ARTICLE IN PRESS
JID:BULSCI AID:103158 /FLA [m1L; v1.317] P.1 (1-58)
Bull. Sci. math. ••• (••••) ••••••
Contents lists available at ScienceDirect
Bulletin des Sciences Mathématiques
www.elsevier.com/locate/bulsci

On backward problems for stochastic fractional reaction equations with standard and fractional Brownian motion

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ARTICLE INFO

Article history:
Received 2 January 2021
Available online xxxx

MSC:
60G15
60G22
60G52
60G57

Keywords:
Fractional differential equation
Fractional reaction equation
Fractional Brownian motion
Inverse problem
Well–posedness
Ill-posedness

ABSTRACT

In this work, we study two final value problems for fractional reaction equation with standard Brownian motion $W(t)$ and fractional Brownian motion $B^H(t)$, for $H \in (\frac{1}{4}, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Firstly, the well-posedness of each problem is investigated under strongly choices of data. We aim to find spaces where we obtain the existence of a unique solution of each problem, and establish some regularity results. Next, since the first problem and the second problem when $H \in (\frac{1}{2}, 1)$ are ill-posed due to the lack of regularity of the terminal condition, a well-known regularization method called Fourier truncation is applied to construct regularized solutions. Furthermore, convergence results of regularized solutions are proposed.

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https://doi.org/10.1016/j.bulsci.2022.103158

1. Introduction

Let $D \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain (with a sufficiently smooth boundary ∂D for $n \geq 2$) and $X := L^2(D)$. Let $A : D(A) \subset X \to X$ be a linear, positive-definite, self-adjoint operator with compact inverse on X. In this paper, we investigate the following fractional reaction-diffusion equation

$$\partial_t u(t) + \partial_t^{1-\alpha} Au(t) = f(t, u(t)) + \text{“stochastic term”}, \tag{1}$$

where $\alpha \in (0,1)$, f is a nonlinear source term, $\partial_t^{1-\alpha}$ is the Riemann-Liouville fractional derivative given by

$$\partial_t^{1-\alpha} v(t) := \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \left(\int_0^t r^{\alpha-1} v(t-r) dr \right).$$

Here $\Gamma(\alpha) = \int_0^\infty e^{-r} r^{\alpha-1} dr$ is the Gamma function [3]. We study two following terminal value problems (TVPs) for fractional reaction-diffusion equations:

- TVP for the stochastic equation driven by standard Brownian motion

$$\begin{cases} \partial_t u(t) + \partial_t^{1-\alpha} Au(t) = f(t, u(t)) + \sigma(t)\dot{W}(t), \quad t \in [0,T], \\ u(t)\big|_{\partial D} = 0, \quad u(T) = g, \end{cases} \tag{2}$$

  where $\{\dot{W}(t)\}_{t\geq 0}$ is an X-valued Wiener process defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and the white noise $\dot{W}(t) = \frac{\partial W(t)}{\partial t}$ describes random effects on transport of particles in medium with memory or particles subject to sticking and trapping.
- TVP for the stochastic equation driven by fractional Brownian motion

$$\begin{cases} \partial_t u(t) + \partial_t^{1-\alpha} Au(t) = f(t, u(t)) + \sigma(t)\dot{B}^H(t), \quad t \in [0,T], \\ u(t)\big|_{\partial D} = 0, \quad u(T) = g, \end{cases} \tag{3}$$

  where $\{\dot{B}^H(t)\}_{t\geq 0}$, with $H \in (\frac{1}{4}, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, is an X-valued fractional Brownian motion (fBm) defined on the filter complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and $\dot{B}^H(t) = \frac{\partial}{\partial t} B^H(t)$ describes the fractional noise.

Notice that if $H = \frac{1}{2}$, then $\dot{B}^H(t)$ becomes standard Brownian motion. Terminal data g is an $\mathcal{F}_0$-measurable random variable, and belongs to X or its subspace. The function σ will be specified later. The motivation of our study for Problem (1) comes from recent results of M. Kovacsz et al. [14]. Different to the work [14], our main purpose in this paper is to investigate well-posed results (the existence, regularity) and ill-posedness results for Problem (2) and (3).

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It is obvious to see that the main equation we study in this paper is the topic of stochastic fractional PDEs. The theory of fractional PDEs which studies derivatives of non-integer orders has been established by famous mathematicians such as Riemann, Liouville, Weyl, Riesz, Caputo and others. Fractional-order derivatives have a wide application in physics and mechanics since it allows us to describe systems, media, and fields that are characterized by non-locality and memory of power-law type. The deterministic counterpart of the model (1) commonly known as subdiffusion, has been extensively studied in the literature over the last few decades. It should be noted that if $\alpha = 1$ then $\frac{\partial^{1-\alpha}}{\partial t}\Delta u$ becomes Δu and the equation

$$\frac{\partial}{\partial t}u(t,x) - \frac{\partial^{1-\alpha}}{\partial t}\Delta u(t,x) = f(t,x,u(t,x)), \tag{4}$$

reduces to the typical heat equation which is used for modeling heat diffusion in homogeneous media. Equation (4) for $0 < \alpha < 1$ is used to describe heat propagation in inhomogeneous media and also describe anomalous diffusion processes and wave propagation in viscoelastic materials [6]. Viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation [29]. As introduced by [8], possible application of (4) is the description of diffusion in special types of porous media. They have recently attracted increasing interest in the physical, chemical and engineering literature [7].

More recently, stochastic partial differential equations (SPDEs) with fractional operators have been received increasingly attentions. The area of SPDEs is interesting to mathematicians since it contains a lot of hard open problems. Stochastic fractional diffusion equation has been studied recently by [11–13,24,26,28,45–50] and the references therein. In [25], M. Foondun and E. Nane considered the asymptotic behavior of the solution for a non-linear time-fractional stochastic heat type equations. L. Chen [9] analyzed moments, Hölder continuity and intermittency of the solution for 1D nonlinear stochastic subdiffusion. Foondun, Liu and Nane [27] obtained some non-existence results for fractional stochastic heat equations driven by colored noise on the multi-dimensional spatial domain. B. Jin et al. [10] developed and analyzed a numerical method for stochastic time-fractional diffusion driven by additive white noise. Q. Du et al. [51] derived a stochastic representation of the solution to a nonlocal-in-time evolution equation. M. Kovacsz et al. [14] studied the numerical approximation of a class of semilinear Volterra integrodifferential equations. For stochastic reaction diffusion equations driven by fractional Brownian motions, We also refer the reader to some interesting papers in fBm for stochastic PDEs, for example, D. Nualart et al. [35–40], K. Lu et al. [15–17], T. Caraballo et al. [18–20] and references therein. However, there appear to be fewer studies in the literature relating to the theoretical analysis of SPDEs with fractional derivative driven by fractional noise.

In this paper, we do not follow the initial value problem, otherwise, we study the terminal value problem for stochastic PDEs. The terminal value problem is to identify the initial data from the given final data. The TVPs stochastic fractional PDEs belong

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to a category of stochastic inverse problems, which refer to inverse problems that involve uncertainties. Looking at the mild solution of deterministic inverse problems and stochastic inverse problems, we obviously see that stochastic inverse problems have substantially more difficulties on top of the existing obstacles due to the randomness and uncertainties. To the best of our knowledge, there have so far been no research results on the effective analysis of the terminal value problem for (1). We can list some few papers on this direction, for instance [21–23]. However, in these papers, the ill-posedness results are not addressed. This is our motivation for studying the ill-posedness and regularization. The main contributions and some challenge problems in the present paper are as follows:

- We have some new difficulties in the techniques and analysis for considering inverse problems for SPDE. For instance, unlike the deterministic PDEs, the solution of a stochastic PDE is usually non-differentiable with respect to the variable with noise (say, the time variable considered in this paper). Also, the usual compactness embedding result does not remain true for the solution spaces related to stochastic PDEs. For the time-integer derivative case, as in [30], A. de Bouard and A. Debussche used Itó formula to deal with the stochastic term and proved the unique existence of the weak solution by Galerkin approximation method. But for the time-fractional derivative with the stochastic case, the stochastic term can not be disposed of by Itó formula. It seems no way to study the weak solution of (1) which motivates us to find other way to study the solution's property. In this paper, we study the existence and uniqueness of the mild solution for (1). The study of finding some suitable spaces for the mild solution is also challenge since the mild solution of Problem (2) and Problem (3) is more complex than the initial value problem.
- Our first results are to derive the well-poseness of Problem (2) and Problem (3) in the sense of Hadamard. A PDE is well-posed (in the sense of Hadamard) if for each choice of data, a solution exists in some sense. For each choice of data, the solution is unique in some space. The map from data to solutions is continuous in some topology. The notion of a well-posed problem is important in applied math. If we were using two Problem (2) and Problem (3) to make predictions about some physical process, we would obviously like Problem (2) and Problem (3) to have a unique solution. And if the solution depends continuously on data and parameters, we don't have to worry about small errors in measurement producing large errors in our predictions. In Section 3, the well-posedness of two problems are investigated under Lipschitz assumption on the nonlinear source term. We attempt to find the spaces our mild solutions belong to in three case of Hurt parameter including $H = \frac{1}{2}, H \in (\frac{1}{4}, \frac{1}{2})$, $H \in (\frac{1}{2}, 1)$. The existence, uniqueness of the solution in each case is obtained by using the fixed point theorem. Additionally, we also discuss some regularity results for each of problems.
- Our second results are to investigate the ill-posedness results in the sense of Hadamard. It is the fact that Problem (2) and Problem (3) when $H \in (\frac{1}{2}, 1)$ are

ill-posed since the solution's behavior does not change continuously with the data (g, σ) (or called instable for short). Therefore, it is required to regularize two problems. For this purpose, we use a well-known method in regularization theory for inverse problems and ill-posed problems that is Fourier truncation. In this way, we control the frequency in such a way that it depends on the error ε appearing in the data (see more details about this method in [43]). Furthermore, the convergence rate of the regularized solution is proposed to show clearly that it tends to the sought solution as $\varepsilon \to 0^+$.

2. Preliminaries

2.1. Notations, Wiener process and fractional Brownian motion (fBm)

Let us begin with some functional spaces used in the present paper. For two given Hilbert spaces X_1, X_2, we denote by $\mathcal{L}(X_1, X_2)$ the space of all bounded linear operators from X_1 to X_2 and $\mathcal{L}(X_1) := \mathcal{L}(X_1, X_1)$. For arbitrary Banach space Y, we define the space

$$L^q(0,T;Y) := \left\{ w : [0,T] \to Y \quad \text{s.t.} \int_0^T \|w(s)\|_Y^q \, ds < \infty \right\}, \quad q > 1,$$

endowed with the norm $\|w\|_{L^q(0,T;Y)} := \left(\int_0^T \|w(s)\|_Y^q \, ds \right)^{1/q}$. By $L^\infty(0,T;Y)$, we denote the space of all essentially bounded functions w on $[0,T]$ with the norm $\|w\|_{L^\infty(0,T;Y)} := \operatorname{ess\,sup}_{0<t<T} \|w(t)\|_Y$. Define by $C([0,T];Y)$ the space of all continuous functions w on $[0,T]$ endowed with the norm $\|w\|_{C([0,T];Y)} := \sup_{0 \le t \le T} \|w(t)\|_Y$. For two positive numbers ϑ_1, ϑ_2 satisfying $\vartheta_1 + \vartheta_2 < 1$, we introduce the space $C^{\vartheta_1,\vartheta_2}([0,T];Y)$, which is a subspace of $C([0,T];Y)$ endowed with the norm

$$\|w\|_{C^{\vartheta_1,\vartheta_2}([0,T];Y)} := \sup_{t \in [0,T]} t^{\vartheta_1} \|w(t)\|_Y + \sup_{0 \le t < t+\theta \le T} t^{\vartheta_1+\vartheta_2} \frac{\|w(t+\theta) - w(t)\|_Y}{\theta^{\vartheta_2}} < \infty.$$

Let $\{\lambda_n\}_{n\ge1}$ be a sequence of eigenvalues of A and $\{e_n\}_{n\ge1}$ be a sequence of eigenfunctions of A, which satisfy $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \lambda_{n+1} \le \ldots$, $\lim_{n\to\infty} \lambda_n = \infty$ and $Ae_n = \lambda_n e_n$. For $\delta > 0$, we define by $\dot{H}^\delta$ the space of all functions $\varphi \in X$ such that $\sum_{n\ge1} \lambda_n^{2\delta} |\langle \varphi, e_n \rangle|^2 < \infty$, with the norm

$$\|\varphi\|_{\dot{H}^\delta} := \Big(\sum_{n\ge1} \lambda_n^{2\delta} |\langle \varphi, e_n \rangle|^2 \Big)^{\frac{1}{2}}.$$

In the case $\delta = 0$, it is clear that $\dot{H}^0 = X = L^2(D)$. Let $\dot{H}^{-\delta}$ stand for the dual space of $\dot{H}^\delta$. We can define the fractional operator (see [32]) $A^\delta : \dot{H}^{\delta/2} \to \dot{H}^{-\delta/2}$ as

$$A^{\delta}\varphi := \sum_{n\geq 1} \lambda_n^{\delta}\langle \varphi, e_n\rangle e_n, \quad \varphi \in \dot{H}^{\delta/2}.$$

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying standard assumptions, namely, it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. We denote by $L^2(\Omega, \dot{H}^{\delta})$, with $\delta \geq 0$, the space of all random variables ϱ taking value on $\dot{H}^{\delta}$ such that $\|\varrho\|_{L^2(\Omega,\dot{H}^{\delta})} := \sqrt{\mathbb{E}\,\|\varrho\|^2_{\dot{H}^{\delta}}} < \infty$. Let Q be an operator defined by $Qe_n = \chi_n e_n$ with finite trace $Tr(Q) = \sum_{n\geq 1}\chi_n < \infty$. The Wiener process $\{W(t)\}_{t\in[0,T]}$ with covariance Q (see [10,33]) can be defined as follows

$$W(t) = \sum_{n\geq 1} Q^{\frac{1}{2}} e_n \beta_n(t) = \sum_{n\geq 1} \chi_n^{\frac{1}{2}} e_n \beta_n(t), \tag{5}$$

where $\beta_n(t)$ are independent one-dimensional Brownian motions. Let $\mathcal{HS}(X, \dot{H}^{\delta})$, be the space of all Hilbert-Schmidt operators $\Psi : X \to \dot{H}^{\delta}$ with the norm $\|\Psi\|_{\mathcal{HS}(X,\dot{H}^{\delta})} := \sqrt{\sum_{n\geq 1}\|\Psi e_n\|^2_{\dot{H}^{\delta}}}$. We denote by $L_0^2(X, \dot{H}^{\delta}))$ the space of all operators $\Psi \in \mathcal{HS}(X, \dot{H}^{\delta})$ such that

$$\|\Psi\|_{L_0^2(X,\dot{H}^{\delta}))} := \left\|\Psi Q^{1/2}\right\|_{\mathcal{HS}(X,\dot{H}^{\delta})} < \infty.$$

For convenience, we define $L_0^2 := L_0^2(X, X)$.

Next, we briefly recall the fractional Brownian motion and the Wiener integral with respect to the fractional Brownian motion.

Definition 2.1. A one-dimensional fractional Brownian motion $\{\beta^H(t)\}_{t\geq 0}$ of Hurst index $H \in (0, 1)$ is a continuous and centered Gaussian process with covariance function

$$R_H(s, r) = \mathbb{E}[\beta^H(s)\beta^H(r)] = \frac{1}{2}\left(s^{2H} + r^{2H} - |s - r|^{2H}\right).$$

In the case of $H = 1/2$, $\{\beta^H(t)\}_{t\geq 0}$ turns out to be the standard Wiener process with covariance function $R_H(s, r) = \min(s, r)$.

For a time interval $[0, T]$, we denote by $\mathcal{E}$ the space of step functions ϕ on $[0, T]$ and by $\mathcal{H}$ the Hilbert space defined as the closure of $\mathcal{E}$ with the scalar product $\langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,r]}\rangle_{\mathcal{H}} = R_H(s, r)$. Consider the kernel [35]

$$K^H(s, r) = c_H \left(\left(\frac{s}{r}\right)^{H-\frac{1}{2}}(s - r)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) r^{\frac{1}{2}-H}\int_r^s u^{H-\frac{3}{2}}(u - r)^{H-\frac{1}{2}}du\right),$$

for $H \in \left(0, \frac{1}{2}\right)$,

and

$$K^H(s,r) = c_H \int_r^s (u-r)^{H-\frac{3}{2}} \left(\frac{u}{r}\right)^{H-\frac{1}{2}} du, \text{ for } H \in \left(\frac{1}{2}, 1\right),$$

where

$$c_H = \begin{cases} \sqrt{\frac{H}{(1-2H)\beta(1-2H,H+1/2)}}, & \text{if } H \in \left(0, \frac{1}{2}\right), \\ \sqrt{\frac{H(2H-1)}{\beta(2-2H,H-1/2)}}, & \text{if } H \in \left(\frac{1}{2}, 1\right), \end{cases}$$

with $\beta(\cdot,\cdot)$ is the Beta function [3]. Noting that if $H \in (0, 1/2)$ then [34]

$$\frac{\partial K^H}{\partial s}(s,r) = c_H (H-1/2) \left(\frac{s}{r}\right)^{H-1/2} (s-r)^{H-3/2}.$$

We refer to [34,35,41] the following form for $\beta^H(t)$

$$\beta^H(t) = \int_0^t K^H(t,s) d\beta(s), \qquad \beta \text{ is the standard Brownian motion.}$$

We now consider the operator $K^*_{H,T} : \mathcal{H} \to \mathbb{L}^2([0,T])$ given by

$$(K^*_{H,T}\phi)(s) = K^H(T,s)\phi(s) + \int_s^T (\phi(u) - \phi(s)) \frac{\partial K^H}{\partial u}(u,s) du.$$

From [34,35,41], it is known that $K^*_{H,T}$ is an isometry between $\mathcal{H}$ and $\mathbb{L}^2([0,T])$. If $H \in (\frac{1}{2}, 1)$ then $K^*_{H,T}$ has a simpler form

$$(K^*_{H,T}\phi)(s) = \int_s^T \phi(u) \frac{\partial K^H}{\partial u}(u,s) du.$$

We refer to [34,35,41] the following relation between the Wiener integral with respect to fBm and the Itô integral with respect to the Wiener process

$$\int_0^T \phi(s) d\beta^H(s) = \int_0^T K^*_{H,T}\phi(s) d\beta(s), \quad \text{for } H \in (0, 1/2) \cup (1/2, 1).$$

Generally, following the standard approach (5) for the case $H = 1/2$, the fBm $\{B^H(t)\}_{t\in[0,T]}$ can be defined as

$$B^H(t) = \sum_{n\geq 1} Q^{\frac{1}{2}} e_n \beta_n^H(t) = \sum_{n\geq 1} \chi_n e_n \beta_n^H(t),$$

where $\beta_n^H(t)$ are independent one-dimensional fractional Brownian motions, and the Wiener integral of $\Psi : [0,T] \to L_0^2$ with respect to fBm can be defined as

$$\int_0^T \Psi(s)dB^H(s) = \sum_{n\geq 1} \int_0^T \Psi(s)Q^{1/2}e_n d\beta_n^H(s) = \sum_{n\geq 1} \int_0^T K_{H,T}^*\big(\Psi Q^{1/2}e_n\big)(s)d\beta_n(s).$$

2.2. Mittag-Leffler function, well-posedness and ill-posedness

We begin with some properties of the Mittag-Leffler E_{α_1,α_2} (see [3], Section 1.2) defined by

$$E_{\alpha_1,\alpha_2}(z) := \sum_{n\geq 0} \frac{z^n}{\Gamma(\alpha_1 n + \alpha_2)}, \quad \alpha_1, \alpha_2 > 0, z \in \mathbb{C}.$$

For short, we set $E_\alpha(z) := E_{\alpha,1}(z)$, $e_\alpha(z) := E_{\alpha,\alpha}(z)$, $\alpha > 0$.

Lemma 2.1 *(see [3]). Given $\alpha \in (0,1)$, $\alpha' > 0$, $\lambda \geq 0$. Then*

$$|E_{\alpha,\alpha'}(-\lambda)| \leq \frac{C_2}{1+\lambda}, \quad |E_\alpha(-\lambda)| \geq \frac{C_1}{1+\lambda}, \quad |e_\alpha(-\lambda)| \leq \min\left(\frac{C_2}{1+\lambda^2}, \frac{C_2}{1+\lambda}\right),$$

where C_1, C_2 are positive constants.

