Stabilisation in Distribution of Hybrid Systems by Intermittent Noise

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***Abstract*— For many stochastic hybrid systems in the real world, it is inappropriate to study if their solutions will converge to an equilibrium state (say, 0 by default) but more appropriate to discuss if the probability distributions of the solutions will converge to a stationary distribution. The former is known as the asymptotic stability of the equilibrium state while the latter the stability in distribution. This paper aims to determine whether or not a stochastic state feedback control can make a given nonlinear hybrid differential equation, which is not stable in distribution, to become stable in distribution. We will refer to this problem as stabilisation in distribution by noise or stochastic stabilisation in distribution. Although the stabilisation by noise in the sense of almost surely exponential stability of the equilibrium state has been well studied, there is little known on the stabilisation in distribution by noise. This paper initiates the study in this direction.**

Key Words: Nonlinear hybrid differential equation, intermittent noise, Brownian motion, Markov chain, stationary distribution, stabilisation.

1. Introduction

System described by stochastic differential equations (SDEs) have been playing a crucial role in the modelling of many practical systems. These practical systems include electric power systems, the control system of a solar thermal central receiver, manufacturing systems, financial systems etc. (see, e.g., [6], [20], [24]). Since systems in the real world often need to run for a long period of time, asymptotic stability and stabilisation are two of most popular topics in the study of SDE systems. There are various concepts of asymptotic stability, for example, in moment, in probability, with probability 1 (i.e., almost surely), or in distribution. These are precisely corresponding to the concepts of convergence of stochastic processes in moment, in probability, almost surely, or in distribution (see, e.g., [8], [10], [14]).

There is an extensive literature on the asymptotic stability and stabilisation (in moment, in probability or almost surely) of an equilibrium state of SDE systems, say 0 by default (see, e.g., [3], [4], [9], [15]). Most of papers on the stabilisation use the feedback controls in the drift term, called deterministic feedback controls for convenience (see, e.g., [9], [19], [23]). Nevertheless, there are some papers where feedback controls driven by Brownian motions, called stochastic feedback controls for convenience, are used (i.e., controls are in the diffusion term). The pioneering work was due to Hasminskii [10, p.229], who stabilized a system by using two white noise sources. The theory on stabilisation by Brownian motion has since then been developed by several authors (see, e.g., [2], [12], [13], [17]). It is noted that all of the existing papers in this area aim to make the stochastically controlled SDEs to be almost surely asymptotically stable (i.e., the solutions of stochastically controlled SDEs will tend to the equilibrium state, namely 0 by default, with probability 1). The reader may wonder why stochastic feedback controls should be used given deterministic ones have been used more widely. Although this was explained in the papers mentioned above, several good points were made in [16] including the volatility-stabilised markets (see, e.g., [5]). We here add one more good point that in ecosystem, the SDE models have revealed another important phenomenon that the environmental noise might make a population become extinct (see, e.g., [14, Chapter 11]). The stochastic stabilisation here is done by nature. It is even more interesting to observe in several countries that the infected number of Covid-19 is currently suppressed by large random interaction between people (no more lockdown in the UK for example).

In contrast to the stability and stabilisation of an equilibrium state, there are much fewer papers on the stability and stabilisation in distribution of SDE systems. The stability in distribution is to study if the probability distributions of the solutions of an SDE system will converge to a probability distribution, known as stationary distribution (see, e.g., [22], [25], [26]). The reason why there are so far much fewer papers on the

This work is entirely theoretical and the results can be reproduced using the methods described in this paper.

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stability and stabilisation in distribution is because the mathematics involved is much more complicated than that used for the study of the asymptotic stability of an equilibrium state but certainly not because the topics are less important. As a matter of fact, there is an urgent need to study them. For example, Covid-19 has been with us for more than 2 years. There are essentially 2 control strategies: one is to suppress infected to 0 but the other is to live with Covid-19. The former is to stabilise the infected to 0 with probability 1 while the latter is to stabilise the distribution of the infected to a stationary distribution (namely, stabilisation in distribution). It should also be pointed out that it is inappropriate to study the asymptotic stability of an equilibrium state for many SDE systems in the real world but more appropriate to study the stability in distribution. For example, for many epidemic/ecological systems under random environment, the stochastic permanence is a more desired control objective than the extinction (see, e.g., [7], [21]).

The aim of this paper is to explore if a stochastic feedback control could be used to make a given unstable system to become stable in distribution. Assume that the given unstable system is described by a hybrid differential equation driven by a continuous-time Markov chain and has the form

$$\dot{x}(t) = f(x(t), r(t)), \tag{1.1}$$

where $x(t)$ is in general referred to as the state and $r(t)$ is regarded as the mode and is modelled by a Markov chain on a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$. (The notation used in this section will be explained in more detail next section). Assume that the given hybrid equation (1.1) is not stable in distribution. The problem we are going to investigate in this paper is:

- **Problem**: Is it possible to design an intermittent stochastic feedback control to make the stochastically controlled SDE
$$dX(t) = f(X(t), r(t))dt + \beta(t)u(X(t), r(t))dB(t) \tag{1.2}$$
to become stable in distribution?

Here $B(t)$ is a Brownian motion and $u(x(t), r(t))dB(t)$ will be referred to as the stochastic feedback control. (These will be explained in Section 2.) Moreover, $\beta : [0, \infty) \to \{0, 1\}$ is defined by

$$\beta(t) = \sum_{k=0}^{\infty} I_{[kh,(k+1-\delta)h)}(t), \quad t \geq 0, \tag{1.3}$$

where $h > 0$, $\delta \in [0, 1)$ are both constants and $I_{[kh,(k+1-\delta)h)}(t)$ is the indicator function of $[kh, (k+1-\delta)h)$, namely it takes 1 when $t \in [kh, (k+1-\delta)h)$ and 0 otherwise. In operation, the stochastic control is switched on during time periods $[0, (1-\delta)h))$, $[h, (2-\delta)h)$, $[2h, (3-\delta)h)$, $\cdots$, while off during $[(1-\delta)h, h)$, $[(2-\delta)h, 2h)$, $[(3-\delta)h, 3h)$, $\cdots$. One of the practical reasons for such an intermittent control is because a controller needs a rest periodically (see, e.g., [11], [27]). The parameter δ is the proportion of rest in one period of h or in long term. In the case when $\delta = 0$, $\beta(t) = 1$ for all $t \geq 0$ so the stochastic control acts without any rest. As mentioned in the second paragraph in this section, stochastic feedback controls have been used to make the solutions of a stochastically controlled SDE to tend to the equilibrium state (e.g., 0) almost surely, but there is so far no paper which has addressed the Problem stated above. We close this section by highlighting the special features of this paper:

