density function analysis

Obviously, model (49) admits a unique quasi-endemic equilibrium point, we wonder what the solution of model (49) with fluctuations looks like around the quasi-endemic equilibrium point. By using

Fokker-Planck equation and stochastic analysis, we derive the expression of the probability density function around the quasi-endemic equilibrium point under some moderate conditions, which reflects the distribution of the density of the solution to model (3).

5.1. Linearization of model (3)

Firstly, let $(u_1, u_2, u_3)^T = (\ln S, \ln V, \ln I)^T$, by Itô's formula, it follows from model (3) that

$$\begin{cases} \mathrm{d}u_1 = \left[\gamma\left(1-\frac{e^{u_1}}{K}\right) + \vartheta e^{u_2-u_1} - \frac{\beta e^{u_3}}{1+a_1e^{u_3}} - \left(\zeta + \frac{\sigma_1^2}{2}\right)\right]\mathrm{d}t + \sigma_1 \mathrm{d}B_1(t), \\ \mathrm{d}u_2 = \left[\zeta e^{u_1-u_2} - \frac{\beta e^{u_3}}{1+a_2e^{u_3}} - \left(\mu+\vartheta+\frac{\sigma_2^2}{2}\right)\right]\mathrm{d}t + \sigma_2 \mathrm{d}B_2(t), \\ \mathrm{d}u_3 = \left[\frac{\beta e^{u_1}}{1+a_1e^{u_3}} + \frac{\beta e^{u_2}}{1+a_2e^{u_3}} - \left(\mu+\delta+\tau+\frac{\sigma_3^2}{2}\right)\right]\mathrm{d}t + \sigma_3 \mathrm{d}B_3(t), \end{cases} \tag{49}$$

assume that

$$R_0^p = \frac{\beta\vartheta\zeta(\zeta+\vartheta+\mu+0.5\sigma_2^2)}{(\vartheta+\mu+0.5\sigma_2^2)[\frac{\gamma}{K}(\vartheta+\mu+0.5\sigma_2^2)(\mu+\delta+\tau+0.5\sigma_3^2)+\beta(\zeta-\gamma+0.5\sigma_1^2)(\zeta+\vartheta+\mu+0.5\sigma_2^2)]} > 1, \tag{50}$$

then there exists a unique quasi-endemic equilibrium point $F^* = (S^*, V^*, I^*) = (e^{u_1^*}, e^{u_2^*}, e^{u_3^*})$, which is determined by the following equations:

$$\begin{cases} \gamma\left(1-\frac{e^{u_1^*}}{K}\right) + \vartheta e^{u_2^*-u_1^*} - \frac{\beta e^{u_3^*}}{1+a_1e^{u_3^*}} - \left(\zeta+\frac{\sigma_1^2}{2}\right) = 0, \\ \zeta e^{u_1^*-u_2^*} - \frac{\beta e^{u_3^*}}{1+a_2e^{u_3^*}} - \left(\mu+\vartheta+\frac{\sigma_2^2}{2}\right) = 0, \\ \frac{\beta e^{u_1^*}}{1+a_1e^{u_3^*}} + \frac{\beta e^{u_2^*}}{1+a_2e^{u_3^*}} - \left(\mu+\delta+\tau+\frac{\sigma_3^2}{2}\right) = 0. \end{cases} \tag{51}$$

We obtain

$$S^* = \frac{bc}{h(m_2+\frac{I^*}{1+a_2I^*})} - \frac{m_1+\frac{I^*}{1+a_1I^*}}{h}, V^* = \frac{bc^2}{h(m_2+\frac{I^*}{1+a_2I^*})} - \frac{c(m_1+\frac{I^*}{1+a_1I^*})}{h(m_2+\frac{I^*}{1+a_2I^*})},$$

and I^* satisfies the following quadratic equation

$$\tilde{F}(I) = g_1I^4 + g_2I^3 + g_3I^2 + g_4I + g_5 = 0, \tag{52}$$

with

$$\begin{aligned} g_1 &= a_1^2a_2^2hm_2^2m_3 + 2a_1^2a_2hm_2m_3 + a_1^2hm_3, \\ g_2 &= a_1^2a_2(2hm_2^2m_3 - bc^2) + a_1a_2^2m_2(2hm_2m_3 - bc) + a_1a_2c(m_2 - b) \\ &\quad + a_1a_2^2cm_1m_2^2 + a_1^2a_2cm_1m_2 + 2a_1^2hm_2m_3 + 4a_1a_2hm_2m_3 \\ &\quad + a_1^2cm_1 + 2a_1a_2m_1m_2 + a_2^2m_2^2 + 2a_1hm_3 + a_1c + a_1m_1 + 2a_2m_2 + 1, \\ g_5 &= hm_2^2m_3 - bc^2 - bcm_2 + cm_1m_2 + m_1m_2^2 \\ &= (hm_2^2m_3 + cm_1m_2 + m_1m_2^2)(1 - R_0^p), \end{aligned}$$

and

$$\begin{aligned} h=\frac{\gamma}{\beta K}>0,\quad b=\frac{\vartheta}{\beta}>0,\quad c=\frac{\zeta}{\beta}>0,\quad m_1=\frac{\zeta+0.5\sigma_1^2-\gamma}{\beta},\\ m_2=\frac{\vartheta+\mu+0.5\sigma_2^2}{\beta}>b>0,\quad m_3=\frac{\mu+\delta+\tau+0.5\sigma_3^2}{\beta}>0. \end{aligned} \tag{53}$$

From (53), we can get $g_1>0$,

$$2hm_2^2m_3-bc^2>hb_2^2m_3-bc^2>b(hb_2m_3-c^2)=b\frac{\gamma\vartheta(\mu+\delta+\tau+0.5\sigma_3^2)-\beta\zeta^2K}{\beta^3K},$$

$$2hm_2m_3-bc>b(hm_3-c)=b\frac{\gamma(\mu+\delta+\tau+0.5\sigma_3^2)-\beta\zeta K}{\beta^2K}.$$

When (50) is valid and

$$\gamma(\mu+\delta+\tau+0.5\sigma_3^2)-\beta\zeta K>0,\quad \gamma\vartheta(\mu+\delta+\tau+0.5\sigma_3^2)-\beta\zeta^2K>0, \tag{54}$$

we derive $g_2>0$ and $g_5<0$, so equation (52) admits a unique positive root.

