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Delay-dependent stability of highly nonlinear neutral stochastic functional differential equations

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Abstract

This paper focuses on the delay-dependent stability of highly nonlinear hybrid neutral stochastic functional differential equations (NSFDEs). The delay dependent stability criteria for a class of highly nonlinear hybrid NSFDEs are derived via the Lyapunov functional. The stabilities discussed in this paper include H_∞ stability, asymptotically stability and exponential stability. A numerical example is given to illustrate the criteria established.

KEYWORDS:
Delay-dependent stability, neutral stochastic systems, highly nonlinear, Markov switching, exponential stability.

1 | INTRODUCTION

Stochastic delay differential equations with continuous-time Markovian chains (known also as hybrid SDDEs) have been developed to model the real-world systems which do not only depend on the present state and the past ones but may also experience abrupt changes in their structures and parameters. One of the important issues in the study of hybrid SDDEs is the automatic control, with consequent emphasis being placed on the analysis of stability.[1,2,3,4,5,6,7,8,9]. Most of stability criteria can only be applied to the hybrid SDDEs where their coefficients are linear or bounded by linear functions[10,11,12,13,14,15,16]. Recently, some new results have been established for highly nonlinear SDDEs. For example, the robust stability for nonlinear hybrid SDDEs is studied by Hu et al[17]. Taking different structures in different modes into account, Fei et al.[18] discussed the structured stability for highly nonlinear hybrid SDDEs.

In many cases, stochastic delay systems may depend on historical states in a time interval. Stochastic functional differential equations (SFDEs) have been used to model these systems. In fact, SFDEs have been widely used in biology, physics, economics and so on. In the past few decades, the theory of SFDEs has attracted a great deal of attention. In particular, many papers have been devoted to the study of stability of highly nonlinear SFDEs. Without the linear growth condition, Wu et al.[19] discussed Razumikhin-type theorems on the asymptotic stability with a general decay rate and its robustness for SFDEs. The critical foundation in developing stability in distribution for highly nonlinear SFDEs has been presented by Wang et al[20]. Under the polynomial growth condition, Mei et al.[21] discussed exponential stabilization by delay feedback control for highly nonlinear hybrid SFDEs with infinite delay.

Recently, there was a great deal of attention to the stability of neutral stochastic differential equations (NSDEs) (see, e.g, related works[22,23,24,25,26,27,28,29,30,31]). One of the important classes of NSDEs is the class of neutral stochastic functional differential equations (NSFDEs). For example, new criteria for the mean square exponential stability of NSFDEs are given by Pham[32].

The asymptotic stability with a general decay rate of nonlinear NSFDEs was analyzed in the work of Wu et al.[33]. Song and Shen[34] studied asymptotic behavior of nonlinear NSFDEs. However, all of these results have the same feature: they are delay independent.

It is well known that the stability criteria can be classified into two categories: delay independent and delay dependent. The delay-independent type is independent of delay size, and it is generally conservative, especially when a delay is short. Although a great deal effort has been devoted to the investigation of this subject, our knowledge of highly nonlinear SDDEs delay-dependent stability criteria is still insufficient. An important breakthrough is due to Fei et al.(see related work[35]) who are first to establish delay-dependent criteria for highly nonlinear hybrid SDDEs. The results were later extended to hybrid stochastic integro-differential delay equations by Fei et al.[36] and hybrid SFDEs by Song et al.[37] respectively. Shen et al.[38] further investigated delay-dependent criterion for a highly nonlinear neutral stochastic delay systems. But they all require drift coefficient to be globally Lipschtiz continuous in the delay component. Moreover, Shen et al.[38] require $0 < \kappa < \frac{\sqrt{2}}{2}$ in contraction mapping of neutral term. Those limitations may exclude many highly nonlinear hybrid SDDEs. Moreover, as far as we know, there's no results on the decay rate of solutions for delay dependent highly nonlinear hybrid SDDEs. In this paper, we will loosen some restrictive conditions in the papers of Shen et al.[38] and Fei et al.[36] to investigate the delay dependent stability of highly nonlinear NSFDEs.

The key contributions of our paper are highlighted below:

1. This paper investigates delay-dependent stability criteria for highly nonlinear hybrid NSFDEs. A significant amount of new mathematics has been developed to deal with the difficulties due to the neutral term.
2. In the papers of Shen et al.[38] and Fei et al.[36], there are some restrictive conditions which may exclude many nonlinear SDDEs. In this paper, we will loosen this restrictive condition to cover a much wider class NSFDEs. Moreover, the delay-dependent exponential stability criterion for highly nonlinear hybrid NSFDEs is established for the first time.
3. The stabilities discussed in this paper include H_∞ stability in L^p, asymptotic stability in $L^{\bar{q}}$ and exponential stability in $L^{\bar{q}}$. These are more general than corresponding results of Shen et al.[38].

The structure of the paper is arranged as follows. In Section 2, by the method of Lyapunov functional, the boundedness and stability of hybrid highly nonlinear NSFDEs are discussed. The delay dependent stability criteria of highly nonlinear neutral stochastic functional differential equations are discussed in Section 3. An example is given to illustrate effectiveness of our theory in Section 4 while the conclusion is made in Section 5.

2 | PRELIMINARY

Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ where a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $C([-\tau, 0]; R^d)$ denote the family of continuous functions φ from $[-\tau, 0] \to R^d$ with the norm $\|\varphi\| = \sup_{-\tau\leq s\leq 0} |\varphi(s)|$. Let $\mathcal{W}([-\tau, 0]; R_+)$ denotes all probability measures on $[-\tau, 0]$. Let $\{r(t), t \geq 0\}$ be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$, and $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$.

Let $\bar{F}(\cdot,\cdot,\cdot) : C([-\tau, 0]; R^d)\times S\times R_+ \to R^d$, $\bar{G}(\cdot,\cdot,\cdot) : C([-\tau, 0]; R^d)\times S\times R_+ \to R^{d\times m}$, $D(\cdot,\cdot,\cdot) : C([-\tau, 0]; R^d)\times S\times R_+ \to R^d$ be Borel measurable functionals and $\eta \in \mathcal{W}([-\tau, 0]; R_+)$ is a probability measure. In this paper, we consider the following hybrid NSFDE

$$d[x(t) - D(x_t, r(t), t)] = \bar{F}(x_t, r(t), t)dt + \bar{G}(x_t, r(t), t)dB(t) \tag{1}$$

on $t \geq 0$ with initial data

$$\{x(t) : -\tau \leq t \leq 0\} = \xi \in C([-\tau, 0]; R^d) \quad \text{and} \quad r(0) = i_0 \in S, \tag{2}$$

where $x_t = \{x(t+u) : -\tau \leq u \leq 0\}$. We also assume that $\bar{F}(0, i, t) = 0$, $\bar{G}(0, i, t) = 0$, $D(0, i, t) = 0$.

Assumption 1. For each integer $L \geq 1$ there is a positive constant $\bar{K}_L$ such that

$$|\bar{F}(\varphi, i, t) - \bar{F}(\bar{\varphi}, i, t)|^2 \vee |\bar{G}(\varphi, i, t) - \bar{G}(\bar{\varphi}, i, t)|^2 \leq \bar{K}_L(\|\varphi - \bar{\varphi}\|^2)$$

for those $\varphi, \bar{\varphi} \in C([-\tau, 0]; R^d)$ with $\|\varphi\| \vee \|\bar{\varphi}\| \leq L$ and all $(i, t) \in S \times R_+$.

Assumption 2. Assume that there is a constant $\kappa_0 \in (0, 1)$ such that

$$|D(\varphi, i, t) - D(\bar{\varphi}, i, t)| \leq \kappa_0 \int_{-\tau}^{0} |\varphi(u) - \bar{\varphi}(u)| d\eta(u) \tag{3}$$

for all $\varphi, \bar{\varphi} \in C([-\tau, 0]; R^d)$ and $(i, t) \in S \times R_+$.

In order to achieve our aim, we extend a lemma in Fei et al.[36] to contain the neutral term case, the results of this lemma will play an important role in the paper.

Lemma 1. For nonnegative integers $b_h \geq b_{h-1} \geq \cdots \geq b_1 \geq 0$, define the quasi polynomial function $H(Y) = a_h|Y|^{b_h} + \cdots + a_1|Y|^{b_1}, Y \in R^d$, where $|Y|$ is Euclidean norm of Y, $a_i \geq 0, i = 1, \cdots, h-1$, and $a_h > 0$. Assume $y(\cdot) : [-\tau, \infty) \to R^d$ is a continuous function, where $y(t) = \zeta(t), t \in [-\tau, 0]$ and $\tau > 0$. Assume that Assumption 2 holds, fix $\varepsilon > 0$ arbitrarily, we have the following properties:
(i)

$$\int_0^T e^{\varepsilon t} \int_{-\tau}^0 H(y(t+u)) d\eta(u) dt \leq e^{\varepsilon\tau} \int_{-\tau}^0 H(\zeta(t)) dt + e^{\varepsilon\tau} \int_0^T e^{\varepsilon t} H(y(t)) dt, \quad \forall T > 0.$$

(ii)

$$\int_0^T e^{\varepsilon t} H\Big(y(t) - D(y_t, r(t), t)\Big) dt \leq \kappa_0 e^{\varepsilon\tau} \int_{-\tau}^0 H(\zeta(t)) dt + C_\tau \int_0^T e^{\varepsilon t} H(y(t)) dt, \quad \forall T > 0,$$

where $C_\tau = \kappa_0 e^{\varepsilon\tau} + (1 - \kappa_0)^{1-b_h}$.