- The key contribution of this paper is to initiate the study of stabilisation in distribution by noise.
- The challenge of this paper lies in the fact that it is much harder mathematically to study if the probability distributions of the solutions will converge to a stationary distribution than to study if the solutions will tend to 0 as most existing papers did.
- The usefulness of this paper is because it is more desired to have the property of stability in distribution for many systems in the real world as we observed in the control of Covid-19.

2. Preliminaries

Throughout this paper, unless otherwise specified, we let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathcal{B}(\mathbb{R}^n)$ denote the family of all Borel measurable sets in $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, then $|x|$ is its Euclidean norm. Let $\mathbb{R}_0^{2n} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$ while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$. If A is a symmetric matrix, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ its largest and smallest eigenvalue, respectively. By $A > 0$ and $A \geq 0$, we mean A is positive and non-negative definite, respectively. If both a, b are real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Let $\mathbb{N}_+$ denote the set of non-negative integers. If G is a set, $I_G(\cdot)$ denotes its indicator function, that is $I_G(x) = 1$ for $x \in G$ and 0 otherwise.

We let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \geq 0$, be a right-continuous irreducible Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$, where $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$.

Let us consider the stochastically controlled system (1.2), where

$$f:\mathbb{R}^n\times\mathbb{S}\to\mathbb{R}^n \quad \text{and} \quad u:\mathbb{R}^n\times\mathbb{S}\to\mathbb{R}^{n\times m},$$

while $\beta(\cdot)$ is defined by (1.3). For the SDE (1.2) to be well defined, we impose the following assumption.

Assumption 2.1: There are constants $a_i>0$, $b_i\geq 0$ and $c_i\geq 0$ ($i\in\mathbb{S}$) such that

$$|f(x,i)-f(y,i)|\leq a_i|x-y|, \tag{2.1}$$

$$|u(x,i)-u(y,i)|\leq b_i|x-y|, \tag{2.2}$$

$$|(x-y)^T(u(x,i)-u(y,i))|\geq c_i|x-y|^2, \tag{2.3}$$

for all $(x,y,i)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}$.

Please note that there are plenty of SDEs in engineering and finance satisfy this assumption, for example, linear SDEs. More clearly, if $f(x,i)=A_ix$ and $u(x,i)=D_ix$ (when $m=1$) with $A_i\in\mathbb{R}^{n\times n}$ and $0<D_i=D_i^T\in\mathbb{R}^{n\times n}$, then Assumption 2.1 is satisfied with $a_i=\|A_i\|$, $b_i=\|D_i\|$ and $c_i=\lambda_{\min}(D_i)$. It is well known (see, e.g., [15], [18]) that under Assumption 2.1, for any given initial data $X(0)=\hat{x}\in\mathbb{R}^n$ and $r(0)=\hat{\imath}\in\mathbb{S}$ at time 0, the SDE (1.2) has a unique global solution on $t\geq 0$, which will be denoted by $X_{\hat{x},\hat{\imath}}(t)$ in this paper in order to highlight the role of the initial data. We also denote by $r_{\hat{\imath}}(t)$ the Markov chain starting from $\hat{\imath}$ at time 0. It is also known that any moment of the solution $X_{\hat{x},\hat{\imath}}(t)$ is finite for all $t\geq 0$. The following lemma will play a fundamental role in this paper.

Lemma 2.2: Under Assumption 2.1,

$$\mathbb{P}(X_{\hat{x},\hat{\imath}}(t)-X_{\hat{y},\hat{\imath}}(t)\neq 0 \text{ for all } t\geq 0)=1 \tag{2.4}$$

for any $\hat{x},\hat{y}\in\mathbb{R}^n$ with $\hat{x}\neq\hat{y}$ and $\hat{\imath}\in\mathbb{S}$.

To concentrate on the overall flow as well as significance of the results, we will defer all the proofs in this and next section to the Appendix.

To discuss the stability in distribution, we need the Markov property, in particular, the time-homogeneous Markov property (see, e.g., [1]). It is well known that the joint process $(X_{\hat{x},\hat{\imath}}(t),r_{\hat{\imath}}(t))$ is a Markov process on $t\geq 0$ (see, e.g., [18]). But due to the intermittent term $\beta(t)$, it is not time-homogeneous. In general, it is not appropriate to study the stability in distribution for a time-inhomogeneous SDE. Fortunately, $\beta(t)$ is a periodic function with its period h. Please note that it is this periodicity that does not only make it possible to study the stability in distribution but also distinguish the results established in this paper significantly from the existing ones, e.g., in [26] (please see Remark 3.5 below). Making use of the periodicity, we observe that $\{(X_{\hat{x},\hat{\imath}}(kh),r_{\hat{\imath}}(kh))\}_{k\in\mathbb{N}_+}$ forms a discrete-time $\mathbb{R}^n\times\mathbb{S}$-valued time-homogeneous Markov process. Its k-step transition probability measure $P(k,\hat{x},\hat{\imath};dy\times\{j\})$ is defined by

$$P(k,\hat{x},\hat{\imath};D\times S)=\mathbb{P}((X_{\hat{x},\hat{\imath}}(kh),r_{\hat{\imath}}(kh))\in D\times S)$$

for any $D\in\mathcal{B}(\mathbb{R}^n)$ and $S\subset\mathbb{S}$. This Markov process will play its important role in this paper. Nevertheless, it only involves the solution at the discrete times kh. We still need to form a time-homogeneous Markov process which involves the solution for all $t\geq 0$. For this purpose, we need a few new notations. Denote $\mathcal{C}_h$ the family of continuous functions ξ from $[0,h]$ to $\mathbb{R}^n$ with norm $\|\xi\|_h=\sup_{s\in[0,h]}|\xi(s)|$. Denote by $\mathcal{P}(\mathcal{C}_h)$ the family of probability measures on $\mathcal{C}_h$. For $P_1,P_2\in\mathcal{P}(\mathcal{C}_h)$, define the Kantorovich metric d_Φ by

