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Enhanced Space-Time Covariance Estimation Based on a System Identification Approach

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***Abstract*—The error inflicted on a space-time covariance estimate due to the availability of only finite data is known to perturb the eigenvalues and eigenspaces of its z-domain equivalent, i.e., the cross-spectral density matrix. In this paper, we show that a significantly more accurate estimate can be obtained if the source signals driving the signal model are also accessible, such that a system identication approach for the source model becomes viable. We demonstrate this improved accuracy in simulations, and discuss its dependencies on the sample size and the signal to noise ratio of the data.**

I. Introduction

For broadband array data in a vector $\mathbf{x}[n] \in \mathbb{C}^M$ with time index $n \in \mathbb{Z}$, signal processing problems are often formulated using second order statistics, such as when aiming to minimise a mean squared error [1], [2]. Since relative delays between signal components are key to addressing the broadband nature of the signals, the space-time covariance matrix $\mathbf{R}[\tau] = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\}$, with $\mathcal{E}\{\cdot\}$ the expectation operator, therefore includes a lag parameter $\tau \in \mathbb{Z}$. Solutions to such problems typically rely on a diagonalisation of $\mathbf{R}[\tau]$. Since the standard eigenvalue decomposition (EVD) can only decouple $\mathbf{R}[\tau]$ for one specific value of τ, an EVD to diagonalise $\mathbf{R}[\tau]$ for all τ, or equivalently its z-transform $\boldsymbol{R}(z) = \sum_\tau \mathbf{R}[\tau]z^{-\tau}$ for all $z \in \mathbb{C}$, is required.

The problem of diagonalising a matrix $\boldsymbol{R}(z)$ is well-understood. An eigenvalue decomposition $\boldsymbol{R}(z) = \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{Q}^{\mathrm{P}}(z)$ exists for almost all analytic matrices [3], [4], such that the diagonal matrix $\boldsymbol{\Lambda}(z)$ contains the eigenvalues, and the $\boldsymbol{Q}(z)$ their corresponding, orthonormal eigenvectors, with $\boldsymbol{Q}^{\mathrm{P}}(z) = \boldsymbol{Q}^{\mathrm{H}}(1/z^*)$ involving the parahermitian, Hermitian, and complex coonjugation operators $\{\cdot\}^{\mathrm{P}}$, $\{\cdot\}^{\mathrm{H}}$, and $\{\cdot\}^*$, respectively [5]. This decomposition can be approximated by various algorithms, including the second order sequential best rotation (SBR2) [6]–[8], sequential matrix diagonalisation (SMD) [9]–[11], and a number of discrete Fourier transform (DFT)-based families of algorithms [12]–[20].

A number of application examples have been successfully addressed by the above algorithms, ranging, e.g., from coding [7], [21], beamforming [22], [23], angle of arrival estimation [24]–[26], speech enhancement [27]–[29], optimum precoder and equaliser resign for MIMO communications systems [30]–[33], and subspace scanning for weak transient signals [34]–[36].

In almost all of these applications, the space-time covariance matrix cannot be obtained via expectations but must be estimated from finite data. The estimate $\hat{\mathbf{R}}[\tau]$ will be prone to estimation errors, and the variance of the unbiased estimator based on N snapshot of data $\mathbf{x}[n]$, $n = 0, \cdots, (N-1)$ has been investigated in [37]. This deviation from the ground truth $\mathbf{R}[\tau]$ will in turn result in a perturbation of the eigenvalues and eigenspaces [38]–[40].

The impact of estimation errors is twofold. Firstly, an estimation error causes imprecision e.g. through subspace leakage for the above applications [41]. Secondly, e.g. overestimating the support of the space-time covariance matrix will result in polynomial matrices of higher order than necessary [43], counteracting many efforts to keep computational complexity low via e.g. numerical efficiency [44]–[47] or trimming of polynomials [48]–[50].