Lemma 2.2 *(see [4]). Given $\alpha \in (0,1)$, $\lambda > 0$. Then*

$$\frac{d}{dt}E_\alpha(-\lambda t^\alpha) = -\lambda t^{\alpha-1} e_\alpha(-\lambda t^\alpha).$$

Next, we aim to give representations for the solutions to Problem (2) and Problem (3). For this purpose, we consider the following deterministic problem

$$\begin{cases} \partial_t u(t) + \partial_t^{1-\alpha} Au(t) = f(t,u(t)), & t \in [0,T], \\ u(t)\big|_{\partial D} = 0, \quad u(T) = g. \end{cases} \tag{6}$$

We refer to [1] the following representation for the solutions to (6), which is obtained by using the Laplace transform method

$$u(t) = E_\alpha(-t^\alpha A)u(0) + \int_0^t E_\alpha(-(t-s)^\alpha A) f(s,u(s))ds, \tag{7}$$

where $E_\alpha(-t^\alpha A)$ is the Mittag-Leffler operator (see [2,5]) defined by

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$$E_\alpha(-t^\alpha A)\varphi := \int_0^\infty M_\alpha(r) e^{-rt^\alpha A}\varphi dr, \qquad \varphi \in X. \tag{8}$$

By the following relation between the Mittag-Leffler function and the Wright-type function (see [31])

$$\int_0^\infty M_\alpha(r) e^{-zr} dr = E_\alpha(-z), \quad z \in \mathbb{C},$$

the formula (8) can be rewritten as

$$E_\alpha(-t^\alpha A)\varphi = \sum_{n\geq 1} E_\alpha(-t^\alpha \lambda_n)\langle \varphi, e_n\rangle e_n, \quad \varphi \in X.$$

Substituting $t = T$, we have from (7) that

$$u(0) = E_\alpha^{-1}(-T^\alpha A)u(T) - \int_0^T E_\alpha^{-1}(-T^\alpha A)E_\alpha(-(T-s)^\alpha A)f(s,u(s))ds, \tag{9}$$

where the operator $E_\alpha^{-1}(-t^\alpha A)$ is defined by

$$E_\alpha^{-1}(-t^\alpha A)\varphi = \sum_{n\geq 1} \frac{1}{E_\alpha(-t^\alpha \lambda_n)}\langle \varphi, e_n\rangle e_n, \quad \varphi \in X.$$

Combining (7), (9), and using the final condition $u(T) = g$, we now obtain

$$u(t) = \mathcal{E}_{\alpha,1}(t)g + \int_0^t \mathcal{E}_{\alpha,2}(t-s)f(s,u(s))ds - \int_0^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)f(s,u(s))ds, \tag{10}$$

where we define $\mathcal{E}_{\alpha,1}(t) := E_\alpha(-t^\alpha A)E_\alpha^{-1}(-T^\alpha A)$ and $\mathcal{E}_{\alpha,2}(t) := E_\alpha(-t^\alpha A)$, for $t \in [0,T]$.

Motivated by (10), we give the definitions of mild solutions of Problem (2) and Problem (3):

Definition 2.2. An X-valued process $u(t)$ is called a mild solution of Problem (2) if it satisfies the equation

$$u(t) = \mathcal{E}_{\alpha,1}(t)g + \int_0^t \mathcal{E}_{\alpha,2}(t-s)f(s,u(s))ds - \int_0^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)f(s,u(s))ds$$

$$+\int_0^t \mathcal{E}_{\alpha,2}(t-s)\sigma(s)dW(s) - \int_0^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)dW(s), \quad \mathbb{P}-a.s. \quad (11)$$

An X-valued process $v(t)$ is called a mild solution of Problem (3) if it satisfies the equation

$$v(t) = \mathcal{E}_{\alpha,1}(t)g + \int_0^t \mathcal{E}_{\alpha,2}(t-s)f(s,v(s))ds - \int_0^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)f(s,v(s))ds$$

$$+\int_0^t \mathcal{E}_{\alpha,2}(t-s)\sigma(s)dB^H(s) - \int_0^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)dB^H(s), \quad \mathbb{P}-a.s. \quad (12)$$

We refer to [42] the following definitions of well-posedness and ill-posedness given by Jacques Hadamard:

Definition 2.3. A problem is called well-posed if it has the following properties

i) a solution exists,
ii) the solution is unique,
iii) the solution is stable (its behavior changes continuously with the data).

If at least one of three properties does not hold, then the problem is called ill-posed.

In the next section, we will study the well-posedness of two problems we are interested in under strongly choices of data. By the assumptions $(g), (\sigma_1)$, we constitute the existence, uniqueness and some regularities of the mild solution $u(t)$ of Problem (2). For Problem (3) in two case $H \in (\frac{1}{4}, \frac{1}{2})$ and $H \in (\frac{1}{2}, 1)$, to obtain similar results, it is required more strict assumptions for the data σ as in $(\sigma_1), (\sigma_2)$.

In the last section, the ill-posedness on $C([0,T], L^2(\Omega, X))$ of Problem (2) and Problem (3) when $H \in (1/2, 1)$ are investigated. This property of two problems comes from the instability of the solutions. Precisely, it is the fact that if we have small errors in the data as in Model (36) then there could be a large change in the solution, i.e. $\|u^\varepsilon - u\|_{C([0,T];L^2(\Omega,X))}$ (res. $\|v^\varepsilon - u\|_{C([0,T];L^2(\Omega,X))}$) does not tend to zero as $\varepsilon \to 0^+$, where $u^\varepsilon = u(g^\varepsilon, \sigma^\varepsilon)$ (res. $v^\varepsilon = v(g^\varepsilon, \sigma^\varepsilon)$) is the solution of each problem with respect to the observed data. Hence, we aim to apply the Fourier truncated method to regularize each problem and then show the convergence results of regularized solutions.

2.3. Properties of solution operators

Lemma 2.3. *Given $\alpha \in (0,1)$ and $\mu_1 \geq 0$. Let μ_2, μ_3, μ_4 satisfy $0 \leq \mu_1 - \mu_2 \leq 1$, $0 \leq \mu_3 - \mu_1 \leq 1$ and $0 \leq \mu_4 - \mu_2 \leq 1$. Then*

a) $\|\mathcal{E}_{\alpha,1}(t)\|_{\mathcal{L}(\dot{H}^{\mu_1+1},\dot{H}^{\mu_1})} \le C_3$, $\|\mathcal{E}_{\alpha,2}(t)\|_{\mathcal{L}(\dot{H}^{\mu_1},\dot{H}^{\mu_1})} \le C_2$, *for* $t \in [0,T]$,

b) $\|\mathcal{E}_{\alpha,1}(t)\|_{\mathcal{L}(\dot{H}^{\mu_2+1},\dot{H}^{\mu_1})} \le C_3 t^{-\alpha(\mu_1-\mu_2)}$, $\|\mathcal{E}_{\alpha,2}(t)\|_{\mathcal{L}(\dot{H}^{\mu_2},\dot{H}^{\mu_1})} \le C_2 t^{-\alpha(\mu_1-\mu_2)}$, *for* $t \in (0,T]$,

c) $\|\mathcal{E}_{\alpha,1}(t_1)\mathcal{E}_{\alpha,2}(t_2)\|_{\mathcal{L}(\dot{H}^{\mu_3},\dot{H}^{\mu_1})} \le C_2 C_3 t_2^{-\alpha(\mu_1+1-\mu_3)}$, *for* $t_1 \in [0,T], t_2 \in (0,T]$,

d) $\|\mathcal{E}_{\alpha,1}(t_1)\mathcal{E}_{\alpha,2}(t_2)\|_{\mathcal{L}(\dot{H}^{\mu_4},\dot{H}^{\mu_1})} \le C_2 C_3 t_1^{-\alpha(\mu_1-\mu_2)} t_2^{-\alpha(\mu_2+1-\mu_4)}$, *for* $t_1 \in (0,T], t_2 \in (0,T]$,

where $C_3 = C_3(\alpha,T) := \frac{C_2}{C_1}(\lambda_1^{-1} + T^\alpha)$.

Proof. For $t \in [0,T]$, it follows from Lemma 2.1 that

$$|E_\alpha(-t^\alpha\lambda_n)| \le C_2, \quad \left|\frac{E_\alpha(-t^\alpha\lambda_n)}{E_\alpha(-T^\alpha\lambda_n)}\right| \le \frac{C_2}{C_1}(1+T^\alpha\lambda_n) \le \frac{C_2}{C_1}(\lambda_1^{-1}+T^\alpha)\lambda_n = C_3\lambda_n. \tag{13}$$

Hence, for $\varphi \in X$ and $t \in [0,T]$, we have

$$\|\mathcal{E}_{\alpha,2}(t)\varphi\|^2_{\dot{H}^{\mu_1}} = \sum_{n\ge1} \lambda_n^{2\mu_1}|E_\alpha(-t^\alpha\lambda_n)|^2\langle\varphi,e_n\rangle^2 \le C_2^2\,\|\varphi\|^2_{\dot{H}^{\mu_1}},$$

$$\|\mathcal{E}_{\alpha,1}(t)\varphi\|^2_{\dot{H}^{\mu_1}} = \sum_{n\ge1} \lambda_n^{2\mu_1}\left|\frac{E_\alpha(-t^\alpha\lambda_n)}{E_\alpha(-T^\alpha\lambda_n)}\right|^2\langle\varphi,e_n\rangle^2 \le C_3^2\,\|\varphi\|^2_{\dot{H}^{\mu_1+1}}.$$

For $t \in (0,T]$, it follows from Lemma 2.1 that

$$|E_\alpha(-t^\alpha\lambda_n)| \le C_2(1+t^\alpha\lambda_n)^{-(\mu_1-\mu_2)} \le C_2 t^{-\alpha(\mu_1-\mu_2)}\lambda_n^{-(\mu_1-\mu_2)},$$
$$\left|\frac{E_\alpha(-t^\alpha\lambda_n)}{E_\alpha(-T^\alpha\lambda_n)}\right| \le \frac{C_2}{C_1}(1+T^\alpha\lambda_n)t^{-\alpha(\mu_1-\mu_2)}\lambda_n^{-(\mu_1-\mu_2)} \le C_3 t^{-\alpha(\mu_1-\mu_2)}\lambda_n^{1-(\mu_1-\mu_2)}.$$

Hence, for $\varphi \in X$ and $t \in (0,T]$, we have

$$\|\mathcal{E}_{\alpha,2}(t)\varphi\|^2_{\dot{H}^{\mu_1}} = \sum_{n\ge1} \lambda_n^{2\mu_1}|E_\alpha(-t^\alpha\lambda_n)|^2\langle\varphi,e_n\rangle^2 \le C_2^2 t^{-2\alpha(\mu_1-\mu_2)}\,\|\varphi\|^2_{\dot{H}^{\mu_2}},$$

$$\|\mathcal{E}_{\alpha,1}(t)\varphi\|^2_{\dot{H}^{\mu_1}} = \sum_{n\ge1} \lambda_n^{2\mu_1}\left|\frac{E_\alpha(-t^\alpha\lambda_n)}{E_\alpha(-T^\alpha\lambda_n)}\right|^2\langle\varphi,e_n\rangle^2 \le C_3^2 t^{-2\alpha(\mu_1-\mu_2)}\,\|\varphi\|^2_{\dot{H}^{\mu_2+1}}.$$

Consequently, part c) is obtained from the estimate for $\mathcal{E}_{\alpha,1}$ in part a) and the estimate for $\mathcal{E}_{\alpha,2}$ in part b). Furthermore, part d) is obtained from the estimate for $\mathcal{E}_{\alpha,1}$ and $\mathcal{E}_{\alpha,2}$ in part b). The proof is completed. □

For short, from now on, $a_1 \lesssim a_2$ (res. $a_1 \gtrsim a_2$) stands for $a_1 \le Ca_2$ (res. $a_1 \ge Ca_2$), where C is a positive constant.

Please cite this article in press as: N.H. Tuan et al., On backward problems for stochastic fractional reaction equations with standard and fractional Brownian motion, Bull. Sci. math. (2022), https://doi.org/10.1016/j.bulsci.2022.103158

Lemma 2.4. *Given $\alpha \in (0,1)$ and $\mu_1 \geq 0$. Let μ_2 satisfy $\frac{1}{2} < \mu_1 - \mu_2 \leq 1$ and $0 \leq \gamma \leq 1$. Then, for $0 < t < t+\theta \leq T$*

$$\left\| \mathcal{E}_{\alpha,1}^{(\theta)}(t) \right\|_{\mathcal{L}(\dot{H}^{2-\mu_1+2\mu_2}, \dot{H}^{\mu_1})} \lesssim \frac{\theta^{\alpha(2\mu_1-2\mu_2-1)\gamma}}{t^{\alpha(2\mu_1-2\mu_2-1)(\gamma+1)}},$$
$$\left\| \mathcal{E}_{\alpha,2}^{(\theta)}(t) \right\|_{\mathcal{L}(\dot{H}^{1-\mu_1+2\mu_2}, \dot{H}^{\mu_1})} \lesssim \frac{\theta^{\alpha(2\mu_1-2\mu_2-1)\gamma}}{t^{\alpha(2\mu_1-2\mu_2-1)(\gamma+1)}},$$

where we define $\mathcal{E}_{\alpha,1}^{(\theta)}(t) := \mathcal{E}_{\alpha,1}(t+\theta) - \mathcal{E}_{\alpha,1}(t)$ and $\mathcal{E}_{\alpha,2}^{(\theta)}(t) := \mathcal{E}_{\alpha,2}(t+\theta) - \mathcal{E}_{\alpha,2}(t)$.

Proof. For $\varphi \in X$ and $t \in (0,T]$, by using Lemma 2.2, we have

$$\begin{aligned} \left\| \mathcal{E}_{\alpha,1}^{(\theta)}(t)\varphi \right\|_{\dot{H}^{\mu_1}}^2 &= \sum_{n\geq 1} \lambda_n^{2\mu_1} \left| \frac{E_\alpha(-(t+\theta)^\alpha \lambda_n) - E_\alpha(-t^\alpha \lambda_n)}{E_\alpha(-T^\alpha \lambda_n)} \right|^2 \langle \varphi, e_n \rangle^2 \\ &= \sum_{n\geq 1} \lambda_n^{2\mu_1} \left| \frac{-\int_t^{t+\theta} \lambda_n s^{\alpha-1} e_\alpha(-\lambda_n s^\alpha) ds}{E_\alpha(-T^\alpha \lambda_n)} \right|^2 \langle \varphi, e_n \rangle^2. \end{aligned}$$

On the other hand, applying Lemma 2.1, we obtain

$$E_\alpha^{-1}(-T^\alpha \lambda_n) \lesssim \lambda_n, \quad e_\alpha(-s^\alpha \lambda_n) \lesssim (1 + s^{2\alpha}\lambda_n^2)^{-(\mu_1-\mu_2)} \leq s^{-2\alpha(\mu_1-\mu_2)} \lambda_n^{-2(\mu_1-\mu_2)},$$

Hence, the following estimate holds

$$\begin{aligned} \left\| \mathcal{E}_{\alpha,1}^{(\theta)}(t)\varphi \right\|_{\dot{H}^{\mu_1}}^2 &\leq \left| \int_t^{t+\theta} s^{-\alpha(2\mu_1-2\mu_2-1)-1} ds \right|^2 \sum_{n\geq 1} \lambda_n^{2(2-\mu_1+2\mu_2)} \langle \varphi, e_n \rangle^2 \\ &\lesssim \left| (t+\theta)^{-\alpha(2\mu_1-2\mu_2-1)} - t^{-\alpha(2\mu_1-2\mu_2-1)} \right|^2 \|\varphi\|_{\dot{H}^{2-\mu_1+2\mu_2}}^2 \\ &= \left| \frac{(t+\theta)^{\alpha(2\mu_1-2\mu_2-1)} - t^{\alpha(2\mu_1-2\mu_2-1)}}{t^{\alpha(2\mu_1-2\mu_2-1)}(t+\theta)^{\alpha(2\mu_1-2\mu_2-1)}} \right|^2 \|\varphi\|_{\dot{H}^{2-\mu_1+2\mu_2}}^2 . \end{aligned}$$

By the observation $0 < \alpha(2\mu_1 - 2\mu_2 - 1) \leq \alpha < 1$ and $\gamma \in [0,1]$, we know that

$$\begin{aligned} &(t+\theta)^{\alpha(2\mu_1-2\mu_2-1)} - t^{\alpha(2\mu_1-2\mu_2-1)} \leq \theta^{\alpha(2\mu_1-2\mu_2-1)}, \\ &(t+\theta)^{\alpha(2\mu_1-2\mu_2-1)} \geq t^{\alpha(2\mu_1-2\mu_2-1)\gamma} \theta^{\alpha(2\mu_1-2\mu_2-1)(1-\gamma)}. \end{aligned}$$

This leads to $\left\| \mathcal{E}_{\alpha,1}^{(\theta)}(t)\varphi \right\|_{\dot{H}^{\mu_1}}^2 \lesssim \frac{\theta^{2\alpha(2\mu_1-2\mu_2-1)\gamma}}{t^{2\alpha(2\mu_1-2\mu_2-1)(\gamma+1)}} \|\varphi\|_{\dot{H}^{2-\mu_1+2\mu_2}}^2$. In a similar way as in above, one arrives at

$$\left\| \mathcal{E}_{\alpha,2}^{(\theta)}(t)\varphi \right\|_{\dot{H}^{\mu_1}}^2 = \sum_{n\geq 1} \lambda_n^{2\mu_1} \left| E_\alpha(-(t+\theta)^\alpha \lambda_n) - E_\alpha(-t^\alpha \lambda_n) \right|^2 \langle \varphi, e_n \rangle^2$$

$$\lesssim \frac{\theta^{2\alpha(2\mu_1-2\mu_2-1)\gamma}}{t^{2\alpha(2\mu_1-2\mu_2-1)(\gamma+1)}} \|\varphi\|^2_{\dot{H}^{1-\mu_1+2\mu_2}}.$$

We now complete the proof. □

3. Existence, uniqueness, regularity of mild solutions

The main purpose of this section is to study the existence, uniqueness, and regularity of mild solutions to Problem (2) and Problem (3). To do this, we make the following assumptions:

(g) $g \in L^2(\Omega, \dot{H}^{\nu_1+1})$, $\nu_1 \geq 0$.

(σ_1) For Problem (2), we assume that $\sigma \in L^p(0,T; L_0^2(X, \dot{H}^{\nu_2}))$, for some $\nu_2 \geq \nu_1$ and $p > 2$ satisfying

$$\nu_2 - \nu_1 \leq 1, \qquad \frac{2p\alpha}{p-2}[1-(\nu_2-\nu_1)] < 1.$$

(σ_2) For Problem (3) when $H \in (\frac{1}{4}, \frac{1}{2})$, we assume that $\sigma : [0,T] \to L_0^2(X, \dot{H}^{\nu_3})$, for some $\nu_3 \geq \nu_1$ satisfying $\alpha[1-(\nu_3-\nu_1)] \in (\frac{1}{2}-H, H)$ and σ satisfies the following Hölder condition

$$\|\sigma(t_1) - \sigma(t_2)\|_{L_0^2(X,\dot{H}^{\nu_3})} \leq \mathcal{C}_\sigma |t_2 - t_1|^\zeta, \qquad t_1, t_2 \in [0,T],$$

where $\mathcal{C}_\sigma$ is a positive constant and $\zeta > \frac{1}{2} - H$.

(σ_3) For Problem (3) when $H \in (\frac{1}{2}, 1)$, we assume $\sigma \in L^\rho(0,T; L_0^2(X, \dot{H}^{\nu_4}))$, for some $\nu_4 \geq \nu_1$ and $\rho > 2$ satisfying

$$\frac{\rho}{\rho-2}\left(\frac{3}{2} - H + \alpha(1-(\nu_4-\nu_1))\right) < 1.$$

(f) $f(\cdot, 0) = 0$ and there exists a positive constant $\mathcal{K}$ such that for $\varphi, \varphi' \in \dot{H}^{\nu_1}$

$$\|f(t,\varphi) - f(t,\varphi')\|_{\dot{H}^{\nu_1}} \leq \mathcal{K}\|\varphi - \varphi'\|_{\dot{H}^{\nu_1}}, \quad t \in [0,T].$$

Assume further that $\mathcal{K}C_2T\left(1 + \frac{C_3^2T^{-2\alpha}}{(1-\alpha)^2}\right)^{\frac{1}{2}} < 1$.

Firstly, we provide some needed estimates for the terms in the right hand sides of the mild formulations (11), (12), under the above assumptions. The proof of the following lemmas can be found in Appendix A.

