Nothing is Set in Stone

For, if it were not so, there would be something disappearing into nothing, which is mathematically absurd.
Aleister Crowley
The Book of Thoth

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Abstract

This note is concerned with the Laplace inversion of a function with two branch points. It employs the Bromwich contour from first principles to correct a result which has been in print for nearly 60 years.

1 Preamble

I admit I have been using the Schaum's outline series of books for both teaching and research from the mid-sixties till now. In particular, I found "Laplace Transforms" [1] particularly useful for teaching, if a little limited for any serious research. However, I have tended to believe that all printed results are correct, until now. The edition of the Schaum's series I purchased as an undergraduate was initially printed in 1965 and on page 248, no. 62, the following is given:

$$\begin{aligned} L^{-1}[\bar{f}(s)] &= L^{-1}\left[\frac{1}{\sqrt{s+a}+\sqrt{s+b}}\right] \\ &= \frac{e^{-bt}-e^{-at}}{2(b-a)\sqrt{\pi t^3}}, \end{aligned} \tag{1.1}$$

where L^{-1} denotes the inverse transform and t denotes time. That this is wrong is immediate if we let $b \to a$ and use L'Hospital's rule. This gives

$$\begin{aligned} \frac{1}{2}L^{-1}\left[\frac{1}{\sqrt{s+a}}\right] &= \lim_{b\to a}\frac{-te^{-bt}}{2\sqrt{\pi t^3}} \\ &= -\frac{e^{-at}}{2\sqrt{\pi t}} \end{aligned} \tag{1.2}$$

which clearly has the wrong sign. For example, to see this all one has to do is evaluate

$$\int_0^\infty \frac{e^{(-a+s)t}}{\sqrt{\pi t}} dt. \tag{1.3}$$

Yet this result has been in print for nearly 60 years and nobody, to my knowledge at any rate, has challenged it.

How I came across this error was because I was using (1.1) as an exemplar for a considerably more complicated problem involving chronoamperometric currents which required an inversion of a Laplace transform with two branch points. I felt that using an appropriate Bromwich contour would provide me with sufficient insight to tackle the more challenging problem. However, having undertaken the task I found that I was unable to achieve the same answer as in the Schaum's series "Laplace Transforms" [1]: by letting $b \to a$, as above, quickly convinced me that I was right and the book contained an error.

The remainder of this note is concerned with the derivation, from first principles, of the correct expression for (1.1).

2 Laplace transform inversion

First of all let us assume, without loss of generality, that $a > b > 0$.

Consider the contour displayed in Figure 1. The Laplace transform has no poles but does have two branch points at $s = -a, -b$, positioned at the centres of the two small circles, respectively. In the contour the small circles both have radius ϵ, while the larger circle has radius R. In the forthcoming analysis we implicitly assume that $\epsilon \to 0$ and $R \to \infty$.

The inverse transform of $\bar{f}(s)$ is given by

$$\begin{aligned} L^{-1}[\bar{f}(s)] &= \frac{1}{2\pi\imath}\int_A^B \bar{f}(s)e^{st}ds \\ &= -\frac{1}{2\pi\imath}\left(\int_{BC}+\int_{CD}+\int_{DE}+\int_{EF}+\int_{FG}+\int_{GH}+\int_{HI}+\int_{IJ}+\int_{JA}\right)\bar{f}(s)e^{st}ds. \end{aligned}$$

It can be readily shown that the integrals over the arcs BC, DE, FG, HI and JA are all zero and so we need only consider the remaining four line segments (See, e.g. [1, Ex 9, page 207]. Let us first consider the integral over CD.

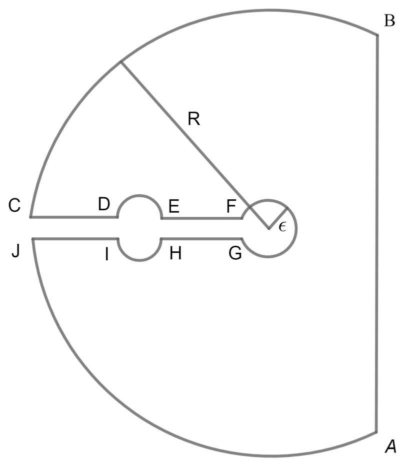

Figure 1: Appropriate contour containing two branch points

We have

$$\begin{aligned}
-\frac{1}{2\pi\imath}\int_{-R}^{-a-\epsilon} e^{st}\bar{f}(s)ds &= -\frac{1}{2\pi\imath}\int_{R}^{a+\epsilon} \frac{e^{-xt}(-dx)}{(u_1e^{\imath\pi})^{1/2}+(u_2e^{\imath\pi})^{1/2}} \\
&= \frac{1}{2\pi}\int_{a}^{\infty} \frac{e^{-xt}dx}{\sqrt{x-a}+\sqrt{x-b}}. \qquad (2.4)
\end{aligned}$$