Therefore, in this paper we aim to enhance the estimate $\hat{\mathbf{R}}[\tau]$ and thus reduce the perturbation of its eigenvalue decomposition, as well as aid in keeping the polynomial orders of all factors low. This is achieved the source signals are accessible, such that the convolutive mixing system that contributes to $\mathbf{x}[n]$ can be estimated via system identication. This type of estimation for $\hat{\mathbf{R}}[\tau]$ is possible e.g. in loudspeaker-microphone setup such as in [26]–[29]. For this purpose, we review the EVD of a space-time covariance matrix in Sec. II. The source model that defined $\mathbf{R}[\tau]$ is introduced in Sec. III together with the unbiased estimator of [37]. Our proposed alternative system identification approach is outlined in Sec. IV, and compared to the unbiased estimator via simulations in Sec. V. Conclusions are drawn in Sec. VI.

II. Parahermitian Matrix EVD and Perturbation

A. Parahermitian Matrix EVD

The diagonalisation of the space-time covariance matrix $\mathbf{R}[\tau]$ was motivated in Sec. I as a way to solve broadband problems. Since the model of $\boldsymbol{R}[\tau]$ in Sec. III typically contains causal, stable system components, the z-transform $\boldsymbol{R}(z) = \sum_\tau \mathbf{R}[\tau]z^{-\tau}$ is analytic in $z \in \mathbb{C}$. To diagonalise $\mathbf{R}[\tau]$ for every lag value τ, or $\boldsymbol{R}(z)$ for every z, a standard EVD is insufficient. Instead, a parahermitian matrix EVD [3], [4]

$$\boldsymbol{R}(z) = \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{Q}^{\mathrm{P}}(z) \qquad (1)$$

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is required, where the diagonal parahermitian matrix $\boldsymbol{\Lambda}(z)$ contains the eigenvalues $\lambda_m(z)$, $m = 1, \ldots, M$. The corresponding eigenvectors form the columns of $\boldsymbol{Q}(z)$, which is paraunitary such that $\boldsymbol{Q}(z)\boldsymbol{Q}^{\mathrm{P}}(z) = \mathbf{I}$. Both $\boldsymbol{\Lambda}(z)$ and $\boldsymbol{Q}(z)$ can be selected to be analytic, such that (1) can be approximated well by Laurent polynomial terms.

Under some circumstances, the ground truth eigenvalues $\lambda_m(z)$, when evaluated on the unit circle, may satisfy spectral majorisation, such that

$$\lambda_1(\mathrm{e}^{\mathrm{j}\Omega}) \geq \lambda_2(\mathrm{e}^{\mathrm{j}\Omega}) \geq \ldots \geq \lambda_M(\mathrm{e}^{\mathrm{j}\Omega}) \ . \tag{2}$$

Though, generally (2) is not a given, and the ground truth eigenvalues of (7) may overlap. Note that the factorisations provided by the SBR2 [6], [7] and SMD [9], [11], [49] families of PEVD algorithm generally encourage (or can even be shown to guarantee [42]) spectral majorisation, thus conflicting with the analytic solution; in particular, the approximation of spectrally majorised eigenvalues can converge very slowly, requiring Laurent polynomials of much high order than for the analytic solution.

B. Perturbation of Eigenvalues

To investigate how a discrepancy between the ground truth $\mathbf{R}[\tau]$ and the estimated $\hat{\mathbf{R}}[\tau]$ perturbs the eigenvalues, recall from [40] that when evaluated at a specific normalised angular frequency Ω_0, the error in the eigenvalues is bounded due to the Hoffman-Wielandt theorem [39]

$$\sum_{m=1}^{M} \left(\hat{\lambda}_m(\mathrm{e}^{\mathrm{j}\Omega_0}) - \lambda_m(\mathrm{e}^{\mathrm{j}\Omega_0})\right)^2 \leq \|\boldsymbol{E}(\mathrm{e}^{\mathrm{j}\Omega_0})\|_{\mathrm{F}}^2 \ , \tag{3}$$

where $\boldsymbol{E}(\mathrm{e}^{\mathrm{j}\Omega_0}) = \boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega_0}) - \hat{\boldsymbol{R}}(\mathrm{e}^{\mathrm{j}\Omega_0})$, and $\hat{\lambda}_m(\mathrm{e}^{\mathrm{j}\Omega_0})$ are the eigenvalues of $\hat{\boldsymbol{R}}(z)$ evaluated for $z = \mathrm{e}^{\mathrm{j}\Omega_0}$. Thus the bin-wise perturbation of the eigenvalues depends directly on the accuracy of the space-time covariance estimate $\hat{\boldsymbol{R}}(z)$. Dependencies similar to (3) can be shown for the eigenspaces.