Lemma 3.1. *Let Assumption (g) be satisfied. Define* $I_1(t) := \mathcal{E}_{\alpha,1}(t)g$. *Then*

i) $\|I_1(t)\|_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim \|g\|_{L^2(\Omega,\dot{H}^{\nu_1+1})}$, *for* $t \in [0,T]$,

ii) $\|I_1(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \lesssim t^{-\alpha r_1} \|g\|_{L^2(\Omega,\dot{H}^{\nu_1+1})}$, *for* $t \in (0,T]$.

Lemma 3.2. *Let Assumption (f) be satisfied. For* $w \in L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))$ *and* $t \in [0,T]$, *we define*

$$I_2(w)(t) := \int_0^t \mathcal{E}_{\alpha,2}(t-s)f(s,w(s))ds, \quad I_3(w)(t) := \int_0^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)f(s,w(s))ds.$$

Then, the following properties hold

i) $\|I_2(w)(t)\|_{L^2(\Omega,\dot{H}^{\nu_1})} \le \mathcal{K}C_2T\|w\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))}$, *for* $t \in [0,T]$,
ii) $\|I_2(w)(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \lesssim t^{-\alpha r_1}\|w\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))}$, *for* $r_1 \in (0,1], t \in (0,T]$,
iii) $\|I_3(w)(t)\|_{L^2(\Omega,\dot{H}^{\nu_1})} \le \mathcal{K}\frac{C_2C_3}{1-\alpha}T^{1-\alpha}\|w\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))}$, *for* $t \in [0,T]$,
iv) $\|I_3(w)(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \lesssim t^{-\alpha r_1}\|w\|^2_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))}$, *for* $r_1 \in (0,1], t \in (0,T]$.

Lemma 3.3. *Let Assumption* (σ_1) *be satisfied. For* $t \in [0,T]$, *we define*

$$I_4(t) := \int_0^t \mathcal{E}_{\alpha,2}(t-s)\sigma(s)dW(s), \quad I_5(t) := \int_0^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)dW(s).$$

Then, the following properties hold

i) $\|I_4(t)\|_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim \|\sigma\|_{L^p(0,T;L^2_0(X,\dot{H}^{\nu_2}))}$, *for* $t \in [0,T]$,
ii) $\|I_4(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \lesssim \|\sigma\|_{L^p(0,T;L^2_0(X,\dot{H}^{\nu_2}))}$, *for* $r_1 \in (0,1], t \in (0,T]$,
iii) $\|I_5(t)\|_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim \|\sigma\|_{L^p(0,T;L^2_0(X,\dot{H}^{\nu_2}))}$, *for* $t \in [0,T]$,
iv) $\|I_5(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \lesssim t^{-\alpha r_1}\|\sigma\|_{L^p(0,T;L^2_0(X,\dot{H}^{\nu_2}))}$, *for* $r_1 \in (0,1], t \in (0,T]$.

Lemma 3.4. *Let Assumption* (σ_2) *be satisfied. For* $t \in [0,T]$, *we define*

$$\overline{I}_4(t) := \int_0^t \mathcal{E}_{\alpha,2}(t-s)\sigma(s)dB^H(s), \quad \overline{I}_5(t) := \int_0^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)dB^H(s).$$

Then, the following properties hold for $H \in (\frac{1}{4},\frac{1}{2})$ *and* $t \in (0,T]$

i) $\|\overline{I}_4(t)\|_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim \mathcal{C}_\sigma + \|\sigma\|_{C((0,T];L^2_0(X,\dot{H}^{\nu_3}))}$,
ii) $\|\overline{I}_5(t)\|_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim \mathcal{C}_\sigma + \|\sigma\|_{C((0,T];L^2_0(X,\dot{H}^{\nu_3}))}$.

Lemma 3.5. *Let Assumption* (σ_2) *be satisfied. Then, the following properties hold for* $H \in (\frac{1}{4},\frac{1}{2})$, $t \in (0,T]$, *and* $r_1 \in (0,1]$

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i) $\left\|\overline{I}_4(t)\right\|_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \lesssim \mathcal{C}_\sigma + \|\sigma\|_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}$,

ii) $\left\|\overline{I}_5(t)\right\|_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \lesssim t^{-\alpha r_1}\left(\mathcal{C}_\sigma + \|\sigma\|_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}\right)$.

Lemma 3.6. *Let Assumption (σ_3) be satisfied. Then, the following properties hold for $H \in (\frac{1}{2}, 1)$ and $t \in (0, T]$*

i) $\left\|\overline{I}_4(t)\right\|_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim \|\sigma\|_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}$,

ii) $\left\|\overline{I}_5(t)\right\|_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim \|\sigma\|_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}$.

Lemma 3.7. *Let Assumption (σ_3) be satisfied. Then, the following properties hold for $H \in (\frac{1}{2}, 1)$, $t \in (0, T]$, and $r_1 \in (0, 1]$*

i) $\left\|\overline{I}_4(t)\right\|_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \lesssim \|\sigma\|_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}$,

ii) $\left\|\overline{I}_5(t)\right\|_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \lesssim t^{-\alpha r_1}\|\sigma\|_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}$.

In what follows, we shall state the main results of this section, including the existence, uniqueness, regularity for the problems we consider.

3.1. Existence, uniqueness, regularity of mild solution to Problem (2)

Theorem 3.1. *Suppose that Assumptions (g), (σ_1), (f) are satisfied. Then*

i) *Problem (2) has a unique mild solution u in $L^\infty(0, T; L^2(\Omega, \dot{H}^{\nu_1}))$.*

ii) *If $0 < r_1 \le 1 < q < \frac{1}{\alpha r_1}$, then $u \in L^q(0, T; L^2(\Omega, \dot{H}^{\nu_1+r_1}))$ and satisfies*

$$\|u\|_{L^q(0,T;L^2(\Omega,\dot{H}^{\nu_1+r_1}))} \lesssim \|g\|_{L^2(\Omega,\dot{H}^{\nu_1+1})} + \|u\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))} + \|\sigma\|_{L^p(0,T;L_0^2(X,\dot{H}^{\nu_2}))}.$$

iii) *If $r_2 \in \left(\frac{\nu_2-\nu_1+1}{2}, 1\right)$, then $u \in C^{\alpha-\eta,\eta}((0, T]; L^2(\Omega, \dot{H}^{\nu_1+r_2}))$, where*

$$\eta := \min\left\{\alpha(1 - r_2), \frac{p-2}{2p} - \alpha[1 - (\nu_2 - \nu_1)]\right\}.$$

Proof. Step 1. Now, we prove that Problem (2) has a unique mild solution $u \in L^\infty(0, T; L^2(\Omega, \dot{H}^{\nu_1}))$ by using the contraction mapping principle. For $w \in L^\infty(0, T; L^2(\Omega, \dot{H}^{\nu_1}))$, we define

$$\mathcal{Z}w(t) := \mathcal{E}_{\alpha,1}(t)g + \int_0^t \mathcal{E}_{\alpha,2}(t-s)f(s, w(s))ds - \int_0^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)f(s, w(s))ds$$

$$+ \int_0^t \mathcal{E}_{\alpha,2}(t-s)\sigma(s)dW(s) - \int_0^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)dW(s)$$

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$$= I_1(t) + I_2(w)(t) + I_3(w)(t) + I_4(t) + I_5(t). \tag{14}$$

Firstly, it can be seen that $\mathcal{Z}$ is well-defined on $L^\infty(0,T; L^2(\Omega, \dot{H}^{\nu_1}))$. Indeed, it follows from the property i) of Lemma 3.1, the properties i), iii) of Lemma 3.2, and the properties i), iii) of Lemma 3.3 that for all $t \in [0,T]$ there holds

$$\|\mathcal{Z}w(t)\|_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim \|g\|_{L^2(\Omega,\dot{H}^{\nu_1+1})} + \|w\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))} + \|\sigma\|_{L^p(0,T;L_0^2(X,\dot{H}^{\nu_2}))}.$$

Next, one can see that $\mathcal{Z}$ is a contraction mapping. Indeed, for $w_1, w_2 \in L^\infty(0,T; L^2(\Omega, \dot{H}^{\nu_1}))$, it follows from (14) that

$$\begin{aligned} &\|\mathcal{Z}w_1(t) - \mathcal{Z}w_2(t)\|_{L^2(\Omega,\dot{H}^{\nu_1})} \\ &\quad \le \left\| \int_0^t \mathcal{E}_{\alpha,2}(t-s)[f(s,w_1(s)) - f(s,w_2(s))]ds \right\|_{L^2(\Omega,\dot{H}^{\nu_1})} \\ &\quad + \left\| \int_0^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)[f(s,w_1(s)) - f(s,w_2(s))]ds \right\|_{L^2(\Omega,\dot{H}^{\nu_1})}. \end{aligned} \tag{15}$$

In the same ways as in the estimates of $I_2(w)(t)$ and $I_3(w)(t)$, one can check that

$$\|\mathcal{Z}w_1(t) - \mathcal{Z}w_2(t)\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} \le \mathcal{K}^2 C_2^2 T^2 \left(1 + \frac{C_3^2 T^{-2\alpha}}{(1-\alpha)^2}\right) \|w_1 - w_2\|^2_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))}, \tag{16}$$

which implies that $\|\mathcal{Z}w_1 - \mathcal{Z}w_2\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))} < \|w_1 - w_2\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))}$. Hence, there exists a unique fixed point u such that $\mathcal{Z}u = u$, which is the unique mild solution of Problem (2).

Step 2. The purpose of this step is to prove that $u \in L^q(0,T; L^2(\Omega, \dot{H}^{\nu_1+r_1}))$. For $t \in (0,T]$, the property ii) of Lemma 3.1, the properties ii), iv) of Lemma 3.2, and the properties ii), iv) of Lemma 3.3 allow that for $t \in (0,T]$ that for all $t \in [0,T]$ there holds

$$\begin{aligned} \|\mathcal{Z}u(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} &\lesssim t^{-\alpha r_1}\big(\|g\|_{L^2(\Omega,\dot{H}^{\nu_1+1})} + \|u\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))} \\ &\quad + \|\sigma\|_{L^p(0,T;L_0^2(X,\dot{H}^{\nu_2}))} \big). \end{aligned}$$

By using the fact that $\int_0^T t^{-\alpha q r_1} dt \lesssim t^{1-\alpha q r_1} \le T^{1-\alpha q r_1}$, for any $q < \frac{1}{\alpha r_1}$, we now conclude that $u \in L^q(0,T; L^2(\Omega, \dot{H}^{\nu_1+r_1}))$.

Step 3. In this step, we will prove that $u \in C^{\alpha-\eta,\eta}((0,T]; L^2(\Omega, \dot{H}^{\nu_1+r_2}))$. To do this, we divide $u(t+\theta) - u(t)$, for $0 < t < t+\theta \le T$, into seven terms as

$$u(t+\theta) - u(t) = \mathcal{E}^{(\theta)}_{\alpha,1}(t)g + \int_0^t \mathcal{E}^{(\theta)}_{\alpha,2}(t-s)f(s,u(s))ds + \int_t^{t+\theta} \mathcal{E}_{\alpha,2}(t+\theta-s)f(s,u(s))ds$$

$$
\begin{aligned}
&- \int_0^T \mathcal{E}_{\alpha,1}^{(\theta)}(t)\mathcal{E}_{\alpha,2}(T-s)f(s,u(s))ds + \int_0^t \mathcal{E}_{\alpha,2}^{(\theta)}(t-s)\sigma(s)dW(s) \\
&+ \int_t^{t+\theta} \mathcal{E}_{\alpha,2}(t+\theta-s)\sigma(s)dW(s) - \int_0^T \mathcal{E}_{\alpha,1}^{(\theta)}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)dW(s) \\
&=: J_1^\theta(t) + J_2^\theta(u)(t) + J_3^\theta(u)(t) + J_4^\theta(u)(t) + J_5^\theta(t) + J_6^\theta(t) + J_7^\theta(t).
\end{aligned}
$$

Applying Lemma 2.4 with $\mu_1 = \nu_1 + r_2, \mu_2 = \nu_1 + \frac{r_2-1}{2}$ and $\gamma = \frac{1}{r_2} - 1$, one has

$$
\left\| \mathcal{E}_{\alpha,1}^{(\theta)}(t) \right\|_{\mathcal{L}(\dot{H}^{\nu_1+1},\dot{H}^{\nu_1+r_2})} \lesssim t^{-\alpha}\theta^{\alpha(1-r_2)}, \quad \left\| \mathcal{E}_{\alpha,2}^{(\theta)}(t) \right\|_{\mathcal{L}(\dot{H}^{\nu_1},\dot{H}^{\nu_1+r_2})} \lesssim t^{-\alpha}\theta^{\alpha(1-r_2)}. \tag{17}
$$

Hence, two first terms $J_1^\theta(t)$, $J_2^\theta(t)$ can be estimated as

$$
\left\| J_1^\theta(t) \right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_2})} = \mathbb{E} \left\| \mathcal{E}_{\alpha,1}^{(\theta)}(t) g \right\|^2_{\dot{H}^{\nu_1+r_2}} \lesssim t^{-2\alpha}\theta^{2\alpha(1-r_2)} \left\| g \right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+1})}, \tag{18}
$$

and

$$
\begin{aligned}
\left\| J_2^\theta(u)(t) \right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_2})} &\le \mathbb{E} \left[\int_0^t \left\| \mathcal{E}_{\alpha,2}^{(\theta)}(t-s) f(s,u(s)) \right\|_{\dot{H}^{\nu_1+r_2}} ds \right]^2 \\
&\lesssim \theta^{2\alpha(1-r_2)} \mathbb{E} \left[\int_0^t (t-s)^{-\alpha} \left\| f(s,u(s)) \right\|_{\dot{H}^{\nu_1}} ds \right]^2
\end{aligned}
$$

On the other hand, by a similar technique as in (54), one can see

$$
\mathbb{E} \left[\int_0^t (t-s)^{-\alpha} \left\| f(s,u(s)) \right\|_{\dot{H}^{\nu_1}} ds \right]^2 \lesssim \left\| u \right\|^2_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))}.
$$

Hence, it is clear that

$$
\left\| J_2^\theta(u)(t) \right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_2})} \lesssim \theta^{2\alpha(1-r_2)} \left\| u \right\|^2_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))}.
$$

By a similar way as in the estimate (53) for $I_2(u)(t)$, we arrive at

$$
\begin{aligned}
\left\| J_3^\theta(u)(t) \right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_2})} &\lesssim \int_t^{t+\theta} (t+\theta-s)^{-\alpha r_2} ds \int_t^{t+\theta} (t+\theta-s)^{-\alpha r_2} \mathbb{E} \left\| f(s,u(s)) \right\|^2_{\dot{H}^{\nu_1}} ds \\
&\lesssim \theta^{2(1-\alpha r_2)} \left\| u \right\|^2_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))},
\end{aligned}
$$

where we have used $\left(\int_t^{t+\theta}(t+\theta-s)^{-\alpha r_2}ds\right)^2 \lesssim \theta^{2(1-\alpha r_2)}$. Applying (17) and Lemma 2.4 with $\mu_1=\nu_1+1$, $\mu_2=\nu_1$, we obtain

$$\begin{aligned}\left\|J_4^\theta(u)(t)\right\|^2_{L^2(\Omega,\dot H^{\nu_1+r_2})} &\le \mathbb{E}\left[\int_0^T \left\|\mathcal{E}^{(\theta)}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)f(s,u(s))\right\|_{\dot H^{\nu_1+r_2}} ds\right]^2 \\ &\lesssim t^{-2\alpha}\theta^{2\alpha(1-r_2)}\mathbb{E}\left[\int_0^T \left\|\mathcal{E}_{\alpha,2}(T-s)f(s,u(s))\right\|_{\dot H^{\nu_1+1}} ds\right]^2 \\ &\lesssim t^{-2\alpha}\theta^{2\alpha(1-r_2)}\mathbb{E}\left[\int_0^T (T-s)^{-\alpha}\left\|f(s,u(s))\right\|_{\dot H^{\nu_1}} ds\right]^2.\end{aligned}$$

Recall that we have proved in (54) that $\mathbb{E}\left[\int_0^T (T-s)^{-\alpha}\left\|f(s,u(s))\right\|_{\dot H^{\nu_1}} ds\right]^2 \lesssim \|u\|^2_{L^\infty(0,T;L^2(\Omega,\dot H^{\nu_1}))}$. Hence

$$\left\|J_4^\theta(u)(t)\right\|^2_{L^2(\Omega,\dot H^{\nu_1+r_2})} \lesssim t^{-2\alpha}\theta^{2\alpha(1-r_2)}\|u\|^2_{L^\infty(0,T;L^2(\Omega,\dot H^{\nu_1}))}. \tag{19}$$

We continue with the fifth term $J_5^\theta(t)$. Noting that the assumption in part iii) of Theorem 3.1 gives us $\frac{1-r_2}{\nu_1-\nu_2+r_2}\in(0,1)$. By using the Itô isometry and Lemma 2.4 with $\mu_1=\nu_1+r_2$, $\mu_2=\frac{\nu_1+\nu_2+r_2-1}{2}$ and $\gamma=\frac{1-r_2}{\nu_1-\nu_2+r_2}$, we obtain

$$\left\|\mathcal{E}^{(\theta)}_{\alpha,2}(t)\right\|_{\mathcal{L}(\dot H^{\nu_2},\dot H^{\nu_1+r_2})} \lesssim \theta^{\alpha(1-r_2)}t^{-\alpha[1-(\nu_2-\nu_1)]},$$

where we note that $2\mu_1-2\mu_2-1=\nu_1-\nu_2+r_2$ and $\gamma+1=\frac{\nu_1-\nu_2+1}{\nu_1-\nu_2+r_2}$. This helps us have the estimate for $J_5^\theta(t)$ as

$$\begin{aligned}\left\|J_5^\theta(t)\right\|^2_{L^2(\Omega,\dot H^{\nu_1+r_2})} &= \int_0^t \left\|A^{\nu_1+r_2}\mathcal{E}^{(\theta)}_{\alpha,2}(t-s)\sigma(s)\right\|^2_{L_0^2} ds \\ &\le \int_0^t \left\|\mathcal{E}^{(\theta)}_{\alpha,2}(t-s)\right\|^2_{\mathcal{L}(\dot H^{\nu_2},\dot H^{\nu_1+r_2})}\left\|A^{\nu_2}\sigma(s)\right\|^2_{L_0^2} ds \\ &\lesssim \theta^{2\alpha(1-r_2)}\int_0^t (t-s)^{-2\alpha[1-(\nu_2-\nu_1)]}\|\sigma(s)\|^2_{L^2(L_0^2(X,\dot H^{\nu_2})} ds.\end{aligned}$$

From (58), we can see that

$$\int_0^t (t-s)^{-2\alpha[1-(\nu_2-\nu_1)]}\|\sigma(s)\|^2_{L_0^2(X,\dot H^{\nu_2})} ds \lesssim \|\sigma\|^2_{L^p(0,T;L_0^2(X,\dot H^{\nu_2}))},$$

which shows clearly that

$$\left\|J_5^\theta(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_2})} \lesssim \theta^{2\alpha(1-r_2)} \|\sigma\|^2_{L^p(0,T;L_0^2(X,\dot{H}^{\nu_2}))}. \tag{20}$$

By using a similar way as in (57), we have an estimate for $J_6^\theta(t)$ as

$$\begin{aligned}\left\|J_6^\theta(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_2})} &= \int_t^{t+\theta} \left\|A^{\nu_1+r_2}\mathcal{E}_{\alpha,2}(t+\theta-s)\sigma(s)\right\|^2_{L_0^2} ds \\ &\lesssim \int_t^{t+\theta} (t+\theta-s)^{-2\alpha[1-(\nu_2-\nu_1)]} \|\sigma(s)\|^2_{L_0^2(X,\dot{H}^{\nu_2})} ds.\end{aligned}$$

The Hölder inequality allows that

$$\begin{aligned}\left\|J_6^\theta(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_2})} &\le \left(\int_t^{t+\theta} (t+\theta-s)^{\frac{-2p\alpha[1-(\nu_2-\nu_1)]}{p-2}} ds\right)^{\frac{p-2}{p}} \left(\int_t^{t+\theta} \|\sigma(s)\|^p_{L_0^2(X,\dot{H}^{\nu_2})} ds\right)^{\frac{2}{p}} \\ &\lesssim \theta^{\frac{p-2}{p}-2\alpha[1-(\nu_2-\nu_1)]} \|\sigma\|^2_{L^p(0,T;L_0^2(X,\dot{H}^{\nu_2}))}.\end{aligned}$$

Using (17), the final term can be estimated as

$$\begin{aligned}\left\|J_7^\theta(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_2})} &= \int_0^T \left\|A^{\nu_1+r_2}\mathcal{E}^{(\theta)}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)\right\|^2_{L_0^2} ds \\ &\lesssim \theta^{2\alpha(1-r_2)} t^{-2\alpha} \int_0^T \left\|A^{\nu_1+1}\mathcal{E}_{\alpha,2}(T-s)\sigma(s)\right\|^2_{L_0^2} ds \\ &\lesssim \theta^{2\alpha(1-r_2)} t^{-2\alpha} \|\sigma\|^2_{L^p(0,T;L_0^2(X,\dot{H}^{\nu_2}))}.\end{aligned} \tag{21}$$

Combining (18), (19), (20), (21), we deduce that

$$t^\alpha \|u(t+\theta)-u(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_2})} \lesssim \theta^\eta, \tag{22}$$

where $\eta = \min\left\{\alpha(1-r_2), \frac{p-2}{2p} - \alpha[1-(\nu_2-\nu_1)]\right\}$.