$$d_\Phi(P_1,P_2)=\sup_{\phi\in\Phi}\Big|\int_{\mathcal{C}_h}\phi(\xi)P_1(d\xi)-\int_{\mathcal{C}_h}\phi(\xi)P_2(d\xi)\Big|$$

where

$$\begin{aligned}\Phi=&\{\phi:\mathcal{C}_h\to\mathbb{R} \text{ satisfying } |\phi(\xi)-\phi(\zeta)|\leq\|\xi-\zeta\|_h\\ &\text{ and } |\phi(\xi)|\leq 1 \text{ for } \xi,\zeta\in\mathcal{C}_h\}.\end{aligned}$$

(Please see, e.g., [8], for the details on the Kantorovich metric d_Φ. Of course our results can also be proved by the Wasserstein metric W_p equivalently.) Moreover, for $k\in\mathbb{N}_+$, define $\widetilde{X}_{\hat{x},\hat{\imath}}(kh)=\{X_{\hat{x},\hat{\imath}}(kh+s):0\leq s\leq h\}$ which is $\mathcal{C}_h$-valued. Denote by $\mathcal{L}(\widetilde{X}_{\hat{x},\hat{\imath}}(kh))$ the probability measure on $\mathcal{C}_h$ generated by $\widetilde{X}_{\hat{x},\hat{\imath}}(kh)$. (Please see again, e.g., [8], for more details about probability measures generated by stochastic processes and Definition 2.3 below.) With these new notations, we see that $\{(\widetilde{X}_{\hat{x},\hat{\imath}}(kh),r_{\hat{\imath}}(kh))\}_{k\in\mathbb{N}_+}$ forms a discrete-time $\mathcal{C}_h\times\mathbb{S}$-valued time-homogeneous Markov process. In fact, the time-homogeneous property follows from the periodic property of $\beta(\cdot)$. Moreover, once $(\widetilde{X}_{\hat{x},\hat{\imath}}(k_1h),r_{\hat{\imath}}(k_1h))$ for some $k_1\in\mathbb{N}_+$ is given, $(X_{\hat{x},\hat{\imath}}(k_1h),r_{\hat{\imath}}(k_1h))$ is known and then $(X_{\hat{x},\hat{\imath}}(t),r_{\hat{\imath}}(t))$ for all $t\geq k_1h$, namely $(\widetilde{X}_{\hat{x},\hat{\imath}}(kh),r_{\hat{\imath}}(kh))$ for all $k\geq k_1$, can be uniquely determined by solving the SDE (1.2) with initial data $(X_{\hat{x},\hat{\imath}}(k_1h),r_{\hat{\imath}}(k_1h))$ at time k_1h, but the information on how the process reaches $(\widetilde{X}_{\hat{x},\hat{\imath}}(k_1h),r_{\hat{\imath}}(k_1h))$ starting from $(\hat{x},\hat{\imath})$ at time 0 is of no further use. These do not only explain the Markov property but also show the following property that

$$\begin{aligned}&\mathbb{E}\phi(\widetilde{X}_{\hat{x},\hat{\imath}}((k+q)h))\\ &\quad=\sum_{j\in\mathbb{S}}\int_{\mathbb{R}^n}\phi(\widetilde{X}_{y,j}(qh))P(k,\hat{x},\hat{\imath};dy\times\{j\})\end{aligned} \tag{2.5}$$

for $\phi\in\Phi$ and $k,q\in\mathbb{N}_+$. It should be emphasised that the formula above uses the transition probability measure of $\{(X_{\hat{x},\hat{\imath}}(kh),r_{\hat{\imath}}(kh))\}_{k\in\mathbb{N}_+}$ but not that of

$\{(\widetilde{X}_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh))\}_{k\in\mathbb{N}_+}$. We can now give the definition of the stability in distribution.

Definition 2.3: The controlled SDE (1.2) is said to be asymptotically stable in distribution if there exists a probability measure $\mu_h \in \mathcal{P}(\mathcal{C}_h)$ such that

$$\lim_{k\to\infty} d_\Phi(\mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}(kh)), \mu_h) = 0$$

for all $(\hat{x}, \hat{i}) \in \mathbb{R}^n \times \mathbb{S}$.

It should be pointed out that in the literature (see, e.g., [26]), the asymptotic stability in distribution is in general defined on the joint process $(\widetilde{X}_{\hat{x},bi}(kh), r_{\hat{i}}(kh))$. On the other hand, given that the law of the Markov chain $r_{\hat{i}}(t)$ is already known to converge to its unique stationary distribution (see, e.g., [1]), our definition here only on $\widetilde{X}_{\hat{x},\hat{i}}(kh)$ is consistent with that in the literature.

3. Stabilisation

Let us begin to discuss the stochastic stabilisation in distribution by imposing a technical assumption.

Assumption 3.1: There is a constant $p \in (0,1)$ such that

$$\mathcal{A} := \text{diag}(\zeta_1 - pa_1, \cdots, \zeta_N - pa_N) - \Gamma \qquad (3.1)$$

is a nonsingular M-matrix, where

$$\zeta_i = 0.5p[(2-p)c_i^2 - b_i^2], \ i \in \mathbb{S} \qquad (3.2)$$

and a_i, b_i, c_i are the same as in Assumption 2.1.