Next, let $x_i=u_i-u_i^*$ for $i=1,2,3$, linearized equations of system (49) are followed

$$\begin{cases} \mathrm{d}x_1=(-a_{11}x_1+a_{12}x_2-a_{13}x_3)\mathrm{d}t+\sigma_1\mathrm{d}B_1(t),\\ \mathrm{d}x_2=(a_{21}x_1-a_{21}x_2-a_{23}x_3)\mathrm{d}t+\sigma_2\mathrm{d}B_2(t),\\ \mathrm{d}x_3=(a_{31}x_1+a_{32}x_2-a_{33}x_3)\mathrm{d}t+\sigma_3\mathrm{d}B_3(t), \end{cases} \tag{55}$$

where

$$\begin{aligned} &a_{11}=\frac{\gamma}{K}e^{u_1^*}+\vartheta e^{u_2^*-u_1^*}>0,\quad a_{12}=\vartheta e^{u_2^*-u_1^*}>0,\quad a_{13}=\frac{\beta e^{u_3^*}}{(1+a_1e^{u_3^*})^2},\\ &a_{21}=\zeta e^{u_1^*-u_2^*},\quad a_{23}=\frac{\beta e^{u_3^*}}{(1+a_2e^{u_3^*})^2},\quad a_{31}=\frac{\beta e^{u_1^*}}{(1+a_1e^{u_3^*})},\\ &a_{32}=\frac{\beta e^{u_2^*}}{(1+a_2e^{u_3^*})},\quad a_{33}=\frac{a_1\beta e^{(u_1^*+u_3^*)}}{(1+a_1e^{u_3^*})^2}+\frac{a_2\beta e^{(u_2^*+u_3^*)}}{(1+a_2e^{u_3^*})^2}. \end{aligned}$$

It is easy to check

$$a_{11}-a_{12}>0,\quad a_{13}-a_{23}>0,\quad a_{33}>a_{13},\quad a_{13}=\frac{\beta e^{u_3^*}}{(1+a_1e^{u_3^*})^2}<1. \tag{56}$$

Let $X=(x_1,x_2,x_3)^{\mathrm{T}}$, $B(t)=(B_1(t),B_2(t),B_3(t))^{\mathrm{T}}$, $M=\mathrm{diag}\{\sigma_1,\sigma_2,\sigma_3\}$ and

$$A=\begin{pmatrix} -a_{11} & a_{12} & -a_{13}\\ a_{21} & -a_{21} & -a_{23}\\ a_{31} & a_{32} & -a_{33} \end{pmatrix}.$$

Therefore, equation (55) can be equivalently rewritten as

$$\mathrm{d}X(t)=AX(t)\mathrm{d}t+M\mathrm{d}B(t).$$

By relative theory in Gardiner [37], a unique density function $\Phi(X)$ around quasi-endemic equilibrium point F^* satisfies the following Fokker-Planck equation:

$$\begin{aligned} &-\sum_{i=1}^{3}\frac{\sigma_i^2}{2}\frac{\partial^2\Phi}{\partial x_i^2}+\frac{\partial}{x_1}[(-a_{11}x_1+a_{12}x_2-a_{13}x_3)\Phi]+\frac{\partial}{x_2}[(a_{21}x_1-a_{21}x_2-a_{23}x_3)\Phi] \\ &+\frac{\partial}{x_3}[(a_{31}x_1+a_{32}x_2-a_{33}x_3)\Phi]=0, \end{aligned} \tag{57}$$

according to Roozen [38], which can be approximated by a Gaussian distribution

$$\Phi(X)=\Phi(x_1,x_2,x_3)=C_0e^{-\frac{1}{2}(x_1,x_2,x_3)Q(x_1,x_2,x_3)^{\mathrm{T}}},$$

where C_0 is a positive constant which is determined by

$$\int_{\mathbb{R}^3}\Phi(x_1,x_2,x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3=1,$$

and the real symmetric inverse matrix Q satisfies the following algebraic equation

$$QM^2Q+QA+A^{\mathrm{T}}Q=0,$$

letting $\Sigma=Q^{-1}$, then we obtain

$$M^2+A\Sigma+\Sigma A^{\mathrm{T}}=0. \tag{58}$$

In addition, we can calculate that the corresponding constant is $C_0=(2\pi)^{-\frac{3}{2}}|\Sigma|^{-\frac{1}{2}}$.

5.2. Density function of stationary distribution

Lemma 5.1. [39] *Let Υ_0 be a symmetric positive definite matrix, such that the three dimensional algebraic equation*

$$G_0^2+A_0\Upsilon_0+\Upsilon_0A_0^T=0, \tag{59}$$

holds, where $G_0=diag\{1,0,0\}$ and

$$A_0=\begin{pmatrix} -c_1 & -c_2 & -c_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and also that $c_1>0,c_3>0$ and $c_1c_2-c_3>0$, then Υ_0 follows

$$\Upsilon_0=\frac{1}{2(c_1c_2-c_3)}\begin{pmatrix} c_2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & \frac{c_1}{c_3} \end{pmatrix}.$$

