A note on sufficient conditions of asymptotic stability in distribution of stochastic differential equations with G-Brownian motion

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Abstract

This paper investigates a sufficient condition of asymptotic stability in distribution of stochastic differential equations driven by G-Brownian motion (G-SDEs). We define the concept of asymptotic stability in distribution under sublinear expectations. Sufficient criteria of the asymptotic stability in distribution based on sublinear expectations are given. Finally, an illustrative example is provided.

Keywords: G-SDEs; sublinear expectation; stability in distribution; Markovian inequality; G-Itô formula.

1. Introduction

Recently, the topic among the stability in distribution of SDEs has been extensively researched. Among them, Du et al. [3] discussed a new sufficient condition for stability in distribution of stochastic differential delay equations with Markovian switching. Li et al. [7] analyzed the stabilization in distribution of hybrid stochastic differential equations by feedback control based on discrete-time state observations while the stabilisation in distribution by delay feedback control for hybrid SDEs is studied in You et al. [9]. In [7] and [9], we can find other related literature on stability in distribution. On the other hand, a great number of work of SDEs driven by G-Brownian motion is studied by many researchers, e.g., Peng [8], Fei et al. [4, 5], references therein.

So far, to the best of our knowledge, the stability in distribution of SDEs driven by G-Brownian motion (G-SDEs) has not been discussed yet. In this paper, we try to investigate the following SDE disturbed by G-Brownian motion

$$dX(t) = f(X(t))dt + g(X(t))dB(t) + h(X(t))d < B > (t) \tag{1.1}$$

on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$.

The main contribution of this paper is as follows.

- The stability in distribution of G-SDEs is discussed under sublinear expectations.
- We provide a criterion of the stability in distribution of G-SDEs.
- New mathematical techniques are employed.

2. Preliminaries

In this section, let us give the concept of sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, where Ω is a given state set and $\mathcal{H}$ a linear space of real valued functions defined on Ω. The concepts of sublinear expectation and G-Brownian motion come from Peng [8].

Below, we assume that G-Brownian motion is one-dimensional process with $G(\alpha) := \frac{1}{2}\hat{\mathbb{E}}[\alpha B(1)^2] = \frac{1}{2}(\bar{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-)$, where $\hat{\mathbb{E}}[B(1)^2] = \bar{\sigma}^2, -\hat{\mathbb{E}}[-B(1)^2] = \underline{\sigma}^2$, $0 < \underline{\sigma} \leq \bar{\sigma} < \infty$. Let $(\mathcal{H}_t)_{t\geq 0}$ be a σ-field filtration generated by G-Brownian motion $(B(t))_{t\geq 0}$. We know that the weakly compact family of probability measures $\mathcal{P}$ characterizes

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the degree of Knightian uncertainty. Especially, if $\mathcal{P}$ is singleton, i.e. $\{P\}$, then the model has no ambiguity. Moreover, the related calculus reduces to a classical one. In this paper, we set

$$\mathcal{P} = \{P^{(\sigma_\cdot)} : (\sigma_t)_{t\geq 0} \text{ is } \mathcal{H}_t\text{-progressively measurable process, and } \sigma_t \in [\underline{\sigma}, \bar{\sigma}] \text{ for each } t \in [0,\infty)\}$$

which is a weakly compact family of probability measures on $(\Omega, \mathcal{B}(\Omega))$. Here $dB(t) = \sigma(t)dw(t)$, and $w(t)$ is a classical standard Brownian motion under probability measure $P^{(\sigma_\cdot)}$ (see, e.g. Fei et al. [4]). Thus, from Fei et al [4, Proposition 2.3], we have $\hat{\mathbb{E}}[X] = \max_{P\in\mathcal{P}} E_P[X]$, where E_P is a classical linear expectation. We now define G-upper capacity $\mathbb{V}(\cdot)$ and G-lower capacity $\mathcal{V}(\cdot)$ by $\mathbb{V}(A) = \sup_{P\in\mathcal{P}} P(A),\ \forall A \in \mathcal{B}(\Omega)$, $\mathcal{V}(A) = \inf_{P\in\mathcal{P}} P(A),\ \forall A \in \mathcal{B}(\Omega)$. By Chen [1, Lemma 2.4], we have $\mathbb{V}(A) = 1 - \mathcal{V}(A^c)$, where A^c is the complement set of event A . The following stochastic processes are based on the generalized nonlinear expectation space be $(\Omega, \mathcal{H}, \hat{\mathbb{E}}, \mathbb{V}, (\mathcal{H}_t)_{t\geq 0})$.