Proof. For $b_i \geq 0$, it is easy to show that

$$\int_0^T e^{\varepsilon t} \int_{-\tau}^0 |y(t+u)|^{b_i} d\eta(u) dt \leq e^{\varepsilon\tau} \int_{-\tau}^0 \int_{-\tau}^T e^{\varepsilon t} |y(t)|^{b_i} dt d\eta(u) = e^{\varepsilon\tau} \int_{-\tau}^0 |\zeta(t)|^{b_i} dt + e^{\varepsilon\tau} \int_0^T e^{\varepsilon t} |y(t)|^{b_i} dt. \tag{4}$$

By the definition of $H(Y)$, we have

$$\int_0^T e^{\varepsilon t} \int_{-\tau}^0 H(y(t+u)) d\eta(u) dt \leq e^{\varepsilon\tau} \int_{-\tau}^0 H(\zeta(t)) dt + e^{\varepsilon\tau} \int_0^T e^{\varepsilon t} H(y(t)) dt, \quad \forall T > 0.$$

Specially, if $\varepsilon = 0$, it is easy to show

$$\int_0^T \int_{-\tau}^0 H(y(t+u)) d\eta(u) dt \leq \int_{-\tau}^0 H(\zeta(t)) dt + \int_0^T H(y(t)) dt, \quad \forall T > 0.$$

Now, let us show the assertion (ii). For $b_i \geq 1$, applying the inequality (see, e.g., Kolmanovskii et al.[39])

$$(\hat{u} + \hat{v})^{b_i} \leq (1 + \imath)^{b_i - 1}(\hat{u}^{b_i} + \imath^{1-b_i} \hat{v}^{b_i}) \quad \forall \hat{u}, \hat{v} \geq 0,\ b_i \geq 1,\ \imath > 0,$$

it is straightforward to get

$$\Big|y(t) - D(y_t, r(t), t)\Big|^{b_i} \leq (1 + \imath)^{b_i - 1}(|y(t)|^{b_i} + \imath^{1-b_i} \Big|D(y_t, r(t), t)\Big|^{b_i}).$$

Setting $\imath = \frac{\kappa_0}{1-\kappa_0}$, by Assumption 2 and the Hölder inequality, we derive

$$\Big|y(t) - D(y_t, r(t), t)\Big|^{b_i} \leq (1 - \kappa_0)^{1-b_i} |y(t)|^{b_i} + \kappa_0^{1-b_i} \Big|D(y_t, r(t), t)\Big|^{b_i}$$

$$\leq (1-\kappa_0)^{1-b_i}|y(t)|^{b_i} + \kappa_0 \int_{-\tau}^{0} |y(t+u)|^{b_i} d\eta(u), \tag{5}$$

which, together with (4), show that

$$\begin{aligned}\int_0^T e^{\varepsilon t}\Big|y(t)-D(y_t,r(t),t)\Big|^{b_i}dt &\leq (1-\kappa_0)^{1-b_i}\int_0^T e^{\varepsilon t}|y(t)|^{b_i}dt + \kappa_0\int_0^T\int_{-\tau}^{0} e^{\varepsilon t}|y(t+u)|^{b_i}d\eta(u)dt \\ &\leq \kappa_0 e^{\varepsilon\tau}\int_{-\tau}^{0}|\zeta(t)|^{b_i}dt + \kappa_0 e^{\varepsilon\tau}\int_0^T e^{\varepsilon t}|y(t)|^{b_i}dt + (1-\kappa_0)^{1-b_i}\int_0^T e^{\varepsilon t}|y(t)|^{b_i}dt \\ &= \kappa_0 e^{\varepsilon\tau}\int_{-\tau}^{0}|\zeta(t)|^{b_i}dt + [\kappa_0 e^{\varepsilon\tau} + (1-\kappa_0)^{1-b_i}]\int_0^T e^{\varepsilon t}|y(t)|^{b_i}dt.\end{aligned}$$

Noting that $0 < 1-\kappa_0 < 1$ and $1-b_i \leq 0$, we have $(1-\kappa_0)^{1-b_i} \leq (1-\kappa_0)^{1-b_h}, \quad \forall b_i \leq b_h$. Thus,

$$\int_0^T e^{\varepsilon t}\Big|y(t)-D(y_t,r(t),t)\Big|^{b_i}dt \leq \kappa_0 e^{\varepsilon\tau}\int_{-\tau}^{0}|\zeta(t)|^{b_i}dt + [\kappa_0 e^{\varepsilon\tau} + (1-\kappa_0)^{1-b_i}]\int_0^T e^{\varepsilon t}|y(t)|^{b_h}dt$$

for all $1 \leq b_i$. Clearly, for $b_i = 0$, the inequality above holds still. By the definition of $H(Y)$, we see that

$$\begin{aligned}\int_0^T e^{\varepsilon t}H\Big(y(t)-D(y_t,r(t),t)\Big)dt &\leq \kappa_0 e^{\varepsilon\tau}\int_{-\tau}^{0}(a_h|\zeta(t)|^{b_h}+\cdots+a_1|\zeta(t)|^{b_1})dt + C_\tau\int_0^T e^{\varepsilon t}(a_h|y(t)|^{b_h}+\cdots+a_1|y(t)|^{b_1})dt \\ &= \kappa_0 e^{\varepsilon\tau}\int_{-\tau}^{0}H(\zeta(t))dt + C_\tau\int_0^T e^{\varepsilon t}H(y(t))dt.\end{aligned}$$

Thus the proof is complete. □

Let $C^{2,1}(R^d \times S \times R_+; R_+)$ denote the family of nonnegative functions $U(x,i,t)$ on $R^d \times S \times R_+$, which are continuously twice differentiable in x and once in t. Let $U_t(x,i,t) = \frac{\partial U(x,i,t)}{\partial t}$, $U_x(x,i,t) = \left(\frac{\partial U(x,i,t)}{\partial x_1},\cdots,\frac{\partial U(x,i,t)}{\partial x_d}\right)$ and $U_{xx}(x,i,t) = \left(\frac{\partial^2 U(x,i,t)}{\partial x_k \partial x_l}\right)_{d\times d}$. Define $\mathbb{L}\bar{U} : C([-\tau,0];R^d)\times S\times R_+ \to R$ by

$$\begin{aligned}\mathbb{L}\bar{U}(\varphi,i,t) &= \bar{U}_t(\varphi(0)-D(\varphi,i,t),i,t) + \bar{U}_x(\varphi(0)-D(\varphi,i,t),i,t)\bar{F}(\varphi,i,t) \\ &+ \frac{1}{2}\text{trace}[\bar{G}^T(\varphi,i,t)\bar{U}_{xx}(\varphi(0)-D(\varphi,i,t),i,t)\bar{G}(\varphi,i,t)] + \sum_{j=1}^{N}\gamma_{ij}\bar{U}(\varphi(0)-D(\varphi,j,t),j,t).\end{aligned}$$

Assumption 3. Assume that Q is a quasi polynomial function. Suppose that there exist nonnegative constants q, a_1, a_2, a_3 with $a_2 > a_3$, $q \geq 2$, and function $\bar{U} \in C^{2,1}(R^d \times S \times R_+; R_+)$, such that

$$|x|^q \leq \bar{U}(x,i,t) \leq Q(x), \quad \forall (x,i,t) \in R^d \times S \times R_+ \tag{6}$$

and

$$\mathbb{L}\bar{U}(\varphi,i,t) \leq a_1 - a_2 Q(\varphi(0)) + a_3\int_{-\tau}^{0} Q(\varphi(u))d\eta(u)$$

for all $(\varphi,i,t) \in C([-\tau,0];R^d)\times S\times R_+$.

With the notations and assumptions introduced in above, we can get the existence and uniqueness of system (1). The following theorem can be proved in the similar way as in theorem 3.2 in the work of Wu et al.[33] or in theorem 1 of Shen et al.[30] Thus, we omit the proof of this theorem.

Theorem 1. Let Assumptions 1, 2 and 3 hold. Then for any initial data (2), we have the following assertions:
(i) There is a unique global solution $x(t)$ to the hybrid NSFDE (1) on $t \in [-\tau,\infty)$.

(ii) The solution has the property that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t EQ(x(s))ds \le \frac{a_1}{a_2-a_3}.$$

The following theorem, which forms the foundation of this paper, shows that the qth moment of the solution of equation (1) is bounded.

Theorem 2. Let Assumptions 1, 2 and 3 hold. Then for any given initial data (2), there is positive constant $q \ge 2$ such that

$$\sup_{-\tau\le t<\infty} E|x(t)|^q < \infty. \tag{7}$$

Proof: Applying the generalized Itô formula to $e^{\varepsilon t}\bar{U}(z(t), r(t), t)$, we get

$$Ee^{\varepsilon t}\bar{U}(z(t), r(t), t) = \bar{U}(z(0), r(0), 0) + E\int_0^t \varepsilon e^{\varepsilon s}\bar{U}(z(s), r(s), s)ds + E\int_0^t e^{\varepsilon s}\mathbb{L}\bar{U}(x_s, r(s), s)ds,$$

where $z(t) = x(t) - D(x_t, r(t), t)$, $0 < \varepsilon < 1$ is sufficiently small such that

$$a_2 - \varepsilon\hat{C}_\tau - a_3 e^{\varepsilon\tau} > 0, \tag{8}$$

where $\hat{C}_\tau$ depends on the highest power of $Q(x)$. From Assumption 3, Lemma 1 and condition (8), we have

$$\begin{aligned}
Ee^{\varepsilon t}|z(t)|^q &\le \bar{U}(z(0), r(0), 0) + \varepsilon E\int_0^t e^{\varepsilon s}Q(z(s))ds + \frac{a_1}{\varepsilon}e^{\varepsilon t} - a_2 E\int_0^t e^{\varepsilon s}Q(x(s))ds + a_3 E\int_0^t e^{\varepsilon s}\int_{-\tau}^0 Q(x(s+u))d\eta(u)ds \\
&\le K_2 + \frac{a_1}{\varepsilon}e^{\varepsilon t} + \varepsilon\hat{C}_\tau E\int_0^t e^{\varepsilon s}Q(x(s))ds - a_2 E\int_0^t e^{\varepsilon s}Q(x(s))ds + a_3 e^{\varepsilon\tau}E\int_0^t e^{\varepsilon s}Q(x(s))ds \\
&\le K_2 + \frac{a_1}{\varepsilon}e^{\varepsilon t},
\end{aligned}$$