In the remainder of this paper we will concentrate on limiting the perturbation in (3) by reducing the error in $\hat{\boldsymbol{R}}(z)$.

III. Signal Model and Space-Time Covariance

A. Source Model

We assume that L zero-mean unit-variance uncorrelated sources $u_\ell[n]$, $\ell = 1, \ldots, L$, contribute to the measurements at M sensors via a matrix $\mathbf{H}[n] \in \mathbb{C}^{M\times L}$ of impulse responses as shown in Fig. 1. This system matrix $\mathbf{H}[n]$ is given as

$$\mathbf{H}[n] = \begin{bmatrix} h_{1,1}[n] & h_{1,2}[n] & \ldots & h_{1,L}[n] \\ h_{2,1}[n] & h_{2,2}[n] & \ldots & h_{2,L}[n] \\ \vdots & & \ddots & \vdots \\ h_{M,1}[n] & h_{M,2}[n] & \ldots & h_{M,L}[n] \end{bmatrix} , \tag{4}$$

where an element $h_{m,\ell}[n]$ is the impulse response connecting the ℓth source to the mth sensor. Using $\mathbf{H}[n]$, the contribution of all L sources at the mth sensor is

$$x_m[n] = \sum_{\ell=1}^{L} h_{m,\ell}[n] * u_\ell[n] + v_m[n] \ , \tag{5}$$

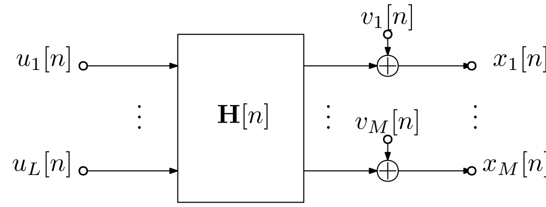

Fig. 1. Source model for the measurement vector $\mathbf{x}[n]$.

where $*$ denotes the convolution operator, and $v_m[n]$ is additive spatially and temporally uncorrelated noise. In matrix notation, for the measurement vector $\mathbf{x}[n] = [x_1[n], \ldots, x_M[n]]^{\mathrm{T}}$ we obtain

$$\mathbf{x}[n] = \mathbf{H}[n] * \mathbf{u}[n] + \mathbf{v}[n] \ , \tag{6}$$

with $\mathbf{u}[n] \in \mathbb{C}^L$ and $\mathbf{v}[n] \in \mathbb{C}^M$ the source signal and noise vectors, respectively, that are defined akin to $\mathbf{x}[n]$. We assume that $\mathbf{H}[n]$ is a finite impulse response system of order L_H.

B. Space-Time Covariance Matrix

With the source covariance $\mathcal{E}\{\mathbf{u}[n]\mathbf{u}^{\mathrm{H}}[n-\tau]\} = \mathbf{I}_L\delta[\tau]$ and the noise covariance $\mathcal{E}\{\mathbf{v}[n]\mathbf{v}^{\mathrm{H}}[n-\tau]\} = \sigma_v^2\mathbf{I}_M\delta[\tau]$, where $\mathcal{E}\{\cdot\}$ is the expectation operator and $\delta[\tau]$ the Kronecker function, the space time covariance $\mathbf{R}[\tau] = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\} \in \mathbb{C}^{M\times M}$ can be tied to the source model of Fig. 1 as

$$\mathbf{R}[\tau] = \sum_n \mathbf{H}[n]\mathbf{H}^{\mathrm{H}}[n-\tau] + \sigma_v^2\mathbf{I}_M\delta[\tau] \ . \tag{7}$$