In next section we will not only explain how to design the control function $u(x,i)$ which meets this assumption but also give a couple of easy-to-check sufficient criteria for it to hold. Meanwhile we just suppose it holds. We need a number of new notations. Define

$$(\theta_1, \cdots, \theta_N)^T = \mathcal{A}^{-1}(1, \cdots, 1)^T. \qquad (3.3)$$

By the theory of M-matrices (see, e.g., [18, Theorem 2.10 on page 68]), $\theta_i > 0$ for all $i \in \mathbb{S}$. Set

$$\underline{\theta} = \min_{1\le i\le N} \theta_i, \ \bar{\theta} = \max_{1\le i\le N} \theta_i, \ \sigma = \max_{1\le i\le N} \zeta_i. \qquad (3.4)$$

Moreover, define

$$\gamma(t) = \frac{1}{\bar{\theta}} - \sigma(1 - \beta(t)) \text{ for } t \ge 0. \qquad (3.5)$$

It should be pointed out that we must have $\sigma > 0$. If not, $\zeta_i \le 0$ for all $i \in \mathbb{S}$ and hence, by the theory of M-matrices (see, e.g., [18, Theorem 2.10 on page 68]), $\mathcal{A} + \text{diag}(-\zeta_1, \cdots, -\zeta_N)$ should be a nonsingular M-matrix. But

$$\mathcal{A}+\text{diag}(-\zeta_1, \cdots, -\zeta_N) = \text{diag}(-pa_1, \cdots, -pa_N) - \Gamma,$$

which cannot be a nonsingular M-matrix by the theory of M-matrices. In other words, we would have a contradiction if $\sigma \le 0$. With these new notations, we can now form a critical parameter

$$\delta^* = 1 \wedge (1/(\sigma\bar{\theta})). \qquad (3.6)$$

In what follows we will require $\delta < \delta^*$ to show the stability in distribution of the controlled system (1.2). Recalling that parameter δ represents the proportion of the rest time of the stochastic control, we see why δ^* is a critical value. The following two lemmas play their key role in the proof of our main theorem.

Lemma 3.2: Let Assumptions 2.1 and 3.1 hold. Let $\delta < \delta^*$. Then for any $(\hat{x}, \hat{y}, \hat{i}) \in \mathbb{R}_0^{2n} \times \mathbb{S}$,

$$\mathbb{E}\|\widetilde{X}_{\hat{x},\hat{i}}(kh) - \widetilde{X}_{\hat{y},\hat{i}}(kh)\|_h^p \le C_1|\hat{x} - \hat{y}|^p e^{-\gamma_1 kh} \quad (3.7)$$

for all $k \in \mathbb{N}_+$, where $\gamma_1 = 1/\bar{\theta} - \sigma\delta > 0$ and C_1 is positive constant independent of the initial data $(\hat{x}, \hat{y}, \hat{i})$.

Lemma 3.3: Let Assumptions 2.1 and 3.1 hold. Let $\delta < \delta^*$. Then for any $(\hat{x}, \hat{i}) \in \mathbb{R}_n \times \mathbb{S}$,

$$\mathbb{E}|X_{\hat{x},\hat{i}}(t)|^p \le C_2(1 + |\hat{x}|^p) \qquad (3.8)$$

for all $t \ge 0$, where C_2 is a positive number independent of the initial data $(\hat{x}, \hat{i})$.

The following theorem is the main result in this paper.

Theorem 3.4: Let Assumptions 2.1 and 3.1 hold. Let $\delta < \delta^*$. Then there exists a unique probability measure $\mu_h \in \mathcal{P}(\mathcal{C}_h)$ such that

$$\lim_{k\to\infty} d_\Phi(\mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}(kh)), \mu_h) = 0 \qquad (3.9)$$

for all $(\hat{x}, \hat{i}) \in \mathbb{R}^n \times \mathbb{S}$. In other words, the SDE (1.2) is asymptotically stable in distribution.

Remark 3.5: We first compare our results with those in [26]. There are several significant differences: (a) The SDE in [26] is time-homogeneous so that the Markov process can be formed in the state space $\mathbb{R}^n \times \mathbb{S}$. But our SDE (1.2) is not time-homogeneous and, in general, a time-inhomogeneous SDE does not possess a stationary distribution. Fortunately, our SDE (1.2) is periodic with period h. Making use of this periodic property, we form a time-homogeneous Markov process in the state space $\mathcal{C}_h \times \mathbb{S}$. (b) As $\mathcal{C}_h \times \mathbb{S}$ is an infinite dimensional space, which is much larger than the finite-dimensional space $\mathbb{R}^n \times \mathbb{S}$, our proofs here are much more complicated than those in [26]. (c) Conditions imposed in [26] are also much different from those in this paper. For instance, the conditions in [26, Theorem 5.1] imply that the given equation (1.1) is already stable in distribution, which is completely different from the fundamental setting in this paper. More precisely, [26, Theorem 5.1] deals with how much stochastic perturbation ($g(x(t), r(t))dB(t)$ there) the given stable equation (1.1) can tolerate without loss of its stability. However, our results address the problem when the given equation (1.1) is not stable in distribution, how a stochastic feedback control can be designed to make the controlled SDE (1.2) to be stable in distribution. We next compare our results with those in [22], [25]. A common feature in [22], [25] is that the second moment of the solution of the underlying system is required to be uniformly bounded in time $t \ge 0$ for any given initial data (see [22, Theorem 3.2] and [25, Lemma III.2]). However, the mean, and hence the second moment of our SDE (1.2) may tend to ∞ as $t \to \infty$ for some given initial data. For example, one of the simplest SDEs in the form of (1.2) is the scalar linear SDE $dX(t) = (1 + X(t))dt + 2\beta(t)X(t)dB(t)$ whose mean $\mathbb{E}X(t) \to \infty$ when $X(0) \neq -1$. In other words, the results in [22], [25] are not applicable to our SDE (1.2).

4. Design of control function

The use of Theorem 3.4 depends on whether the control function $u(x, i)$ can be designed to meet Assumptions 2.1 and 3.1. In order to design $u(x, i)$ more easily, we first present an easy-to-check sufficient criterion for Assumption 3.1 to hold. We recall that $r(t)$ is an irreducible Markov chain in the finite state space $\mathbb{S}$. Hence, it has a unique stationary distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_N) \in \mathbb{R}^{1\times N}$ which can be determined by solving the linear equation $\pi\Gamma = 0$ subject to $\sum_{i\in\mathbb{S}} \pi_i = 1$ and $\pi_i > 0$ for all $i \in \mathbb{S}$ (see, e.g., [1]). Th following Proposition was proved in [17].