where $K_2 = \bar{U}(z(0), r(0), 0) + \varepsilon e^{\varepsilon\tau}\int_{-\tau}^0 Q(\xi(s))ds + a_3 e^{\varepsilon\tau}\int_{-\tau}^0 Q(\xi(s))ds$. It implies that

$$\sup_{0\le t<\infty} E|z(t)|^q < \infty. \tag{9}$$

Similar to the discussion of (5), we have that for any $t \ge 0$,

$$\begin{aligned}
\sup_{0\le s\le t} E|x(s)|^q &\le (1-\kappa_0)^{1-q}\sup_{0\le s\le t} E|z(s)|^q + \sup_{0\le s\le t}\kappa_0 E\int_{-\tau}^0 |x(s+u)|^q d\eta(u) \\
&\le (1-\kappa_0)^{1-q}\sup_{0\le s\le t} E|z(s)|^q + \kappa_0\|\xi\|^q + \sup_{0\le s\le t}\kappa_0 E|x(s)|^q.
\end{aligned} \tag{10}$$

Thus, we can get

$$(1-\kappa_0)\sup_{0\le s\le t} E|x(s)|^q \le (1-\kappa_0)^{1-q}\sup_{0\le s\le t} E|z(s)|^q + \kappa_0\|\xi\|^q.$$

Letting $t \to \infty$, it yields

$$(1-\kappa_0)\sup_{0\le s<\infty} E|x(s)|^q \le (1-\kappa_0)^{1-q}\sup_{0\le s<\infty} E|z(s)|^q + \kappa_0\|\xi\|^q.$$

Making use of (9), we clearly have

$$\sup_{0\le t<\infty} E|x(t)|^q \le \frac{1}{(1-\kappa_0)^q}\sup_{0\le t<\infty} E|z(t)|^q + \frac{\kappa_0}{1-\kappa_0}\|\xi\|^q < \infty,$$

which implies the assertion (7). Thus the proof is complete. □

3 | DELAY-DEPENDENT STABILITY OF NSFDES

In this section, we will investigate the delay-dependent stability of NSFDEs. The neutral term D was allowed to depend on the mode and time before. In this situation, the stability analysis will become very complicated. In order to make our stability analysis more understandable, we will only consider the simple case in this paper where the neutral term D is independent of the mode and time. Moreover, we assume $\bar{F}(x_t, r(t), t) = f(x(t), r(t), t) + F(x_t, r(t), t)$, $\bar{G}(x_t, r(t), t) = g(x(t), r(t), t) + G(x_t, r(t), t)$. That is we only consider the following highly nonlinear hybrid NSFDE

$$d[x(t) - D(x_t)] = [f(x(t), r(t), t) + F(x_t, r(t), t)]dt + [g(x(t), r(t), t) + G(x_t, r(t), t)]dB(t). \tag{11}$$

The key technique used in this paper is the method of Lyapunov functionals. To define the Lyapunov functionals, we introduce two segments $\bar{x}_t := \{x(t+s) : -2\tau \le s \le 0\}$ and $\bar{r}_t := \{r(t+s) : -2\tau \le s \le 0\}$ for $t \ge 0$. For $\bar{x}_t$ and $\bar{r}_t$ to be well defined for $0 \le t < 2\tau$, we set $x(s) = \xi(-\tau)$ for $s \in [-2\tau, -\tau)$ and $r(s) = r_0$ for $s \in [-2\tau, 0)$. To study the delay dependent stability of the NSFDE (11), we need to impose the following assumptions which is named as polynomial growth condition.

Assumption 4. Assume that there exist three constants $K > 0$, $q_1 \ge 1$ and $q_2 \ge 1$ such that

$$|f(x, i, t) + F(\varphi, i, t)| \le K[1 + |x|^{q_1} + \int_{-\tau}^{0} |\varphi(u)|^{q_1} d\eta(u)],$$

$$|g(x, i, t) + G(\varphi, i, t)| \le K[1 + |x|^{q_2} + \int_{-\tau}^{0} |\varphi(u)|^{q_2} d\eta(u)]$$

for all $(x, i, t) \in R^d \times S \times R_+$ and $(\varphi, i, t) \in C([-\tau, 0]; R^d) \times S \times R_+$.

In the paper of Fei et al.[36], there is a restrictive condition arranged as Assumption 3.3 which requires the drift coefficient to be globally Lipschtiz continuous in the delay component. This condition may exclude many nonlinear SDDEs, for example, the one to be discussed in Example 6 where

$$F(x_t, 1, t) = -\frac{1}{\tau}\int_{-\tau}^{0} x(t+u)du + (\frac{1}{\tau}\int_{-\tau}^{0} x(t+u)du)^3,$$

$$F(x_t, 2, t) = -\frac{1}{\tau}\int_{-\tau}^{0} x(t+u)du + 0.5(\frac{1}{\tau}\int_{-\tau}^{0} x(t+u)du)^3.$$

In this paper, we replace the restrictive condition with the following assumption which will cover a much wider class SDDEs.

Assumption 5. Assume that F can be decomposed as

$$F(x_t, i, t) = F_1(x_t, i, t) + F_2(x_t, i, t),$$

and, moreover, there is a positive number β such that

$$|F_1(\varphi, i, t) - F_1(\psi, i, t)| \le \beta \int_{-\tau}^{0} |\varphi(u) - \psi(u)| d\eta(u) \tag{12}$$

for all $\varphi, \psi \in C([-\tau, 0]; R^d)$ and $(i, t) \in S \times R_+$.

Remark 1. From Assumption 5, we can see that F can be decomposed as globally Lipschtiz continuous term F_1 and highly nonlinear term F_2, this decomposition enables us to arrange the highly nonlinear term of the delay component into F_2 but leave the globally Lipschitz continuous term in F_1.

The Lyapunov functional used in this paper has the form

$$V(\bar{x}_t, \bar{r}_t, t) = U(x(t) - D(x_t), r(t), t) + \varrho \int_{-\tau}^{0}\int_{t+s}^{t} \Pi(v) dv ds \tag{13}$$

for $t \geq 0$, where $\Pi(t) = \tau|f(x(t), r(t), t) + F(x_t, r(t), t)|^2 + |g(x(t), r(t), t) + G(x_t, r(t), t)|^2$, $U \in C^{2,1}(R^d \times S \times R_+; R_+)$ such that

$$\lim_{|x|\to\infty} [\inf_{(i,t)\in S\times R_+} U(x, i, t)] = \infty,$$

and ϱ is a positive number to be determined later while we set

$$f(x, i, t) = f(x, i, 0), \quad g(x, i, t) = g(x, i, 0), \quad F(x_t, i, t) = F(x_t, i, 0), \quad G(x_t, i, t) = G(x_t, i, 0)$$

for $x \in R^d$, $x_t \in C([-\tau, 0]; R^d)$ and $(i, t) \in S\times[-2\tau, 0)$. Applying the generalized Itô formula[5] to $U(x(t)-D(x_t), r(t), t)$, we get

$$\begin{aligned} dU(x(t) - D(x_t), r(t), t) = \Big(& U_t(x(t) - D(x_t), r(t), t) + U_x(x(t) - D(x_t), r(t), t)(f(x(t), r(t), t) + F(x_t, r(t), t)) \\ &+ \frac{1}{2}\text{trace}[(g(x(t), r(t), t) + G(x_t, r(t), t))^T U_{xx}(x(t) - D(x_t), r(t), t)(g(x(t), r(t), t) + G(x_t, r(t), t))] \\ &+ \sum_{j=1}^{N} \gamma_{r(t),j} U(x(t) - D(x_t), r(t), t)\Big)dt + dM(t) \end{aligned}$$

for $t \geq 0$, where $M(t)$ is a continuous local martingale with $M(0) = 0$. Define a mapping $\pi_\tau : R^d \to C([-\tau, 0]; R^d)$ by $\pi_\tau(x)(s) = x$ for $s \in [-\tau, 0]$. Rearranging terms gives

$$dU(x(t) - D(x_t), r(t), t) = \Big(U_x(x(t) - D(x_t), r(t), t)[F_1(x_t, r(t), t) - F_1(\pi_\tau(x(t)), r(t), t)] + \mathcal{L}U(x_t, r(t), t)\Big)dt + dM(t),$$

where the function $\mathcal{L}U : C([-\tau, 0]; R^d) \times S \times R_+ \to R$ is defined by

$$\begin{aligned} \mathcal{L}U(\varphi, i, t) =& U_t(\varphi(0) - D(\varphi), i, t) + U_x(\varphi(0) - D(\varphi), i, t)[f(\varphi(0), i, t) + F_1(\pi_\tau(\varphi(0)), i, t) + F_2(\varphi, i, t)] \\ &+ \frac{1}{2}\text{trace}[(g(\varphi(0), i, t) + G(\varphi, i, t))^T U_{xx}(\varphi(0) - D(\varphi), i, t)(g(\varphi(0), i, t) + G(\varphi, i, t))] \\ &+ \sum_{j=1}^{N} \gamma_{ij} U(\varphi(0) - D(\varphi), j, t). \end{aligned}$$

Lemma 2. With the notation above, $V(\bar{x}_t, \bar{r}_t, t)$ is an Itô process on $t \geq 0$ with its Itô differential

$$dV(\bar{x}_t, \bar{r}_t, t) = LV(\bar{x}_t, \bar{r}_t, t)dt + dM(t),$$

where $M(t)$ is a continuous local martingale with $M(0) = 0$ and

$$\begin{aligned} LV(\bar{x}_t, \bar{r}_t, t) =& U_x(x(t) - D(x_t), r(t), t)[F_1(x_t, r(t), t) - F_1(\pi_\tau(x(t)), r(t), t)] \\ &+ \mathcal{L}U(x_t, r(t), t) + \varrho\tau\Pi(t) - \varrho \int_{t-\tau}^{t} \Pi(v)dv. \end{aligned}$$

3.1 | Delay-dependent asymptotic stability

To study the delay-dependent asymptotic stability of the NSFDE (11), we need to impose a new assumption.