Lemma 5.2. [39] *Let Υ_1 be a symmetric positive semi-definite matrix, such that the three-dimensional algebraic equation*

$$G_0^2+\tilde{A}_0\Upsilon_1+\Upsilon_1\tilde{A}_0^T=0, \tag{60}$$

holds, where $G_0 = diag\{1, 0, 0\}$ *and*

$$\tilde{A}_0 = \begin{pmatrix} -d_1 & -d_2 & -d_3 \\ 1 & 0 & 0 \\ 0 & 0 & d_{33} \end{pmatrix},$$

and also that $d_1 > 0, d_2 > 0$, *thus* Υ_1 *takes the form*

$$\Upsilon_1 = \mathrm{diag}\Big\{\frac{1}{2d_1}, \frac{1}{2d_1 d_2}, 0\Big\}.$$

Theorem 5.3. *For any initial value* $(S(0), V(0), I(0)) \in \mathbb{R}^3_+$, *if*

$$R_0^p > 1, \quad 2a_1 + a_1^2 > a_2, \quad \gamma\theta(\mu + \delta + \tau + 0.5\sigma_3^2) > \max\{\zeta, \theta\}\beta\zeta K, \tag{61}$$

then model (3) *has a probability density function*

$$\Phi(S, V, I) = (2\pi)^{-\frac{3}{2}}|\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}\left(\ln\frac{S}{S^*}, \ln\frac{V}{V^*}, \ln\frac{I}{I^*}\right)\Sigma^{-1}\left(\ln\frac{S}{S^*}, \ln\frac{V}{V^*}, \ln\frac{I}{I^*}\right)^T},$$

the special form of positive definite matrix Σ *is given as follows.*

Proof. By the finite independent superposition principle, equation (58) can be written as the sum of the solutions of the following algebraic sub-equations,

$$M_k^2 + A\Sigma_k + \Sigma_k A^{\mathrm{T}} = 0, \quad k = 1, 2, 3, \tag{62}$$

where $M_1 = \mathrm{diag}(\sigma_1, 0, 0), M_2 = \mathrm{diag}(0, \sigma_2, 0), M_3 = \mathrm{diag}(0, 0, \sigma_3)$, clearly $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3, M^2 = M_1^2 + M_2^2 + M_3^2$.

Firstly, we prove that A is a Hurwitz matrix. Equivalently, the characteristic polynomial of matrix A is

$$\varphi_A(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3, \tag{63}$$

by (56), we find that

$$\begin{aligned} &p_1 = a_{11} + a_{21} + a_{33} > 0, \\ &p_2 = (a_{11} - a_{12})a_{21} + a_{11}a_{33} + a_{13}a_{31} + a_{21}a_{33} + a_{23}a_{32} > 0, \\ &p_3 = (a_{11} - a_{12})a_{21}a_{33} + a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{31} + a_{13}a_{21}a_{32} > 0, \end{aligned} \tag{64}$$

and

$$\begin{aligned} p_1p_2 - p_3 = &(a_{11} - a_{12})a_{11}a_{21} + (a_{11} - a_{12})a_{21}^2 + a_{11}^2a_{33} + (a_{11}a_{13} - a_{12}a_{23})a_{31} \\ &+ 2a_{11}a_{21}a_{33} + a_{11}a_{33}^2 + (a_{21}a_{33}^2 + a_{21}a_{32}a_{23} - a_{13}a_{21}a_{32}) \\ &+ a_{13}a_{31}a_{33} + a_{21}a_{23}a_{32} + a_{23}a_{32}a_{33} > 0. \end{aligned} \tag{65}$$

Since

$$a_{11}a_{13} - a_{12}a_{23} > (a_{11} - a_{12})a_{23} > 0, \quad a_{21}a_{33}^2 - a_{13}a_{21}a_{32} > a_{21}a_{13}(a_{33} - a_{32}),$$

and by (55), direct substitution gives that

$$a_{21}a_{13}(a_{33}-a_{32})+a_{21}a_{32}a_{23}=a_{21}\beta^2e^{u_2^*+u_3^*}\frac{2a_1e^{u_3^*}+a_1^2e^{2u_3^*}-a_2e^{u_3^*}}{(1+a_2e^{u_3^*})^3(1+a_1e^{u_3^*})^2}$$
$$>\frac{a_{21}\beta^2e^{u_2^*+2u_3^*}}{(1+a_2e^{u_3^*})^3(1+a_1e^{u_3^*})^2}(2a_1+a_1^2-a_2)>0,$$

from (64), we can get A is a Hurwitz matrix when $2a_1+a_1^2>a_2>a_1>1$.

Now we will prove that Σ is positive definite by three steps.

Step 1. We consider the algebraic equation

$$M_1^2+A\Sigma_1+\Sigma_1A^{\mathrm{T}}=0, \tag{66}$$

and choose J_1 such that $A_1=J_1AJ_1^{-1}$, where

$$J_1=\begin{pmatrix}1&0&0\\0&1&0\\0&-\frac{a_{31}}{a_{21}}&1\end{pmatrix},\quad A_1=\begin{pmatrix}-a_{11}&a_{12}-\frac{a_{13}a_{31}}{a_{21}}&-a_{13}\\a_{21}&-a_{21}-\frac{a_{23}a_{31}}{a_{21}}&-a_{23}\\0&k_1&-a_{33}+\frac{a_{23}a_{31}}{a_{21}}\end{pmatrix}, \tag{67}$$

with

$$k_1=a_{31}+a_{32}-\frac{a_{31}a_{33}}{a_{21}}+\frac{a_{31}^2a_{23}}{a_{21}^2},$$

the relevant discussions will be given by the value of k_1 into two cases:

Case 1. If $k_1\neq0$, by Zhou et al [39], we choose H_1 such that $B_1=H_1A_1H_1^{-1}$, where the standardized transformation matrix is

$$H_1=\begin{pmatrix}a_{21}k_1&-(a_{21}+a_{33})k_1&\Delta_1\\0&k_1&-\frac{a_{21}a_{33}-a_{23}a_{31}}{a_{21}}\\0&0&1\end{pmatrix}, \tag{68}$$

with

$$\Delta_1=-\frac{a_{21}^2a_{23}k_1-a_{21}^2a_{33}^2+2a_{21}a_{23}a_{31}a_{33}-a_{23}^2a_{31}^2}{a_{21}^2}.$$

By direct calculation, one obtains

$$B_1=\begin{pmatrix}-y_1&-y_2&-y_3\\1&0&0\\0&1&0\end{pmatrix},$$

where

$$y_1=a_{11}+a_{21}+a_{33},$$
$$y_2=a_{11}a_{21}+a_{11}a_{33}+a_{13}a_{31}+a_{21}a_{33}+a_{23}a_{32}-a_{12}a_{21},$$
$$y_3=a_{11}(a_{21}a_{33}+a_{23}a_{32})+a_{13}a_{21}(a_{31}+a_{32})+a_{12}a_{23}a_{31}-a_{12}a_{21}a_{33}.$$