Proposition 4.1: Assumption 3.1 holds if

$$\sum_{i\in\mathbb{S}} \pi_i(a_i + 0.5b_i^2 - c_i^2) < 0. \tag{4.1}$$

and, moreover, there is some $\hat{j} \in \mathbb{S}$ for which

$$\gamma_{i\hat{j}} > 0 \text{ for all } i \in \mathbb{S} \text{ but } i \neq \hat{j}. \tag{4.2}$$

In the remaining part of this section we will show how the control function $u(x, i)$ can be designed to satisfy Assumptions 2.1 and 3.1 in various situations. It should be emphasised that δ^* can be computed by (3.6) once Assumptions 2.1 and 3.1 are satisfied by $u(x, i)$ to be designed.

We always assume that the coefficient $f(x, i)$ of the given equation (1.1) satisfies condition (2.1). Due to the page limit, we only explain how to design a linear control function, namely

$$u(x, i) = (A_{1i}x, A_{2i}x, \cdots, A_{mi}x) \tag{4.3}$$

$(x, i) \in \mathbb{R}^n \times \mathbb{S}$, where $A_{ki} \in \mathbb{R}^{n\times n}$ is symmetric and non-negative definite for $i \in \mathbb{S}$ and $k = 1, 2, \cdots, m$. It is straightforward to see that that $u(x, i)$ satisfies Assumption 2.1 with

$$b_i^2 = \sum_{k=1}^{m} \|A_{ki}\|^2 \text{ and } c_i^2 = \sum_{k=1}^{m} \lambda_{\min}^2(A_{ki}). \tag{4.4}$$

In other words, What we need to do is to refine the choices of A_{ki} for Assumption 3.1 to hold. We discuss a number of useful cases.

Case 1. For $i \in \mathbb{S}$ and $1 \le k \le m$, choose symmetric matrices $\bar{A}_{ki}$ such that

$$\sqrt{2}\lambda_{\min}(\bar{A}_{ki}) > \|\bar{A}_{ki}\|. \tag{4.5}$$

Obviously, there are lots of such matrices. Choose a positive number α sufficiently large so that

$$0.5\alpha^2\Big(2\sum_{k=1}^{m} \lambda_{\min}^2(\bar{A}_{ki}) - \sum_{k=1}^{m} \|\bar{A}_{ki}\|^2\Big) > a_i \tag{4.6}$$

for all $i \in \mathbb{S}$. This guarantees that there is a $p \in (0, 1)$ sufficiently small for which

$$0.5\alpha^2\Big((2-p)\sum_{k=1}^{m} \lambda_{\min}^2(\bar{A}_{ki}) - \sum_{k=1}^{m} \|\bar{A}_{ki}\|^2\Big) > a_i \tag{4.7}$$

for all $i \in \mathbb{S}$. Let us now set $A_{ki} = \alpha\bar{A}_{ki}$. Noting that ζ_i defined by (3.2) has the form

$$\zeta_i = 0.5\alpha^2 p\Big((2-p)\sum_{k=1}^{m} \lambda_{\min}^2(\bar{A}_{ki}) - \sum_{k=1}^{m} \|\bar{A}_{ki}\|^2\Big),$$

we see $\zeta_i > pa_i$ for all $i \in \mathbb{S}$. By the theory of M-matrices (see, e.g., [18, Theorem 2.10 on page 68]), we see that $\mathcal{A}$ defined by (3.1) is a nonsingular M-matrix. In other words, Assumption 3.1 holds if A_{ki}'s are designed as above.

Case 2. Observe that the arguments above still hold as long as $2\sum_{k=1}^{m} \lambda_{\min}^2(\bar{A}_{ki}) > \sum_{k=1}^{m} \|\bar{A}_{ki}\|^2$ for all $i \in \mathbb{S}$ but it is unnecessary for (4.5) to hold for every $i \in \mathbb{S}$ and $1 \le k \le m$. This gives us an opportunity to design the control function to fit into various situations in the real world. For example, we may let $\bar{A}_{ki} = 0$ for all $k = 2, \cdots, m$ but only need $0.5\alpha^2\Big((2-p)\lambda_{\min}^2(\bar{A}_{1i}) - \|\bar{A}_{1i}\|^2\Big) > a_i$ for all $i \in \mathbb{S}$. This is equivalent to the situation when $m = 1$. In other words, we may only use a scalar Brownian motion as the noise source to achieve the stochastic stabilisation in distribution.

Case 3. The observation in Case 2 also reveals another useful situation, where a different and independent scalar Brownian motion is used in different mode $i \in \mathbb{S}$. In terms of mathematics, we have that $m = N$ and $A_{ki} = 0$ in (4.3) for all $k \neq i$. In this situation, we may choose $\bar{A}_{ii}$'s and α for which $0.5\alpha^2\Big((2-p)\lambda_{\min}^2(\bar{A}_{ii}) - \|\bar{A}_{ii}\|^2\Big) > a_i$ and then set $A_{ii} = \alpha\bar{A}_{ii}$ for all $i \in \mathbb{S}$.