Assumption 6. Let U_1, U_2 are nonnegative coefficient quasi polynomial functions. Assume that there are positive constants α_k $(k = 1, 2, 3, 4)$ and ρ_j $(j = 1, 2, 3)$, as well as function $U \in C^{2,1}(R^d \times S \times R_+; R_+)$, such that

$$\alpha_2 < \alpha_1, \quad \alpha_4 \leq \alpha_3 \tag{14}$$

and

$$\begin{aligned} &\mathcal{L}U(\varphi, i, t) + \rho_1|U_x(\varphi(0) - D(\varphi), i, t)|^2 + \rho_2|f(\varphi(0), i, t) + F(\varphi, i, t)|^2 + \rho_3|g(\varphi(0), i, t) + G(\varphi, i, t)|^2 \\ &\leq -\alpha_1 U_1(\varphi(0)) + \alpha_2 \int_{-\tau}^{0} U_1(\varphi(u))d\eta(u) - \alpha_3 U_2(\varphi(0)) + \alpha_4 \int_{-\tau}^{0} U_2(\varphi(u))d\eta(u), \end{aligned} \tag{15}$$

for all $(\varphi, i, t) \in C([-\tau, 0], R^d) \times S \times R_+$.

Theorem 3. Let Assumptions 1, 2, 5 and 6 hold. Assume that

$$\tau \le \frac{2(1-\kappa_0)^2\rho_1\rho_3}{\beta^2} \wedge \frac{(1-\kappa_0)\sqrt{2\rho_1\rho_2}}{\beta}. \tag{16}$$

Then for any given initial data (2), the solution of the hybrid NSFDE (11) has the properties that

$$\int_0^\infty EU_1(x(t))dt < \infty,$$

$$\sup_{0\le t<\infty} EU(x(t)-D(x_t), r(t), t) < \infty. \tag{17}$$

Proof: Fix the initial data $\xi \in C([-\tau,0];R^d)$ and $r_0 \in S$ arbitrarily. Let $\hat{k} > 0$ be a sufficiently large integer such that $\|\xi\| < \hat{k}$. For each integer $k > \hat{k}$, define the stopping time

$$v_k = \inf\{t \ge 0 : |x(t)| \ge k\}, \tag{18}$$

where throughout this paper we set $\inf \emptyset = \infty$. By Theorem 1, we can see v_k is increasing as $k \to \infty$ and $\lim_{k\to\infty} v_k = \infty$ a.s. By the generalized Itô formula we obtain from Lemma 2 that

$$EV(\bar{x}_{t\wedge v_k}, \bar{r}_{t\wedge v_k}, t\wedge v_k) = V(\bar{x}_0, \bar{r}_0, 0) + E\int_0^{t\wedge v_k} LV(\bar{x}_s, \bar{r}_s, s)ds \tag{19}$$

for any $t \ge 0$ and $k \ge \hat{k}$. Let $\varrho = \frac{\beta^2}{2\rho_1(1-\kappa_0)^2}$. By Assumption 6 and the Hölder inequality, it is easy to see that

$$U_x(x(t)-D(x_t), r(t), t)[F_1(x_t, r(t), t) - F_1(\pi_\tau(x(t)), r(t), t)]$$
$$\le \rho_1|U_x(x(t)-D(x_t), r(t), t)|^2 + \frac{\beta^2}{4\rho_1}\int_{-\tau}^0 |x(t)-x(t+s)|^2 d\eta(s).$$

By condition (16), it is easy to see that $\varrho\tau^2 \le \rho_2$ and $\varrho\tau \le \rho_3$. It then follows from Lemma 2 that

$$LV(\bar{x}_s, \bar{r}_s, s) \le \mathcal{L}U(x_s, r(s), s) + \rho_1|U_x(x(s)-D(x_s), r(s), s)|^2 + \rho_2|f(x(s), r(s), s) + F(x_s, r(s), s)|^2$$
$$+ \rho_3|g(x(s), r(s), s) + G(x_s, r(s), s)|^2 + \frac{\beta^2}{4\rho_1}\int_{-\tau}^0 |x(s)-x(s+v)|^2 d\eta(v) - \frac{\beta^2}{2\rho_1(1-\kappa_0)^2}\int_{s-\tau}^s \Pi(v)dv.$$

By Assumption 6, we then have

$$LV(\bar{x}_s, \bar{r}_s, s) \le -\alpha_1 U_1(x(s)) + \alpha_2\int_{-\tau}^0 U_1(x(s+v))d\eta(v) - \alpha_3 U_2(x(s)) + \alpha_4\int_{-\tau}^0 U_2(x(s+v))d\eta(v)$$
$$+ \frac{\beta^2}{4\rho_1}\int_{-\tau}^0 |x(s)-x(s+v)|^2 d\eta(v) - \frac{\beta^2}{2\rho_1(1-\kappa_0)^2}\int_{s-\tau}^s \Pi(v)dv.$$

Substituting this into (19) implies

$$EV(\bar{x}_{t\wedge v_k}, \bar{r}_{t\wedge v_k}, t\wedge v_k) \le V(\bar{x}_0, \bar{r}_0, 0) + W_1 + W_2 - W_3, \tag{20}$$

where

$$W_1 = E\int_0^{t\wedge v_k}\Big[-\alpha_1 U_1(x(s)) + \alpha_2\int_{-\tau}^0 U_1(x(s+u))d\eta(u) - \alpha_3 U_2(x(s)) + \alpha_4\int_{-\tau}^0 U_2(x(s+u))d\eta(u)\Big]ds,$$

$$W_2 = \frac{\beta^2}{4\rho_1}E\int_0^{t\wedge v_k}\int_{-\tau}^0 |x(s)-x(s+v)|^2 d\eta(v)ds, \quad W_3 = \frac{\beta^2}{2\rho_1(1-\kappa_0)^2}E\int_0^{t\wedge v_k}\int_{s-\tau}^s \Pi(v)dvds.$$

Noting that

$$\int_0^{t\wedge\nu_k}\int_{-\tau}^0 U_1(x(s+u))d\eta(u)ds \le \int_{-\tau}^0 U_1(\xi(s))ds + \int_0^{t\wedge\nu_k} U_1(x(s))ds$$

and

$$\int_0^{t\wedge\nu_k}\int_{-\tau}^0 U_2(x(s+u))d\eta(u)ds \le \int_{-\tau}^0 U_2(\xi(s))ds + \int_0^{t\wedge\nu_k} U_2(x(s))ds.$$

These imply

$$W_1 \le \alpha_2\int_{-\tau}^0 U_1(\xi(s))ds + \alpha_4\int_{-\tau}^0 U_2(\xi(s))ds - \bar\alpha_1 E\int_0^{t\wedge\nu_k} U_1(x(s))ds - \bar\alpha_2 E\int_0^{t\wedge\nu_k} U_2(x(s))ds,$$

where $\bar\alpha_1 = \alpha_1 - \alpha_2 > 0, \bar\alpha_2 = \alpha_3 - \alpha_4 \ge 0$ by condition (14). Substituting this into (20) yields

$$\bar\alpha_1 E\int_0^{t\wedge\nu_k} U_1(x(s))ds \le C_1 + W_2 - W_3, \tag{21}$$

where C_1 is a constant defined by

$$C_1 = V(\bar x_0, \bar r_0, 0) + \alpha_2\int_{-\tau}^0 U_1(\xi(s))ds + \alpha_4\int_{-\tau}^0 U_2(\xi(s))ds.$$

Applying the classical Fatou lemma and let $k\to\infty$ in (21) to obtain

$$\bar\alpha_1 E\int_0^{t} U_1(x(s))ds \le C_1 + \bar W_2 - \bar W_3, \tag{22}$$

where

$$\bar W_2 = \frac{\beta^2}{4\rho_1}E\int_0^t\int_{-\tau}^0 |x(s)-x(s+v)|^2 d\eta(v)ds, \quad \bar W_3 = \frac{\beta^2}{2\rho_1(1-\kappa_0)^2}E\int_0^t\int_{s-\tau}^s \Pi(v)dvds.$$

By the well-known Fubini theorem, we have

$$\bar W_2 = \frac{\beta^2}{4\rho_1}\int_0^t\int_{-\tau}^0 E|x(s)-x(s+v)|^2 d\eta(v)ds.$$

For $t\in[0,\tau]$, we have

$$\bar W_2 \le \frac{\beta^2}{2\rho_1}\int_0^\tau\int_{-\tau}^0 (E|x(s)|^2 + E|x(s+v)|^2)d\eta(v)ds \le \frac{\tau\beta^2}{\rho_1}\Big(\sup_{-\tau\le v\le\tau} E|x(v)|^2\Big) =: C_2.$$

For $t>\tau$, we have

$$\bar W_2 \le C_2 + \frac{\beta^2}{4\rho_1}\int_\tau^t\int_{-\tau}^0 E|x(s)-x(s+v)|^2 d\eta(v)ds.$$