Moreover, algebraic equation (66) is equivalently transformed into

$$(H_1J_1)M_1^2(H_1J_1)^{\mathrm{T}} + B_1(H_1J_1)\Sigma_1(H_1J_1)^{\mathrm{T}} + (H_1J_1)\Sigma_1(H_1J_1)^{\mathrm{T}}B_1^{\mathrm{T}} = 0,$$

letting

$$\Theta_1 = \varrho_1^{-2}(H_1J_1)\Sigma_1(H_1J_1)^{\mathrm{T}}, \quad \varrho_1 = a_{21}k_1\sigma_1,$$

algebraic equation (66) is converted as

$$G_0^2 + B_1\Theta_1 + \Theta_1B_1^{\mathrm{T}} = 0. \tag{69}$$

Noting that A has all negative real-part eigenvalues, then B_1 is a Hurwitz matrix. By Lemma 5.1, Θ_1 is positive definite and takes the form

$$\Theta_1 = \frac{1}{2(y_1y_2 - y_3)}\begin{pmatrix} y_2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & \dfrac{y_1}{y_3} \end{pmatrix}.$$

Therefore, $\Sigma_1 = \varrho_1^2(H_1J_1)^{-1}\Theta_1[(H_1J_1)^{\mathrm{T}}]^{-1}$.

Case 2. If $k_1 = 0$, we choose $\hat{H}_1$ such that $\hat{B}_1 = \hat{H}_1A_1\hat{H}_1^{-1}$, where

$$\hat{H}_1 = \begin{pmatrix} a_{21} & -\dfrac{a_{21}^2 + a_{23}a_{31}}{a_{21}} & -a_{23} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{B}_1 = \begin{pmatrix} -b_1 & -b_2 & -b_3 \\ 1 & 0 & 0 \\ 0 & 0 & -\dfrac{a_{33}a_{21} - a_{23}a_{31}}{a_{21}} \end{pmatrix}.$$

One can equivalently transform (66) into

$$(\hat{H}_1J_1)M_1^2(\hat{H}_1J_1)^{\mathrm{T}} + \hat{B}_1(\hat{H}_1J_1)\Sigma_1(\hat{H}_1J_1)^{\mathrm{T}} + (\hat{H}_1J_1)\Sigma_1(\hat{H}_1J_1)^{\mathrm{T}}\hat{B}_1^{\mathrm{T}} = 0,$$

letting

$$\hat{\Theta}_1 = \hat{\varrho}_1^{-2}(\hat{H}_1J_1)\Sigma_1(\hat{H}_1J_1)^{\mathrm{T}}, \quad \hat{\varrho}_1 = a_{21}\sigma_1,$$

algebraic equation by Lemma 5.2, (66) becomes

$$G_0^2 + \hat{B}_1\hat{\Theta}_1 + \hat{\Theta}_1\hat{B}_1^{\mathrm{T}} = 0, \tag{70}$$

with

$$\hat{\Theta}_1 = \mathrm{diag}\Big\{\frac{1}{2b_1}, \frac{1}{2b_1b_2}, 0\Big\}. \tag{71}$$

Therefore, $\Sigma_1 = \hat{\varrho}_1^2(\hat{H}_1J_1)^{-1}\hat{\Theta}_1[(\hat{H}_1J_1)^{\mathrm{T}}]^{-1}$.

Step 2. For the algebraic equation

$$M_2^2 + A\Sigma_2 + \Sigma_2A^{\mathrm{T}} = 0, \tag{72}$$

we find the corresponding elimination matrix J_2 such that $A_2 = J_2 A J_2^{-1}$, where

$$J_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\dfrac{a_{12}}{a_{32}} & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -a_{21} & -a_{23} + \dfrac{a_{12}a_{21}}{a_{32}} & a_{21} \\ a_{32} & -a_{33} + \dfrac{a_{12}a_{31}}{a_{32}} & a_{31} \\ 0 & k_2 & -a_{11} - \dfrac{a_{12}a_{31}}{a_{32}} \end{pmatrix},$$

with

$$k_2 = -a_{13} + \frac{a_{12}}{a_{32}}\Big(a_{33} - a_{11} - \frac{a_{12}a_{31}}{a_{32}}\Big).$$

Similarly, the following two cases are considered.

Case 1. If $k_2 \neq 0$, let $B_2 = H_2 A_2 H_2^{-1}$, where

$$H_2 = \begin{pmatrix} a_{32}k_2 & -(a_{11}+a_{33})k_2 & \Delta_2 \\ 0 & k_2 & -a_{11} - \dfrac{a_{12}a_{31}}{a_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \tag{73}$$

with

$$\Delta_2 = \frac{a_{11}^2 a_{32}^2 + 2a_{11}a_{12}a_{31}a_{32} + a_{12}^2 a_{31}^2 + a_{31}a_{32}^2 k_2}{a_{32}^2}.$$

In fact, one can equivalently transform (72) into

$$(H_2J_2)M_2^2(H_2J_2)^{\mathrm{T}} + B_2[(H_2J_2)\Sigma_2(H_2J_2)^{\mathrm{T}}] + [(H_2J_2)\Sigma_2(H_2J_2)^{\mathrm{T}}]B_2^{\mathrm{T}} = 0,$$

letting

$$\Theta_2 = \varrho_2^{-2}(H_2J_2)\Sigma_2(H_2J_2)^{\mathrm{T}}, \quad \varrho_2 = a_{32}k_2\sigma_2,$$

which by Lemma 5.1 can be simplified as

$$G_0^2 + B_2\Theta_2 + \Theta_2 B_2^{\mathrm{T}} = 0, \tag{74}$$

with

$$B_2 = \begin{pmatrix} -q_1 & -q_2 & -q_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Theta_2 = \frac{1}{2(q_1q_2 - q_3)} \begin{pmatrix} q_2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & \dfrac{q_1}{q_3} \end{pmatrix}.$$

In other words, $\Sigma_2 = \varrho_2^2(H_2J_2)\Theta_2[(H_2J_2)^{\mathrm{T}}]^{-1}$.