Definition 2.1. (i). *The distribution $\mathfrak{F}_\xi$ generated by d-dimensional random variable ξ in $\mathcal{H}$ is defined by*

$$\mathfrak{F}_\xi(A) = \mathbb{V}\{\omega \in \Omega : \xi(\omega) \in A\} = \hat{\mathbb{E}}[\mathbf{1}_{\{\omega\in\Omega:\xi(\omega)\in A\}}],\ \forall A \in \mathcal{B}(\mathbb{R}^d).$$

(ii). *For random variables ξ and η, we denote their distributions by $\mathfrak{F}_\xi$ and $\mathfrak{F}_\eta$, respectively. Define the distance of distributions of random variables ξ and η as follows $d_{\mathbb{T}}(\mathfrak{F}_\xi, \mathfrak{F}_\eta) = \sup_{\phi\in\mathbb{T}} |\hat{\mathbb{E}}[\phi(\xi)] - \hat{\mathbb{E}}[\phi(\eta)]|$, where $\mathbb{T} = \{\phi : \mathbb{R}^d \to \mathbb{R} : |\phi(x) - \phi(y)| \leq |x - y|$ and $|\phi(\cdot)| \leq 1\}$.*
(iii). *For the stochastic process $(x(t))_{t\geq 0}$ on sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}}, \mathbb{V}, (\mathcal{H}_t)_{t\geq 0})$, we denote the distribution of $x(t)$ by $\mathfrak{F}_{x(t)}$ for each $t \in [0,\infty)$. If there is a distribution $\nu(\cdot)$ of the random variable such that $d_{\mathbb{T}}(\mathfrak{F}_{x(t)}, \nu) = 0,$ as $t \to \infty$, then, the stochastic process $(x(t))_{t\geq 0}$ is called the (asymptotic) stability in distribution. We also call the process $(x(t))_{t\geq 0}$ converges weakly to the distribution ν.*

Lemma 2.2. *For two random variables ξ, η in $\mathcal{H}$, we have $d_{\mathbb{T}}(\mathfrak{F}_\xi, \mathfrak{F}_\eta) \leq \sup_{\phi\in\mathbb{T}} |\hat{\mathbb{E}}[\phi(\xi) - \phi(\eta)]|$.*

Proof: For two random variables ξ, η in $\mathcal{H}$, decompose $\mathbb{T} = \mathbb{T}^+ \cup \mathbb{T}^-$, where $\mathbb{T}^+ = \{\phi \in \mathbb{T} : \hat{\mathbb{E}}\phi(\xi) - \hat{\mathbb{E}}\phi(\eta) \geq 0\}$, and $\mathbb{T}^- = \{\phi \in \mathbb{T} : \hat{\mathbb{E}}\phi(\xi) - \hat{\mathbb{E}}\phi(\eta) < 0\}$. By the property of sublinear expectation, we easily deduce our claim. □

Denote the family of capacities on $\mathbb{R}^d$ by $C(\mathcal{B}(\mathbb{R}^d))$. It is easy to know that the metric $d_{\mathbb{T}}$ on $C(\mathcal{B}(\mathbb{R}^d))$ is a distance, and $(C(\mathcal{B}(\mathbb{R}^d)), d_{\mathbb{T}})$ is a Polish space.

From [2, Proposition 2.1], we have the following inequality.

Lemma 2.3. *Let ξ be a nonnegative random variable in $\mathcal{H}$, and φ: $\mathbb{R}_+ \to \mathbb{R}_+$ a nondecreasing function. Then we have, for any $\varepsilon > 0$, $\mathbb{V}\{\xi \geq \varepsilon\} \leq \frac{\hat{\mathbb{E}}[\varphi(\xi)]}{\varphi(\varepsilon)}$.*

Now we consider the stochastic differential equation (1.1) with initial value $X(0) = x$, where $(B(t))_{t\geq 0}$ is the G-Brownian motion in $\mathbb{R}$ on the generalized nonlinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}}, \mathbb{V}, (\mathcal{H}_t)_{t\geq 0})$, and

$$f : \mathbb{R}^d \to \mathbb{R}^d,\ g : \mathbb{R}^d \to \mathbb{R}^d,\ h : \mathbb{R}^d \to \mathbb{R}^d.$$

For Eq. (1.1), we provide the following locally Lipschitzian and linear growth conditions.

Assumption 2.4. *The functions f, g, h are locally Lipschitzian, i.e., for each k, there exists $b_k > 0$ such that*

$$|f(x) - f(y)| \vee |g(x) - g(y)| \vee |h(x) - h(y)| \leq b_k|x - y|$$

for those $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq k$, and the linear growth condition $|f(x)| + |g(x)| + |h(x)| \leq b(1 + |x|)$ for all $x \in \mathbb{R}^d$, and $b > 0$.