Noting that, for $v\in[-\tau,0]$,

$$\begin{aligned}|x(s)-x(s+v)| &\le |[x(s)-D(x_s)] - [x(s+v)-D(x_{s+v})]| + |D(x_s)-D(x_{s+v})|\\ &\le \kappa_0\int_{-\tau}^0 |x(s+u)-x(s+v+u)|d\eta(u) + |\int_{s+v}^s [f(x(u),r(u),u) + F(x_u,r(u),u)]du\end{aligned}$$

$$+\int_{s+v}^{s}[g(x(u),r(u),u)+G(x_u,r(u),u)]dB(u)|.$$

Hence

$$\begin{aligned}E|x(s)-x(s+v)|^2 &\le (1+\theta)\kappa_0^2\int_{-\tau}^{0}E|x(s+u)-x(s+v+u)|^2d\eta(u)\\&+(1+\frac{1}{\theta})E|\int_{s+v}^{s}[f(x(u),r(u),u)+F(x_u,r(u),u)]du+\int_{s+v}^{s}[g(x(u),r(u),u)+G(x_u,r(u),u)]dB(u)|^2\\&\le(1+\theta)\kappa_0^2\int_{-\tau}^{0}E|x(s+u)-x(s+v+u)|^2d\eta(u)+2(1+\frac{1}{\theta})E\int_{s-\tau}^{s}\Pi(u)du,\end{aligned}$$

where θ is a positive constant. Letting $\theta=\frac{1}{\kappa_0}-1$, we have that

$$\int_{-\tau}^{0}E|x(s)-x(s+v)|^2d\eta(v)\le\kappa_0\int_{-\tau}^{0}\int_{-\tau}^{0}E|x(s+u)-x(s+v+u)|^2d\eta(u)d\eta(v)+\frac{2}{1-\kappa_0}E\int_{s-\tau}^{s}\Pi(u)du.$$

This implies

$$\begin{aligned}\int_{\tau}^{t}\int_{-\tau}^{0}E|x(s)-x(s+v)|^2d\eta(v)ds&\le\kappa_0\int_{\tau}^{t}\int_{-\tau}^{0}\int_{-\tau}^{0}E|x(s+u)-x(s+v+u)|^2d\eta(u)d\eta(v)ds+\frac{2}{1-\kappa_0}E\int_{\tau}^{t}\int_{s-\tau}^{s}\Pi(u)duds\\&\le\kappa_0\int_{0}^{t}\int_{-\tau}^{0}E|x(s)-x(s+v)|^2d\eta(v)ds+\frac{2}{1-\kappa_0}E\int_{\tau}^{t}\int_{s-\tau}^{s}\Pi(u)duds.\end{aligned}$$

Noting that $0<\kappa_0<1$, it follows that

$$\int_{\tau}^{t}\int_{-\tau}^{0}E|x(s)-x(s+v)|^2d\eta(v)ds\le\frac{\kappa_0}{1-\kappa_0}\int_{0}^{\tau}\int_{-\tau}^{0}E|x(s)-x(s+v)|^2d\eta(v)ds+\frac{2}{(1-\kappa_0)^2}E\int_{\tau}^{t}\int_{s-\tau}^{s}\Pi(u)duds.$$

Hence

$$\bar{W}_2\le C_2+\frac{\kappa_0\beta^2\tau}{\rho_1(1-\kappa_0)}\sup_{-\tau\le s\le\tau}E|x(s)|^2+\bar{W}_3=C_3+\bar{W}_3,$$

where $C_3=C_2+\frac{\kappa_0\beta^2\tau}{\rho_1(1-\kappa_0)}\sup_{-\tau\le s\le\tau}E|x(s)|^2$. Substituting this into (22) yields

$$\bar{\alpha}_1E\int_{0}^{t}U_1(x(s))ds\le C_1+C_3.$$

Letting $t\to\infty$ gives $E\int_0^\infty U_1(x(s))ds\le\frac{1}{\bar{\alpha}_1}(C_1+C_3)<\infty$. Similarly, we can see from (20) that

$$EU(x(t\wedge\nu_k)-D(x_{t\wedge\nu_k}),r(t\wedge\nu_k),t\wedge\nu_k)\le C_1+W_1+W_2-W_3.$$

Letting $k\to\infty$ we get

$$EU(x(t)-D(x_t),r(t),t)\le C_1+C_3<\infty,$$

which shows

$$\sup_{0\le t<\infty}EU(x(t)-D(x_t),r(t),t)<\infty.$$

Thus the proof is complete. □

Next, we will state a corollary which gives a criterion on H_∞-stability.

Corollary 1. Let the conditions of Theorem 3 hold. If there moreover exists a pair of positive constants c and $p < q$ such that

$$c|x|^p \le U_1(x), \quad \forall (x,t) \in R^n \times R_+,$$

then for any given initial data (2), the solution of the NSFDE (11) satisfies

$$\int_0^\infty E|x(t)|^p dt < \infty. \tag{23}$$

That is, the NSFDE (11) is H_∞-stable in L^p.

This corollary follows from Theorem 3 obviously. However, it does not follow from (23) that $\lim_{t\to\infty} E|x(t)|^p = 0$. Next, we will discuss asymptotically stable of the NSFDE (11).

Theorem 4. Let the conditions of Corollary 1 hold. If, moreover,

$$p \ge 2 \quad \text{and} \quad (p + q_1 - 1) \vee (p + 2q_2 - 2) \le q,$$

then the solution of the NSFDE (11) satisfies

$$\lim_{t\to\infty} E|x(t)|^{\bar q} = 0, \quad \forall \bar q \in [p, q) \tag{24}$$

for any initial data (2). That is, the NSFDE (11) is asymptotically stable in $L^{\bar q}$.

Proof: Fix the initial data (2) arbitrarily. For any $0 \le t_1 < t_2 < \infty$, by the Itô formula, we get

$$\begin{aligned}
&E|x(t_2) - D(x_{t_2})|^p - E|x(t_1) - D(x_{t_1})|^p \\
&= E\int_{t_1}^{t_2} \Big(p|x(t) - D(x_t)|^{p-2}(x(t) - D(x_t))^T [f(x(t), r(t), t) + F(x_t, r(t), t)] \\
&+ \frac{p}{2}|x(t) - D(x_t)|^{p-2}|g(x(t), r(t), t) + G(x_t, r(t), t)|^2 \\
&+ \frac{p(p-2)}{2}|x(t) - D(x_t)|^{p-4}|(x(t) - D(x_t))^T[g(x(t), r(t), t) + G(x_t, r(t), t)]|^2 \Big) dt.
\end{aligned}$$

This implies

$$\begin{aligned}
&\Big| E|x(t_2) - D(x_{t_2})|^p - E|x(t_1) - D(x_{t_1})|^p \Big| \\
&\le E\int_{t_1}^{t_2} \Big(p|x(t) - D(x_t)|^{p-1}|f(x(t), r(t), t) + F(x_t, r(t), t)| \\
&+ \frac{p(p-1)}{2}|x(t) - D(x_t)|^{p-2}|g(x(t), r(t), t) + G(x_t, r(t), t)|^2 \Big) dt \\
&\le E\int_{t_1}^{t_2} \Big(pK|x(t) - D(x_t)|^{p-1}\big[1 + |x(t)|^{q_1} + \int_{-\tau}^{0} |x(t+u)|^{q_1} d\eta(u)\big] \\
&+ \frac{3p(p-1)K^2}{2}|x(t) - D(x_t)|^{p-2}\big[1 + |x(t)|^{2q_2} + \int_{-\tau}^{0} |x(t+u)|^{2q_2} d\eta(u)\big] \Big) dt.
\end{aligned}$$

By inequalities

$$|x(t) - D(x_t)|^p \le 2^{p-1}(|x(t)|^p + \kappa_0^p \int_{-\tau}^{0} |x(t+u)|^p d\eta(u)),$$

$$|x(t)|^{p-1} \int_{-\tau}^{0} |x(t+u)|^{q_1} d\eta(u) \le |x(t)|^{p+q_1-1} + \int_{-\tau}^{0} |x(t+u)|^{p+q_1-1} d\eta(u),$$

$$|x(t)|^{p-1} \le 1 + |x(t)|^q.$$

We can obtain

$$\left|E|x(t_2) - D(x_{t_2})|^p - E|x(t_1) - D(x_{t_1})|^p\right| \le C_4(t_2 - t_1),$$

where $C_4 = 2^{p+1}[pK + 3p(p-1)K^2/2](1 + \sup_{-\tau\le t<\infty} E|x(t)|^q) < \infty$. Thus we have $E|x(t) - D(x_t)|^p$ is uniformly continuous in t on R_+. By (23) we can obtain

$$\int_0^\infty Ex(t) - D(x_t)|^p dt \le \int_0^\infty 2^{p-1} E\Big(|x(t)|^p + \kappa_0^p \int_{-\tau}^0 |x(t+u)|^p d\eta(u)\Big) dt$$
$$\le 2^{p-1}(1+\kappa_0^p)\int_0^\infty E|x(t)|^p dt + 2^{p-1}\kappa_0^p\|\xi\| < \infty,$$

so we can obtain $\lim_{t\to\infty} E|x(t) - D(x_t)|^p = 0$. By Assumption 2 and inequality

$$(a+b)^p \le (1+\iota)^{p-1}(a^p + \iota^{1-p}b^p), \quad \forall a, b \ge 0, p \ge 1, \iota > 0,$$

we have

$$E|x(t)|^p \le E[(1+\iota)^{p-1}(|x(t) - D(x_t)|^p + \iota^{1-p}\kappa_0^p \int_{-\tau}^0 |x(t+u)|^p d\eta(u))].$$

Setting $\iota = \kappa_0/(1-\kappa_0)$, then

$$E|x(t)|^p \le (\frac{1}{1-\kappa_0})^{p-1} E|x(t) - D(x_t)|^p + \kappa_0 E\int_{-\tau}^0 |x(t+u)|^p d\eta(u).$$