Case 2. If $k_2 = 0$, we choose $\hat{H}_2$ such that $\hat{B}_2 = \hat{H}_2 A_2 \hat{H}_2^{-1}$, where $\hat{H}_2$ and $\hat{B}_2$ are given by

$$\hat{H}_2 = \begin{pmatrix} a_{32} & \dfrac{a_{12}a_{31} - a_{32}a_{33}}{a_{32}} & a_{31} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{B}_2 = \begin{pmatrix} -w_1 & -w_2 & -w_3 \\ 1 & 0 & 0 \\ 0 & 0 & -a_{11} - \dfrac{a_{12}a_{31}}{a_{32}} \end{pmatrix}.$$

One can equivalently transform (72) into

$$(\hat{H}_2J_2)M_2^2(\hat{H}_2J_2)^{\mathrm{T}} + \hat{B}_2[(\hat{H}_2J_2)\Sigma_2(\hat{H}_2J_2)^{\mathrm{T}}] + [(\hat{H}_2J_2)\Sigma_2(\hat{H}_2J_2)^{\mathrm{T}}]\hat{B}_2^{\mathrm{T}} = 0,$$

letting

$$\hat{\Theta}_2 = \hat{\varrho}_2^{-2}(\hat{H}_2 J_2)\Sigma_2(\hat{H}_2 J_2)^{\mathrm{T}}, \quad \hat{\varrho}_2 = a_{32}\sigma_2,$$

which by Lemma 5.2 is simplified as

$$G_0^2 + \hat{B}_2\hat{\Theta}_2 + \hat{\Theta}_2\hat{B}_2^{\mathrm{T}} = 0, \tag{75}$$

with

$$\hat{\Theta}_2 = \mathrm{diag}\Big\{\frac{1}{2w_1}, \frac{1}{2w_1 w_2}, 0\Big\}.$$

In other words, $\Sigma_2 = \hat{\varrho}_2^2(\hat{H}_2 J_2)\hat{\Theta}_2[(\hat{H}_2 J_2)^{\mathrm{T}}]^{-1}$.

Step 3. For the following algebraic equation

$$M_3^2 + A\Sigma_3 + \Sigma_3 A^{\mathrm{T}} = 0, \tag{76}$$

we find the corresponding elimination matrix J_3 such that $A_3 = J_3 A J_3^{-1}$, where

$$J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -\dfrac{a_{23}}{a_{13}} & 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -a_{33} & a_{31} + \dfrac{a_{32}a_{23}}{a_{13}} & a_{32} \\ -a_{13} & -a_{11} + \dfrac{a_{23}a_{12}}{a_{13}} & a_{12} \\ 0 & k_3 & -a_{21} - \dfrac{a_{23}a_{12}}{a_{13}} \end{pmatrix},$$

with

$$k_3 = a_{21} + \frac{a_{23}a_{11}}{a_{13}} + \frac{a_{23}(-a_{23}a_{12} - a_{13}a_{21})}{a_{13}^2}.$$

Similarly, the following two cases are discussed.

Case 1. If $k_3 \neq 0$, we find H_3 such that $B_3 = H_3 A_3 H_3^{-1}$, where

$$H_3 = \begin{pmatrix} -a_{13}k_3 & -(a_{11} + a_{21})k_3 & \Delta_3 \\ 0 & k_3 & -\dfrac{a_{12}a_{23} + a_{13}a_{21}}{a_{13}} \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$\Delta_3 = \frac{a_{12}^2 a_{23}^2 + a_{12}a_{13}^2 k_3 + 2a_{12}a_{13}a_{21}a_{23} + a_{13}^2 a_{21}^2}{a_{13}^2}.$$

So, (76) is equivalently transformed into

$$(H_3 J_3)M_3^2(H_3 J_3)^{\mathrm{T}} + B_3[H_3 J_3 \Sigma_3 (H_3 J_3)^{\mathrm{T}}] + [(H_3 J_3)\Sigma_3(H_3 J_3)^{\mathrm{T}}]B_3^{\mathrm{T}} = 0,$$

letting

$$\Theta_3 = \varrho_3^{-2}(H_3 J_3)\Sigma_3(H_3 J_3)^{\mathrm{T}}, \quad \varrho_3 = a_{13}k_3\sigma_3,$$

by Lemma 5.1, thus (76) is simplified as

$$G_0^2 + B_3\Theta_3 + \Theta_3 B_3^{\mathrm{T}} = 0, \tag{77}$$

where

$$B_3=\begin{pmatrix}-s_1 & -s_2 & -s_3\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},\quad \Theta_3=\frac{1}{2(s_1s_2-s_3)}\begin{pmatrix}s_2 & 0 & -1\\ 0 & 1 & 0\\ -1 & 0 & \frac{s_1}{s_3}\end{pmatrix}.$$

In other words, $\Sigma_3=\varrho_3^2(H_3J_3)\Theta_3[(H_3J_3)^{\mathrm{T}}]^{-1}$.

Case 2. If $k_3=0$, we choose $\hat{H}_3$ such that $\hat{B}_3=\hat{H}_3A_3\hat{H}_3^{-1}$, where

$$\hat{H}_3=\begin{pmatrix}-a_{13} & \frac{-a_{11}a_{13}+a_{12}a_{23}}{a_{13}} & a_{21}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad \hat{B}_3=\begin{pmatrix}-z_1 & -z_2 & -z_3\\ 1 & 0 & 0\\ 0 & 0 & -a_{11}-\frac{a_{12}a_{31}}{a_{32}}\end{pmatrix}.$$