Under Assumption 2.4, we deduce Eq. (1.1) has a unique continuous solution $X(t)$ on $t \geq 0$ from Peng [8] or Fei et al. [5].

Let $C^2(\mathbb{R}^d; \mathbb{R}_+)$ denote the all nonnegative functions $V(x)$ on $\mathbb{R}^d$ being continuously twice differential in x. For $V \in C^2(\mathbb{R}^d; \mathbb{R}_+)$, we define G-operator $\mathbb{L} : \mathbb{R}^d \to \mathbb{R}$ by $\mathbb{L}V(x) = V_x(x)f(x) + G(2V_x(x)h(x) + g^\top(x)V_{xx}(x)g(x))$.

For the convenience of the reader, we provide the following lemma.

Lemma 2.5. *Let $V \in C^2(\mathbb{R}^d; \mathbb{R}_+)$ and τ_1, τ_2 be bounded stopping times such that $0 \leq \tau_1 \leq \tau_2$ q.s. If both $V(X(t))$ and $\mathbb{L}(V(X(t)))$ are bounded on $t \in [\tau_1, \tau_2]$ quasi surely, then we have*

$$\hat{\mathbb{E}}[V(X(\tau_2))] \leq \hat{\mathbb{E}}[V(X(\tau_1))] + \hat{\mathbb{E}}\left[\int_{\tau_1}^{\tau_2} \mathbb{L}V(X(s))ds\right].$$

Proof: From G-Itô formula from Peng [8, Proposition 3.6.3], we easily derive the claim. □

3. Asymptotic stability in distribution

In this section, we will set up some sufficient criteria on the asymptotic stability in distribution the solution process $X^x(t)$ of SDE (1.1) with initial value $X(0) = x \in \mathbb{R}^d$. We now give the following definition.

Definition 3.1. *SDE (1.1) is called to have property (P1) if for any $x \in \mathbb{R}^d$ and any $\varepsilon > 0$, there exists a positive constant H such that $\mathbb{V}\{|X^x(t)| \geq H\} < \varepsilon,\ \forall t \geq 0$. SDE (1.1) is called to have property (P2) if for any $\varepsilon > 0$ and any compact subset K of $\mathbb{R}^d$, there exists a $T = T(\varepsilon, K) > 0$ such that $\mathcal{V}\{|X^x(t) - X^y(t)| < \varepsilon\} \geq 1 - \varepsilon,\ \forall t \geq T$ for $x, y \in K$.*

It is well to know that property (P1) guarantees that for each $\varepsilon > 0$ there exists a compact subset $K = K(\varepsilon, x)$ of $\mathbb{R}^d$ such that $\mathcal{V}(X^x(t) \in K) \geq 1 - \varepsilon$. We now state our main result.

Theorem 3.2. *Let Assumption 2.4 hold. If SDE (1.1) have properties (P1) and (P2), then SDE (1.1) is asymptotically stable in distribution.*

In order to complete the proof of this theorem, we need to prove three lemmas.

Lemma 3.3. *Let Assumption 2.4 be satisfied and SDE (1.1) have property (P2). Then, for each compact subset K of $\mathbb{R}^d$, we have $\lim_{t\to\infty} d_{\mathbb{T}}(\mathfrak{F}_{X^x(t)}, \mathfrak{F}_{X^y(t)}) = 0$ uniformly in $x, y \in K$.*

Proof. Set $A_t = \{|X^x(t) - X^y(t)| < \varepsilon/4\}$. Thus, there is $T > 0$ such that for $t \geq T$, we get $\mathbb{V}(A_t^c) = 1 - \mathcal{V}(A_t) \leq \varepsilon/4$ by property (P2). Moreover, we have, for each $P \in \mathcal{P}$ and $\forall t \geq T$,

$$\begin{aligned}
&|E_P[\phi(X^x(t))] - E_P[\phi(X^y(t))]| \leq E_P[2 \wedge |X^x(t) - X^y(t)|] \\
&\leq \hat{\mathbb{E}}[\mathbf{1}_{A_t} 2 \wedge |X^x(t) - X^y(t)|] + \hat{\mathbb{E}}[\mathbf{1}_{A_t^c} 2 \wedge |X^x(t) - X^y(t)|] < \frac{\varepsilon}{2} + 2\mathbb{V}(A_t^c) \leq \varepsilon.
\end{aligned}$$