Each element of $\mathbf{R}[\tau]$ is a cross-correlation

$$r_{\ell,m}[\tau] = \mathcal{E}\{x_\ell[n]x_m^*[n-\tau]\} \tag{8}$$

$$= \sum_n \sum_{k=1}^{L} h_{\ell,k}[n]h_{m,k}^*[n-\tau] + \sigma_v^2\delta[\tau]\delta[l-m] \ . \tag{9}$$

C. Unbiased Estimation

In applications, $\mathbf{R}[\tau]$ typically has to be estimated from finite data, leading to an estimated space-time covariance matrix $\hat{\mathbf{R}}[\tau]$. If N measurements $\mathbf{x}[n]$, $n = 0, \ldots, (N-1)$ are available, then an un-biased estimator for (8) can be defined as

$$\hat{r}_{\ell,m}[\tau] = \begin{cases} \frac{1}{N-|\tau|}\sum_{n=0}^{N-|\tau|-1} x_\ell[n+\tau]x_m^*[n], \ \tau \geq 0 \ , \\ \frac{1}{N-|\tau|}\sum_{n=0}^{N-|\tau|-1} x_\ell[n]x_m^*[n-\tau], \ \tau < 0 \ . \end{cases} \tag{10}$$

The variance of the estimator (10) is derived in [37] which forms the average power of the estimation error. It depends on both $\mathbf{R}[\tau]$ and N, and for the variance of one element $\hat{r}_{\ell,m}[\tau]$, we can state [37]

$$\mathrm{var}\{\hat{r}_{\ell,m}[\tau]\} = \frac{1}{(N-|\tau|)^2}\sum_{t=-N+|\tau|+1}^{N-|\tau|-1}(N-|\tau|-|t|) \cdot \left(r_{\ell,\ell}[t]r_{m,m}^*[t] + \bar{r}_{\ell,m}[\tau+t]\bar{r}_{\ell,m}^*[\tau-t]\right), \tag{11}$$

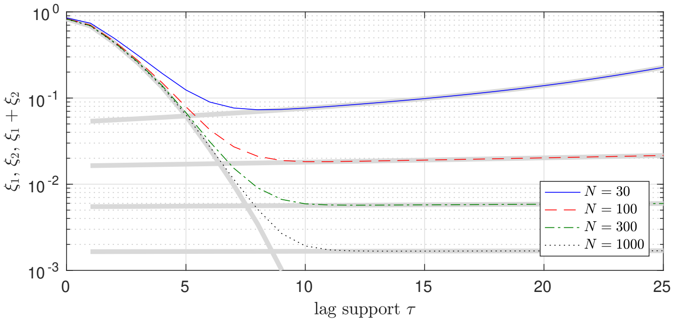

Fig. 2. Overall error ξ when estimating $\mathbf{H}[n]$ from data, in dependence of the number of lags τ, with truncation and estimation error terms ξ_1 and ξ_2, respectively, underlaid in grey for different sample sizes N.

where $\bar{r}_{\ell,m}[\tau] = \mathcal{E}\{x_\ell[n]x_m[n-\tau]\}$ is the complementary cross-correlation sequence. The overall estimation error

$$\xi = \sum_\tau \|\mathcal{E}\left\{\mathbf{R}[\tau] - \hat{\mathbf{R}}[\tau]\right\}\|_{\mathrm{F}}^2 \;, \tag{12}$$

with $\|\cdot\|_{\mathrm{F}}$ the Frobenius norm, can be minimised by judiciously setting the lag support [43].

Example. For some space-time matrix $\mathbf{R}[\tau] \in \mathbb{C}^{20\times 20}$ of polynomial order 88, Fig. 2 show the truncation error ξ_1 as well as the estimation error ξ_2, which make up the error term $\xi = \xi_1 + \xi_2$ in (12). Note that an increase of the sample size N reduces the estimation error, and increases the optimum lag support, i.e., the value for τ where ξ takes on its minimum in Fig. 2.