Now, in the same way as in Step 2 (noting that r_1 is replaced by r_2), one easily see that

$$t^{\alpha r_2} \|u(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_2})} \lesssim \|g\|_{L^2(\Omega,\dot{H}^{\nu_1+1})} + \|u\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))} + \|\sigma\|_{L^p(0,T;L_0^2(X,\dot{H}^{\nu_2}))}.$$

This leads to

Please cite this article in press as: N.H. Tuan et al., On backward problems for stochastic fractional reaction equations with standard and fractional Brownian motion, Bull. Sci. math. (2022), https://doi.org/10.1016/j.bulsci.2022.103158

$$\begin{aligned} t^{\alpha-\eta} \|u(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_2})} &= t^{\alpha r_2} t^{\alpha(1-r_2)-\eta} \|u(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_2})} \\ &\lesssim t^{\alpha r_2} \|u(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_2})} \\ &\lesssim \|g\|_{L^2(\Omega,\dot{H}^{\nu_1+1})} + \|u\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))} + \|\sigma\|_{L^p(0,T;L_0^2(X,\dot{H}^{\nu_2}))} . \end{aligned} \tag{23}$$

From (22) and (23), we conclude that $u \in C^{\alpha-\eta,\eta}((0,T]; L^2(\Omega, \dot{H}^{\nu_1+r_2}))$. □

By choosing the two parameters $\nu_1 = \nu_2 = 0$, we directly obtain the following corollary:

Corollary 3.1. *Let us consider Problem (2) when $\alpha \in (0, \frac{1}{2})$. In this case, by assumptions for g, σ, f as $g \in L^2(\Omega, \dot{H}^1)$, $\sigma \in L^p(0,T; L^2(\Omega, L_0^2))$, for some $p > \frac{2}{1-2\alpha}$, $f(\cdot, 0) = 0$ and*

$$\|f(t,\varphi) - f(t,\varphi')\|_X \le \mathcal{K} \|\varphi - \varphi'\|_X , \quad \textit{where} \quad \mathcal{K} C_2 T \left(1 + \frac{C_3^2 T^{-2\alpha}}{(1-\alpha)^2}\right)^{\frac{1}{2}} < 1,$$

then Problem (2) has a unique mild solution u satisfying

$$v \in L^\infty(0,T; L^2(\Omega, X)) \cap L^q(0,T; L^2(\Omega, \dot{H}^{r_1})) \cap C^{\alpha-\eta,\eta}((0,T]; L^2(\Omega, \dot{H}^{r_2})),$$

where $0 < r_1 \le 1 < q < \frac{1}{\alpha r_1}$, $\frac{1}{2} < r_2 < 1$, and $\eta := \min\left\{\alpha(1-r_2), \frac{p-2}{2p} - \alpha\right\}$.

3.2. Existence, uniqueness, regularity of mild solution to Problem (3) when $H \in (\frac{1}{4}, \frac{1}{2})$

Theorem 3.2. *Let $H \in (\frac{1}{4}, \frac{1}{2})$. Suppose that Assumptions (g), (σ_2), (f) are satisfied. Then*

i) *Problem (3) has a unique mild solution v in $L^\infty(0,T; L^2(\Omega, \dot{H}^{\nu_1}))$.*
ii) *If $0 < r_1 \le 1 < q < \frac{1}{\alpha r_1}$, then $v \in L^q(0,T; L^2(\Omega, \dot{H}^{\nu_1+r_1}))$ and satisfies*

$$\begin{aligned} \|v\|_{L^q(0,T;L^2(\Omega,\dot{H}^{\nu_1+r_1}))} \lesssim \mathcal{C}_\sigma + \|g\|_{L^2(\Omega,\dot{H}^{\nu_1+1})} + \|v\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))} \\ + \|\sigma\|_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))} . \end{aligned}$$

Proof. **Step 1**. This step is aimed to prove Problem (3) when $H \in (\frac{1}{4}, \frac{1}{2})$ has a unique mild solution $v \in L^\infty(0,T; L^2(\Omega, \dot{H}^{\nu_1}))$ by using the contraction mapping principle. For $w \in L^\infty(0,T; L^2(\Omega, \dot{H}^{\nu_1}))$, we define

$$\begin{aligned} \overline{\mathcal{Z}} w(t) := \mathcal{E}_{\alpha,1}(t) g + \int_0^t \mathcal{E}_{\alpha,2}(t-s) f(s, w(s)) ds - \int_0^T \mathcal{E}_{\alpha,1}(t) \mathcal{E}_{\alpha,2}(T-s) f(s, w(s)) ds \\ + \int_0^t \mathcal{E}_{\alpha,2}(t-s) \sigma(s) dB^H(s) - \int_0^T \mathcal{E}_{\alpha,1}(t) \mathcal{E}_{\alpha,2}(T-s) \sigma(s) dB^H(s) \end{aligned}$$

$$= \quad I_1(t) + I_2(w)(t) + I_3(w)(t) + \overline{I}_4(t) + \overline{I}_5(t). \tag{24}$$

Firstly, we prove that $\overline{\mathcal{Z}}$ is well-defined on $L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))$. Indeed, it follows from the property i) of Lemma 3.1, the properties i), iii) of Lemma 3.2, and the properties i), ii) of Lemma 3.4 that for all $t \in (0,T]$ there holds

$$\|\mathcal{Z}w(t)\|_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim \|g\|_{L^2(\Omega,\dot{H}^{\nu_1+1})} + \|w\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))} + \mathcal{C}_\sigma + \|\sigma\|_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))} .$$

In addition, by exactly the same way as in (15)-(16), one can prove that there exists a unique fixed point v such that $\overline{\mathcal{Z}}v = v$, which is the unique mild solution of Problem (3) when $H \in (\frac{1}{4}, \frac{1}{2})$.

Step 2. It can be seen that $v \in L^q(0,T;L^2(\Omega,\dot{H}^{\nu_1+r_1}))$. Indeed, by the property ii) of Lemma 3.1, the properties ii), iv) of Lemma 3.2, and Lemma 3.4, we have

$$\begin{aligned}&\|v(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_1})}\\ &\lesssim t^{-\alpha r_1}\left(\|g\|_{L^2(\Omega,\dot{H}^{\nu_1+1})} + \|v\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))} + \mathcal{C}_\sigma + \|\sigma\|_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}\right),\end{aligned}$$

which follows that $v \in L^q(0,T;L^2(\Omega,\dot{H}^{\nu_1+r_1}))$. □

By choosing the two parameters $\nu_1 = \nu_2 = 0$, we directly obtain the following corollary:

Corollary 3.2. *Let us consider Problem (3) when $\alpha \in (\frac{1}{2} - H, H)$. In this case, by assumptions for g, σ, f as $g \in L^2(\Omega, \dot{H}^1)$, σ satisfies*

$$\|\sigma(t_1) - \sigma(t_2)\|_{L_0^2} \le \mathcal{C}_\sigma |t_2 - t_1|^\zeta, \qquad t_1, t_2 \in (0,T],$$

where $\zeta > \frac{1}{2} - H$, $f(\cdot, 0) = 0$ and

$$\|f(t,\varphi) - f(t,\varphi')\|_X \le \mathcal{K}\|\varphi - \varphi'\|_X, \quad \text{where} \quad \mathcal{K}C_2T\left(1 + \frac{C_3^2 T^{-2\alpha}}{(1-\alpha)^2}\right)^{\frac{1}{2}} < 1,$$

then Problem (3) when $H \in (\frac{1}{4}, \frac{1}{2})$ has a unique mild solution v satisfying

$$v \in L^\infty(0,T;L^2(\Omega,X)) \cap L^q(0,T;L^2(\Omega,\dot{H}^{r_1})),$$

where $0 < r_1 \le 1 < q < \frac{1}{\alpha r_1}$.

Remark 3.1. It can be observed that the stochastic term $\overline{I}_5(t)$ contains the operator $\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)$, which is bounded by a singular kernel $(T-s)^{-\alpha(1-(\nu_3-\nu_1))}$ (see (62)). Due to the appearance of this kernel, to guarantee the existence of the mild solution, H is required to belong to $(\frac{1}{4}, \frac{1}{2})$ instead of the whole interval $(0, \frac{1}{2})$. In more details, we need $\alpha[1-(\nu_3-\nu_1)] < H$ and $\alpha[1-(\nu_3-\nu_1)] > \frac{1}{2}-H$ to ensure (63)-(64), and (70)-(73) hold true. This is the reason why H need to be belonged to $(\frac{1}{4},\frac{1}{2})$ and we can not gain similar results for the non-covered case.

3.3. Existence, uniqueness, regularity of mild solution to Problem (3) when $H \in (\frac{1}{2},1)$

Theorem 3.3. *Let $H \in (\frac{1}{2},1)$. Suppose that Assumptions (g), (σ_3), (f) are satisfied. Then*

i) *Problem (3) has a unique mild solution v in $L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))$.*
ii) *If $0 < r_1 \le 1 < q < \frac{1}{\alpha r_1}$, then $v \in L^q(0,T;L^2(\Omega,\dot{H}^{\nu_1+r_1}))$ and satisfies*

$$\|v\|_{L^q(0,T;L^2(\Omega,\dot{H}^{\nu_1+r_1}))} \lesssim \|g\|_{L^2(\Omega,\dot{H}^{\nu_1+1})} + \|v\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))} + \|\sigma\|_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}.$$

iii) *If $r_3 \in \left(\frac{\nu_4-\nu_1+1}{2},1\right)$, then $v \in C^{\alpha-\vartheta,\vartheta}((0,T];L^2(\Omega,\dot{H}^{\nu_1+r_3}))$, where*

$$\vartheta := \min\left\{\alpha(1-r_3), \frac{\rho-2}{2\rho} - \alpha[1-(\nu_4-\nu_1)]\right\}.$$

Proof. Step 1. Our goal in this step is to prove Problem (3) when $H \in (\frac{1}{2},1)$ has a unique mild solution $v \in L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))$ by using the contraction mapping principle.

For $w \in L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))$, we also define $\overline{\mathcal{Z}}$ as in (24) and aim to prove that $\overline{\mathcal{Z}}$ is well-defined on $L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))$ in the case $H \in (\frac{1}{2},1)$. From the property i) of Lemma 3.1, the properties i), iii) of Lemma 3.2, and Lemma 3.6, we conclude that $\overline{\mathcal{Z}}$ is well-defined on $L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))$ when $H \in (\frac{1}{2},1)$ under the assumptions as in Theorem 3.3.

Furthermore, by exactly the same way as in (15)-(16), one easily see that there exists a unique fixed point v such that $\overline{\mathcal{Z}}v = v$, which is the unique mild solution of Problem (3) when $H \in (\frac{1}{2},1)$.

Step 2. Now, we aim to prove that $v \in L^q(0,T;L^2(\Omega,\dot{H}^{\nu_1+r_1}))$. From the property ii) of Lemma 3.1, the properties ii), iv) of Lemma 3.2, and Lemma 3.7, we conclude that

$$\begin{aligned}&\|v(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_1})}\\ &\quad\lesssim t^{-\alpha r_1}\left(\|g\|_{L^2(\Omega,\dot{H}^{\nu_1+1})} + \|v\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))} + \|\sigma\|^2_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}\right),\end{aligned} \quad (25)$$

which follows that $v \in L^q(0,T;L^2(\Omega,\dot{H}^{\nu_1+r_1}))$.

Step 3. In this step, we will prove that $v \in C^{\alpha-\vartheta,\vartheta}((0,T];L^2(\Omega,\dot{H}^{\nu_1+r_2}))$. For $0 < t < t+\theta \le T$, we have

$$v(t+\theta)-v(t) = \mathcal{E}_{\alpha,1}^{(\theta)}(t)g + \int_0^t \mathcal{E}_{\alpha,2}^{(\theta)}(t-s)f(s,v(s))ds + \int_t^{t+\theta} \mathcal{E}_{\alpha,2}(t+\theta-s)f(s,v(s))ds$$

$$- \int_0^T \mathcal{E}_{\alpha,1}^{(\theta)}(t)\mathcal{E}_{\alpha,2}(T-s)f(s,v(s))ds + \int_0^{t+\theta} \mathcal{E}_{\alpha,2}(t+\theta-s)\sigma(s)dB^H(s)$$
$$- \int_0^t \mathcal{E}_{\alpha,2}(t-s)\sigma(s)dB^H(s) - \int_0^T \mathcal{E}_{\alpha,1}(t+\theta)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)dB^H(s)$$
$$+ \int_0^T \mathcal{E}_{\alpha,1}(t+\theta)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)dB^H(s).$$

From (18)-(19), it is obvious that the following estimates for four first terms hold

$$\left\|\mathcal{E}_{\alpha,1}^{(\theta)}(t)g\right\|_{L^2(\Omega,\dot{H}^{\nu_1+r_3})} \lesssim t^{-\alpha}\theta^{\alpha(1-r_3)}\left\|g\right\|_{L^2(\Omega,\dot{H}^{\nu_1+1})}, \tag{26}$$
$$\left\|\int_0^t \mathcal{E}_{\alpha,2}^{(\theta)}(t-s)f(s,v(s))ds\right\|_{L^2(\Omega,\dot{H}^{\nu_1+r_3})} \lesssim \theta^{\alpha(1-r_3)}\left\|v\right\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))},$$
$$\left\|\int_t^{t+\theta} \mathcal{E}_{\alpha,2}(t+\theta-s)f(s,v(s))ds\right\|_{L^2(\Omega,\dot{H}^{\nu_1+r_3})} \lesssim \theta^{\alpha(1-r_3)}\left\|v\right\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))},$$
$$\left\|\int_0^T \mathcal{E}_{\alpha,1}^{(\theta)}(t)\mathcal{E}_{\alpha,2}(T-s)f(s,v(s))ds\right\|_{L^2(\Omega,\dot{H}^{\nu_1+r_3})} \lesssim t^{-\alpha}\theta^{\alpha(1-r_3)}\left\|g\right\|_{L^2(\Omega,\dot{H}^{\nu_1+1})}. \tag{27}$$

Now, we only need to estimate four last terms. We would like to note that the technique we have used to estimate the stochastic integrals with respect to Wiener process, namely $J_5^{(\theta)}(t), J_6^{(\theta)}(t), J_7^{(\theta)}(t)$ in proof of Theorem 3.1, can not be applied here.

For the sake of convenience, we set

$$S_5^{(\theta)}(t) := \int_0^{t+\theta} \mathcal{E}_{\alpha,2}(t+\theta-s)\sigma(s)dB^H(s) - \int_0^t \mathcal{E}_{\alpha,2}(t-s)\sigma(s)dB^H(s),$$
$$S_6^{(\theta)}(t) := \int_0^T \mathcal{E}_{\alpha,1}(t+\theta)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)dB^H(s) - \int_0^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)dB^H(s),$$

and give explicit representations for two above terms firstly. For the first term, it can be observed that

$$S_5^{(\theta)}(t) = \sum_{n\geq 1}\int_0^{t+\theta} \mathcal{E}_{\alpha,2}(t+\theta-s)\sigma(s)Q^{\frac{1}{2}}e_n d\beta_n^H(s)$$

$$
\begin{aligned}
&-\sum_{n\geq 1}\int_0^t \mathcal{E}_{\alpha,2}(t-s)\sigma(s)Q^{\frac{1}{2}}e_n d\beta_n^H(s)\\
&=\int_0^{t+\theta} K_{H,t+\theta}^*\left(\mathcal{E}_{\alpha,2}(t+\theta-\cdot)\sigma(\cdot)Q^{\frac{1}{2}}e_n\right)(s)d\beta_n(s)\\
&-\int_0^{t} K_{H,t}^*\left(\mathcal{E}_{\alpha,2}(t-\cdot)\sigma(\cdot)Q^{\frac{1}{2}}e_n\right)(s)d\beta_n(s).
\end{aligned}
$$

By the formula of $K_{H,t+\theta}^*$ and $K_{H,t}^*$, one can see

$$
\begin{aligned}
S_5^{(\theta)}(t) &= \sum_{n\geq 1}\int_0^{t+\theta}\left(\int_s^{t+\theta}\mathcal{E}_{\alpha,2}(t+\theta-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\frac{\partial K^H}{\partial \tilde{s}}(\tilde{s},s)d\tilde{s}\right)d\beta_n(s)\\
&-\sum_{n\geq 1}\int_0^{t}\left(\int_s^{t}\mathcal{E}_{\alpha,2}(t-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\frac{\partial K^H}{\partial \tilde{s}}(\tilde{s},s)d\tilde{s}\right)d\beta_n(s)\\
&= \sum_{n\geq 1}\int_t^{t+\theta}\left(\int_s^{t+\theta}\mathcal{E}_{\alpha,2}(t+\theta-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\frac{\partial K^H}{\partial \tilde{s}}(\tilde{s},s)d\tilde{s}\right)d\beta_n(s)\\
&+\sum_{n\geq 1}\int_0^{t}\left(\int_t^{t+\theta}\mathcal{E}_{\alpha,2}(t+\theta-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\frac{\partial K^H}{\partial \tilde{s}}(\tilde{s},s)d\tilde{s}\right)d\beta_n(s)\\
&+\sum_{n\geq 1}\int_0^{t}\left(\int_s^{t}\mathcal{E}_{\alpha,2}^{(\theta)}(t-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\frac{\partial K^H}{\partial \tilde{s}}(\tilde{s},s)d\tilde{s}\right)d\beta_n(s)\\
&=: S_{51}^{(\theta)}(t)+S_{52}^{(\theta)}(t)+S_{53}^{(\theta)}(t).
\end{aligned}
$$

Similarly, for the second term, it is clear that

$$
\begin{aligned}
S_6^{(\theta)}(t) &= \int_0^{T} K_{H,T}^*\left(\mathcal{E}_{\alpha,1}(t+\theta)\mathcal{E}_{\alpha,2}(T-\cdot)\sigma(\cdot)Q^{\frac{1}{2}}e_n\right)(s)d\beta_n(s)\\
&-\int_0^{T} K_{H,T}^*\left(\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-\cdot)\sigma(\cdot)Q^{\frac{1}{2}}e_n\right)(s)d\beta_n(s)\\
&= \sum_{n\geq 1}\int_0^{T}\left(\int_s^{T}\mathcal{E}_{\alpha,1}(t+\theta)\mathcal{E}_{\alpha,2}(T-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\frac{\partial K^H}{\partial \tilde{s}}(\tilde{s},s)d\tilde{s}\right)d\beta_n(s)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{n\geq 1}\int_0^T\left(\int_s^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\frac{\partial K^H}{\partial\tilde{s}}(\tilde{s},s)d\tilde{s}\right)d\beta_n(s)\\
&=\sum_{n\geq 1}\int_0^T\left(\int_s^T \mathcal{E}^{(\theta)}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\frac{\partial K^H}{\partial\tilde{s}}(\tilde{s},s)d\tilde{s}\right)d\beta_n(s).
\end{aligned}
$$

Now, we are ready to estimate $S_5^{(\theta)}(t)$ and $S_6^{(\theta)}(t)$.