Since ϕ, P are arbitrary, we obtain that $\sup_{\phi\in\mathbb{T}} |\hat{\mathbb{E}}[\phi(X^x(t)) - \phi(X^y(t))]| < \varepsilon,\ \forall t \geq T$, which shows, by Lemma 2.2, $d_{\mathbb{T}}(\mathfrak{F}_{X^x(t)}, \mathfrak{F}_{X^y(t)}) < \varepsilon, \forall t \geq T$ for all $x, y \in \mathbb{R}^d$. Thus, we complete the proof. □

Lemma 3.4. *Let Assumption 2.4 be satisfied and SDE (1.1) have properties (P1) and (P2). Then, for any $x \in \mathbb{R}^d$, $\{\mathfrak{F}_{X^x(t)}(\cdot) : t \geq 0\}$ is Cauchy in the capacity space $C(\mathcal{B}(\mathbb{R}^d))$ with metric $d_{\mathbb{T}}$.*

Proof. Fix $x \in \mathbb{R}^d$ arbitrarily. We need to prove that for any $\varepsilon > 0$, there is a $T > 0$ such that

$$d_{\mathbb{T}}(\mathfrak{F}_{X^x(t+s)}, \mathfrak{F}_{X^x(t)}) < \varepsilon, \forall t \geq T,\ s > 0.$$

By Lemma 2.2, we only need to show

$$\sup_{\phi\in\mathbb{T}} |\hat{\mathbb{E}}[\phi(X^x(t+s)) - \phi(X^x(t))]| < \varepsilon,\ \forall t \geq T,\ s > 0. \tag{3.1}$$

In fact, for any $\phi \in \mathbb{T}$ and $t, s > 0$, and each $P \in \mathcal{P}$, we get

$$\begin{aligned}
&|E_P[\phi(X^x(t+s))] - E_P[\phi(X^x(t))]| \leq |E_P[E_P[\phi(X^x(t+s))]|\mathcal{H}_s]] - E_P[\phi(X^x(t))]| \\
&\leq \left| \int_{\mathbb{R}^d} E_P[\phi(X^z(t))] p(s, x, dz) - E_P[\phi(X^x(t))] \right| \leq \int_{\mathbb{R}^d} |E_P[\phi(X^z(t))] - E_P[\phi(X^x(t))]| p(s, x, dz) \\
&\leq 2\mathbb{V}(X^x(s) \in B_H^c) + \int_{B_H} |E_P[\phi(X^z(t))] - E_P[\phi(X^x(t))]| p(s, x, dz),
\end{aligned} \tag{3.2}$$

where $B_H = \{x \in \mathbb{R}^d : |x| \leq H\}$ and $B_H^c = \mathbb{R}^d \setminus B_H$, and $p(s, x, dz)$ denotes the transition probability of solution process $X^x(s)$ with $X^x(0) = x$ under probability measure P. From property (P1), there exists a positive number H sufficiently large such that

$$\mathbb{V}(X^x(s) \in B_H^c) < \frac{\varepsilon}{4},\ \forall s > 0. \tag{3.3}$$

Moreover, by Lemma 3.3, there exists a $T(P) > 0$ such that

$$\sup_{\phi\in\mathbb{T}} |E_P[\phi(X^z(t))] - E_P[\phi(X^x(t))]| \leq \frac{\varepsilon}{2},\ \forall t \geq T(P) \tag{3.4}$$

for any $z \in B_H$. Thus, substituting (3.4) and (3.3) into (3.2), due to weak compactness of $\mathcal{P}$, we can take $\bar{T} = \max_{P\in\mathcal{P}} T(P) < \infty$ such that $|\hat{\mathbb{E}}[\phi(X^x(t+s)) - \phi(X^x(t))]| \le \varepsilon,\ \forall t \ge \bar{T},\ s > 0$. Since ϕ is arbitrary, the inequality (3.1) holds. Thus, the proof is complete. □

Proof of Theorem 3.2. From Lemma 3.4, $\{\mathfrak{F}_{X^x(t)} : t \ge 0\}$ is Cauchy in the space $C(\mathcal{B}(\mathbb{R}^d))$ with metric $d_{\mathbb{T}}$. Thus, there exists a unique $\nu \in C(\mathcal{B}(\mathbb{R}^d))$ such that $\lim_{t\to\infty}(\mathfrak{F}_{X^0(t)}, \nu) = 0$. Moreover, by Lemma 3.3, for each $x \in K$, where K is a compact subset of $\mathbb{R}^d$, we obtain $\lim_{t\to\infty} d_{\mathbb{T}}(\mathfrak{F}_{X^x(t)}, \nu) \le \lim_{t\to\infty}[d_{\mathbb{T}}(\mathfrak{F}_{X^0(t)}, \nu) + d_{\mathbb{T}}(\mathfrak{F}_{X^x(t)}, \mathfrak{F}_{X^0(t)})] = 0$. Thus, we prove the claim. □

The following proposition characterizes an equivalent definition of stability in distribution under sublinear expectation. Here, weak convergence of capacities is defined by $F^P_{\xi_n} \xrightarrow{w} F^P_{\xi}$ for each $P \in \mathcal{P}$ as $n \to \infty$, where, $F^P_{\xi_n}$ and F^P_{ξ} denote the distribution functions of random variables ξ_n and ξ under probability P, respectively. Here, $\xrightarrow{w}$ stands for weak convergence of probability distribution.