IV. Estimation via System Identification

In case we have significantly more access to the system in Fig. 1 and in addition to $\mathbf{x}[n]$ are able to acquire N samples of the source vector $\mathbf{u}[n]$, we can obtain an estimate for $\mathbf{R}[\tau]$ directly via (7), such that

$$\hat{\mathbf{R}}[\tau] = \sum_n \hat{\mathbf{H}}[n]\hat{\mathbf{H}}^{\mathrm{H}}[n-\tau] + \hat{\sigma}_v^2 \mathbf{I}_M \delta[\tau] \;, \tag{13}$$

where $\hat{\mathbf{H}}[n]$ is an estimate of the convolutive mixing system $\mathbf{H}[n]$, which we can obtain via adaptive system identification [2]. The estimate for the noise variance, $\hat{\sigma}_v^2$, can be reached via the minimum mean squared error. We outline these two steps below, followed by some thoughts on how to optimise the lag support in combination with the convolution operation in (13). Because with $\mathbf{u}[n]$, we know significantly more about our system, we also expect (13) to significantly exceed the estimate via (10) based on only $\mathbf{x}[n]$.

A. Adaptive System Identification

Various approaches can be used to perform system identification, including the least mean square and recursive least squares algorithms [2]. In order to operate analogously to the estimation of statistics over N time instances in Sec. III-C, we here select the Wiener solution to identify M separate L-channel adaptive filter problems based on (5),

$$\hat{x}_m[n] = \sum_{\ell=1}^{L} \hat{\mathbf{h}}_{m,\ell}^{\mathrm{H}} \mathbf{u}_\ell[n] = \hat{\mathbf{w}}_m^{\mathrm{H}} \mathbf{y}[n] \;. \tag{14}$$

In (14), $\hat{\mathbf{h}}_{m,\ell}^* \in \mathbb{C}^{L_f}$ contains the L_f estimated coefficients of the impulse response $h_{m,\ell}[n]$, and $\mathbf{u}_\ell[n] = [u_\ell[n], \ldots, u_\ell[n-L_f+1]]^{\mathrm{T}}$ is a tap delay line vector. For compactness of the mean square error problem

$$\hat{\mathbf{w}}_{m,\text{opt}} = \min_{\hat{\mathbf{w}}_m} \mathcal{E}\left\{|\mathbf{x}_m[n] - \hat{\mathbf{x}}_m[n]|^2\right\} \;, \tag{15}$$

we can further define

$$\hat{\mathbf{w}}_m = \begin{bmatrix} \hat{\mathbf{h}}_{m,1} \\ \vdots \\ \hat{\mathbf{h}}_{m,L} \end{bmatrix} \;, \qquad \mathbf{y}[n] = \begin{bmatrix} \mathbf{u}_1[n] \\ \vdots \\ \mathbf{u}_L[n] \end{bmatrix} \;, \tag{16}$$

as utilised in (14). With a sample covariance matrix $\hat{\mathbf{R}}$ and a vector $\hat{\mathbf{p}}_m$ estimating the quantities $\mathcal{E}\left\{\mathbf{y}[n]\mathbf{y}^{\mathrm{H}}[n]\right\}$ and $\mathcal{E}\{\mathbf{y}[n]x_m[n]\}$ over N time instances, we obtain [1], [2]

$$\hat{\mathbf{w}}_{m,\text{opt}} = \hat{\mathbf{R}}^{-1}\hat{\mathbf{p}}_m \tag{17}$$

as the minimum mean square error estimate of the coefficients in the mth row of $\mathbf{H}[n]$.

B. Minimum Mean Squared Error

In the ideal case where $\hat{\mathbf{w}}_{m,\text{opt}}$ accurately reflects the approriate coefficients of $\mathbf{H}[n]$, the variance estimate $\hat{\sigma^2}_v$ is equivalent to the minimum mean square error,

$$\hat{\sigma}_{v,m}^2 = \hat{\sigma}_{x_m}^2 - \hat{\mathbf{p}}_m^{\mathrm{H}}\hat{\mathbf{R}}^{-1}\hat{\mathbf{p}}_m \;, \tag{18}$$

where $\hat{\sigma}_{x_m}^2$ is the power estimated over the N samples of $x_m[n]$. Since we need to perform M multichannel adaptive filter calculations, $\hat{\sigma}_v^2$ can be averaged over the M different evaluations of (18), such that $\hat{\sigma}_v^2 = \frac{1}{M}\sum_m \hat{\sigma}_{v,m}^2$.