We equivalently transform (76) into

$$(\hat{H}_3J_3)M_3^2(\hat{H}_3J_3)^{\mathrm{T}}+\hat{B}_3[(\hat{H}_3J_3)\Sigma_3(\hat{H}_3J_3)^{\mathrm{T}}]+[(\hat{H}_3J_3)\Sigma_3(\hat{H}_3J_3)^{\mathrm{T}}]\hat{B}_3^{\mathrm{T}}=0,$$

letting

$$\hat{\Theta}_3=\hat{\varrho}_3^{-2}(\hat{H}_3J_3)\Sigma_3(\hat{H}_3J_3)^{\mathrm{T}},\quad \hat{\varrho}_3=a_{13}\sigma_3,$$

so, (76) is simplified as

$$G_0^2+\hat{B}_3\hat{\Theta}_3+\hat{\Theta}_3\hat{B}_3^{\mathrm{T}}=0, \tag{78}$$

with

$$\hat{\Theta}_3=\begin{pmatrix}\frac{1}{2z_1} & 0 & 0\\ 0 & \frac{1}{2z_1z_2} & 0\\ 0 & 0 & 0\end{pmatrix}.$$

In other words, $\Sigma_3=\hat{\varrho}_3^2(\hat{H}_3J_3)\hat{\Theta}_3[(\hat{H}_3J_3)^{\mathrm{T}}]^{-1}$.

6. Numerical simulations

Milstein's higher order method for stochastic differential equations was established in [40]. In this section, we adopt Milstein's method to write the equations of discretization to model (3), the details are suggested to read [10, 17, 22, 31], and other methods for simulations are suggested to read [18, 41] and references therein.

Example 6.1. We present numerical simulations to illustrate our main theoretical results about persistence. We assume that $a_1<a_2$, and let the initial values of model (3) be $S(0)=2, V(0)=0.5, I(0)=0.5$, and other parameters be $\gamma=0.7, K=5, \beta=0.5, \tau=0.25, \delta=0.15, \zeta=0.07, \vartheta=0.005345, \mu=0.002, \sigma_1=0.05, \sigma_2=0.05, \sigma_3=0.05, a_1=1.5, a_2=4$. It is easy to check that conditions of Theorem 2.2 are satisfied as follows:

$$R_0^s=1.3492>1,\quad 0.002=\mu>0.5(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)=0.00125,\quad 1.26=2(\gamma-\zeta)>\sigma_1^2=0.0025.$$

Figure 1 shows that the susceptible, the vaccinated and the infected are persistent in the mean for a long run. Figure 2 presents the persistence of the infected when a_1 and a_2 vary, the corresponding simulations reveal that the density of the infected decreases when a_1 and a_2 increase. The persistence for the susceptible and the vaccinated could be found in Figure 3 as ϑ varies and Figure 4 as ζ varies. More precisely, the density of the susceptible increases, and the density of the vaccinated decreases as ϑ increases in Figure 3. The density of the susceptible decreases, and the density of the vaccinated increases as ζ increases in Figure 4.

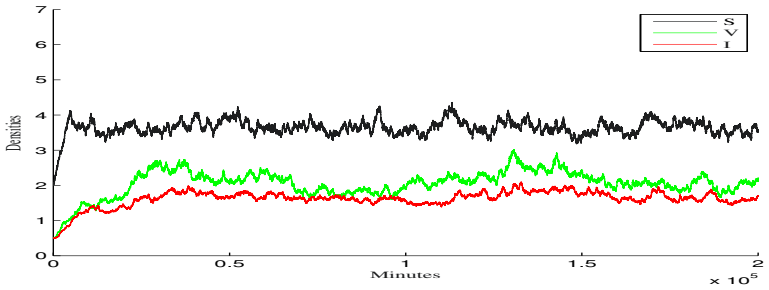

Figure 1: Persistence in the mean of the susceptible, the vaccinated and the infected.

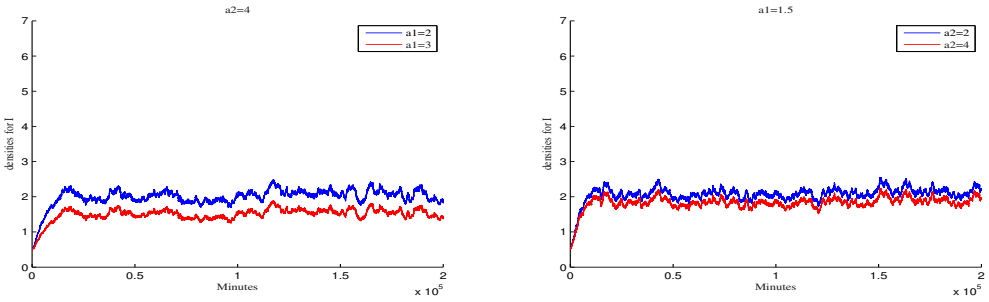

Figure 2: Persistence in the mean of the infected, as a_1 and a_2 increase.

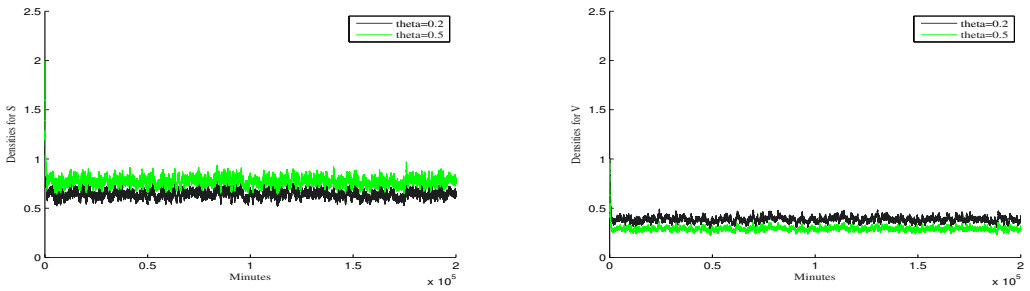

Figure 3: Densities of the susceptible and the vaccinated as ϑ increases.