Proposition 3.5. *Weak convergence of capacities implies $d_{\mathbb{T}}(\mathfrak{F}_{\xi_n}, \mathfrak{F}_{\xi}) = 0$ as $n \to \infty$.*

Proof. By Ikeda and Watanabe [6, Proposition 2.5 in Chapter I], together with Lemma 2.2, we get easily our claims. □

Above proposition helps us to intuitively understand the stability in distribution under sublinear expectation.

4. Sufficient criteria for properties (P1) and (P2)

Theorem 3.2 above depends on properties (P1) and (P2). For applications of Theorem 3.2, we will establish sufficient criteria. Property (P1) is associated with boundedness while property (P2) is concerned uniformly asymptotic stability. Thus, the importance of this section is clear.

Let $\mathcal{K}$ denote the family of increasing functions $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\kappa(0) = 0$, and $\mathcal{K}_\infty$ the family of functions $\kappa \in \mathcal{K}$ such that $\kappa(s) \to \infty$ as $s \to \infty$.

Assumption 4.1. *Suppose that there are functions $V \in C^2(\mathbb{R}^d; \mathbb{R}_+)$, $\kappa \in \mathcal{K}_\infty$ positive numbers μ and α_1 such that*

$$\kappa(|x|) \le V(x), \tag{4.1}$$

$$\mathbb{L}V(x) \le -\alpha_1 V(x) + \mu \tag{4.2}$$

for all $x \in \mathbb{R}^d$.

We now give a criterion for property (P1).

Proposition 4.2. *Let Assumption 4.1 hold. Then SDE (1.1) has property (P1).*

Proof. Fix any $x \in \mathbb{R}^d$, and denote $X^x(t) = X(t)$. Let k be a positive integer. Define the stopping time $\tau_k = \inf\{t > 0 : |X(t)| \ge k\}$. Clearly, $\tau_k \to \infty$ q.s. as $k \to \infty$. Let $t_k = \tau_k \wedge t$ for each t. The G-Itô formula shows that

$$\hat{\mathbb{E}}[e^{\alpha_1 t_k} V(X(t_k))] \le V(x) + \hat{\mathbb{E}}\Big[\int_0^{t_k} e^{\alpha_1 s}\mathbb{L}V(X(s))ds\Big] + \alpha_1\hat{\mathbb{E}}\Big[\int_0^{t_k} e^{\alpha_1 s} V(X(s))ds\Big].$$

In terms of (4.1) and (4.2), we have $\hat{\mathbb{E}}[e^{\alpha_1 t_k} V(X(t_k))] \le V(x) + \mu\int_0^{t_k} e^{\alpha_1 s}ds = V(x) + \frac{\mu}{\alpha_1}[e^{\alpha_1 t_k} - 1]$. Letting $k \to \infty$, we have

$$\hat{\mathbb{E}}[V(X(t))] \le V(x) + \frac{\mu}{\alpha_1}, \tag{4.3}$$

which, together with (4.1), shows $\hat{\mathbb{E}}[\kappa(|X(t)|)] \le C,\ \forall t \ge 0$, where C denotes the right-hand term of (4.3). Thus, by Lemma 2.3, we have

$$\mathbb{V}\{|X(t)| \ge H\} \le \frac{\hat{\mathbb{E}}[\kappa(|X(t)|)]}{\kappa(H)} \le \frac{C}{\kappa(H)},\ \forall t \ge 0.$$

Therefore, for any $\varepsilon > 0$, taking sufficiently large H such that $C/\kappa(H) < \varepsilon$, we prove the result. □

Next, to establish a criterion for property (P2), we consider the difference between two solutions of (1.1) starting from different initial values x, y

$$\begin{aligned} X^x(t) - X^y(t) =& x - y + \int_0^t [f(X^x(s)) - f(X^y(s))]ds + \int_0^t [g(X^x(s)) - g(X^y(s))]dB(s) \\ &+ \int_0^t [h(X^x(s)) - h(X^y(s))]d < B > (s). \end{aligned} \tag{4.4}$$

For a given function $W \in C^2(\mathbb{R}^d; \mathbb{R}_+)$, define G-Itô operator $\mathcal{L}W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ associated with SDE (4.4) by

$$\begin{aligned} \mathcal{L}W(x, y) =& W_x(x - y)[f(x) - f(y)] \\ &+ G(2W_x(x - y)[h(x) - h(y)] + \mathrm{trace}([g(x) - g(y)]^\top W_{xx}(x - y)[g(x) - g(y)])). \end{aligned} \tag{4.5}$$

We now give the following assumption for establishing property (P2).