Moreover, letting $t \to \infty$, it gives that

$$\lim_{t\to\infty}\sup E|x(t)|^p \le \kappa_0 \lim_{t\to\infty}\sup E|x(t)|^p \quad a.s.$$

This, together with the Theorem 2, yields

$$\lim_{t\to\infty} E|x(t)|^p = 0. \tag{25}$$

Let us now fix any $\bar{q} \in (p, q)$, for a constant $\bar{\theta} \in (0, 1)$, the Hölder inequality shows

$$E|x(t)|^{\bar{q}} \le (E|x(t)|^p)^{\bar{\theta}}(E|x(t)|^{(\bar{q}-p\bar{\theta})/(1-\bar{\theta})})^{1-\bar{\theta}}.$$

Letting $\bar{\theta} = \frac{q-\bar{q}}{q-p}$, then we can obtain

$$E|x(t)|^{\bar{q}} \le (E|x(t)|^p)^{(q-\bar{q})/(q-p)}(E|x(t)|^q)^{(\bar{q}-p)/(q-p)} \le C_5^{(\bar{q}-p)/(q-p)}(E|x(t)|^p)^{(q-\bar{q})/(q-p)}, \tag{26}$$

where $C_5 := \sup_{-\tau\le t<\infty} E|x(t)|^q < \infty$. This, along with (25), implies the assertion (24). Thus the proof is complete. □

3.2 | Delay-dependent exponential stability

Asymptotic stability discussed above shows that the solution of the NSFDE (11) will tend to zero in $L^{\bar{q}}$ asymptotically but does not show the rate of decay. In this subsection, we will take a further step to show the solution of the NSFDE (11) will tend to zero exponentially fast if the delay is small enough. In order to get exponential stability criteria, we need some stronger conditions than those in Assumption 7 and Theorem 3.

Assumption 7. Assume that U_1, U_2 are nonnegative coefficient quasi polynomial functions. Assume more that there are positive constants α_k $(k = 1, 2, 3, 4)$ and ρ_j $(j = 1, 2, 3)$, as well as the function $U \in C^{2,1}(R^d \times S \times R_+; R_+)$, such that

$$\alpha_2 < \alpha_1, \quad \alpha_4 < \alpha_3, \quad U(x, i, t) \le U_1(x)$$

and

$$\mathcal{L}U(\varphi, i, t) + \rho_1|U_x(\varphi(0) - D(\varphi), i, t))|^2$$
$$+ \rho_2|f(\varphi(0), i, t) + F(\varphi, i, t)|^2 + \rho_3|g(\varphi(0), i, t) + G(\varphi, i, t)|^2$$

$$\leq -\alpha_1 U_1(\varphi(0)) + \alpha_2 \int_{-\tau}^{0} U_1(\varphi(u))d\eta(u) - \alpha_3 U_2(\varphi(0)) + \alpha_4 \int_{-\tau}^{0} U_2(\varphi(u))d\eta(u),$$

for all $\varphi(0) \in R^d$, $(\varphi, i, t) \in C([-\tau, 0]; R^d) \times S \times R_+$.

Theorem 5. Let Assumptions 1, 2, 5 and 7 hold. Assume also that

$$\tau \leq \frac{(1-\kappa_0)^2\rho_1\rho_3}{\beta^2} \wedge \frac{(1-\kappa_0)\sqrt{\rho_1\rho_2}}{\beta}. \tag{27}$$

Then for any given initial data (2), the solution of the hybrid NSFDE (11) has the property that

$$\limsup_{t\to\infty} \frac{\log EU(x(t)-D(x_t), r(t), t)}{t} < 0. \tag{28}$$

Proof: We will use the same Lyapunov functional $V(\bar{x}_t, \bar{r}_t, t)$ as defined by (13) and same stopping times as defined by (18) with $\varrho = \frac{\beta^2}{\rho_1(1-\kappa_0)^2}$. By the Itô formula, we can show that

$$E(e^{\lambda(t\wedge\nu_k)}V(\bar{x}_{(t\wedge\nu_k)}, \bar{r}_{(t\wedge\nu_k)}, t\wedge\nu_k)) = V(\bar{x}_0, \bar{r}_0, 0) + E\int_0^{t\wedge\nu_k} e^{\lambda s}(\lambda V(\bar{x}_s, \bar{r}_s, s) + LV(\bar{x}_s, \bar{r}_s, s))ds$$

for any $t \geq 0$, where λ is a sufficiently small positive number to be determined later. By Assumption 7, we have

$$E(e^{\lambda(t\wedge\nu_k)}V(\bar{x}_{(t\wedge\nu_k)}, \bar{r}_{(t\wedge\nu_k)}, t\wedge\nu_k)) \leq V(\bar{x}_0, \bar{r}_0, 0) + E\int_0^{t\wedge\nu_k} \lambda e^{\lambda s}U_1(x(s) - D(x_s))ds$$
$$+ E\int_0^{t\wedge\nu_k} e^{\lambda s}LV(\bar{x}_s, \bar{r}_s, s)ds + E\int_0^{t\wedge\nu_k} \lambda e^{\lambda s}\int_{-\tau}^{0}\int_{s+u}^{s} \Pi(v)dvduds. \tag{29}$$

By Lemma 1, we have

$$\int_0^{t\wedge\nu_k} e^{\lambda s}U_1(x(s) - D(x_s))ds \leq \kappa_0 e^{\lambda\tau}\int_{-\tau}^{0} U_1(\xi(s))ds + \bar{C}_\tau \int_0^{t\wedge\nu_k} e^{\lambda s}U_1(x(s))ds, \tag{30}$$

where $\bar{C}_\tau$ depends on the highest power of U_1. It is easy to see

$$E\int_0^{t\wedge\nu_k} e^{\lambda s}\int_{-\tau}^{0}\int_{s+u}^{s} \Pi(v)dvd\eta(u)ds \leq E\tau\int_0^{t\wedge\nu_k} e^{\lambda s}\int_{s-\tau}^{s} \Pi(v)dvds. \tag{31}$$

As we did in the proof of Theorem 3, we can show that

$$LV(\bar{x}_s, \bar{r}_s, s) \leq -\alpha_1 U_1(x(s)) + \alpha_2\int_{-\tau}^{0} U_1(x(s+v))d\eta(v) - \alpha_3 U_2(x(s)) + \alpha_4\int_{-\tau}^{0} U_2(x(s+v))d\eta(v)$$
$$+ \frac{\beta^2}{4\rho_1}\int_{-\tau}^{0} |x(s) - x(s+v)|^2 d\eta(v) - \frac{\beta^2}{\rho_1(1-\kappa_0)^2}\int_{s-\tau}^{s} \Pi(v)dv. \tag{32}$$

Substituting (30)-(32) into (29), as we did in the proof of Theorem 3, we can show that

$$E(e^{\lambda(t\wedge\nu_k)}V(\bar{x}_{t\wedge\nu_k}, \bar{r}_{t\wedge\nu_k}, t\wedge\nu_k)) \leq C_6 + (\lambda\bar{C}_\tau - a_1 - a_2 e^{\lambda\tau})E\int_0^{t\wedge\nu_k} e^{\lambda s}U_1(x(s))ds + (a_4 e^{\lambda\tau} - a_3)E\int_0^{t\wedge\nu_k} e^{\lambda s}U_2(x(s))ds$$
$$+ \frac{\beta^2}{4\rho_1}E\int_0^{t\wedge\nu_k}\int_{-\tau}^{0} |x(s) - x(s+v)|^2 d\eta(v)ds + (\frac{\lambda\tau\beta^2}{\rho_1(1-\kappa_0)^2} - \frac{\beta^2}{\rho_1(1-\kappa_0)^2})E\int_0^{t\wedge\nu_k} e^{\lambda s}\int_{s-\tau}^{s} \Pi(v)dvds,$$

where C_6 is a positive constant. We can now choose a sufficiently small λ such that

$$\lambda \bar{C}_\tau - a_2 e^{\lambda\tau} \leq a_1,\ a_4 e^{\lambda\tau} \leq a_3 \text{ and } \lambda\tau \leq \frac{1}{2}.$$

Consequently, by condition (27), we have

$$E(e^{\lambda(t\wedge\nu_k)} V(\bar{x}_{t\wedge\nu_k}, \bar{r}_{t\wedge\nu_k}, t\wedge\nu_k)) \leq C_6 + \frac{\beta^2}{4\rho_1} E\int_0^{t\wedge\nu_k} e^{\lambda s}\int_{-\tau}^0 |x(s)-x(s+v)|^2 d\eta(v)ds - \frac{\beta^2}{2\rho_1(1-\kappa_0)^2} E\int_0^{t\wedge\nu_k} e^{\lambda s}\int_{s-\tau}^s \Pi(v)dvds.$$

Now, letting $k\to\infty$ yields

$$e^{\lambda t} EV(\bar{x}_t, \bar{r}_t, t) \leq C_6 + \frac{\beta^2}{4\rho_1} E\int_0^t e^{\lambda s}\int_{-\tau}^0 |x(s)-x(s+v)|^2 d\eta(v)ds - \frac{\beta^2}{2\rho_1(1-\kappa_0)^2} E\int_0^t e^{\lambda s}\int_{s-\tau}^s \Pi(v)dvds.$$

As we did in the proof of Theorem 3, we can show that

$$e^{\lambda t} EV(\bar{x}_t, \bar{r}_t, t) \leq C_6.$$

This along with definition of $V(\bar{x}_t, \bar{r}_t, t)$, implies that

$$e^{\lambda t} EU(x(t) - D(x_t), r(t), t) \leq C_6, \tag{33}$$

which implies the required assertion (28).