The integral along EF provides

$$\begin{aligned}
-\frac{1}{2\pi\imath}\int_{-a+\epsilon}^{-b-\epsilon} e^{st}\bar{f}(s)ds &= -\frac{1}{2\pi\imath}\int_{a}^{b} \frac{e^{-xt}(-dx)}{(u_1)^{1/2}+(u_2e^{\imath\pi})^{1/2}} \\
&= -\frac{1}{2\pi\imath}\int_{b}^{a} \frac{e^{-xt}dx}{\sqrt{a-x}+\imath\sqrt{x-b}}. \qquad (2.5)
\end{aligned}$$

Now consider the integral along GH. We get

$$\begin{aligned}
-\frac{1}{2\pi\imath}\int_{-b-\epsilon}^{-a+\epsilon} e^{st}\bar{f}(s)ds &= -\frac{1}{2\pi\imath}\int_{b}^{a} \frac{e^{-xt}(-dx)}{(u_1)^{1/2}+(u_2e^{-\imath\pi})^{1/2}} \\
&= \frac{1}{2\pi\imath}\int_{b}^{a} \frac{e^{-xt}dx}{\sqrt{a-x}-\imath\sqrt{x-b}}. \qquad (2.6)
\end{aligned}$$

Finally consider the integral over IJ from which we obtain

$$\begin{aligned}
-\frac{1}{2\pi\imath}\int_{-a-\epsilon}^{-R} e^{st}\bar{f}(s)ds &= -\frac{1}{2\pi\imath}\int_a^\infty \frac{e^{-xt}(-dx)}{(u_1e^{-\imath\pi})^{1/2}+(u_2e^{-\imath\pi})^{1/2}} \\
&= \frac{1}{2\pi}\int_a^\infty \frac{e^{-xt}dx}{\sqrt{x-a}+\sqrt{x-b}}. \qquad (2.7)
\end{aligned}$$

This then allows us to write

$$\begin{aligned}
L^{-1}\left[\frac{1}{\sqrt{s+a}+\sqrt{s+b}}\right] &= \frac{1}{\pi}\int_a^\infty \frac{e^{-xt}dx}{\sqrt{x-a}+\sqrt{x-b}} \\
&- \frac{1}{2\pi\imath}\left\{\int_b^a \frac{e^{-xt}dx}{\sqrt{a-x}+\imath\sqrt{x-b}} - \int_b^a \frac{e^{-xt}dx}{\sqrt{a-x}-\imath\sqrt{x-b}}\right\}
\end{aligned}$$

and, after a little algebraic manipulation,

$$\begin{aligned}
L^{-1}\left[\frac{1}{\sqrt{s+a}+\sqrt{s+b}}\right] &= \frac{1}{\pi(a-b)}\int_a^\infty e^{-xt}(-\sqrt{x-a}+\sqrt{x-b})dx \\
&+ \frac{1}{\pi(a-b)}\int_b^a e^{-xt}\sqrt{x-b}\,dx \\
&= \frac{1}{\pi(a-b)}\left(\int_b^\infty e^{-xt}\sqrt{x-b}\,dx - \int_a^\infty e^{-xt}\sqrt{x-a}\,dx\right).
\end{aligned}$$

Now we have the following identity:

$$\int_\alpha^\infty e^{-xt}\sqrt{x-\alpha}\,dx = \frac{\sqrt{\pi}e^{-\alpha t}}{2\sqrt{t^3}}. \qquad (2.8)$$

Thus, using (2.8) with $\alpha = a$ and $\alpha = b$ we obtain

$$L^{-1}\left[\frac{1}{\sqrt{s+a}+\sqrt{s+b}}\right] = \frac{e^{-bt}-e^{-at}}{2(a-b)\sqrt{\pi t^3}}, \qquad (2.9)$$

which, of course, differs from (1.1) by a minus sign.

3 Concluding remarks

The Schaum series can be a useful addition when preparing undergraduate lecture notes. For a deeper understanding, however, this author recommends Doetsch [3]. For applications of Laplace transforms there is the old favourite by Carslaw and Jaeger [2]. A recent paper ([4]) by the author and his colleagues showed how apparently novel mathematical identities could be derived by applying different transforms to the same diffusion problem. An even more recent work [5] was concerned with obtaining the current at a cylindrical electrode in electrochemistry: this resulted in a generalization of the Jaeger integral [6] of Carslaw and Jaeger fame.

There are very few, if any books dealing with multiple branch points. However, the author with his colleagues ([7]) published a paper on drug diffusion in 2013 which provided details in the appendix as to how one inverts a Laplace transform which has 3 branch points.

The typographical error made by Murray Speigel [1] is understandable: his form, although incorrect, has symmetry and arguably appears more elegant and natural. However, errors do occur in books and the reader should always be reminded that nothing is set in stone.

References

[1] M. R. Spiegel, *Schaum's series of Theory and Problems of Laplace Transforms,* McGraw-Hill, New York, 1965.

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