Assumption 4.3. *Suppose that there exist functions $W \in C^2(\mathbb{R}^d; \mathbb{R}_+)$, $\nu_1 \in \mathcal{K}_\infty$, $\nu_2 \in \mathcal{K}$ such that*

$$W(0) = 0, \tag{4.6}$$

$$\nu_1(|x|) \le W(x), \ \forall x \in \mathbb{R}^d, \tag{4.7}$$

$$\mathcal{L}W(x, y) \le -\nu_2(|x - y|), \ \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{4.8}$$

Proposition 4.4. *Let Assumption 4.3 hold. Then SDE (1.1) has property (P2).*

Proof. For each $\varepsilon \in (0, 1)$, in terms of continuity of W and (4.6), we can select $\beta \in (0, \varepsilon)$ sufficiently small for

$$\frac{\sup_{|u|\le\beta} W(u)}{\nu_1(\varepsilon)} < \frac{\varepsilon}{2}. \tag{4.9}$$

Let K be a compact subset of $\mathbb{R}^d$ and fix any $x, y \in K$. Define now the stopping times $\tau_\beta = \inf\{t \ge 0 : |X^x(t) - X^y(t)| \le \beta\}$ and $\tilde{\tau}_\gamma = \inf\{t \ge 0 : |X^x(t) - X^y(t)| \ge \gamma\}$, where $\gamma > \beta$. Set $t_\gamma = \tilde{\tau}_\gamma \wedge t$. By the G-Itô formula, (4.7) and (4.8), we derive that for each $t > 0$,

$$\begin{aligned} \nu_1(\gamma)\mathbb{V}\{\tilde{\tau}_\gamma \le t\} &\le \hat{\mathbb{E}}[\mathbf{1}_{\{t_\gamma \le t\}} W(X^x(t_\gamma) - X^y(t_\gamma))] \le \hat{\mathbb{E}}[W(X^x(t_\gamma) - X^y(t_\gamma))] \\ &\le W(x - y) + \hat{\mathbb{E}}\left[\int_0^{t_\gamma} \mathcal{L}W(X^x(s), X^y(s))ds\right] \le W(x - y). \end{aligned}$$

Moreover, we have $\mathbb{V}\{\tilde{\tau}_\gamma \le t\} \le \frac{W(x-y)}{\nu_1(\gamma)}$. For $(x, y) \in K \times K$, the function $W(x - y)$ is bounded, which implies that there is a $\gamma = \gamma(K, \varepsilon) > 0$ such that

$$\mathbb{V}\{\tilde{\tau}_\gamma < \infty\} \le \frac{\varepsilon}{4}. \tag{4.10}$$

Fix the γ. Let $t_\beta = \tau_\beta \wedge \tilde{\tau}_\gamma \wedge t$. Through the G-Itô formula and (4.8), we can deduce that for each $t > 0$, each $P \in \mathcal{P}$,

$$0 \le E_P[W(X^x(t_\beta) - X^y(t_\beta))] = W(x - y) + E_P\left[\int_0^{t_\beta} \mathcal{L}W(X^x(s), X^y(s))ds\right] \le W(x - y) - \nu_2(\beta)E_P(\tau_\beta \wedge \tilde{\tau}_\gamma \wedge t).$$

This shows $E_P(\tau_\beta \wedge \tilde{\tau}_\gamma \wedge t) \le \frac{W(x-y)}{\nu_2(\beta)}$, which derive $\hat{\mathbb{E}}(\tau_\beta \wedge \tilde{\tau}_\gamma \wedge t) \le \frac{W(x-y)}{\nu_2(\beta)}$. Thus, from above inequality we get

$$t\mathbb{V}\{\tau_\beta \wedge \tilde{\tau}_\gamma \ge t\} = \hat{\mathbb{E}}[(\tau_\beta \wedge \tilde{\tau}_\gamma \wedge t)\mathbf{1}_{\{\tau_\beta \wedge \tilde{\tau}_\gamma \ge t\}}] \le \hat{\mathbb{E}}(\tau_\beta \wedge \tilde{\tau}_\gamma \wedge t) \le \frac{W(x - y)}{\nu_2(\beta)}.$$