Corollary 2. Let the conditions of Theorem 5 hold and $\kappa_0 e^\tau < 1$. If there moreover exists a pair of positive constants c and $p \geq 2$ such that

$$c|x|^p \leq U(x,i,t), \quad \forall (x,i,t) \in R^d \times S \times R_+, \tag{34}$$

then for any given initial data (2), the solution of the NSFDE (11) satisfies

$$\lim_{t\to\infty}\sup \frac{1}{t}\log(E|x(t)|^{\bar{q}}) < 0, \quad \forall \bar{q} \in [p,q). \tag{35}$$

That is, the NSFDE (11) is almost surely exponentially stable in $L^{\bar{q}}$.

Proof: For $T > \tau$, by (10), we have

$$\begin{aligned}\sup_{0\leq t\leq T} e^{\lambda t} E|x(t)|^p &\leq (1-\kappa_0)^{1-p} \sup_{0\leq t\leq T} e^{\lambda t} E|x(t) - D(x_t)|^p + \sup_{0\leq t\leq T} \kappa_0 e^{\lambda t} E\int_{-\tau}^0 |x(t+u)|^p d\eta(u) \\ &\leq (1-\kappa_0)^{1-p} \sup_{0\leq t\leq T} e^{\lambda t} E|x(t) - D(x_t)|^p + \kappa_0 (\sup_{0\leq t\leq T} e^{\lambda t} E|x(t)|^p + \sup_{-\tau\leq t\leq 0} E|\xi(t)|^p).\end{aligned}$$

Which implies that

$$\sup_{0\leq t\leq T} e^{\lambda t} E|x(t)|^p \leq \frac{(1-\kappa_0)^{1-p}}{1-\kappa_0 e^\tau} \sup_{0\leq t\leq T} e^{\lambda t} E|x(t) - D(x_t)|^p + \frac{\kappa_0 e^\tau}{1-\kappa_0 e^\tau} \sup_{-\tau\leq t\leq 0} E|\xi(t)|^p.$$

Letting $T\to\infty$, it then follows from (33) and (34) that

$$\sup_{0\leq t<\infty} e^{\lambda t} E|x(t)|^p \leq C_7,$$

where C_7 is a positive constant. It implies that

$$E|x(t)|^p \leq C_7 e^{-\lambda t}$$

This, along with (26), can show that

$$E|x(t)|^{\bar{q}} \leq C_5^{(\bar{q}-p)/(q-p)} C_7{}^{(q-\bar{p})/(q-p)} e^{-\bar{\lambda} t},$$

where $\bar{\lambda} = \lambda(q-\bar{q})/(q-p)$, which implies the assertion (35). □

4 | AN EXAMPLE

In this section, we will give an example to illustrate our theory. Although our example is scalar highly nonlinear hybrid neutral stochastic functional differential equations whose probability measure η is uniform distribution on $[-\tau, 0]$, it will supports our theory fully.

Example 6. Consider a scalar hybrid NSFDE

$$d[x(t) - D(x_t)] = [f(x(t), r(t), t) + F(x_t, r(t), t)]dt + [g(x(t), r(t), t) + G(x_t, r(t), t)]dB(t), \tag{36}$$

where $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with its generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix},$$

and the coefficients are defined by

$$\begin{aligned}
&f(x,1,t) = -6x^3, \quad f(x,2,t) = -4x^3, \quad F_1(x_t,1,t) = -\frac{1}{\tau}\int_{-\tau}^{0} x(t+u)du,\\
&F_1(x_t,2,t) = -\frac{1}{\tau}\int_{-\tau}^{0} x(t+u)du, \quad F_2(x_t,1,t) = (\frac{1}{\tau}\int_{-\tau}^{0} x(t+u)du)^3,\\
&F_2(x_t,2,t) = 0.5(\frac{1}{\tau}\int_{-\tau}^{0} x(t+u)du)^3, \quad g(x,1,t) + G(x_t,1,t) = \frac{1}{2\tau}\int_{-\tau}^{0} x(t+u)du,\\
&g(x,2,t) + G(x_t,2,t) = \frac{1}{2\tau}\int_{-\tau}^{0} x(t+u)du, \quad D(x_t) = \frac{0.1}{\tau}\int_{-\tau}^{0} x(t+u)du.
\end{aligned} \tag{37}$$

For the sake of simplicity, we define $\varpi(x_t) = \frac{1}{\tau}\int_{-\tau}^{0} x(t+u)du$, $\bar{\varpi}_k(x_t) = \frac{1}{\tau}\int_{-\tau}^{0} |x(t+u)|^k du$, $k \geq 1$.

Before applying our theory, we consider two case: $\tau = 5$ and $\tau = 0.01$ for all $t \geq 0$. In the case of $\tau = 0.01$, let the initial data $x(u) = 2 + \cos(u)$ for $u \in [-0.01, 0]$, $r(0) = 2$. The sample paths of the Markov chain and the solution of the NSFDE are shown in Figure 1, which indicates that the NSFDE is asymptotically stable. In the case of $\tau = 5$, let the initial data $x(u) = 2 + \cos(u)$ for $u \in [-5, 0]$, $r(0) = 2$. The sample paths of the Markov chain and the solution of the NSFDE (36) are plotted in Figure 2, which indicates that the NSFDE is unstable.

We can see coefficients defined by (37) satisfy Assumption 4 with $q_1 = 3$ and $q_2 = 2$. Define $\bar{U}(x, i, t) = |x|^6$ for $(x, i, t) \in R \times S \times R_+$. It is easy to show that

$$\mathbb{L}\bar{U}(x_t, i, t) = 6(x - 0.1\varpi(x_t))^5[f(x, i, t) + F(x_t, i, t)] + 15(x - 0.1\varpi(x_t))^4|g(x, i, t) + G(x_t, i, t)|^2.$$

Applying the inequalities $(a + b)^p \leq (1 + \iota)^{p-1}(a^p + \iota^{1-p}b^p)$, here setting $\iota = \kappa_0/(1 - \kappa_0) = 1/9$, and $a^\rho b^{1-\rho} \leq \rho a + (1 - \rho)b$, we can obtain

$$\mathbb{L}\bar{U}(x_t, 1, t) \leq -13.651x^8 + 7.958\bar{\varpi}_8(x_t) + 7.626x^6 + 2.13\bar{\varpi}_6(x_t)$$

and

$$\mathbb{L}\bar{U}(x_t, 2, t) \leq -8.905x^8 + 5.192\bar{\varpi}_8(x_t) + 7.626x^6 + 2.13\bar{\varpi}_6(x_t).$$

Thus, we have

$$\begin{aligned}
\mathbb{L}\bar{U}(x_t, i, t) &\leq -8.905x^8 + 7.958\bar{\varpi}_8(x_t) + 7.626x^6 + 2.13\bar{\varpi}_6(x_t)\\
&\leq c_1 - 8.5(1 + x^8) + 8(1 + \bar{\varpi}_8(x_t)),
\end{aligned}$$

where $c_1 = \sup_{x,y\in R}[1 + 7.626x^6 + 2.13\bar{\varpi}_6(x_t) - 0.405x^8 - 0.224\bar{\varpi}_8(x_t)] < \infty$. Therefore, Assumption 3 is satisfied with $Q(x) = 1 + x^8$, $c_2 = 8.5$, $c_3 = 8$. From Theorem 2, solution of the NSFDE (36) has the that

$$\sup_{-\tau\leq t<\infty} E|x(t)|^6 < \infty.$$

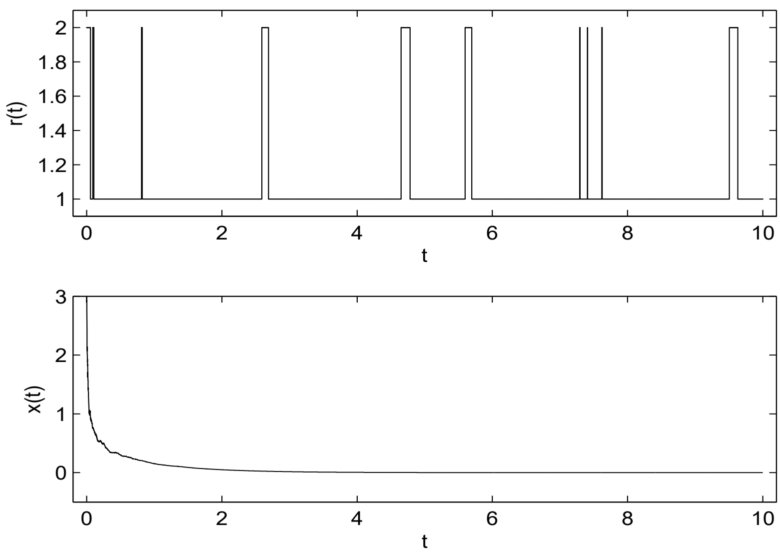

Figure 1: The computer simulation of the sample paths of the Markov chain and the NSFDE (36) with $\tau = 0.01$ using the truncation method [40].

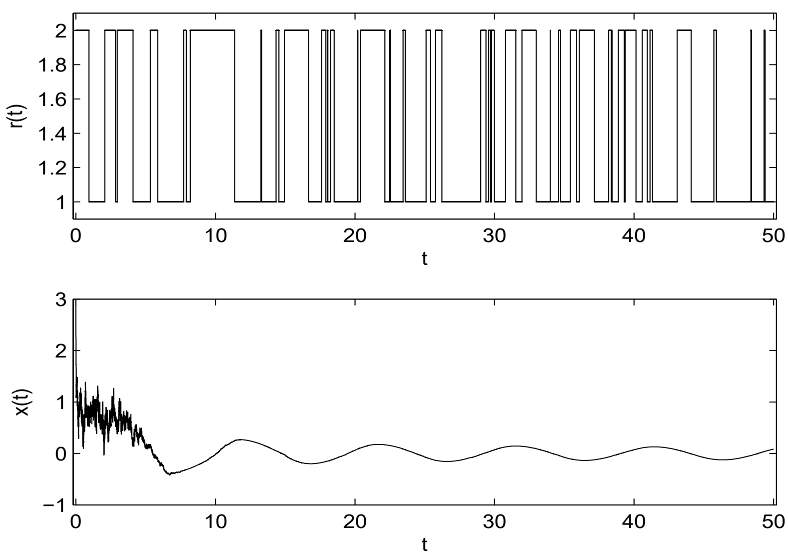

Figure 2: The computer simulation of the sample paths of the Markov chain and the NSFDE (36) with $\tau = 5$ using the truncation method.