• *Estimating $S_5^{(\theta)}(t)$*. By using a similar way as in (101), we obtain the following estimate for the first term

$$
\begin{aligned}
&\left\|S_{51}^{(\theta)}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_3})}\\
&=\sum_{n\geq 1}\int_t^{t+\theta}\left\|\int_s^{t+\theta}\mathcal{E}_{\alpha,2}(t+\theta-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\frac{\partial K^H}{\partial\tilde{s}}(\tilde{s},s)d\tilde{s}\right\|^2_{\dot{H}^{\nu_1+r_3}}ds\\
&\lesssim\sum_{n\geq 1}\int_t^{t+\theta}\left\|\int_s^{t+\theta}\mathcal{E}_{\alpha,2}(t+\theta-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right\|^2_{\dot{H}^{\nu_1+r_3}}ds\\
&\lesssim\sum_{n\geq 1}\int_t^{t+\theta}\left[\int_s^{t+\theta}(t+\theta-s)^{-\alpha(1-(\nu_4-\nu_1))}\left\|\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_4}}\left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right]^2ds.
\end{aligned}
$$

Using the Hölder inequality, we obtain

$$
\begin{aligned}
\left\|S_{51}^{(\theta)}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_3})}\lesssim&\int_t^{t+\theta}\left(\int_s^{t+\theta}\left(\frac{s}{\tilde{s}}\right)^{1-2H}(\tilde{s}-s)^{H-\frac{3}{2}}(t+\theta-\tilde{s})^{-\alpha(1-(\nu_4-\nu_1))}d\tilde{s}\right)\times\\
&\times\left(\int_s^{t+\theta}(\tilde{s}-s)^{H-\frac{3}{2}}(t+\theta-\tilde{s})^{-\alpha(1-(\nu_4-\nu_1))}\|\sigma(\tilde{s})\|^2_{L_0^2(X,\dot{H}^{\nu_4})}d\tilde{s}\right)ds.
\end{aligned}
$$

By a similar technique as in (97)-(98), we arrive at

$$
\begin{aligned}
&\left\|S_{51}^{(\theta)}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_3})}\\
&\lesssim\|\sigma\|^2_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}\int_t^{t+\theta}s^{1-2H}(t+\theta-s)^{\frac{\rho-2}{\rho}+1-2\left[\frac{3}{2}-H+\alpha(1-(\nu_4-\nu_1))\right]}ds.
\end{aligned}
$$

Consider the integral in the right-hand side, the identity (96) allows that

$$\begin{aligned}
&\int_t^{t+\theta} s^{1-2H}(t+\theta-s)^{\frac{\rho-2}{\rho}+1-2\left[\frac{3}{2}-H+\alpha(1-(\nu_4-\nu_1))\right]}ds \\
&= \int_0^{\theta} (\tau+t)^{1-2H}(\theta-\tau)^{\frac{\rho-2}{\rho}+1-2\left[\frac{3}{2}-H+\alpha(1-(\nu_4-\nu_1))\right]}d\tau \\
&\le \int_0^{\theta} \tau^{1-2H}(\theta-\tau)^{\frac{\rho-2}{\rho}+1-2\left[\frac{3}{2}-H+\alpha(1-(\nu_4-\nu_1))\right]}d\tau \\
&\lesssim \theta^{\frac{\rho-2}{\rho}-2\alpha(1-(\nu_4-\nu_1))}.
\end{aligned}$$

Hence

$$\left\|S_{51}^{(\theta)}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_3})} \lesssim \theta^{\frac{\rho-2}{\rho}-2\alpha(1-(\nu_4-\nu_1))}\left\|\sigma\right\|^2_{L^\rho(0,T;L^2_0(X,\dot{H}^{\nu_4}))}. \tag{28}$$

By a similar way as in above, one gets

$$\begin{aligned}
&\left\|S_{52}^{(\theta)}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_3})} \\
&= \sum_{n\ge 1}\int_0^{t+\theta}\left\|\int_t^{t+\theta}\mathcal{E}_{\alpha,2}(t+\theta-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\frac{\partial K^H}{\partial \tilde{s}}(\tilde{s},s)d\tilde{s}\right\|^2_{\dot{H}^{\nu_1+r_3}} ds \\
&\lesssim \sum_{n\ge 1}\int_0^{t+\theta}\left[\int_t^{t+\theta}(t+\theta-s)^{-\alpha(1-(\nu_4-\nu_1))}\left\|\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_4}}\left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right]^2 ds \\
&\lesssim \int_0^{t+\theta}\left(\int_t^{t+\theta}\left(\frac{s}{\tilde{s}}\right)^{1-2H}(\tilde{s}-s)^{H-\frac{3}{2}}(t+\theta-\tilde{s})^{-\alpha(1-(\nu_4-\nu_1))}d\tilde{s}\right)\times \\
&\quad \times\left(\int_t^{t+\theta}(\tilde{s}-s)^{H-\frac{3}{2}}(t+\theta-\tilde{s})^{-\alpha(1-(\nu_4-\nu_1))}\left\|\sigma(\tilde{s})\right\|^2_{L^2_0(X,\dot{H}^{\nu_4})}d\tilde{s}\right)ds.
\end{aligned}$$

Estimating two integrals in the right-hand side by using the property (96), one arrives at

$$\begin{aligned}
&\int_t^{t+\theta}\left(\frac{s}{\tilde{s}}\right)^{1-2H}(\tilde{s}-s)^{H-\frac{3}{2}}(t+\theta-\tilde{s})^{-\alpha(1-(\nu_4-\nu_1))}d\tilde{s} \\
&\lesssim s^{1-2H}\int_t^{t+\theta}(\tilde{s}-s)^{H-\frac{3}{2}}(t+\theta-\tilde{s})^{-\alpha(1-(\nu_4-\nu_1))}d\tilde{s}
\end{aligned}$$

$$\lesssim s^{1-2H}\int_0^\theta \tau^{H-\frac{3}{2}}(\theta-\tau)^{-\alpha(1-(\nu_4-\nu_1))}d\tau$$
$$\lesssim s^{1-2H}\theta^{H-\frac{1}{2}-\alpha(1-(\nu_4-\nu_1))},$$

and

$$\int_t^{t+\theta}(\tilde{s}-s)^{H-\frac{3}{2}}(t+\theta-\tilde{s})^{-\alpha(1-(\nu_4-\nu_1))}\|\sigma(\tilde{s})\|^2_{L_0^2(X,\dot{H}^{\nu_4})}d\tilde{s}$$
$$\lesssim \int_0^\theta \tau^{H-\frac{3}{2}}(\theta-\tau)^{-\alpha(1-(\nu_4-\nu_1))}\|\sigma(\tau+t)\|^2_{L_0^2(X,\dot{H}^{\nu_4})}d\tau$$
$$\lesssim \left(\int_0^\theta \tau^{-\frac{\rho(3/2-H)}{\rho-2}}(\theta-\tau)^{-\frac{\rho\alpha[1-(\nu_4-\nu_1)]}{\rho-2}}d\tau\right)^{\frac{\rho-2}{\rho}}\left(\int_0^\theta \|\sigma(\tau+t)\|^\rho_{L_0^2(X,\dot{H}^{\nu_4})}d\tau\right)^{\frac{2}{\rho}}$$
$$\lesssim \theta^{\frac{\rho-2}{\rho}-\left[\frac{3}{2}-H+\alpha(1-(\nu_4-\nu_1))\right]}\|\sigma\|^2_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}.$$

Hence, it is obvious that

$$\left\|S_{52}^{(\theta)}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_3})}\lesssim \theta^{\frac{\rho-2}{\rho}-2[1-H+\alpha(1-(\nu_4-\nu_1))]}\|\sigma\|^2_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}\int_0^{t+\theta}s^{1-2H}ds$$
$$\lesssim \theta^{\frac{\rho-2}{\rho}-2[1-H+\alpha(1-(\nu_4-\nu_1))]}\|\sigma\|^2_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}(t+\theta)^{2-2H}$$
$$\lesssim \theta^{\frac{\rho-2}{\rho}-2\alpha(1-(\nu_4-\nu_1))}\|\sigma\|^2_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}. \quad (29)$$

Now, we aim to estimate $S_{53}^{(\theta)}(t)$ by using the following property

$$\left\|\mathcal{E}_{\alpha,2}^{(\theta)}(t)\right\|_{\mathcal{L}(\dot{H}^{\nu_4},\dot{H}^{\nu_1+1})}\lesssim \theta^{\alpha(1-r_3)}t^{-\alpha(1-(\nu_4-\nu_1))}, \quad (30)$$

which is obtained by applying Lemma 2.4 with $\mu_1=\nu_1+r_3$, $\mu_2=\frac{\nu_1+\nu_4+r_3-1}{2}$ and $\gamma=\frac{1-r_3}{\nu_1-\nu_4+r_3}$. In this way, we arrive at

$$\left\|S_{53}^{(\theta)}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_3})}$$
$$\lesssim \sum_{n\ge 1}\int_0^t\left\|\int_s^t \mathcal{E}_{\alpha,2}^{(\theta)}(t-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right\|^2_{\dot{H}^{\nu_1+r_3}}ds$$
$$\lesssim \sum_{n\ge 1}\int_0^t\left[\int_s^t\left\|\mathcal{E}_{\alpha,2}^{(\theta)}(t-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1+r_3}}\left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right]^2ds$$

$$\leq \sum_{n\geq 1}\int_0^t \left[\int_s^t \lambda_1^{-(1-r_3)} \left\|\mathcal{E}_{\alpha,2}^{(\theta)}(t-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1+1}} \left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right]^2 ds$$

$$\leq \theta^{2\alpha(1-r_3)}\sum_{n\geq 1}\int_0^t \left[\int_s^t (t-s)^{-\alpha(1-(\nu_4-\nu_1))} \left\|\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_4}} \left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right]^2 ds.$$

Recall that we have proved in (101)-(102) that

$$\sum_{n\geq 1}\int_0^t \left[\int_s^t (t-s)^{-\alpha(1-(\nu_4-\nu_1))} \left\|\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_4}} \left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right]^2 ds$$
$$\lesssim \|\sigma\|^2_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}.$$

Hence, it is clear that

$$\left\|S_{53}^{(\theta)}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_3})} \lesssim \theta^{2\alpha(1-r_3)}\|\sigma\|^2_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}. \tag{31}$$

Combining (28), (29), (31), we conclude that

$$\left\|S_5^{(\theta)}(t)\right\|_{L^2(\Omega,\dot{H}^{\nu_1+r_3})} \lesssim \theta^{\vartheta}\|\sigma\|_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}, \tag{32}$$

where $\vartheta = \min\left\{\alpha(1-r_3), \frac{\rho-2}{2\rho}-\alpha[1-(\nu_4-\nu_1)]\right\}$.

• *Estimating $S_6^{(\theta)}(t)$*. Using the property (30) and applying part b) of Lemma 2.3 with $\mu_1=\nu_1+1$, $\mu_2=\nu_4$, we obtain

$$\left\|S_6^{(\theta)}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_3})}$$
$$\lesssim \sum_{n\geq 1}\int_0^T \left\|\int_s^T \mathcal{E}_{\alpha,1}^{(\theta)}(t)\mathcal{E}_{\alpha,2}(T-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right\|^2_{\dot{H}^{\nu_1+r_3}} ds$$
$$\lesssim \frac{\theta^{2\alpha(1-r_3)}}{t^{2\alpha(1-(\nu_4-\nu_1))}}\sum_{n\geq 1}\int_0^T \left[\int_s^T \left\|\mathcal{E}_{\alpha,2}(T-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1+r_3}} \left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right]^2 ds$$
$$\lesssim \frac{\theta^{2\alpha(1-r_3)}}{t^{2\alpha(1-(\nu_4-\nu_1))}}\sum_{n\geq 1}\int_0^T \left[\int_s^T \left\|\mathcal{E}_{\alpha,2}(T-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1+1}} \left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right]^2 ds$$

$$\lesssim \frac{\theta^{2\alpha(1-r_3)}}{t^{2\alpha(1-(\nu_4-\nu_1))}}$$
$$\times \sum_{n\geq 1} \int_0^T \left[\int_s^T (T-\tilde{s})^{\alpha(1-(\nu_4-\nu_1))} \left\| \sigma(\tilde{s}) Q^{\frac{1}{2}} e_n \right\|_{\dot{H}^{\nu_4}} \left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H} (\tilde{s}-s)^{H-\frac{3}{2}} d\tilde{s} \right]^2 ds$$

Recall that we have proved in (95)-(98)

$$\sum_{n\geq 1} \int_0^T \left[\int_s^T (T-\tilde{s})^{\alpha(1-(\nu_4-\nu_1))} \left\| \sigma(\tilde{s}) Q^{\frac{1}{2}} e_n \right\|_{\dot{H}^{\nu_4}} \left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H} (\tilde{s}-s)^{H-\frac{3}{2}} d\tilde{s} \right]^2 ds$$
$$\lesssim \|\sigma\|^2_{L^\rho(0,T;L^2_0(X,\dot{H}^{\nu_4}))}.$$

Hence, it is obvious that

$$\left\| S_6^{(\theta)}(t) \right\|_{L^2(\Omega,\dot{H}^{\nu_1+r_3})} \lesssim \theta^{\alpha(1-r_3)} t^{-\alpha(1-(\nu_4-\nu_1))} \|\sigma\|_{L^\rho(0,T;L^2_0(X,\dot{H}^{\nu_4}))}. \tag{33}$$

Now, combining (26)-(27) and (32)-(33), we conclude that

$$t^\alpha \left\| v(t+\theta) - v(t) \right\|_{L^2(\Omega,\dot{H}^{\nu_1+r_3})} \lesssim \theta^\vartheta. \tag{34}$$

On the other hand, in the same way employed to obtain (25), one obtains

$$t^{\alpha r_3} \|v(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_3})} \lesssim \|g\|_{L^2(\Omega,\dot{H}^{\nu_1+1})} + \|v\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))} + \|\sigma\|_{L^\rho(0,T;L^2_0(X,\dot{H}^{\nu_4}))}.$$

From two latter results, it is clear to see that

$$\begin{aligned} t^{\alpha-\vartheta} \|v(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_3})} &\lesssim t^{\alpha r_3} \|v(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_3})} \\ &\lesssim \|g\|_{L^2(\Omega,\dot{H}^{\nu_1+1})} + \|v\|_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))} + \|\sigma\|_{L^\rho(0,T;L^2_0(X,\dot{H}^{\nu_4}))}. \end{aligned} \tag{35}$$

From (34), (35), we conclude that $v \in C^{\alpha-\vartheta,\vartheta}((0,T];L^2(\Omega,\dot{H}^{\nu_1+r_3}))$. □

By choosing the two parameters $\nu_1 = \nu_2 = 0$, we directly obtain the following corollary:

Corollary 3.3. *Let us consider Problem (3) when $\alpha \in (0, H-\frac{1}{2})$. In this case, by assumptions for g, σ, f as $g \in L^2(\Omega, \dot{H}^1)$, $\sigma \in L^\rho(0,T;L^2(\Omega,L^2_0))$, for some $\rho > \frac{2}{H-1/2-\alpha}$, $f(\cdot,0) = 0$ and*

$$\|f(t,\varphi) - f(t,\varphi')\|_X \leq \mathcal{K} \|\varphi - \varphi'\|_X, \quad \textit{where} \quad \mathcal{K} C_2 T \left(1 + \frac{C_3^2 T^{-2\alpha}}{(1-\alpha)^2}\right)^{\frac{1}{2}} < 1,$$

then Problem (3) when $H \in (\frac{1}{2}, 1)$ has a unique mild solution v satisfying

$$u \in L^\infty(0,T; L^2(\Omega, X)) \cap L^q(0,T; L^2(\Omega, \dot{H}^{r_1})) \cap C^{\alpha-\eta,\eta}((0,T]; L^2(\Omega, \dot{H}^{r_3})),$$

where $0 < r_1 \leq 1 < q < \frac{1}{\alpha r_1}$, $\frac{1}{2} < r_3 < 1$, *and* $\vartheta := \min\left\{\alpha(1-r_2), \frac{\rho-2}{2\rho} - \alpha\right\}$.

4. Regularization results on $C([0,T]; L^2(\Omega, \dot{H}^\delta))$

In Section 3, we have investigated the well-poseness of Problem (2) and Problem (3) when $H \in (\frac{1}{2}, 1)$ in the case g, σ, f satisfy strongly assumptions $(g), (\sigma_1), (f)$. However, does the well-poseness results are obtained if $g \in L^2(\Omega, X), \sigma \in L^p(0,T; L_0^2)$ and f satisfies Assumption (f) with $\nu_1 = \delta$ instead? Due to the lack of regularity of the terminal condition, it can be observed that the two problems are ill-posed. Even the purely deterministic equation fails to be well-posed because of a lack of continuity for rough terminal conditions, see [44]. Hence, it is required to regularize the two problems.

Physically, we can not obtain the exact data (g, ψ) in most of situations. Additionally, small errors always appear in the observations. By this reason, from now on, we assume that the exact data (g, ψ) is contaminated by observed data $(g^\varepsilon, \psi^\varepsilon)$ satisfying the following model

$$\|g^\varepsilon - g\|_{L^2(\Omega,X)} + \|\sigma^\varepsilon - \sigma\|_{L^p(0,T;L_0^2)} < \varepsilon, \quad p \geq 2, \tag{36}$$

where $\varepsilon > 0$ is the noisy level. In what follows, we shall apply a regularization method called Fourier truncation to construct regularized solution for the two problems. After that, convergence rates of regularized solutions are also investigated.

4.1. Regularization result for Problem (2)

In this subsection, we aim construct a regularized solution for Problem (2) by using the observed data $(g^\varepsilon, \sigma^\varepsilon)$ and a regularization method called Fourier truncated method. The idea here is as follows. To obtain a regularized solution, we replace the solution operators $\mathcal{E}_{\alpha,1}(t), \mathcal{E}_{\alpha,2}(t)$ by new operators denoted by $\widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t), \widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(t)$, with $N(\varepsilon) \in \mathbb{Z}_+$ (called regularized parameter), defined as

$$\widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t)\varphi := \sum_{n \leq N(\varepsilon)} \frac{E_\alpha(-t^\alpha \lambda_n)}{E_\alpha(-T^\alpha \lambda_n)} \langle \varphi, e_n \rangle e_n, \quad \widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(t)\varphi := \sum_{n \leq N(\varepsilon)} E_\alpha(-t^\alpha \lambda_n) \langle \varphi, e_n \rangle e_n,$$

$$\varphi \in X,$$

which are the truncated versions of $\mathcal{E}_{\alpha,1}(t), \mathcal{E}_{\alpha,2}(t)$ respectively. In this way, the regularized solution is constructed as

$$u^{N(\varepsilon)}(t) = \widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,1}(t) g^\varepsilon + \int_0^t \mathcal{E}^{N(\varepsilon)}_{\alpha,2}(t-s) f(s, u^{N(\varepsilon)}(s)) ds$$
$$- \int_0^T \widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,1}(t) \widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,2}(T-s) f(s, u^{N(\varepsilon)}(s)) ds$$
$$+ \int_0^t \widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,2}(t-s) \sigma^\varepsilon(s) dW(s) - \int_0^T \widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,1}(t) \widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,2}(T-s) \sigma^\varepsilon(s) dW(s). \quad (37)$$

In the following theorem, we will show the convergence result of the regularized solution mentioned above.

Theorem 4.1. *Let $\delta \geq 0$, $N(\varepsilon) \in \mathbb{Z}_+$. Assume that (36) holds for $p = 2$, f satisfies Assumption (f) with $\nu_1 = \delta$, and there exist constants $\mathcal{M}_u > 0$, $\delta_+ > \delta$ such that $\|u(t)\|_{L^2(\Omega, \dot{H}^{\delta_+})} \leq \mathcal{M}_u$, for all $0 \leq t \leq T$. Then, Equation (37) has a unique solution $u^{N(\varepsilon)} \in C([0,T]; L^2(\Omega, \dot{H}^\delta))$ satisfying*

$$\left\| u^{N(\varepsilon)} - u \right\|_{C([0,T]; L^2(\Omega, \dot{H}^\delta))} \lesssim \lambda^{\delta+1}_{N(\varepsilon)} \varepsilon + \lambda^{\delta-\delta_+}_{N(\varepsilon)} \mathcal{M}_u. \quad (38)$$

Corollary 4.1. *Let $N(\varepsilon)$ be the largest positive integer number such that $\lambda_{N(\varepsilon)} \leq \varepsilon^{-\frac{\kappa}{\delta+1}}$, for some positive constant $\kappa < \delta + 1$. Then*

$$\left\| u^{N(\varepsilon)} - u \right\|_{C([0,T]; L^2(\Omega, \dot{H}^\delta))} \lesssim \varepsilon^{1-\frac{\kappa}{\delta+1}} + \varepsilon^{\frac{\kappa}{\delta+1}(\delta^+ - \delta)} \longrightarrow 0, \quad \text{as} \ \ \varepsilon \to 0^+.$$

Remark 4.1. It can be seen from Theorem 3.1 the assumption “$u(t)$ is bounded in $L^2(\Omega, \dot{H}^{\delta_+})$” is fulfilled if the terminal function $g \in L^2(\Omega, \dot{H}^{\delta_+ + 1})$, f satisfies Assumption (f) with $\nu_1 = \delta_+$, and $\sigma \in L^p(0,T; L^2_0(X, \dot{H}^{\delta_*}))$, for some $\delta_* \in [\delta_+, \delta_+ + 1]$, $p > 2$, and $\frac{2p\alpha}{p-2}[1 - (\delta_* - \delta_+)] < 1$.