Moreover, there exists a positive constant $T = T(K, \varepsilon)$ such that $\mathcal{V}\{\tau_\beta \wedge \tilde{\tau}_\gamma \le T\} > 1 - \frac{\varepsilon}{4}$. From (4.10), we get

$$1 - \frac{\varepsilon}{4} < \mathcal{V}\{\tau_\beta \wedge \tilde{\tau}_\gamma \le T\} \le \mathcal{V}\{\tau_\beta \le T\} + \mathcal{V}\{\tilde{\tau}_\gamma < \infty\} \le \mathcal{V}\{\tau_\beta \le T\} + \frac{\varepsilon}{4},$$

which deduce $\mathcal{V}\{\tau_\beta \le T\} \ge 1 - \frac{\varepsilon}{2}$. Thus, by Chen [1, Lemma 2.4] we have

$$\mathbb{V}\{\tau_\beta > T\} \le \frac{\varepsilon}{2}. \tag{4.11}$$

We now define the stopping times $\rho = \inf\{t \geq \tau_\beta \wedge T : |X^x(t) - X^y(t)| \geq \varepsilon\}$. From (4.8), we know

$$\mathcal{L}W(X^x(\tau_\beta \wedge t), X^y(\tau_\beta \wedge t)) \leq -\nu_2(|X^x(\tau_\beta \wedge t) - X^y(\tau_\beta \wedge t)|) \leq 0.$$

Thus, from the G-Itô formula, we get

$$\begin{aligned} &\hat{\mathbb{E}}[\mathbf{1}_{\{\tau_\beta \leq T\}} W(X^x(\rho \wedge t) - X^y(\rho \wedge t))] \leq \hat{\mathbb{E}}[\mathbf{1}_{\{\tau_\beta \leq T\}} W(X^x(\tau_\beta \wedge t) - X^y(\tau_\beta \wedge t))] \\ &+ \hat{\mathbb{E}}\left[\int_{\tau_\beta \wedge t}^{\rho \wedge t} \mathbf{1}_{\{\tau_\beta \leq T\}} \mathcal{L}W(X^x(s), X^y(s)) ds\right] \leq \hat{\mathbb{E}}[\mathbf{1}_{\{\tau_\beta \leq T\}} W(X^x(\tau_\beta \wedge t) - X^y(\tau_\beta \wedge t))]. \end{aligned} \tag{4.12}$$

For $t > T$, from (4.12), we have

$$\begin{aligned} &\mathbb{V}(\{\tau_\beta \leq T\} \cap \{\rho \leq t\})\nu_1(\varepsilon) \leq \hat{\mathbb{E}}[\mathbf{1}_{\{\tau_\beta \leq T, \rho \leq t\}} W(X^x(\rho \wedge t) - X^y(\rho \wedge t))] \\ &\leq \hat{\mathbb{E}}[\mathbf{1}_{\{\tau_\beta \leq T\}} W(X^x(\rho \wedge t) - X^y(\rho \wedge t))] \leq \hat{\mathbb{E}}[\mathbf{1}_{\{\tau_\beta \leq T\}} W(X^x(\tau_\beta \wedge t) - X^y(\tau_\beta \wedge t))] \\ &= \hat{\mathbb{E}}[\mathbf{1}_{\{\tau_\beta \leq T\}} W(X^x(\tau_\beta) - X^y(\tau_\beta))] \leq \mathbb{V}\{\tau_\beta \leq T\} \sup_{|u| \leq \beta} W(u) \leq \sup_{|u| \leq \beta} W(u), \end{aligned}$$

which, together with (4.9), get

$$\mathbb{V}(\{\tau_\beta \leq T\} \cap \{\rho \leq t\}) < \frac{\varepsilon}{2}. \tag{4.13}$$

By (4.11) and (4.13), we get $\mathbb{V}\{\rho \leq t\} \leq \mathbb{V}(\{\tau_\beta \leq T\} \cap \{\rho \leq t\}) + \mathbb{V}\{\tau_\beta > T\} < \varepsilon$. Letting $t \to \infty$ we obtain $\mathbb{V}\{\rho < \infty\} < \varepsilon$, which shows that for any $(x, y) \in K \times K$, we have $\mathcal{V}\{|X^x(t) - X^y(t)| < \varepsilon\} > 1 - \varepsilon$, $\forall t \geq T$. Thus, we complete the proof. □

5. Examples

In this section, we provide two examples for illustrating our conclusions.