Example 6.2. The extinction will be discussed here, let the initial values of model (3) be $S(0) = 2, V(0) = 1, I(0) = 1$, and other parameters be $\gamma = 0.1, K = 4, \beta = 0.5, \tau = 0.5$, $\delta = 0.5, \zeta = 0.8$, $\vartheta = 0.5$,

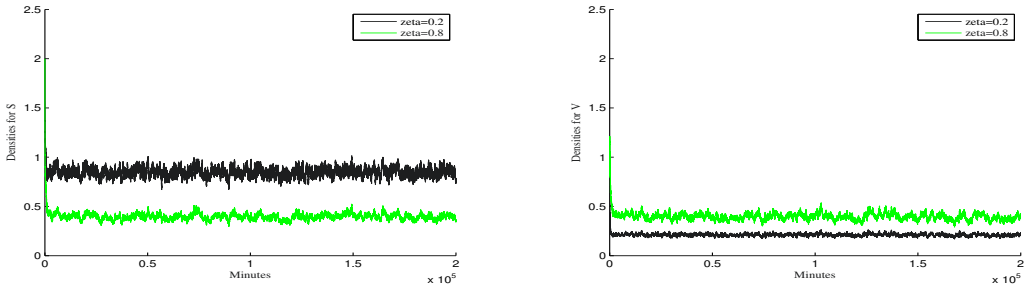

Figure 4: Densities of the susceptible and the vaccinated as ζ increases.

$\mu = 0.3, \sigma_1 = 0.05, \sigma_2 = 0.05, \sigma_3 = 0.05, a_1 = 3, a_2 = 4$. It is easy to check that conditions of Theorem 4.1 are satisfied as follows:

$$R_0^e = 0.2562 < 1, \quad 0.3 = \mu > 0.5(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2) = 0.00125.$$

Thus corresponding simulations reveal that the extinction will occur for a long run. The density of the infected exponentially tends to the extinction as β decreases and as $\delta, \tau, \mu, \zeta, \sigma_3$ increase as well in Figures 5-7.

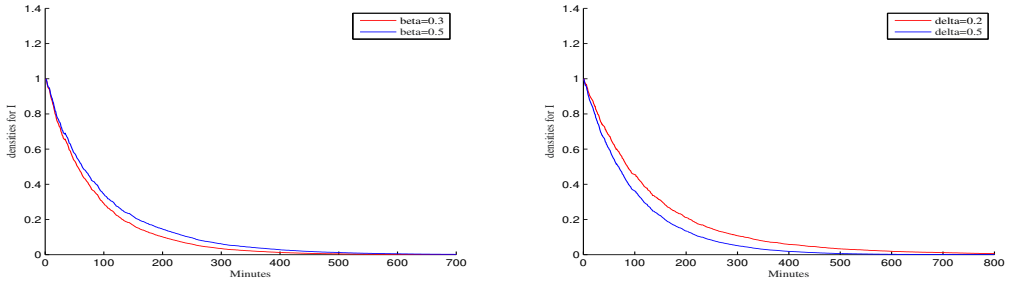

Figure 5: Extinction of the infected with an exponential rate as β and δ increase.

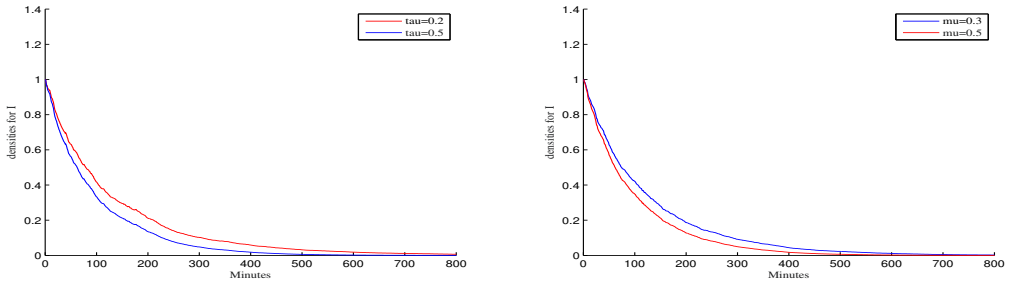

Figure 6: Extinction of the infected with an exponential rate as τ and μ increase.

Example 6.3. Let the initial values of model (3) be $S(0) = 2, V(0) = 1, I(0) = 1$ and other parameters be $\gamma = 0.5, K = 1, \beta = 0.3, \tau = 0.35, \delta = 0.08, \zeta = 0.8, \vartheta = 0.2, \mu = 0.08, \sigma_1 = 0.05, \sigma_2 =$

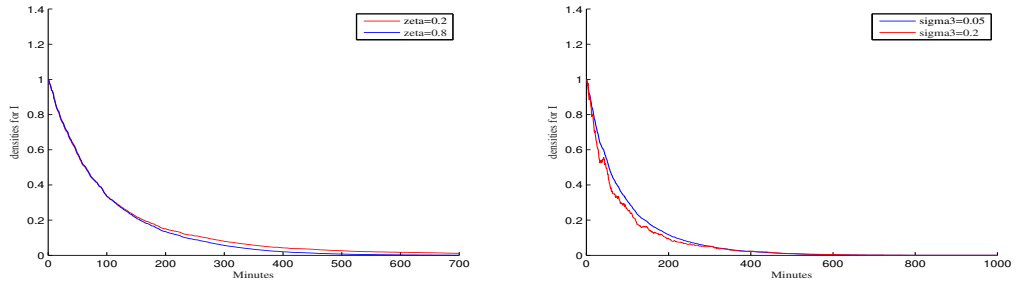

Figure 7: Extinction of the infected with an exponential rate as ζ and σ_3 increase.