Proof. Firstly, we prove that Equation (37) has a unique solution $u^{N(\varepsilon)} \in C([0,T]; L^2(\Omega, \dot{H}^\delta))$ by using the contraction mapping principle. Defining

$$\Lambda w(t) = \widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,1}(t) g^\varepsilon + \int_0^t \mathcal{E}^{N(\varepsilon)}_{\alpha,2}(t-s) f(s, w(s)) ds - \int_0^T \widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,1}(t) \widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,2}(T-s) f(s, w(s)) ds$$
$$+ \int_0^t \widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,2}(t-s) \sigma^\varepsilon(s) dW(s) - \int_0^T \widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,1}(t) \widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,2}(T-s) \sigma^\varepsilon(s) dW(s).$$

For $w \in C([0,T]; L^2(\Omega, \dot{H}^\delta))$, one can see that

$$\left\|\Lambda w_1(t)-\Lambda w_2(t)\right\|_{L^2(\Omega,\dot{H}^\delta)} \le \left\|\int_0^t \mathcal{E}_{\alpha,2}^{N(\varepsilon)}(t-s)\left[f(s,w_1(s))-f(s,w_2(s))\right]ds\right\|_{L^2(\Omega,\dot{H}^\delta)} + \left\|\int_0^T \widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t)\widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(T-s)\left[f(s,w_1(s))-f(s,w_2(s))\right]ds\right\|_{L^2(\Omega,\dot{H}^\delta)}. \tag{39}$$

In a similar way as (52)-(54), one arrives at

$$\|\Lambda w_1(t)-\Lambda w_2(t)\|^2_{C([0,T];L^2(\Omega,\dot{H}^\delta))} \le \mathcal{K}^2 C_2^2 T^2\left(1+\frac{C_3^2T^{-2\alpha}}{(1-\alpha)^2}\right)\|w_1-w_2\|^2_{C([0,T];L^2(\Omega,\dot{H}^\delta))}. \tag{40}$$

It follows that Equation (37) has a unique solution in $C([0,T];L^2(\Omega,\dot{H}^\delta))$.

Next, we show that the error estimate (38) holds. For the sake of calculation, let us set

$$y^{N(\varepsilon)}(t) := \widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t)g + \int_0^t \mathcal{E}_{\alpha,2}^{N(\varepsilon)}(t-s)f(s,u(s))ds - \int_0^T \widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t)\widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(T-s)f(s,u(s))ds + \int_0^t \widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(t-s)\sigma(s)dW(s) - \int_0^T \widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t)\widetilde{\mathcal{E}}_{\alpha,2}(T-s)\sigma^\varepsilon(s)dW(s). \tag{41}$$

To contribute (38), we estimate $\left\|u^{N(\varepsilon)}(t)-y^{N(\varepsilon)}(t)\right\|_{L^2(\Omega,\dot{H}^\delta)}$ and $\|y^{N(\varepsilon)}(t)-u(t)\|_{L^2(\Omega,\dot{H}^\delta)}$ firstly. It follows from (37), (41) that

$$\begin{aligned}
&\left\|u^{N(\varepsilon)}(t)-y^{N(\varepsilon)}(t)\right\|_{L^2(\Omega,\dot{H}^\delta)} \\
&\le \left\|\widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t)\left(g^\varepsilon-g\right)\right\|_{L^2(\Omega,\dot{H}^\delta)} \\
&+ \left\|\int_0^t \mathcal{E}_{\alpha,2}^{N(\varepsilon)}(t-s)\left(f(s,u^{N(\varepsilon)}(s))-f(s,u(s))\right)ds\right\|_{L^2(\Omega,\dot{H}^\delta)} \\
&+ \left\|\int_0^T \widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t)\widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(T-s)\left(f(s,u^{N(\varepsilon)}(s))-f(s,u(s))\right)ds\right\|_{L^2(\Omega,\dot{H}^\delta)} \\
&+ \left\|\int_0^t \widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(t-s)\left(\sigma^\varepsilon(s)-\sigma(s)\right)dW(s)\right\|_{L^2(\Omega,\dot{H}^\delta)}
\end{aligned}$$

$$+\left\| \int_0^T \widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t) \widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(T-s) \left(\sigma^\varepsilon(s)-\sigma(s)\right) dW(s) \right\|_{L^2(\Omega,\dot{H}^\delta)}$$
$$=: \Pi_1(t)+\Pi_2(t)+\Pi_3(t)+\Pi_4(t)+\Pi_5(t).$$

By the observation (13), we have the following properties, which will be used throughout the present proof

$$\left\| \mathcal{E}_{\alpha,1}^{N(\varepsilon)}(t) \right\|_{\mathcal{L}(X,\dot{H}^\delta)} \leq C_3 \lambda_{N(\varepsilon)}^{\delta+1}, \quad \left\| \mathcal{E}_{\alpha,2}^{N(\varepsilon)}(t) \right\|_{\mathcal{L}(X,\dot{H}^\delta)} \leq C_2 \lambda_{N(\varepsilon)}^{\delta}, \quad t \in [0,T]. \tag{42}$$

Using the first property, we get

$$\Pi_1^2(t) = \mathbb{E} \left\| \widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t) \left(g^\varepsilon - g\right) \right\|_{\dot{H}^\delta}^2 \leq C_3^2 \lambda_{N(\varepsilon)}^{2(\delta+1)} \mathbb{E} \left\| g^\varepsilon - g \right\|_X^2 = C_3^2 \lambda_{N(\varepsilon)}^{2(\delta+1)} \left\| g^\varepsilon - g \right\|_{L^2(\Omega,X)}^2 . \tag{43}$$

For $\Pi_2(t)$ and $\Pi_3(t)$, in the same way as in (39)-(40), one arrives at

$$\Pi_2^2(t)+\Pi_3^2(t) \leq \mathcal{K}^2 C_2^2 T^2 \left(1+\frac{C_3^2 T^{-2\alpha}}{(1-\alpha)^2}\right) \left\| u^{N(\varepsilon)} - u \right\|_{C([0,T];L^2(\Omega,\dot{H}^\delta))}^2 . \tag{44}$$

For $\Pi_4(t)$ and $\Pi_5(t)$, the Itô isometry and the properties in (42) allow that

$$\Pi_4^2(t) = \int_0^t \left\| A^\delta \widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(t-s) \left(\sigma^\varepsilon(s)-\sigma(s)\right) \right\|_{L_0^2}^2 ds$$
$$\leq C_2^2 \lambda_{N(\varepsilon)}^{2\delta} \int_0^t \left\| \left(\sigma^\varepsilon(s)-\sigma(s)\right) \right\|_{L_0^2}^2 ds \leq C_2^2 \lambda_{N(\varepsilon)}^{2\delta} \left\| \sigma^\varepsilon - \sigma \right\|_{L^2(0,T;L_0^2)}^2 ,$$

and

$$\Pi_5^2(t) = \int_0^T \left\| A^\delta \widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t) \widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(T-s) \left(\sigma^\varepsilon(s)-\sigma(s)\right) \right\|_{L_0^2}^2 ds$$
$$\leq C_2^2 C_3^2 \lambda_{N(\varepsilon)}^{2(\delta+1)} \int_0^T \left\| \left(\sigma^\varepsilon(s)-\sigma(s)\right) \right\|_{L_0^2}^2 ds$$
$$= C_2^2 C_3^2 \lambda_{N(\varepsilon)}^{2(\delta+1)} \left\| \sigma^\varepsilon - \sigma \right\|_{L^2(0,T;L_0^2)}^2 . \tag{45}$$

Now, from (43)-(45) and Assumption (36), we deduce that

$$\left\|u^{N(\varepsilon)}(t)-y^{N(\varepsilon)}(t)\right\|_{L^2(\Omega,\dot{H}^\delta)} \le \left(C_2+C_3(C_2+1)\lambda_{N(\varepsilon)}\right)\lambda^{\delta}_{N(\varepsilon)}\varepsilon + \mathcal{K}C_2T\left(1+\frac{C_3^2T^{-2\alpha}}{(1-\alpha)^2}\right)^{\frac{1}{2}}\left\|u^{N(\varepsilon)}-u\right\|_{C([0,T];L^2(\Omega,\dot{H}^\delta))}. \tag{46}$$

Additionally, since $y^{N(\varepsilon)}(t)-u(t)=\sum_{n>N(\varepsilon)}\langle u(t),e_n\rangle e_n$, one can see

$$\left\|y^{N(\varepsilon)}(t)-u(t)\right\|^2_{L^2(\Omega,\dot{H}^\delta)} = \sum_{n>N(\varepsilon)}^{\infty}\lambda_n^{2\delta}\mathbb{E}|\langle u(t),e_n\rangle|^2 \le \lambda^{2(\delta-\delta_+)}_{N(\varepsilon)}\|u(t)\|^2_{L^2(\Omega,\dot{H}^{\delta_+})}. \tag{47}$$

Combining (46), (47) and using $\|u(t)\|_{L^2(\Omega,\dot{H}^{\delta_+})}\le\mathcal{M}_u$, we conclude that

$$\begin{aligned}\left\|u^{N(\varepsilon)}(t)-u(t)\right\|_{L^2(\Omega,\dot{H}^\delta)} &\le \left\|u^{N(\varepsilon)}(t)-y^{N(\varepsilon)}(t)\right\|_{L^2(\Omega,\dot{H}^\delta)} + \left\|y^{N(\varepsilon)}(t)-u(t)\right\|_{L^2(\Omega,\dot{H}^\delta)}\\ &\le \left(C_2+C_3(C_2+1)\lambda_{N(\varepsilon)}\right)\lambda^{\delta}_{N(\varepsilon)}\varepsilon+\lambda^{\delta-\delta_+}_{N(\varepsilon)}\mathcal{M}_u\\ &\quad+\mathcal{K}C_2T\left(1+\frac{C_3^2T^{-2\alpha}}{(1-\alpha)^2}\right)^{\frac{1}{2}}\left\|u^{N(\varepsilon)}-u\right\|_{C([0,T];L^2(\Omega,\dot{H}^\delta))},\end{aligned}$$

which shows clearly that $\left\|u^{N(\varepsilon)}-u\right\|_{C([0,T];L^2(\Omega,\dot{H}^\delta))}\lesssim\lambda^{\delta+1}_{N(\varepsilon)}\varepsilon+\lambda^{\delta-\delta_+}_{N(\varepsilon)}\mathcal{M}_u$. We now complete the proof. □

4.2. Regularization result for Problem (3)

By the Fourier truncated method, a regularized solution for Problem (3) when $H\in(\frac{1}{2},1)$ can be constructed as

$$\begin{aligned}v^{N(\varepsilon)}(t) &= \widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,1}(t)g^\varepsilon+\int_0^t\mathcal{E}^{N(\varepsilon)}_{\alpha,2}(t-s)f(s,v^{N(\varepsilon)}(s))ds\\ &\quad-\int_0^T\widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,1}(t)\widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,2}(T-s)f(s,v^{N(\varepsilon)}(s))ds\\ &\quad+\int_0^t\widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,2}(t-s)\sigma^\varepsilon(s)dB^H(s)-\int_0^T\widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,1}(t)\widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,2}(T-s)\sigma^\varepsilon(s)dB^H(s),\end{aligned}$$

where $N(\epsilon)$ is a positive integer number depending on the noisy level ε. The convergence result of the regularized solution $v^{N(\varepsilon)}(t)$ is stated in the following theorem.

Theorem 4.2. *Let $H\in(\frac{1}{2},1)$, $\delta\ge0$, $N(\varepsilon)\in\mathbb{Z}_+$. Assume that (36) holds for some $p>\frac{2}{H-\frac{1}{2}}$, f satisfies Assumption (f) with $\nu_1=\delta$, and there exist constants $\mathcal{M}_u>0$,*

$\delta_+ > \delta$ such that $\|u(t)\|_{L^2(\Omega,\dot H^{\delta_+})} \le \mathcal{M}_u$, for all $0 \le t \le T$. Then, Equation (37) has a unique solution $v^{N(\varepsilon)} \in C([0,T];L^2(\Omega,\dot H^{\delta}))$ satisfying

$$\left\|v^{N(\varepsilon)} - v\right\|_{C([0,T];L^2(\Omega,\dot H^{\delta}))} \lesssim \lambda_{N(\varepsilon)}^{\delta+1}\varepsilon + \lambda_{N(\varepsilon)}^{\delta-\delta_+}\mathcal{M}_u. \tag{48}$$

Corollary 4.2. *Let $N(\varepsilon)$ be the largest positive integer number such that $\lambda_{N(\varepsilon)} \le \varepsilon^{-\frac{\kappa}{\delta+1}}$, for some positive constant $\kappa < \delta+1$. Then*

$$\left\|v^{N(\varepsilon)} - v\right\|_{C([0,T];L^2(\Omega,\dot H^{\delta}))} \lesssim \varepsilon^{1-\frac{\kappa}{\delta+1}} + \varepsilon^{\frac{\kappa}{\delta+1}(\delta^+-\delta)} \longrightarrow 0, \quad \text{as } \ \varepsilon \to 0^+.$$

Remark 4.2. It can be seen from Theorem 3.3 that the assumption "$u(t)$ is bounded in $L^2(\Omega,\dot H^{\delta_+})$" is fulfilled if the terminal function $g \in L^2(\Omega,\dot H^{\delta_++1})$, f satisfies Assumption (f) with $\nu_1 = \delta_+$, and $\sigma \in L^\rho(0,T;L_0^2(X,\dot H^{\delta_*}))$, for some $\delta_* \ge \delta_+$, $\rho > 2$, and $\frac{\rho}{\rho-2}\big(\frac{3}{2} - H + \alpha(1-\delta_*+\delta_+)\big) < 1$.

Proof. Firstly, by a similar way as in the proof of Theorem 4.1, one also see that (37) has a unique solution in $C([0,T];L^2(\Omega,\dot H^{\delta}))$. Next, to show that the error estimate (48) holds, we set

$$\begin{aligned}\overline{y}^{N(\varepsilon)}(t) &:= \widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t)g + \int_0^t \mathcal{E}_{\alpha,2}^{N(\varepsilon)}(t-s)f(s,v(s))ds - \int_0^T \widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t)\widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(T-s)f(s,v(s))ds \\ &\quad + \int_0^t \widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(t-s)\sigma(s)dB^H(s) - \int_0^T \widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t)\widetilde{\mathcal{E}}_{\alpha,2}(T-s)\sigma^\varepsilon(s)dB^H(s),\end{aligned}$$

and then estimate $\left\|v^{N(\varepsilon)}(t) - \overline{y}^{N(\varepsilon)}(t)\right\|_{L^2(\Omega,\dot H^{\delta})}$ and $\left\|\overline{y}^{N(\varepsilon)}(t) - v(t)\right\|_{L^2(\Omega,\dot H^{\delta})}$ separately. From the formulas of $v^{N(\varepsilon)}(t)$ and $\overline{y}^{N(\varepsilon)}(t)$, it is clear that

$$\begin{aligned}&\left\|v^{N(\varepsilon)}(t) - \overline{y}^{N(\varepsilon)}(t)\right\|_{L^2(\Omega,\dot H^{\delta})} \\ &\le \left\|\widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t)\left(g^\varepsilon - g\right)\right\|_{L^2(\Omega,\dot H^{\delta})} \\ &+ \left\|\int_0^t \mathcal{E}_{\alpha,2}^{N(\varepsilon)}(t-s)\left(f(s,u^{N(\varepsilon)}(s)) - f(s,u(s))\right)ds\right\|_{L^2(\Omega,\dot H^{\delta})} \\ &+ \left\|\int_0^T \widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t)\widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(T-s)\left(f(s,u^{N(\varepsilon)}(s)) - f(s,u(s))\right)ds\right\|_{L^2(\Omega,\dot H^{\delta})} \\ &+ \left\|\int_0^t \widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(t-s)\left(\sigma^\varepsilon(s) - \sigma(s)\right)dB^H(s)\right\|_{L^2(\Omega,\dot H^{\delta})}\end{aligned}$$

$$+\left\|\int_0^T \widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t)\widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(T-s)\left(\sigma^\varepsilon(s)-\sigma(s)\right)dB^H(s)\right\|_{L^2(\Omega,\dot{H}^\delta)}$$
$$=: \Pi_1(t)+\Pi_2(t)+\Pi_3(t)+\overline{\Pi}_4(t)+\overline{\Pi}_5(t).$$

Recall the $\Pi_1(t), \Pi_2(t), \Pi_3(t)$ have been estimate in (43) and (44). We now continue to estimate two last terms $\overline{\Pi}_4(t), \overline{\Pi}_5(t)$. For $\overline{\Pi}_4(t)$, we have

$$\begin{aligned}
|\overline{\Pi}_4(t)|^2 &= \Big\|\sum_{n\geq 1}\int_0^t K_{H,t}^*\left(\widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(t-\cdot)\left(\sigma^\varepsilon(\cdot)-\sigma(\cdot)\right)Q^{\frac12}e_n\right)(s)d\beta_n(s)\Big\|^2_{L^2(\Omega,\dot{H}^\delta)}\\
&= \Big\|c_H\sum_{n\geq 1}\int_0^t\Big(\int_s^t \widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(t-\tilde{s})\left(\sigma^\varepsilon(\tilde{s})-\sigma(\tilde{s})\right)\\
&\quad\times Q^{\frac12}e_n\left(\frac{s}{\tilde{s}}\right)^{\frac12-H}(\tilde{s}-s)^{H-\frac32}d\tilde{s}\Big)d\beta_n(s)\Big\|^2_{L^2(\Omega,\dot{H}^\delta)}\\
&\lesssim \sum_{n\geq 1}\int_0^t\Big\|\int_s^t \widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(t-\tilde{s})\left(\sigma^\varepsilon(\tilde{s})-\sigma(\tilde{s})\right)Q^{\frac12}e_n\left(\frac{s}{\tilde{s}}\right)^{\frac12-H}(\tilde{s}-s)^{H-\frac32}d\tilde{s}\Big\|^2_{\dot{H}^\delta}ds\\
&\lesssim \sum_{n\geq 1}\int_0^t\Big[\int_s^t \Big\|\widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(t-\tilde{s})\left(\sigma^\varepsilon(\tilde{s})-\sigma(\tilde{s})\right)Q^{\frac12}e_n\Big\|_{\dot{H}^\delta}s^{\frac12-H}(\tilde{s}-s)^{H-\frac32}d\tilde{s}\Big]^2ds.
\end{aligned}$$

Using the proper (42) and the Hölder inequality, we deduce that

$$\begin{aligned}
|\overline{\Pi}_5(t)|^2 &\lesssim \sum_{n\geq 1}\int_0^t\Big[\int_s^t \lambda_{N(\varepsilon)}^\delta\Big\|\left(\sigma^\varepsilon(\tilde{s})-\sigma(\tilde{s})\right)Q^{\frac12}e_n\Big\|_X s^{\frac12-H}(\tilde{s}-s)^{H-\frac32}d\tilde{s}\Big]^2ds\\
&\lesssim \lambda_{N(\varepsilon)}^{2\delta}\int_0^t\left(\int_s^t s^{1-2H}(\tilde{s}-s)^{H-\frac32}d\tilde{s}\right)\left(\int_s^t \|(\sigma^\varepsilon(\tilde{s})-\sigma(\tilde{s}))\|^2_{L_0^2}(\tilde{s}-s)^{H-\frac32}d\tilde{s}\right)ds\\
&\lesssim \lambda_{N(\varepsilon)}^{2\delta}\int_0^t s^{1-2H}\left(\int_s^t(\tilde{s}-s)^{\frac{p(H-\frac32)}{p-2}}d\tilde{s}\right)^{\frac{p-2}{p}}\left(\int_s^t\left\|\sigma^\varepsilon(\tilde{s})-\sigma(\tilde{s})\right\|^p_{L_0^2}d\tilde{s}\right)^{\frac2p}ds\\
&\lesssim \lambda_{N(\varepsilon)}^{2\delta}\left\|\sigma^\varepsilon-\sigma\right\|^2_{L^p(0,T;L_0^2)}.
\end{aligned}\tag{49}$$