Case 4. We now consider a situation where the state $X(t)$ cannot be observed in some modes, whence the stochastic control cannot be used in these modes. Without loss of any generality, we let $\mathbb{S}_1 = \{1, 2, \cdots, N_1\}$ contain these modes ($1 \le N_1 < N$). Mathematically speaking, we are forced to set $A_{ki} = 0$ for $i \in \mathbb{S}_1$, $1 \le k \le m$, whence $b_i = c_i = 0$ for $i \in \mathbb{S}_1$. What we need to do is to design matrices A_{ki} for $N_1 + 1 \le i \le N$ and $1 \le k \le m$. To establish a simple criterion, we impose an additional condition: there is some $\hat{j} \in \mathbb{S}$ for which (4.2) holds. Recall that $\pi = (\pi_1, \pi_2, \cdots, \pi_N) \in \mathbb{R}^{1\times N}$ is the unique stationary distribution of the Markov chain with all $\pi_i > 0$ (please see Proposition 4.1). Choose symmetric positive definite matrices $\bar{A}_{ki}$ for $N_1 + 1 \le i \le N$ and $1 \le k \le m$ so that $\sum_{k=1}^m \lambda^2_{\min}(\bar{A}_{ki}) > 0.5 \sum_{k=1}^m \|\bar{A}_{ki}\|^2$. Then choose a positive number α so large that $\sum_{i=N_1+1}^N \pi_i \alpha^2 \sum_{k=1}^m \big(0.5\|\bar{A}_{ki}\|^2 - \lambda^2_{\min}(\bar{A}_{ki})\big) + \sum_{i=1}^N \pi_i a_i < 0$. Now set $A_{ki} = \alpha \bar{A}_{ki}$. Recalling (4.4), we see $b_i^2 = \alpha^2 \sum_{k=1}^m \|\bar{A}_{ki}\|^2$ and $c_i^2 = \alpha^2 \sum_{k=1}^m \lambda^2_{\min}(\bar{A}_{ki})$ for $N_1 + 1 \le i \le N$. Consequently, $\sum_{i=1}^N \pi_i a_i + \sum_{i=N_1+1}^N \pi_i(0.5b_i^2 - c_i^2) < 0$. That is $\sum_{i=1}^N \pi_i(a_i + 0.5b_i^2 - c_i^2) < 0$ if we recall that $b_i = c_i = 0$ for $i \in \mathbb{S}_1$. By Proposition 4.1, we see Assumption 3.1 is satisfied as long as A_{ki}'s are designed as above.

Remark 4.2: In particular, the discussions above show that any unstable n-dimensional system of the form $\dot{x}(t) = g_{r(t)} + G_{r(t)}x(t)$ can be stabilised by any of the stochastic feedback controls described above, where $G_i \in \mathbb{R}^{n\times n}$ and $g_i \in \mathbb{R}^n$ for $i \in \mathbb{S}$. There are lots of such hybrid linear systems in applications (see, e.g., [3], [14]). But we could not discuss an example here due to the page limit.

5. Conclusion

In this paper we have discussed if the probability distributions of the solutions to the stochastically controlled system will converge to a stationary distribution. We refer the problem as to the stabilisation in distribution by noise. Although this is an important and useful problem, there is so far little known on it due to the mathematical difficulty. But we have successfully tackled the problem in this paper.

6. Appendix

Due to the page limit, we will only be able to outline the proofs.

Proof of Lemma 2.2. If (2.4) were false, there would exist some $(\hat{x}, \hat{y}, \hat{i}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}$ with $\hat{x} \ne \hat{y}$ such that $\mathbb{P}(\tau < \infty) > 0$, where $\tau = \inf\{t \ge 0 : X_{\hat{x},\hat{i}}(t) - X_{\hat{y},\hat{i}}(t) = 0\}$, in which and throughout this paper we set $\inf \emptyset = \infty$. We can then find a pair of positive numbers R and T such that $\mathbb{P}(\Omega_1) > 0$, where

$$\Omega_1 = \{\omega \in \Omega : \tau(\omega) \le T \text{ and } \sup_{0\le t\le \tau(\omega)} (X_{\hat{x},\hat{i}}(t,\omega)| \vee |X_{\hat{y},\hat{i}}(t,\omega)|) \le R - 1\}.$$

By Assumption 2.1, $|f(x,i) - f(y,i)| \vee |u(x,i) - u(y,i)| \le h_1|x - y|$ for $(x,y,i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}$, where $h_1 = \max_{i\in\mathbb{S}}(a_i \vee b_i)$. Set $h_2 = 2h_1 + 4h_1^2$ and define the Lyapunov function $V_1(z,t) = e^{-h_2 t}|z|^{-2}$ for $(z,t) \in (\mathbb{R}^n - \{0\}) \times \mathbb{R}_+$. For any $\varepsilon \in (0, |\hat{x} - \hat{y}|)$, define a stopping time

$$\tau_\varepsilon = \inf\{t \ge 0 : |X_{\hat{x},\hat{i}}(t) - X_{\hat{y},\hat{i}}(t)| \le \varepsilon \text{ or } |X_{\hat{x},\hat{i}}(t)| \wedge |X_{\hat{y},\hat{i}}(t)| \ge R\}.$$

Set $Z(t) = X_{\hat{x},\hat{i}}(t) - X_{\hat{y},\hat{i}}(t)$. By Itô formula (see, e.g., [14]), it is not very difficult to show that $\mathbb{E}\big[e^{-h_2(\tau_\varepsilon\wedge T)}|Z(\tau_\varepsilon \wedge T)|^{-2}\big] \le |\hat{x} - \hat{y}|^{-2}$. Noting that $\tau_\varepsilon \le T$ and $|Z(\tau_\varepsilon)| = \varepsilon$ whenever $\omega \in \Omega_1$, we see from the inequality above that $\mathbb{E}\big[e^{-h_2 T}\varepsilon^{-2} I_{\Omega_1}\big] \le |\hat{x} - \hat{y}|^{-2}$. This implies $\mathbb{P}(\Omega_1) \le \varepsilon^2 |\hat{x} - \hat{y}|^{-2} e^{h_2 T}$. Letting $\varepsilon \to 0$ yields that $\mathbb{P}(\Omega_1) = 0$, which contradicts $\mathbb{P}(\Omega_1) > 0$. We therefore must have the required assertion (2.4). □

Proof of Lemma 3.2. Fix $\hat{x}, \hat{y}, \hat{i}$ arbitrarily and let $Z(t) = X_{\hat{x},\hat{i}}(t) - X_{\hat{y},\hat{i}}(t)$. By Lemma 2.2, $Z(t) \ne 0$ for all $t \ge 0$ with probability 1. Define a Lyapunov function $V_2(z,i,t) = \theta_i |z|^p \Psi(t)$ for $(z,i,t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where $\Psi(t) = \exp(\int_0^t \gamma(s)ds)$. By the generalised Itô formula (see, e.g., [18, Theorem 1.45 on page 48]) and Assumption 2.1 and definition (3.3) of θ_i's, it is not very difficult to show that

$$\mathbb{E}V_2(Z(t), r_{\hat{i}}(t), t) - \theta_{\hat{i}}|Z(0)|^p \le 0. \tag{6.1}$$

for $t \ge 0$. This implies

$$\underline{\theta}\mathbb{E}|Z(t)|^p \le \bar{\theta}|Z(0)|^p \Psi^{-1}(t) \tag{6.2}$$

for all $t \ge 0$. Let k be the integer part of t/h, whence $kh \le t < (k+1)h$. By the definitions of $\gamma(t)$ and $\beta(t)$, we can derive

$$\Psi^{-1}(t) \le \exp\Big((-\frac{t}{\theta} + \sigma\delta h(k+1)\Big) \le e^{-\gamma_1 t + \sigma h}.$$