Example 5.1. Let $B(t)$ be a scalar G-Brownian motion, and α and σ be constants. Consider the Ornstein-Uhlenbeck process

$$dX(t) = \alpha X(t)dt + \sigma dB(t), \ t \geq 0 \tag{5.1}$$

with initial value $X(0) = x_0 \in \mathbb{R}^d$. SDE (5.1) has a unique solution

$$X(t) = e^{\alpha t} x_0 + \sigma \int_0^t e^{\alpha(t-s)} dB(s). \tag{5.2}$$

We observe that when $\alpha < 0$, the distribution of the solution $X(t)$ converges to the G-normal distribution $\nu \sim N(0, [\underline{\sigma}^2/(2|\alpha|), \bar{\sigma}^2/(2|\alpha|)])$ as $t \to \infty$ for any x_0, but when $\alpha \geq 0$, the distribution will not converge. Thus, SDE (5.2) is asymptotically stable in distribution if $\alpha < 0$ but it is not if $\alpha \geq 0$.

In fact, in order to apply Propositions 4.2 and 4.4, we set $V(x) = x^2$. Thus, we get $\mathbb{L}V(x) = 2\alpha V(x) + \sigma^2\bar{\sigma}^2$. Moreover, Assumption 4.1 holds as $\alpha < 0$. On the other hand, we obtain

$$\mathcal{L}V(x, y) = 2\alpha(x - y)^2 + \sigma^2\bar{\sigma}^2 = 2\alpha V(x - y) + \sigma^2\bar{\sigma}^2,$$

which implies Assumption 4.3 holds as $\alpha < 0$. Then by Propositions 4.2 and 4.4, we can also get stability in distribution of SDE (5.1) as $\alpha < 0$.

(**Algorithm and simulation**) In order to illustrate stability in distribution of solution of SDE (5.1), we provide an algorithm as follows. We select $\underline{\sigma} = \sigma_0 < \sigma_1 < \cdots < \sigma_m = \bar{\sigma}$ such that $\sigma_{i+1} - \sigma_i = \sigma_i - \sigma_{i-1}, i = 1, \cdots, m$. Let h be a small positive number. For any $t > 0$, there a positive integer k such that $t \in [(k-1)h, kh)$. The discrete approximation solution of SDE (5.2) with probability measure $P^{(\sigma_i)}$ can be expressed as

$$X^j(kh, \sigma_i) = e^{\alpha(k-1)h} x_0 + e^{\alpha(k-1)h} \sum_{\ell=0}^{k-1} e^{-\alpha \ell h} \sigma_i \Delta w_\ell^j,$$

where for each ℓ ($\ell = 0, \cdots, k-1$), Δw_ℓ^j ($j = 1, \cdots, n$) are random numbers from the normal distribution $\Delta w \sim N(0, h)$. Let $\nu_i^j (j = 1, \cdots, n)$ are random numbers from normal distribution $\nu_i \sim N(0, \sigma_i^2/(2|\alpha|))$. Define the error of two distributions of random variables $X(t)$ and ν as follows

$$e(kh) = \frac{1}{(m+1)n} \sum_{i=0}^{m} \sum_{j=1}^{n} \Big((X^j(kh, \sigma_i))^2 - (\nu_i^j)^2\Big).$$

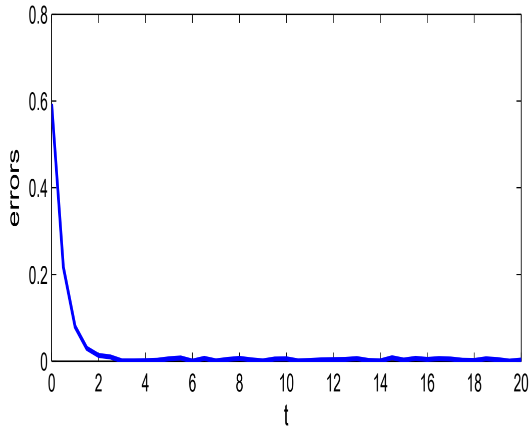

Fig. 1. The computer simulation of the errors $|e(kh)|$ with $\alpha = -1, \underline{\sigma} = 0.8, \bar{\sigma} = 1, x_0 = 1, h = 0.001, n = 1000, m = 20$.

Obviously, for each $\sigma_i (i = 0, 1, \cdots, m)$, $E_{P(\sigma_i)}[X(t)] \to E_{P(\sigma_i)}[\nu]$ as $t \to \infty$. If the error $|e(kh)| \to 0$ as $k \to \infty$, then we claim stability in distribution of solution of SDE (5.1). Indeed, a simulation figure (see Figure 1) shows this assertion.

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