Similarly, one has the following estimate for $\overline{\Pi}_5(t)$

$$|\overline{\Pi}_5(t)|^2 = \Big\|\sum_{n\geq 1}\int_0^T K_{H,T}^*\left(\widetilde{\mathcal{E}}_{\alpha,1}^{N(\varepsilon)}(t)\widetilde{\mathcal{E}}_{\alpha,2}^{N(\varepsilon)}(T-\cdot)\left(\sigma^\varepsilon(\cdot)-\sigma(\cdot)\right)Q^{\frac12}e_n\right)(s)d\beta_n(s)\Big\|^2_{L^2(\Omega,\dot{H}^\delta)}$$

$$
\begin{aligned}
&\lesssim \sum_{n\geq 1}\int_0^T \left\| \int_s^T \widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,1}(t)\widetilde{\mathcal{E}}^{N(\varepsilon)}_{\alpha,2}(T-\tilde{s})\left(\sigma^\varepsilon(\tilde{s})-\sigma(\tilde{s})\right)Q^{\frac12}e_n\left(\frac{s}{\tilde{s}}\right)^{\frac12-H}(\tilde{s}-s)^{H-\frac32}d\tilde{s}\right\|^2_{\dot{H}^\delta} ds\\
&\lesssim \sum_{n\geq 1}\int_0^T \left[\int_s^T \lambda^{\delta+1}_{N(\varepsilon)}\left\|\left(\sigma^\varepsilon(\tilde{s})-\sigma(\tilde{s})\right)Q^{\frac12}e_n\right\|_X s^{\frac12-H}(\tilde{s}-s)^{H-\frac32}d\tilde{s}\right]^2 ds\\
&\lesssim \lambda^{2(\delta+1)}_{N(\varepsilon)}\int_0^T s^{1-2H}\left(\int_s^T(\tilde{s}-s)^{\frac{p(H-\frac32)}{p-2}}d\tilde{s}\right)^{\frac{p-2}{p}}\left(\int_s^T\left\|\sigma^\varepsilon(\tilde{s})-\sigma(\tilde{s})\right\|^p_{L^2_0}d\tilde{s}\right)^{\frac2p} ds\\
&\lesssim \lambda^{2(\delta+1)}_{N(\varepsilon)}\left\|\sigma^\varepsilon-\sigma\right\|^2_{L^p(0,T;L^2_0)}. \qquad (50)
\end{aligned}
$$

Since $\overline{y}^{N(\varepsilon)}(t)-v(t)=\sum_{n>N(\varepsilon)}\langle v(t),e_n\rangle e_n$, it is obvious that

$$
\left\|\overline{y}^{N(\varepsilon)}(t)-v(t)\right\|^2_{L^2(\Omega,\dot{H}^\delta)}=\sum_{n>N(\varepsilon)}^{\infty}\lambda_n^{2\delta}\mathbb{E}|\langle v(t),e_n\rangle|^2\leq\lambda^{2(\delta-\delta_+)}_{N(\varepsilon)}\left\|v(t)\right\|^2_{L^2(\Omega,\dot{H}^{\delta_+})}. \qquad (51)
$$

Now, from (43), (44), (49), (50), (51), we deduce that

$$
\begin{aligned}
\left\|v^{N(\varepsilon)}-v\right\|_{C([0,T];L^2(\Omega,\dot{H}^\delta))} &\lesssim \left\|v^{N(\varepsilon)}(t)-\overline{y}^{N(\varepsilon)}(t)\right\|_{L^2(\Omega,\dot{H}^\delta)}+\left\|\overline{y}^{N(\varepsilon)}(t)-v(t)\right\|_{L^2(\Omega,\dot{H}^\delta)}\\
&\lesssim \lambda^{\delta+1}_{N(\varepsilon)}\varepsilon+\lambda^{\delta-\delta_+}_{N(\varepsilon)}\mathcal{M}_u.
\end{aligned}
$$

This completes the proof. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The authors would like to thank the Associate Editor and the Reviewer for their time and comments that helped improve our manuscript. Renhai Wang was supported by China Postdoctoral Science Foundation under grant numbers 2020TQ0053 and 2020M680456. Nguyen Huy Tuan is supported by Van Lang University.

Appendix A

Proof of Lemma 3.1. i) For $t\in[0,T]$, by applying part a) of Lemma 2.3, we directly obtain

$$\|I_1(t)\|^2_{L^2(\Omega,\dot H^{\nu_1})} = \mathbb{E}\,\|\mathcal{E}_{\alpha,1}(t)g\|^2_{\dot H^{\nu_1}} \lesssim \mathbb{E}\,\|g\|^2_{\dot H^{\nu_1+1}} = \|g\|^2_{L^2(\Omega,\dot H^{\nu_1+1})}.$$

ii) For $t \in (0,T]$, applying part b) of Lemma 2.3 with $\mu_1 = \nu_1 + r_1$, $\mu_2 = \nu_1$, we obtain

$$\|I_1(t)\|^2_{L^2(\Omega,\dot H^{\nu_1+r_1})} = \mathbb{E}\,\|\mathcal{E}_{\alpha,1}(t)g\|^2_{\dot H^{\nu_1+r_1}} \lesssim t^{-2\alpha r_1}\mathbb{E}\,\|g\|^2_{\dot H^{\nu_1+1}} \lesssim t^{-2\alpha r_1}\,\|g\|^2_{L^2(\Omega,\dot H^{\nu_1+1})}.$$

The proof is complete. □

Proof of Lemma 3.2. i) For $t \in [0,T]$, by using the Hölder inequality and part a) of Lemma 2.3, we have

$$\|I_2(w)(t)\|^2_{L^2(\Omega,\dot H^{\nu_1})} \le t\int_0^t \mathbb{E}\,\|\mathcal{E}_{\alpha,2}(t-s)f(s,w(s))\|^2_{\dot H^{\nu_1}}\,ds \le C_2^2 t\int_0^t \mathbb{E}\,\|f(s,w(s))\|^2_{\dot H^{\nu_1}}\,ds,$$

which associated with the Lipschitz condition of f leads to

$$\begin{aligned}\|I_2(w)(t)\|^2_{L^2(\Omega,\dot H^{\nu_1})} &\le \mathcal{K}^2C_2^2 t\int_0^t \mathbb{E}\,\|w(s)\|^2_{\dot H^{\nu_1}}\,ds\\ &\le \mathcal{K}^2C_2^2T^2\,\|w\|^2_{L^\infty(0,T;L^2(\Omega,\dot H^{\nu_1}))}.\end{aligned}\tag{52}$$

ii) For $t \in (0,T]$, by applying part b) of Lemma 2.3 with $\mu_1 = \nu_1 + r_1$ and $\mu_2 = \nu_1$, we have

$$\begin{aligned}\|I_2(w)(t)\|^2_{L^2(\Omega,\dot H^{\nu_1+r_1})} &\le \mathbb{E}\left[\int_0^t \|\mathcal{E}_{\alpha,2}(t-s)f(s,w(s))\|_{\dot H^{\nu_1+r_1}}\,ds\right]^2\\ &\lesssim \mathbb{E}\left[\int_0^t (t-s)^{-\alpha r_1}\,\|f(s,w(s))\|_{\dot H^{\nu_1}}\,ds\right]^2.\end{aligned}$$

Using the Hölder inequality and the Lipschitz condition of f, we get

$$\begin{aligned}\|I_2(w)(t)\|^2_{L^2(\Omega,\dot H^{\nu_1+r_1})} &\lesssim \int_0^t (t-s)^{-\alpha r_1}ds\int_0^t (t-s)^{-\alpha r_1}\mathbb{E}\,\|f(s,w(s))\|^2_{\dot H^{\nu_1}}\,ds\\ &\lesssim t^{-2\alpha r_1}\,\|w\|^2_{L^\infty(0,T;L^2(\Omega,\dot H^{\nu_1}))},\end{aligned}\tag{53}$$

where we have used $\left(\int_0^t (t-s)^{-\alpha r_1}ds\right)^2 \lesssim t^{2(1-\alpha r_1)} \le T^2 t^{-2\alpha r_1}$.
iii) Applying part c) of Lemma 2.3 with $\mu_1 = \mu_3 = \nu_1$, we get

$$\|I_3(w)(t)\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} \leq \mathbb{E}\left[\int_0^T \|\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)f(s,w(s))\|_{\dot{H}^{\nu_1}}\,ds\right]^2$$
$$\leq C_2^2C_3^2\mathbb{E}\left[\int_0^T (T-s)^{-\alpha}\|f(s,w(s))\|_{\dot{H}^{\nu_1}}\,ds\right]^2.$$

This together with the Hölder inequality and the Lipschitz condition of f gives us

$$\|I_3(w)(t)\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} \leq C_2^2C_3^2\int_0^T (T-s)^{-\alpha}ds\int_0^T (T-s)^{-\alpha}\mathbb{E}\|f(s,w(s))\|^2_{\dot{H}^{\nu_1}}\,ds$$
$$\leq \mathcal{K}^2\frac{C_2^2C_3^2}{1-\alpha}T^{1-\alpha}\int_0^T (T-s)^{-\alpha}\|w(s)\|^2_{L^2(\Omega,\dot{H}^{\nu_1})}\,ds$$
$$\leq \mathcal{K}^2\frac{C_2^2C_3^2}{(1-\alpha)^2}T^{2(1-\alpha)}\|w\|^2_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))}. \tag{54}$$

iv) Applying part d) of Lemma 2.3 with $\mu_1=\nu_1+r_1$ and $\mu_2=\mu_4=\nu_1$, we have

$$\|I_3(w)(t)\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \leq \mathbb{E}\left[\int_0^T \|\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)f(s,w(s))\|_{\dot{H}^{\nu_1+r_1}}\,ds\right]^2$$
$$\lesssim t^{-2\alpha r_1}\mathbb{E}\left[\int_0^T (T-s)^{-\alpha}\|f(s,w(s))\|_{\dot{H}^{\nu_1}}\,ds\right]^2.$$

Estimating the integral in the right-hand side by the same way as in ii), we arrive at

$$\|I_3(w)(t)\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \lesssim t^{-2\alpha r_1}\|w\|^2_{L^\infty(0,T;L^2(\Omega,\dot{H}^{\nu_1}))}. \tag{55}$$

The proof is complete. □

Proof of Lemma 3.3. i) The property in part a) of Lemma 2.3 leads to the following estimate for $t\in[0,T]$

$$\|I_4(t)\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} = \int_0^t \|A^{\nu_1}\mathcal{E}_{\alpha,2}(t-s)\sigma(s)\|^2_{L_0^2}\,ds$$
$$\leq \lambda_1^{-2(\nu_2-\nu_1)}\int_0^t \|A^{\nu_2}\mathcal{E}_{\alpha,2}(t-s)\sigma(s)\|^2_{L_0^2}\,ds$$

$$\lesssim \int_0^t \|\mathcal{E}_{\alpha,2}(t-s)\|^2_{\mathcal{L}(\dot{H}^{\nu_2},\dot{H}^{\nu_2})} \|A^{\nu_2}\sigma(s)\|^2_{L_0^2} ds \lesssim \int_0^t \|\sigma(s)\|^2_{L_0^2(X,\dot{H}^{\nu_2})} ds.$$

Since $\int_0^t \|\sigma(s)\|^2_{L_0^2(X,\dot{H}^{\nu_2})} ds \le T^{\frac{p-2}{p}} \|\sigma\|^2_{L^p(0,T;L_0^2(X,\dot{H}^{\nu_2}))}$, which follows from the Hölder inequality, we deduce that

$$\|I_4(t)\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim \|\sigma\|^2_{L^p(0,T;L_0^2(X,\dot{H}^{\nu_2}))}. \tag{56}$$

ii) For $t \in (0,T]$, the term $I_4(t)$ can be bounded as

$$\begin{aligned}\|I_4(t)\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} &\le \int_0^t \left\|A^{\nu_1+r_1}\mathcal{E}_{\alpha,2}(t-s)\sigma(s)\right\|^2_{L_0^2} ds \\ &\le \int_0^t \lambda_1^{-2(1-r_1)} \left\|A^{\nu_1+1}\mathcal{E}_{\alpha,2}(t-s)\sigma(s)\right\|^2_{L_0^2} ds \\ &\lesssim \int_0^t \|\mathcal{E}_{\alpha,2}(t-s)\|^2_{\mathcal{L}(\dot{H}^{\nu_2},\dot{H}^{\nu_1+1})} \|A^{\nu_2}\sigma(s)\|^2_{L_0^2} ds.\end{aligned}$$

On the other hand, applying part b) of Lemma 2.3 with $\mu_1 = \nu_1 + 1$ and $\mu_2 = \nu_2$, one has

$$\|\mathcal{E}_{\alpha,2}(t-s)\|_{\mathcal{L}(\dot{H}^{\nu_2},\dot{H}^{\nu_1+1})} \lesssim (t-s)^{-\alpha(\nu_1+1-\nu_2)}.$$

This together with the Hölder inequality allows that

$$\begin{aligned}\|I_4(t)\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} &\lesssim \int_0^t (t-s)^{-2\alpha(\nu_1+1-\nu_2)} \|\sigma(s)\|^2_{L_0^2(X,\dot{H}^{\nu_2})} ds \\ &\lesssim \left(\int_0^t (t-s)^{\frac{-2p\alpha[1-(\nu_2-\nu_1)]}{p-2}} ds\right)^{\frac{p-2}{p}} \left(\int_0^t \|\sigma(s)\|^p_{L^2(\Omega,L_0^2(X,\dot{H}^{\nu_2}))} ds\right)^{\frac{2}{p}} \\ &\lesssim \|\sigma\|^2_{L^p(0,T;L_0^2(X,\dot{H}^{\nu_2}))},\end{aligned} \tag{57}$$

where we note that $\frac{2p\alpha[1-(\nu_2-\nu_1)]}{p-2} < 1$.
iii) For $t \in [0,T]$, by using the Itô isometry and applying part c) of Lemma 2.3 with $\mu_1 = \nu_1$, $\mu_3 = \nu_2$ (noting that $0 \le \nu_2 - \nu_1 \le 1$ following from assumption (σ_1), the term $I_5(t)$ can be estimated as

Please cite this article in press as: N.H. Tuan et al., On backward problems for stochastic fractional reaction equations with standard and fractional Brownian motion, Bull. Sci. math. (2022), https://doi.org/10.1016/j.bulsci.2022.103158

$$\begin{aligned}\|I_5(t)\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} &= \int_0^T \|A^{\nu_1}\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)\|^2_{L_0^2}\, ds \\ &\leq \int_0^T \|\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\|^2_{\mathcal{L}(\dot{H}^{\nu_2},\dot{H}^{\nu_1})} \|A^{\nu_2}\sigma(s)\|^2_{L_0^2}\, ds \\ &\lesssim \int_0^T (T-s)^{-2\alpha(\nu_1+1-\nu_2)} \|\sigma(s)\|^2_{L_0^2(X,\dot{H}^{\nu_2})}\, ds.\end{aligned}$$

By the same way employed to obtain (57), we arrive at

$$\|I_5(t)\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim \|\sigma\|^2_{L^p(0,T;L_0^2(X,\dot{H}^{\nu_2}))}. \tag{58}$$

iv) For $t \in (0,T]$, by using the Itô isometry and applying part d) of Lemma 2.3 with $\mu_1 = \nu_1 + r_1$, $\mu_2 = \nu_1$, $\mu_4 = \nu_2$, the term $I_5(t)$ can be estimated as

$$\begin{aligned}\|I_5(t)\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} &= \int_0^T \left\|A^{\nu_1+r_1}\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)\right\|^2_{L_0^2}\, ds \\ &\leq \int_0^T \|\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\|^2_{\mathcal{L}(\dot{H}^{\nu_2},\dot{H}^{\nu_1+r_1})} \|A^{\nu_2}\sigma(s)\|^2_{L_0^2}\, ds \\ &\lesssim t^{-2\alpha r_1} \int_0^T (T-s)^{-2\alpha(\nu_1+1-\nu_2)} \|\sigma(s)\|^2_{L_0^2(X,\dot{H}^{\nu_2})}\, ds \\ &\lesssim t^{-2\alpha r_1} \|\sigma\|^2_{L^p(0,T;L_0^2(X,\dot{H}^{\nu_2}))}.\end{aligned} \tag{59}$$

The proof is complete. □

Proof of Lemma 3.4. Firstly, let us present explicit representations for two terms $\overline{I}_4(t), \overline{I}_5(t)$. For $\overline{I}_5(t)$, it can be seen that

$$\begin{aligned}\overline{I}_5(t) &= \sum_{n\geq 1}\int_0^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)Q^{\frac{1}{2}}e_n d\beta_n^H(s) \\ &= \sum_{n\geq 1}\int_0^T K^*_{H,T}\left(\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-\cdot)\sigma(\cdot)Q^{\frac{1}{2}}e_n\right)(s) d\beta_n(s) \\ &= \sum_{n\geq 1}\int_0^T K^H(T,s)\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)Q^{\frac{1}{2}}e_n d\beta_n(s)\end{aligned}$$

$$+\sum_{n\geq 1}\int_0^T\left(\int_s^T\mathcal{E}_{\alpha,1}(t)\left(\mathcal{E}_{\alpha,2}(T-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n-\mathcal{E}_{\alpha,2}(T-s)\sigma(s)Q^{\frac{1}{2}}e_n\right)\right.$$
$$\left.\times\frac{\partial K^H}{\partial\tilde{s}}(\tilde{s},s)d\tilde{s}\right)d\beta_n(s).$$

This together with the formula of K^H and $\frac{\partial K^H}{\partial\tilde{s}}$ yields

$$\begin{aligned}\overline{I}_5(t)&=c_H\sum_{n\geq 1}\int_0^T\left(\frac{T}{s}\right)^{H-\frac{1}{2}}(T-s)^{H-\frac{1}{2}}\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)Q^{\frac{1}{2}}e_nd\beta_n(s)\\&-\tilde{c}_H\sum_{n\geq 1}\int_0^T s^{\frac{1}{2}-H}\left(\int_s^T\tilde{s}^{H-\frac{3}{2}}(\tilde{s}-s)^{H-\frac{1}{2}}d\tilde{s}\right)\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)Q^{\frac{1}{2}}e_nd\beta_n(s)\\&+c_H\sum_{n\geq 1}\int_0^T\left(\int_s^T\mathcal{E}_{\alpha,1}(t)\tilde{\mathcal{E}}_{\alpha,2}(T,\tilde{s},s)Q^{\frac{1}{2}}e_n\left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right)d\beta_n(s)\\&=:\overline{I}_{51}(t)+\overline{I}_{52}(t)+\overline{I}_{53}(t),\end{aligned}\tag{60}$$

where we define $\tilde{\mathcal{E}}_{\alpha,2}(t,\tilde{s},s):=\mathcal{E}_{\alpha,2}(t-\tilde{s})\sigma(\tilde{s})-\mathcal{E}_{\alpha,2}(t-s)\sigma(s)$, $t\in[0,T]$, for short.

For $\overline{I}_4(t)$, we also have the following formulation

$$\begin{aligned}\overline{I}_4(t)&=\sum_{n\geq 1}\int_0^t\mathcal{E}_{\alpha,2}(t-s)\sigma(s)Q^{\frac{1}{2}}e_nd\beta_n^H(s)\\&=\sum_{n\geq 1}\int_0^t K_{H,t}^*\left(\mathcal{E}_{\alpha,2}(t-\cdot)\sigma(\cdot)Q^{\frac{1}{2}}e_n\right)(s)d\beta_n(s)\\&=\sum_{n\geq 1}\int_0^T K^H(t,s)\mathcal{E}_{\alpha,2}(t-s)\sigma(s)Q^{\frac{1}{2}}e_nd\beta_n(s)\\&+\sum_{n\geq 1}\int_0^t\left(\int_s^t\left(\mathcal{E}_{\alpha,2}(t-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n-\mathcal{E}_{\alpha,2}(t-s)\sigma(s)Q^{\frac{1}{2}}e_n\right)\frac{\partial K^H}{\partial\tilde{s}}(\tilde{s},s)d\tilde{s}\right)d\beta_n(s),\end{aligned}$$

which associated with the formula of K^H and $\frac{\partial K^H}{\partial\tilde{s}}$ yields

$$\overline{I}_4(t)=c_H\sum_{n\geq 1}\int_0^t\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}}\mathcal{E}_{\alpha,2}(t-s)\sigma(s)Q^{\frac{1}{2}}e_nd\beta_n(s)$$

$$
\begin{aligned}
&- \tilde{c}_H \sum_{n\geq 1} \int_0^t s^{\frac{1}{2}-H} \left(\int_s^t \tilde{s}^{H-\frac{3}{2}} (\tilde{s}-s)^{H-\frac{1}{2}} d\tilde{s} \right) \mathcal{E}_{\alpha,2}(t-s)\sigma(s) Q^{\frac{1}{2}} e_n d\beta_n(s) \\
&+ \sum_{n\geq 1} \int_0^t \left(\int_s^t \tilde{\mathcal{E}}_{\alpha,2}(t,\tilde{s},s) Q^{\frac{1}{2}} e_n \left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H} (\tilde{s}-s)^{H-\frac{3}{2}} d\tilde{s} \right) d\beta_n(s) \\
&=: \overline{I}_{41}(t) + \overline{I}_{42}(t) + \overline{I}_{43}(t). \qquad (61)
\end{aligned}
$$

Next, we estimate two terms $\overline{I}_5(t)$, $\overline{I}_4(t)$ separately.