C. Filter Length and Lag Support

Using the elements of the system matrix $\hat{\mathbf{H}}[n]$ identified via Sec. IV-A and the noise variance as discussed in Sec. IV-B, we can estimate $\mathbf{R}[\tau]$ using (13). Similar to the un-biased estimator, two terms contribute to the error ζ_{SI} defined akin to (12) between $\mathbf{H}[n]$ and $\hat{\mathbf{H}}[n]$: (i) a truncation term in case the adaptive filter length L_f falls short of the ground truth system length L_H; and (ii) a perturbation term that impacts on the coefficients of $\hat{\mathbf{w}}_{m,\text{opt}}$ in (17), which grows with the number of coefficients. Therefore, we expect to find an optimum length $L_{f,\text{opt}}$, where the two error terms are in balance.

Example. We perform an experiment with an ensemble consisting of 300 instances of a parahermitian matrix $\mathbf{R}[\tau] \in \mathbb{C}^{2\times 2}$ with $L_H = 30$. For the noise variance σ_v^2, we define an average SNR at the sensors,

$$\mathrm{SNR} = \frac{\sum_n \|\mathbf{H}[n]\|_{\mathrm{F}}^2}{M\sigma_v^2} \;, \tag{19}$$

where the numerator reflects the total power due to the sources and the denominator the total power due to the additive noise

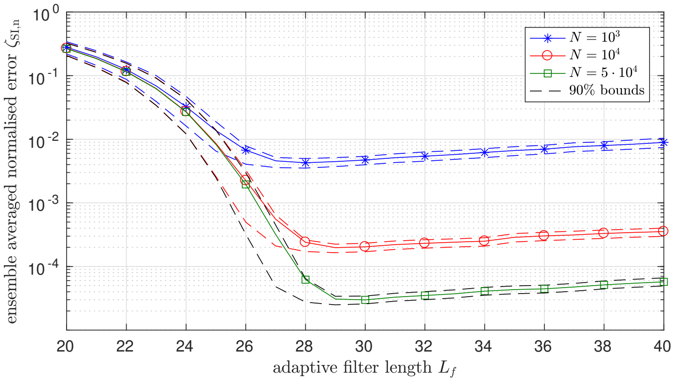

Fig. 3. Ensemble results for ζ when obtaining $\hat{\mathbf{R}}[\tau]$ in dependence of adaptive filter length, L_f.

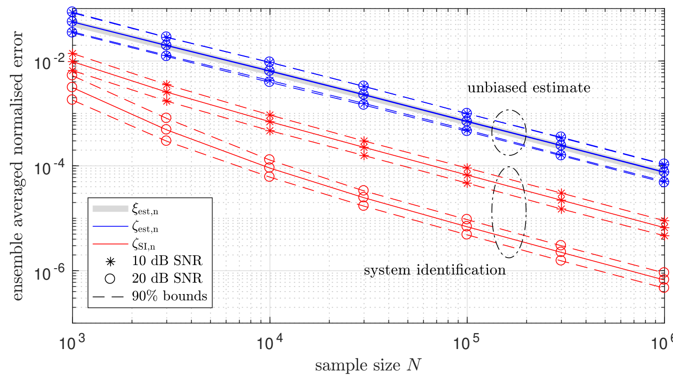

Fig. 4. Comparison of estimation methods via an ensemble of $\mathbf{R}[\tau] \in \mathbb{C}^{2\times 2}$, showing the theoretical and measured error via the unbiased estimator, $\xi_{\mathrm{est,n}}$ and $\zeta_{\mathrm{est,n}}$, respectively, as well as the measured error using the system identification approach, $\zeta_{\mathrm{SI,n}}$.

at the sensors measuring $\mathbf{x}[n]$. In the experiments, we set σ_v^2 to provide an SNR of 20 dB. The ensemble instances are identified with L_f varying from 20 to 40 using various sequence lengths N. The results are illustrated in terms of normalised errors $\zeta_{\mathrm{SI,n}} = \zeta_{\mathrm{SI}} / \sum_n \|\mathbf{H}[n]\|_{\mathrm{F}}^2$ in Fig. 3, which highlights the above trade-off: while for low values of L_f, the truncation error dominates, the error at higher values of L_f increases to the noisy coefficients in the adaptation process.