Substituting this into (6.2) yields

$$\mathbb{E}|Z(t)|^p \le \bar{C}_1 |Z(0)|^p e^{-\gamma_1 t}, \quad \forall t \ge 0, \tag{6.3}$$

where $\bar{C}_1 = (\bar{\theta}/\underline{\theta})e^{\sigma h}$. It is then very standard to show the assertion (3.7) by the Burkholder-Davis-Gundy inequality etc. □

Proof of Lemma 3.3. It follows from Assumption 2.1 that there is a constant $K_1 > 0$ such that

$$\begin{aligned} 2x^T f(x,i) &\le 2a_i|x|^2 + K_1|x|, \\ |u(x,i)|^2 &\le b_i^2|x|^2 + K_1(|x| + 1), \\ |x^T u(x,i)|^2 &\ge c_i^2|x|^4 - K_1(|x|^3 + |x|^2) \end{aligned} \tag{6.4}$$

for all $(x,i) \in \mathbb{R}^n \times \mathbb{S}$. Define a Lyapunov function $V_3(x,i,t) = \theta_i(1+|x|^2)^{0.5p}\Psi(t)$ for $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where $\Psi(t)$ has bee defined in the proof of Lemma 3.2. Once again, by the generalised Itô formula as well as (3.5), (6.4) and $p \in (0,1)$, it is easy to show

$$\underline{\theta}\Psi(t)\mathbb{E}|X_{\hat{x},\hat{i}}(t)|^p \le \bar{\theta}(1+|\hat{x}|^p) + K_2\int_0^t \Psi(s)ds,$$

where K_2 is a positive number. But, we can also show that $e^{\gamma_1 t-\sigma h} \le \Psi(t) \le e^{\gamma_1 t+\sigma h}$ for $t \ge 0$, where γ_1 was defined in the statement of Lemma 3.2. Hence

$$\begin{aligned}&\underline{\theta}\mathbb{E}|X_{\hat{x},\hat{i}}(t)|^p \exp(\gamma_1 t - \sigma h)\\ &\le \bar{\theta}(1+|\hat{x}|^p) + (K_2/\gamma_1)\exp(\gamma_1 t + \sigma h).\end{aligned}$$

This implies the assertion (3.8). □

Proof of Theorem 3.4. Step 1. We first claim that for any compact subset G of $\mathbb{R}^n$,

$$\lim_{k\to\infty} d_\Phi(\mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}(kh)), \mathcal{L}(\widetilde{X}_{\hat{y},\hat{j}}(kh))) = 0 \qquad (6.5)$$

uniformly in $\hat{x},\hat{y} \in G$ and $\hat{i},\hat{j} \in \mathbb{S}$. Note that $\{r(kh)\}_{k\in\mathbb{N}_+}$ is a discrete-time ergodic Markov chain with its one-step transition probability matrix $e^{h\Gamma}$. Define the stopping time

$$\kappa_{\hat{i}\hat{j}} = \inf\{kh : r_{\hat{i}}(kh) = r_{\hat{j}}(kh),\ k \in \mathbb{N}_+\}.$$

Then $\kappa_{\hat{i}\hat{j}} < \infty$ a.s. (see, e.g., [1]). Hence, for any $\varepsilon \in (0,1)$, there is a positive number $T_1 > 0$ such that

$$\mathbb{P}(\kappa_{\hat{i}\hat{j}} \le T_1) > 1 - \varepsilon/6, \quad \forall \hat{i},\hat{j} \in \mathbb{S}. \qquad (6.6)$$

Recalling a known result ([18, p. 99, Theorem 3.24]) that $\sup_{(\hat{x},\hat{i})\in G\times\mathbb{S}} \mathbb{E}\Big(\sup_{0\le t\le T_1} |X_{\hat{x},\hat{i}}(t)|^2\Big) < \infty$, we see there is a sufficiently large $\rho > 0$ such that

$$\mathbb{P}(\Omega_{\hat{x},\hat{i}}) > 1 - \varepsilon/12, \quad \forall (\hat{x},\hat{i}) \in G \times \mathbb{S}, \qquad (6.7)$$

where $\Omega_{\hat{x},\hat{i}} = \{\omega \in \Omega : \sup_{0\le t\le T_1} |X_{\hat{x},\hat{i}}(t,\omega)| \le \rho\}$. We now fix $\hat{x},\hat{y} \in G$ and $\hat{i},\hat{j} \in \mathbb{S}$ arbitrarily. For any $\phi \in \Phi$ and $k \in \mathbb{N}_+$ with $kh \ge T_1$, we have

$$|\mathbb{E}\phi(\widetilde{X}_{\hat{x},\hat{i}}(kh)) - \mathbb{E}\phi(\widetilde{X}_{\hat{y},\hat{j}}(kh))| \le \frac{\varepsilon}{3} + \bar{J}(kh), \qquad (6.8)$$

where

$$\bar{J}(kh) = \mathbb{E}\Big(I_{\{\kappa_{\hat{i}\hat{j}}\le T_1\}}|\phi(\widetilde{X}_{\hat{x},\hat{i}}(kh)) - \phi(\widetilde{X}_{\hat{y},\hat{j}}(kh))|\Big).$$