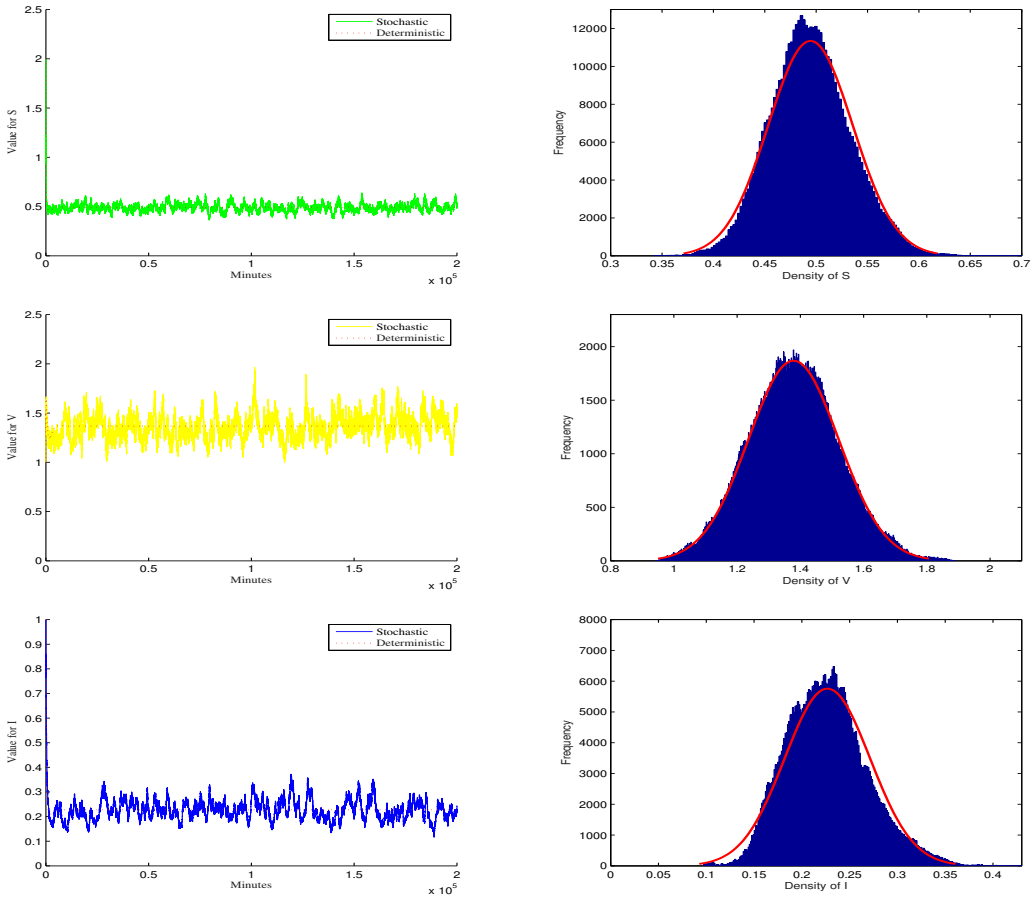

Figure 8: Solid lines for model (3), dashed dots for model (2) on left panel. Distributions of the densities to model (3) with the intensities $(\sigma_1, \sigma_2, \sigma_3) = (0.05, 0.05, 0.05)$ on right panel.

$0.05, \sigma_3 = 0.05, a_1 = 3, a_2 = 4$. It is easy to check that the equilibrium point is $F^* = (0.494, 1.373, 0.025)$, and $k_1 = 0.5015 \neq 0$, $k_2 = -1.3981 \neq 0$, $k_3 = 0.2727 \neq 0$, and also that conditions (61) are satisfied as follows:

$$R_0^p = 1.08797 > 1, \quad 2a_1 + a_1^2 - a_2 = 8 > 0,$$
$$\gamma(\mu + \delta + \tau + 0.5\sigma_3^2) - \beta\xi K = 0.041525 > 0, \quad \gamma\vartheta(\mu + \delta + \tau + 0.5\sigma_3^2) - \beta\xi^2 K = 0.015625 > 0.$$

By Theorem 5.3, we obtain that model (3) has a unique stationary distribution, which is ergodic.

Remark 6.1. The main results for model (1) and model (3) had shown that the corresponding sample paths of the stochastic models had less impacts on the persistence and the stability, compared with those in the deterministic models. That was to say, both model (1) and model (3) admitted the persistence in the mean and the ergodic stationary distributions under some moderate conditions. Precisely, $\sigma_1, \sigma_2, \sigma_3$ did not take great impacts on the persistence for the corresponding sample paths in Figure 2 in [1] and Figure 1 of our paper. Further, under the condition $R_0^s > 1$, the parameters $\vartheta, \zeta, a_1, a_2$ in Figures 2-4 change the densities of model (3).

Remark 6.2. The extinction was derived in Theorem 2 of [1] and Theorem 4.1 under the corresponding suitable conditions, which are demonstrated in Figure 1 of [1] and in Figures 5-7 at Example 6.2. In addition, we found that the parameters $\beta, \delta, \tau, \mu, \sigma_3, \zeta$ in model (3) played some roles on the extinction as $R_0^e < 1$ was valid. Especially, the parameters δ, τ, μ had great impacts on the time for the infected whose density declined to zero.

7. Conclusions and discussions

In this paper, we study the dynamic behaviors of a stochastic SVIR model with the saturation incidence. Two critical thresholds R_0^s and R_0^e by constructing appropriate Lyapunov functions are obtained, we further prove the existence of a unique ergodic stationary distribution when $R_0^s > 1$ holds, and the extinction of infectious diseases for a long time when $R_0^e < 1$ holds. Further, the numerical simulations show that the parameters of model (3) have the crucial impacts on the persistence and the extinction as demonstrated in Theorem 2.2 and Theorem 4.1 respectively. Meanwhile, we find the expression of the probability density function in Theorem 5.3 around the quasi-endemic equilibrium point by applying the asymptotic analysis and Fokker-Planck equation when $R_0^p > 1$ holds. Figure 8 presents the sample paths and the distributions of the densities of the susceptible, the vaccinated and the infected.

Model (1) assumes that the vaccinated return to the recovered and do not get infected any more. According to the mechanism of the spreading of COVID-19, we further propose the assumption of model (3) that the vaccinated lose their immunities and return to the susceptible. Moreover, model (1) governs the bilinear incidence to describe transmission rates, but for the pandemic COVID-19, the infected produce the crowdedness effect when the number of the infected is large enough, so we modify the bilinear incidence into the saturation incidence by using the saturated constants a_1 and a_2. We notice that, the persistence and the extinction are studied in both model (1) and model (3), the research results show that the time that the density of the infected tends to zero is less when the intensities of the white noises are larger. Meanwhile, we also provide the expression of the probability density function in Theorem 5.3 in this paper, instead of the existence of a nontrivial periodic solution of model (1).

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