In addition, the ensemble optimum depends on the filter length. In Fig. 3, note that $L_{\mathrm{f,opt}}$ is 28, 29 and 30 for $N = 1e3$, $1e4$ and $5e4$ respectively. The filter length, for which the minimum is reached, therefore converges towards the ground truth support L_H.

V. Simulations and Comparison

This section provides a comparison of the two approaches to obtain $\hat{\mathbf{R}}[\tau]$ discussed in this paper, and an assessment of the consequences for the perturbation of its eigenvalues.

A. Performance Metric

The performance metric for a comparison of both methods is given as

$$\zeta = \frac{\sum_\tau \|\mathbf{R}[\tau] - \hat{\mathbf{R}}[\tau]\|_{\mathrm{F}}^2}{\sum_\tau \|\mathbf{R}[\tau]\|_{\mathrm{F}}^2} \, . \tag{20}$$

Note that the numerator of this metric relates to the bin-wise perturbation bound on the eigenvalues in (3) via Parseval's theorem [51]. The normalisation by the Frobenius norm of the ground-truth ensures that the metric can be applied to extract ensemble results for different instances of $\mathbf{R}[\tau]$.

B. Scenario and Parameters

To compare both methods, we employ an ensemble of 500 random instances of $\mathbf{R}[\tau] \in \mathbb{C}^{2\times 2}$ with moderately large support $L_H = 30$. The estimates are made over various sample sizes N ranging from 10^3 to 10^6 and noise levels of 10 and 20 dB SNR according to (19). The optimal lag support for the unbiased estimator is selected on the basis of the lowest value of ζ by varying the lag support between 1 and 29 because $\tau_{\mathrm{opt}} < \tau_{\mathrm{gt}} = 30$. In contrast, the support value for SI estimate is set equal to $\tau_{\mathrm{gt}} = 30$.

C. Ensemble Results

Fig. 4 shows the ensemble results for the experiment. The normalised error $\zeta_{\mathrm{est},n}$ is adopted from (20) for the unbiased estimator based on (10); likewise, $\zeta_{\mathrm{SI,n}}$ is the normalised error for the system identification approach. For each case, curves for 10 dB and 20 dB SNR are shown, together with the bounds within which 90% of the ensemble results fall. Further, the theoretical normalised variance for the unbiased estimator, a normalised version of (11), is underlaid in grey, and matches the simulation results well.

We firstly observe that the unbiased estimator, which treats measurement noise as part of the data, is independent of the SNR. In contrast, the noise terms acts as observation noise for the system identication approach, which therefore yields increased accuracy as the SNR grows. All curves converge with approximately $1/N$, but the system identification approach generally is capable of reaching better accuracy than the unbiased estimator. This is due to the additional information that in this case is known for the system — the source signals $u_\ell[n]$. In contrast, for lower SNR, the system identification performance will drop below that of the unbiased estimator, as the known signals $u_\ell[n]$ will be dwarfed by the unknown observation noise $v_m[n]$ which then start to dominate.

VI. Conclusions

We have compared the unbiased estimator with a system identification approach for the estimation of a space time covariance matrix. The latter can be exploited in case the source signals are known, and consists of the identification of the convolutive mixing system by a Wiener filter approach, and the estimation of the additive noise power via the minimum mean square error of the Wiener filter. An ensemble experiment carried out at various noise levels demonstrates that the system identification approach performs significantly better than the unbiased estimator for reasonable to high SNRs. This is important, as the enhanced accuracy results in a lower bin-wise perturbation of the eigenvalue decomposition of this matrix, which is key to formulating and solving a number of relevant broadband array problems.

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