Set $\Omega_1 = \Omega_{\hat{x},\hat{i}} \cap \Omega_{\hat{y},\hat{j}} \cap \{\kappa_{\hat{i}\hat{j}} \le T_1\}$. By the time-homogeneous Markov property, it is not very difficult to show that

$$\begin{aligned}\bar{J}(kh) \le \varepsilon/3 + 2\mathbb{E}\Big(I_{\Omega_1}\mathbb{E}\|\widetilde{X}_{w,l}(kh - \kappa_{\hat{i}\hat{j}})\\ - \widetilde{X}_{z,l}(kh - \kappa_{\hat{i}\hat{j}})\|_h^p\Big),\end{aligned} \qquad (6.9)$$

where $w = X_{\hat{x},\hat{i}}(\kappa_{\hat{i}\hat{j}})$, $z = X_{\hat{y},\hat{j}}(\kappa_{\hat{i}\hat{j}})$ and $l = r_{\hat{i}}(\kappa_{\hat{i}\hat{j}}) = r_{\hat{j}}(\kappa_{\hat{i}\hat{j}})$. Observing that for any given $\omega \in \Omega_1$, $|w|\vee|z| \le \rho$, we can apply Lemma 3.2 to see that there is another positive constant T_2 such that

$$\mathbb{E}\|\widetilde{X}_{w,l}(kh - \kappa_{\hat{i}\hat{j}}) - \widetilde{X}_{z,l}(kh - \kappa_{\hat{i}\hat{j}})\|_h^p \le \frac{\varepsilon}{6}$$

for all $kh \ge T_1 + T_2$. Substituting this into (6.9) yields that $\bar{J}(kh) \le 2\varepsilon/3$ for all $kh \ge T_1 + T_2$. This, together with (6.8), implies that

$$|\mathbb{E}\phi(\widetilde{X}_{\hat{x},\hat{i}}(kh) - \mathbb{E}\phi(\widetilde{X}_{\hat{y},\hat{j}}(kh))| \le \varepsilon \qquad (6.10)$$

for all $kh \ge T_1 + T_2$. Since ϕ is arbitrary, we must have

$$d_\Phi(\mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}(kh)), \mathcal{L}(\widetilde{X}_{\hat{y},\hat{j}}(kh))) \le \varepsilon, \quad \forall kh \ge T_1 + T_2$$

for all $\hat{x},\hat{y} \in G$ and $\hat{i},\hat{j} \in \mathbb{S}$. This proves (6.5).

Step 2. We next claim that for any $(\hat{x},\hat{i}) \in \mathbb{R}^n \times \mathbb{S}$, $\{\mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}(kh))\}_{k\in\mathbb{N}_+}$ is a Cauchy sequence in $\mathcal{P}(\mathcal{C}_h)$ with metric d_Φ. In other words, we need to show that for any $\varepsilon > 0$, there is an integer $k_0 > 0$ such that

$$d_\Phi(\mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}((v+q)h)), \mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}(qh))) \le \varepsilon \qquad (6.11)$$

for all integers $q \ge k_0$ and $v \ge 1$. Let $\varepsilon \in (0,1)$ be arbitrarily. By Lemma 3.3, there is a $\bar{\rho} > 0$ such that

$$\mathbb{P}\{\omega \in \Omega : |X_{\hat{x},\hat{i}}(vh,\omega)| \le \bar{\rho}\} > 1 - \varepsilon/4 \qquad (6.12)$$

for any integer $v \ge 1$. For any $\phi \in \Phi$, we can then derive, using (2.5) and (6.12), that

$$\begin{aligned}&|\mathbb{E}\phi(\widetilde{X}_{\hat{x},\hat{i}}((v+q)h)) - \mathbb{E}\phi(\widetilde{X}_{\hat{x},\hat{i}}(qh))|\\
=&|\mathbb{E}(\mathbb{E}[\phi(\widetilde{X}_{\hat{x},\hat{i}}((v+q)h))|\mathcal{F}_{vh}]) - \mathbb{E}\phi(\widetilde{X}_{\hat{x},\hat{i}}(qh))|\\
=&\Big|\sum_{j\in\mathbb{S}}\int_{\mathbb{R}^n} \mathbb{E}\phi(\widetilde{X}_{y,j}(qh))P(v,\hat{x},\hat{i};dy\times\{j\})\\
&\quad - \mathbb{E}\phi(\widetilde{X}_{\hat{x},\hat{i}}(qh))\Big|\\
\le&\sum_{j\in\mathbb{S}}\int_{\mathbb{R}^n} |\mathbb{E}\phi(\widetilde{X}_{y,j}(qh)) - \mathbb{E}\phi(\widetilde{X}_{\hat{x},\hat{i}}(qh))|\\
&\quad \times P(v,\hat{x},\hat{i};dy\times\{j\})\\
\le&\frac{\varepsilon}{2} + \sum_{j\in\mathbb{S}}\int_{B_{\bar{\rho}}} d_\Phi(\mathcal{L}(\widetilde{X}_{y,j}(qh)), \mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}(qh)))\\
&\quad \times P(v,\hat{x},\hat{i};dy\times\{j\}),\end{aligned}$$

where $B_{\bar{\rho}} = \{x \in \mathbb{R}^n : |x| \le \bar{\rho}\}$. But, by (6.5), there is a positive integer k_0 such that

$$d_\Phi(\mathcal{L}(\widetilde{X}_{\hat{y},\hat{j}}(qh)), \mathcal{L}(\widetilde{X}_{\hat{x},\hat{i}}(qh))) \le \frac{\varepsilon}{2}, \quad \forall q \ge k_0$$

whenever $(y,j) \in B_{\bar{\rho}} \times \mathbb{S}$. We therefore obtain

$$|\mathbb{E}\phi(\widetilde{X}_{\hat{x},\hat{i}}((v+q)h)) - \mathbb{E}\phi(\widetilde{X}_{\hat{x},\hat{i}}(qh))| \le \varepsilon$$

for $q \ge k_0$ and $v \ge 1$. As this holds for any $\phi \in \Phi$, we must have (6.11) as claimed.

Step 3. Recalling the well-known fact that the weak convergence of probability measures is a metric concept (see, e.g., [8, Proposition 2.5]), we observe from Step 2 that there is a unique $\mu_h \in \mathcal{P}(\mathcal{C}_h)$ such that $\lim_{k\to\infty} d_\Phi(\mathcal{L}(\widetilde{X}_{0,1}(kh)), \mu_h) = 0$. This, together with (6.5), implies the assertion (3.9) immediately. The proof is complete. □

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