This is a peer-reviewed, author's accepted manuscript of the following research article:
Finkelstein, M., & Cha, J. H. (2022). Discussion of
"An overview of some classical models and discussion of the signature-based models of preventive maintenance".
Applied Stochastic Models in Business and Industry. https://doi.org/10.1002/asmb.2712

Discussion of Signature-based Models of Preventive Maintenance.

Ji Hwan Cha[1,a] and Maxim Finkelstein[b,c]

[a]Department of Statistics, Ewha Womans University, Seoul, 120-750, Rep. of Korea.

e-mail: jhcha@ewha.ac.kr

[b] Department of Mathematical Statistics, University of the Free State, Bloemfontein, SA
E-mail: FinkelM@ufs.ac.za

[c]University of Strathclyde, Glasgow, UK

First of all, we want to congratulate Asadi, Hashemi and Balakrishnan for this very interesting discussion that combines the detailed survey of the basic preventive maintenance models with a special emphasis on signature-based models and presentation and analysis of some potential directions of research in this area. The paper has around 50 pages of a journal format length, but one cannot say that it is 'too long' as it is versatile, well written and considers many interesting topics. Therefore, we think that this paper is a useful tool on general methods employed in preventive maintenance and specifically in signature-based reasoning.

The survey is mostly focused on age-based maintenance. In accordance with the definition and usefulness of signatures the described models are mostly applied to coherent systems of i.i.d. or exchangeable components. In what follows, we would like to outline the new approach to preventive maintenance that hopefully can be generalized/modified using tools described in this valuable paper. It is mostly related to the first part of the paper describing conventional age-based maintenance; however, modified signature-based methods can be also employed in the further research (e.g., for populations of items operating in a random environment modeled by a shock process).

This approach refers to the PM actions with respect to the whole population of degrading items. At each instant of chronological time, a population of i.i.d. items (e.g., cars of the same make, road machines, etc) can be characterized by a random variable of age. If this age is relatively large (in the appropriate stochastic sense), then to decrease it, some actions can be undertaken. Denote by $N(x,t)$ the age-specific population size at time t, for a given *static* population, i.e., the number of items of age x at time t. Let X_t denote a random age at a chronological time t of an item, which is picked out at random (with equal probabilities) from a population [1,2]

$$\pi_t(x) = \frac{N(x,t)}{\int_0^t N(u,t)du}, \; 0 \le x \le t \tag{1}$$

to be called the "*age composition*" (the denominator defines the size of a population).

Consider now the *dynamic* population. Assume now items with the lifetime CDF $F(x)$ are manufactured/incepted into operation with rate $B(t)$ at time t meaning that $B(t)$ is the number of items incepted into operation in the small unit interval of time. Then it is easy to see that [1]

---

[1] Corresponding Author

$$\pi_t(x) = \frac{B(t-x)\overline{F}(x)}{\int_0^t B(t-u)\overline{F}(u)du} \cdot I(0 \le x \le t), \tag{2}$$

where $I(0 \le x \le t)$ is the corresponding indicator.

Setting $B(t) = B, t \to \infty,$ we arrive at the well-known density of the equilibrium distribution:

$$\pi_\infty(x) = \frac{\overline{F}(x)}{\int_0^\infty \overline{F}(u)du} \equiv \frac{\overline{F}(x)}{\mu}, \tag{3}$$

where μ denotes the mean that corresponds to the Cdf $F(x)$.

The main objective of conventional PM for degrading items is to decrease the consequences of sudden failures of aged items by replacing/repairing them preventively at the optimally obtained times. In the case of populations, we can do the same, but just discarding an item. At some instances, it can be considered as a replacement, but it should be noted that the replacement is 'taken' from the pool of manufactured items, which is finite. Thus, now we have the fixed resources, which makes the situation different from the conventional PM, as discarding an item comes at a price of reducing the population of operating items.

Let Δ denote the truncation age at which items should be discarded. Then the initial distribution $F(x)$ is substituted by

$$F_\Delta(x) = \begin{cases} F(x), 0 \le x < \Delta, \\ 1, \qquad x \ge \Delta. \end{cases} \tag{4}$$

It can be easily seen that $F_\Delta(x) \ge F(x)$ and therefore, the lifetime after truncation is obviously smaller than that before it in the sense of the usual stochastic order.

Setting $B(t) = B, t \to \infty,$ similar to (3), we arrive at the new equilibrium density

$$\pi_{\infty,\Delta}(x) = \begin{cases} \dfrac{\overline{F}(x)}{\mu_\Delta}, & 0 \le x < \Delta \\ 0, & x \ge \Delta \end{cases} \tag{5}$$

where μ_Δ denotes the mean lifetime that corresponds to (4). Intuitively, it is clear that truncation should result in a smaller random age $X_{t,\Delta}$. It can be easily shown that this is in the sense of the likelihood ratio ordering, i.e., $X_{t,\Delta} \le_{lr} X_t$. Thus the main goal is achieved: population age becomes smaller. But at what

Let us choose an item at random from a stationary population with the truncation level Δ . Denote the failure rate of an item by $\lambda(x)$ and define an average population failure rate (PFR) as the following mixture:

$$\Lambda(\Delta) = \int_0^\Delta \lambda(x)\pi_{\infty,\Delta}(x)dx. \tag{6}$$

Thus, $\Lambda(\Delta)dt$ defined by (6) can be interpreted as the probability of failure in $[t, t+dt)$ of an item chosen at random from the population defined by the age composition (5). It can be easily seen that when $\lambda(x)$ is increasing this function is monotonically increasing with Δ to the 'nontruncated' limit. Thus, it is better to have smaller Δ, as it decreases the probability of failure of a chosen item. But, as was discussed, this comes at a price of a smaller number of items in a population. Note that the size of a population is proportional to μ and μ_Δ in (3) and (5), accordingly. This effect can be taken into account by considering the *relative* PFR when $\Lambda(\Delta)$ in (6), similar to our discussions of (1) and (3) is divided by the size of a population $B\mu_\Delta$, i.e.,

$$\Lambda_r(\Delta) = \frac{\int_0^\Delta \lambda(x)\pi_{\infty,\Delta}(x)dx}{B\mu_\Delta} = \frac{\int_0^\Delta \lambda(x)\bar{F}(x)dx}{B(\mu_\Delta)^2} \quad . \tag{7}$$

From general intuitive considerations it follows that the optimal solution of the following optimization problem exists for items with increasing failure rates, i.e.,

$$\Lambda_r(\Delta^*) = \min_{\Delta>0} \Lambda_r(\Delta) . \tag{8}$$

Thus, the optimal solution is achieving the minimum of the relative PFR, which means that the probability of failure of an item from the defined by (8) stationary population that is chosen at random is minimal for the truncation value Δ^*. We see that our reasoning differs from the conventional optimal PM framework. It also needs further detailed consideration and illustration via the real-life examples.

To conclude: the area of PM modeling is still rapidly developing using new methodology and settings. The paper by Asadi, Hashemi and Balakrishnan contributes to this process providing the up-to-date picture of current research and possible future directions. In the current note we have briefly discussed one of the possible future developments.

References

1. Cha, J.H., Finkelstein, M., 2018. On stochastic comparisons for population age structure and remaining lifetime. Statistical papres 59, 199-213.

2. Hazra, N., Finkelstein, M., Cha, J.H., 2018. Stochastic ordering for populations of manufactured items. Test 27, 173-196.