To verify Assumption 6, we define

$$U(x,i,t) = \begin{cases} 2x^2 + x^4, & \text{if } i = 1, \\ 2x^2 + 2x^4, & \text{if } i = 2 \end{cases}$$

for $(x,i,t) \in R \times S \times R_+$. Then

$$\begin{aligned} \mathcal{L}U(x_t,1,t) = & [4(x - 0.1\varpi(x_t)) + 4(x - 0.1\varpi(x_t))^3](-x + \varpi^3(x_t) - 6x^3) \\ & +0.5(1 + 3(x - 0.1\varpi(x_t))^2)\varpi^4(x_t) + (x - 0.1\varpi(x_t))^4 \end{aligned}$$

and

$$\mathcal{L}U(x_t,2,t) = [4(x-0.1\varpi(x_t)) + 8(x-0.1\varpi(x_t))^3](-x+\frac{1}{2}\varpi^3(x_t)-4x^3) \\ +0.5(1+6(x-0.1\varpi(x_t))^2)\varpi^4(x_t) - 2(x-0.1\varpi(x_t))^4.$$

We can show that

$$\mathcal{L}U(x_t,i,t) \le \begin{cases} -3.8x^2+0.2\bar{\varpi}_2(x_t)-22.927x^4+4.103\bar{\varpi}_4(x_t)-14.968x^6+6.832\bar{\varpi}_6(x_t), & \text{if } i=1, \\ -3.8x^2+0.2\bar{\varpi}_2(x_t)-20.498x^4+2.806\bar{\varpi}_4(x_t)-20.414x^6+8.36\bar{\varpi}_6(x_t), & \text{if } i=2. \end{cases} \tag{38}$$

Moreover,

$$\begin{aligned} |U_x(x-D(x_t),i,t)|^2 &= \begin{cases} 16(x-0.1\varpi(x_t))^2+32(x-0.1\varpi(x_t))^4+16(x-0.1\varpi(x_t)^6, & \text{if } i=1, \\ 16(x-0.1\varpi(x_t))^2+64(x-0.1\varpi(x_t))^4+64(x-0.1\varpi(x_t))^6, & \text{if } i=2. \end{cases} \\ &\le \begin{cases} 17.8x^2+1.6\bar{\varpi}_2(x_t)+43.9x^4+3.2\bar{\varpi}_4(x_t)+27.1x^6+1.6\bar{\varpi}_6(x_t), & \text{if } i=1, \\ 17.8x^2+1.6\bar{\varpi}_2(x_t)+87.8x^4+6.4\bar{\varpi}_4(x_t)+81.3x^6+6.4\bar{\varpi}_6(x_t), & \text{if } i=2. \end{cases} \end{aligned} \tag{39}$$

$$\begin{aligned} &|f(x,i,t)+F(x_t,i,t)|^2 \\ &= \begin{cases} |\varpi(x_t)-\varpi^3(x_t)+6x^3|^2 \le \bar{\varpi}_2(x_t)+9x^4+\bar{\varpi}_4(x_t)+42x^6+7\bar{\varpi}_6(x_t), & \text{if } i=1, \\ |\varpi(x_t)-\frac{1}{2}\varpi^3(x_t)+4x^3|^2 \le \bar{\varpi}_2(x_t)+6x^4+\bar{\varpi}_4(x_t)+18x^6+2.25\bar{\varpi}_6(x_t), & \text{if } i=2. \end{cases} \end{aligned} \tag{40}$$

$$|g(x,1,t)+G(x_t,1,t)|^2 = |g(x,2,t)+G(x_t,2,t)|^2 \le 0.25\bar{\varpi}_4(x_t). \tag{41}$$

Setting $\rho_1 = 0.05$, $\rho_2 = 0.1$, $\rho_3 = 4$, using (38)-(41), we obtain that

$$\begin{aligned} &\mathcal{L}U(x_t,i,t)+\rho_1|U_x(x-D(x_t,i,t))|^2+\rho_2|f(x,i,t)+F(x_t,i,t)|^2+\rho_3|g(x,i,t)+G(x_t,i,t)|^2 \\ &\le \begin{cases} -2.91x^2+0.38\bar{\varpi}_2(x_t)-19.827x^4+5.363\bar{\varpi}_4(x_t)-9.108x^6+7.612\bar{\varpi}_6(x_t), & \text{if } i=1, \\ -2.91x^2+0.38\bar{\varpi}_2(x_t)-15.508x^4+4.226\bar{\varpi}_4(x_t)-14.549x^6+8.9\bar{\varpi}_6(x_t), & \text{if } i=2. \end{cases} \end{aligned}$$

This implies

$$\begin{aligned} &\mathcal{L}U(x_t,i,t)+\rho_1|U_x(x-D(x_t),i,t)|^2+\rho_2|f(x,i,t)+F(x_t,i,t)|^2+\rho_3|g(x,i,t)+G(x_t,i,t)|^2 \\ &\le -2.91x^2+0.38\bar{\varpi}_2(x_t)-15.508x^4+5.363\bar{\varpi}_4(x_t)-9.108x^6+8.9\bar{\varpi}_6(x_t) \\ &\le -9.1(0.2x^2+x^6)+8.9(0.2\bar{\varpi}_2(x_t)+\bar{\varpi}_6(x_t))-15.5x^4+5.4\bar{\varpi}_4(x_t). \end{aligned}$$

Letting $U_1(x) = 0.2x^2+x^6$, $U_2(x) = x^4$, $\alpha_1 = 9.1$, $\alpha_2 = 8.9$, $\alpha_3 = 15.5$, $\alpha_4 = 5.4$, we get condition (15). Noting that $\kappa_0 = 0.1$, $\beta = 1$, then condition (16) becomes $\tau \le 0.09$. By Theorem 3, we can therefore conclude that the solution of the NSFDE (36) has the properties that

$$\int_0^\infty (x^2(t)+x^6(t))dt < \infty \quad a.s. \quad \text{and} \quad \int_0^\infty E(x^2(t)+x^6(t))dt < \infty.$$

Moreover, as $|x(t)|^p \le x^2(t)+x^6(t)$ for any $p \in [2,6]$, we have

$$\int_0^\infty E|x(t)|^p dt < \infty.$$

Recalling $q_1 = 3,\ q_2 = 2$ and $q = 6$, we see that for $p = 4$, all the conditions of Theorem 4 are satisfied and hence we have

$$\lim_{t\to\infty} E|x(t)|^4 = 0.$$

It is also easy to see that if we set $c = 1$, we have got $q_1 = 3,\ q_2 = 2$, $p = 4$, then all the conditions of Corollary 1 are satisfied too. We perform a computer simulation with the time delay $\tau = 0.09$ for all $t \ge 0$ and the initial date $x(u) = 2+\cos(u)$ for $u \in [-0.09, 0]$ and $r(0) = 2$. The sample paths of the Markovian chain and the solution of the NSFDE (36) are plotted in Figure 3. The simulation supports our theoretical results.

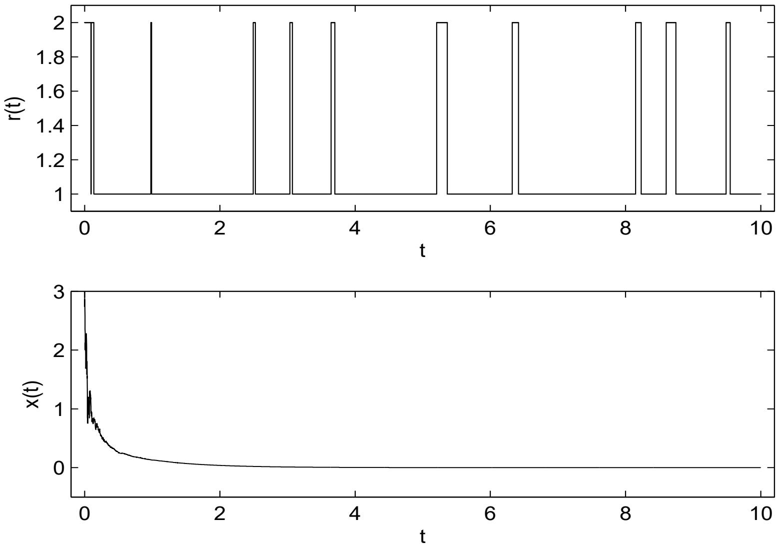

Figure 3: The computer simulation of the sample paths of the Markov chain and the NSFDE (36) with $\tau = 0.09$ using the truncation method.

5 | CONCLUSION

In this study, some criteria for delay dependent stability of highly nonlinear neutral stochastic functional differential systems have been investigated. We point out that the existing results on the delay-dependent stability of hybrid NSDEs require the coefficients of the underlying NSDEs satisfy the linear growth condition. On the other hand, many hybrid NSDE models in the real world do not fulfill this linear growth condition. There is hence a need to develop a new theory on the delay-dependent stability for the highly nonlinear NSDE models. In this paper, we consider delay-dependent stability of a class of highly nonlinear NSFDEs, the H_∞ stability in L^p, asymptotic stability in $L^{\bar{q}}$ and exponential stability are discussed in this paper. The key technique used in this paper is the method of Lyapunov functionals. A numerical example is given to show the effectiveness of the proposed theory.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability Statement

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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