- *Estimating* $\overline{I}_5(t)$. We begin with the estimate for $\overline{I}_{51}(t)$ as

$$
\begin{aligned}
\left\| \overline{I}_{51}(t) \right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} &\lesssim \sum_{n\geq 1} \int_0^T \left\| \left(\frac{T}{s}\right)^{H-\frac{1}{2}} (T-s)^{H-\frac{1}{2}} \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s) Q^{\frac{1}{2}} e_n \right\|^2_{\dot{H}^{\nu_1}} ds \\
&\lesssim \sum_{n\geq 1} \int_0^T (T-s)^{2H-1} \left\| \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s) Q^{\frac{1}{2}} e_n \right\|^2_{\dot{H}^{\nu_1}} ds \\
&\lesssim \sum_{n\geq 1} \int_0^T (T-s)^{2H-1} \left\| \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s) \right\|^2_{\mathcal{L}(\dot{H}^{\nu_3},\dot{H}^{\nu_1})} \left\| \sigma(s) Q^{\frac{1}{2}} e_n \right\|^2_{\dot{H}^{\nu_3}} ds.
\end{aligned}
$$

Applying part c) of Lemma 2.3 with $\mu_1 = \nu_1$, $\mu_3 = \nu_3$ and using the Hölder inequality, we get

$$
\left\| \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s) \right\|_{\mathcal{L}(\dot{H}^{\nu_3},\dot{H}^{\nu_1})} \lesssim (T-s)^{-\alpha(1-(\nu_3-\nu_1))}. \qquad (62)
$$

It follows that

$$
\begin{aligned}
\left\| \overline{I}_{51}(t) \right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} &\lesssim \int_0^T (T-s)^{-[1-2H+2\alpha(1-(\nu_3-\nu_1))]} \left\| \sigma(s) \right\|^2_{L_0^2(X,\dot{H}^{\nu_3})} ds \\
&\lesssim \|\sigma\|^2_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))} \int_0^T (T-s)^{-[1-2H+2\alpha(1-(\nu_3-\nu_1))]} ds \\
&\lesssim \|\sigma\|^2_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}, \qquad (63)
\end{aligned}
$$

where we note that

$$
\int_0^T (T-s)^{-[1-2H+2\alpha(1-(\nu_3-\nu_1))]} ds \lesssim T^{2[H-\alpha(1-(\nu_3-\nu_1))]}, \qquad (64)
$$

since $\alpha[1-(\nu_3-\nu_1)] < H$. For $\overline{I}_{52}(t)$, we can see that

$$
\begin{aligned}
&\left\|\overline{I}_{52}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})}\\
&\lesssim \sum_{n\geq 1}\int_0^T \mathbb{E}\left\| s^{\frac{1}{2}-H}\left(\int_s^T \tilde{s}^{H-\frac{3}{2}}(\tilde{s}-s)^{H-\frac{1}{2}}d\tilde{s}\right)\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)Q^{\frac{1}{2}}e_n\right\|^2_{\dot{H}^{\nu_1}} ds\\
&\lesssim \sum_{n\geq 1}\int_0^T s^{1-2H}\left(\int_s^T \tilde{s}^{H-\frac{3}{2}}(\tilde{s}-s)^{H-\frac{1}{2}}d\tilde{s}\right)^2\\
&\quad\times \left\|\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\right\|^2_{\mathcal{L}(\dot{H}^{\nu_3},\dot{H}^{\nu_1})}\left\|\sigma(s)\right\|^2_{L^2_0(X,\dot{H}^{\nu_3})} ds.
\end{aligned}
$$

Using the property (62), one also get

$$
\begin{aligned}
&\left\|\overline{I}_{52}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})}\\
&\lesssim \sum_{n\geq 1}\int_0^T s^{1-2H}\left(\int_s^T \tilde{s}^{H-\frac{3}{2}}(\tilde{s}-s)^{H-\frac{1}{2}}d\tilde{s}\right)^2 (T-s)^{-2\alpha(1-(\nu_3-\nu_1))}\left\|\sigma(s)\right\|^2_{L^2_0(X,\dot{H}^{\nu_3})} ds.
\end{aligned}
$$

For the sake of calculation, we provide a useful estimate for $\overline{I}_{52}(t)$ as follows

$$
s^{1-2H}\left(\int_s^t \tilde{s}^{H-\frac{3}{2}}(\tilde{s}-s)^{H-\frac{1}{2}}d\tilde{s}\right)^2 \leq s^{1-2H}\left(\int_s^\infty \tilde{s}^{H-\frac{3}{2}}(\tilde{s}-s)^{H-\frac{1}{2}}d\tilde{s}\right)^2 \lesssim s^{-(1-2H)},
\quad \text{for } s<\tilde{s}<t. \tag{65}
$$

We can prove it by using the generalized binomial expansion as

$$
\begin{aligned}
& s^{1-2H}\left(\int_s^t \tilde{s}^{H-\frac{3}{2}}(\tilde{s}-s)^{H-\frac{1}{2}}d\tilde{s}\right)^2\\
&= s^{1-2H}\left(\int_s^t \tilde{s}^{H-\frac{3}{2}}\sum_{j\in\mathbb{N}}\binom{H-\frac{1}{2}}{j}\tilde{s}^{H-\frac{1}{2}-j}(-1)^j s^j d\tilde{s}\right)^2\\
&= s^{1-2H}\left(\sum_{j\in\mathbb{N}}\binom{H-\frac{1}{2}}{j}(-1)^j s^j\int_s^t \tilde{s}^{2H-2-j}d\tilde{s}\right)^2\\
&\lesssim s^{1-2H}\left(\sum_{j\in\mathbb{N}}\binom{H-\frac{1}{2}}{j}(-1)^j s^j\left(s^{2H-1-j}-t^{2H-1-j}\right)\right)^2\\
&\lesssim s^{-(1-2H)}.
\end{aligned}
$$

By (65), we can estimate $\overline{I}_{52}(t)$ as

$$
\begin{aligned}
\left\|\overline{I}_{52}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} &\lesssim \sum_{n\geq 1}\int_0^T \omega(s,T)\left\|\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\right\|^2_{\mathcal{L}(\dot{H}^{\nu_3},\dot{H}^{\nu_1})}\left\|\sigma(s)\right\|^2_{L^2_0(X,\dot{H}^{\nu_3})}ds \\
&\lesssim \sum_{n\geq 1}\int_0^T s^{-(1-2H)}(T-s)^{-2\alpha(1-(\nu_3-\nu_1))}\left\|\sigma(s)\right\|^2_{L^2_0(X,\dot{H}^{\nu_3})}ds \\
&\lesssim \left\|\sigma\right\|^2_{C((0,T];L^2_0(X,\dot{H}^{\nu_3}))}\int_0^T s^{-(1-2H)}(T-s)^{-2\alpha(1-(\nu_3-\nu_1))}ds \\
&\lesssim \left\|\sigma\right\|^2_{C((0,T];L^2_0(X,\dot{H}^{\nu_3}))}, \qquad (66)
\end{aligned}
$$

where we have used the fact that

$$
\int_0^T s^{-(1-2H)}(T-s)^{-2\alpha(1-(\nu_3-\nu_1))}ds = T^{2H-2\alpha(1-(\nu_3-\nu_1))}\frac{\Gamma\big(2H\big)\Gamma\big(1-2\alpha(1-(\nu_3-\nu_1))\big)}{\Gamma\big(2H+1-2\alpha(1-(\nu_3-\nu_1))\big)},
$$

with Γ is the Gamma function. For $\overline{I}_{53}(t)$, we can see that

$$
\begin{aligned}
\left\|\overline{I}_{53}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} &\lesssim \sum_{n\geq 1}\int_0^T \left\|\int_s^T \mathcal{E}_{\alpha,1}(t)\tilde{\mathcal{E}}_{\alpha,2}(T,\tilde{s},s)Q^{\frac{1}{2}}e_n\left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right\|^2_{\dot{H}^{\nu_1}}ds \\
&\lesssim \sum_{n\geq 1}\int_0^T \left(\int_s^T \left\|\mathcal{E}_{\alpha,1}(t)\tilde{\mathcal{E}}_{\alpha,2}(T,\tilde{s},s)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1}}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right)^2 ds.
\end{aligned}
$$

In what follows, we state some useful properties to estimate $\left\|\mathcal{E}_{\alpha,1}(t)\tilde{\mathcal{E}}_{\alpha,2}(T,\tilde{s},s)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1}}$. We observe that

$$
\begin{aligned}
\left\|\tilde{\mathcal{E}}_{\alpha,2}(T,\tilde{s},s)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1+1}} &\leq \left\|\mathcal{E}_{\alpha,2}(T-\tilde{s})\left(\sigma(\tilde{s})-\sigma(s)\right)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1+1}} \\
&\quad+\left\|\left(\mathcal{E}_{\alpha,2}(T-\tilde{s})-\mathcal{E}_{\alpha,2}(T-s)\right)\sigma(s)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1+1}} \\
&\leq \left\|\mathcal{E}_{\alpha,2}(T-\tilde{s})\right\|_{\mathcal{L}(\dot{H}^{\nu_3},\dot{H}^{\nu_1+1})}\left\|\left(\sigma(\tilde{s})-\sigma(s)\right)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_3}} \\
&\quad+\left\|\mathcal{E}_{\alpha,2}(T-\tilde{s})-\mathcal{E}_{\alpha,2}(T-s)\right\|_{\mathcal{L}(\dot{H}^{\nu_3},\dot{H}^{\nu_1+1})}\left\|\sigma(s)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_3}}. \qquad (67)
\end{aligned}
$$

Applying part b) of Lemma 2.3 with $\mu_1=\nu_1+1$, $\mu_2=\nu_3$, and Lemma 2.4 with $\mu_1=\nu_1+1$, $\mu_2=\frac{\nu_3+\nu_1}{2}$, $\gamma=1$, one has

$$
\left\|\mathcal{E}_{\alpha,2}(t-s)\right\|_{\mathcal{L}(\dot{H}^{\nu_2},\dot{H}^{\nu_1+1})}\lesssim (t-s)^{-\alpha(\nu_1+1-\nu_3)},
$$

$$\|\mathcal{E}_{\alpha,2}(T-\tilde{s})-\mathcal{E}_{\alpha,2}(T-s)\|_{\mathcal{L}(\dot{H}^{\nu_3},\dot{H}^{\nu_1+1})} \lesssim (\tilde{s}-s)^{\alpha[1-(\nu_3-\nu_1)]}(T-\tilde{s})^{-2\alpha[1-(\nu_3-\nu_1)]}. \tag{68}$$

From (67), (68) and the fact that $\|\mathcal{E}_{\alpha,1}(t)\|_{\mathcal{L}(\dot{H}^{\nu_1+1},\dot{H}^{\nu_1})} \leq C_3$, which is obtained by part a) of Lemma 2.3, one can see

$$\begin{aligned}\left\|\mathcal{E}_{\alpha,1}(t)\tilde{\mathcal{E}}_{\alpha,2}(T,\tilde{s},s)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1}} &\lesssim \left\|\tilde{\mathcal{E}}_{\alpha,2}(T,\tilde{s},s)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1+1}} \\ &\lesssim (T-\tilde{s})^{-\alpha[1-(\nu_3-\nu_1)]}\left\|\left(\sigma(\tilde{s})-\sigma(s)\right)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_3}} \\ &\quad + (\tilde{s}-s)^{\alpha[1-(\nu_3-\nu_1)]}(T-\tilde{s})^{-2\alpha[1-(\nu_3-\nu_1)]}\left\|\sigma(s)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_3}}.\end{aligned} \tag{69}$$

This associated with assumption (σ_2) helps us estimate $\overline{I}_{53}(t)$ as

$$\begin{aligned}&\left\|\overline{I}_{53}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} \\ &\lesssim \sum_{n\geq 1}\int_0^T\left(\int_s^T (T-s)^{-\alpha[1-(\nu_3-\nu_1)]}\left\|\left(\sigma(\tilde{s})-\sigma(s)\right)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_3}}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right)^2 ds \\ &+ \sum_{n\geq 1}\int_0^T\left(\int_s^T (\tilde{s}-s)^{\alpha[1-(\nu_3-\nu_1)]-(\frac{3}{2}-H)}(T-\tilde{s})^{-2\alpha[1-(\nu_3-\nu_1)]}\left\|\sigma(s)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_3}}d\tilde{s}\right)^2 ds \\ &\lesssim \mathcal{C}_\sigma^2\int_0^T\left(\int_s^T (T-s)^{-\alpha[1-(\nu_3-\nu_1)]}(\tilde{s}-s)^{\zeta-(\frac{3}{2}-H)}d\tilde{s}\right)^2 ds \\ &+ \|\sigma\|^2_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}\int_0^T\left(\int_s^T (T-\tilde{s})^{-2\alpha[1-(\nu_3-\nu_1)]}(\tilde{s}-s)^{\alpha[1-(\nu_3-\nu_1)]-(\frac{3}{2}-H)}d\tilde{s}\right)^2 ds.\end{aligned} \tag{70}$$

Now, it is required to bound the latter integrals. Since $\zeta > \frac{1}{2}-H > 0$, one can see that

$$\begin{aligned}\int_0^T\left(\int_s^T (T-s)^{-\alpha[1-(\nu_3-\nu_1)]}(\tilde{s}-s)^{\zeta-(\frac{3}{2}-H)}d\tilde{s}\right)^2 ds &\lesssim \int_0^T (T-s)^{2\zeta-(1-2H)-2\alpha[1-(\nu_3-\nu_1)]}ds \\ &\lesssim \int_0^T (T-s)^{-[(1-2H)+2\alpha(1-(\nu_3-\nu_1))]}ds,\end{aligned} \tag{71}$$

Similarly, since $\alpha(1-(\nu_3-\nu_1)) > \frac{1}{2}-H$, we also have

$$\int_0^T \left(\int_s^T (T-\tilde{s})^{-2\alpha[1-(\nu_3-\nu_1)]} (\tilde{s}-s)^{\alpha[1-(\nu_3-\nu_1)]-(\frac{3}{2}-H)} d\tilde{s} \right)^2 ds$$

$$= \int_0^T \left((T-s)^{-2\alpha[1-(\nu_3-\nu_1)]+\alpha[1-(\nu_3-\nu_1)]-(\frac{3}{2}-H)+1} \right)^2 ds \times$$

$$\times \left| B\Big(-2\alpha[1-(\nu_3-\nu_1)]+1, \alpha[1-(\nu_3-\nu_1)] - (\frac{3}{2}-H)+1 \Big) \right|^2$$

$$\lesssim \int_0^T (T-s)^{-[(1-2H)+2\alpha(1-(\nu_3-\nu_1))]} ds. \tag{72}$$

From (70), (71), (72), and noting $(1-2H)+2\alpha(1-(\nu_3-\nu_1)) < 1$, we deduce that

$$\left\| \overline{I}_{53}(t) \right\|_{L^2(\Omega, \dot{H}^{\nu_1})} \lesssim \mathcal{C}_\sigma + \|\sigma\|_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}. \tag{73}$$

Combining (63), (66), (73), it is obvious that

$$\left\| \overline{I}_5(t) \right\|_{L^2(\Omega, \dot{H}^{\nu_1})} \lesssim \mathcal{C}_\sigma + \|\sigma\|_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}. \tag{74}$$

• Estimating $\overline{I}_4(t)$. We now continue to bound $\overline{I}_4(t)$ by a quite similar way as in the estimate for $\overline{I}_5(t)$. For $\overline{I}_{41}(t)$, we can see

$$\begin{aligned}\left\| \overline{I}_{41}(t) \right\|^2_{L^2(\Omega, \dot{H}^{\nu_1})} &\lesssim \sum_{n\geq 1} \int_0^t \left(\frac{t}{s}\right)^{2H-1} (t-s)^{2H-1} \left\| \mathcal{E}_{\alpha,2}(t-s)\sigma(s)Q^{\frac{1}{2}}e_n \right\|^2_{\dot{H}^{\nu_1}} ds \\ &\leq \sum_{n\geq 1} \int_0^t \left(\frac{t}{s}\right)^{2H-1} (t-s)^{2H-1} \lambda_1^{-2(\nu_3-\nu_1)} \left\| \mathcal{E}_{\alpha,2}(t-s)\sigma(s)Q^{\frac{1}{2}}e_n \right\|^2_{\dot{H}^{\nu_3}} ds \\ &\lesssim \int_0^t (t-s)^{2H-1} \left\| \mathcal{E}_{\alpha,2}(t-s) \right\|^2_{\mathcal{L}(\dot{H}^{\nu_3},\dot{H}^{\nu_3})} \left\| \sigma(s) \right\|^2_{L_0^2(X,\dot{H}^{\nu_3})} ds.\end{aligned}$$

On the other hand, part a) of Lemma 2.3 has showed that

$$\left\| \mathcal{E}_{\alpha,2}(t-s) \right\|_{\mathcal{L}(\dot{H}^{\nu_3},\dot{H}^{\nu_3})} \leq C_2. \tag{75}$$

This helps us bound $\left\| \overline{I}_{41}(t) \right\|_{L^2(\Omega, \dot{H}^{\nu_1})}$ as

$$\left\| \overline{I}_{41}(t) \right\|^2_{L^2(\Omega, \dot{H}^{\nu_1})} \lesssim \sum_{n\geq 1} \int_0^t (t-s)^{2H-1} \left\| \sigma(s) \right\|^2_{L_0^2(X,\dot{H}^{\nu_3})} ds \lesssim \|\sigma\|^2_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}, \tag{76}$$

where we note that $\int_0^t (t-s)^{2H-1} ds \lesssim t^{2H} \le T^{2H}$. For $\overline{I}_{42}(t)$, it is obvious that

$$\begin{aligned}
&\left\|\overline{I}_{42}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})}\\
&\lesssim \sum_{n\ge1}\int_0^t \left\| s^{\frac12-H}\left(\int_s^t \tilde{s}^{H-\frac32}(\tilde{s}-s)^{H-\frac12}d\tilde{s}\right)\mathcal{E}_{\alpha,2}(t-s)\sigma(s)Q^{\frac12}e_n\right\|^2_{\dot{H}^{\nu_1}} ds\\
&\le \sum_{n\ge1}\int_0^t \lambda_1^{-2(\nu_3-\nu_1)}\left\| s^{\frac12-H}\left(\int_s^t \tilde{s}^{H-\frac32}(\tilde{s}-s)^{H-\frac12}d\tilde{s}\right)\mathcal{E}_{\alpha,2}(t-s)\sigma(s)Q^{\frac12}e_n\right\|^2_{\dot{H}^{\nu_3}} ds\\
&\lesssim \sum_{n\ge1}\int_0^t s^{1-2H}\left(\int_s^t \tilde{s}^{H-\frac32}(\tilde{s}-s)^{H-\frac12}d\tilde{s}\right)^2 \left\|\mathcal{E}_{\alpha,2}(t-s)\right\|^2_{\mathcal{L}(\dot{H}^{\nu_3},\dot{H}^{\nu_3})}\left\|\sigma(s)Q^{\frac12}e_n\right\|^2_{\dot{H}^{\nu_3}} ds.
\end{aligned}$$

By (65) and (75), we deduce that

$$\left\|\overline{I}_{42}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim \int_0^t s^{-(1-2H)}\left\|\sigma(s)\right\|^2_{L_0^2(X,\dot{H}^{\nu_3})} \lesssim \left\|\sigma\right\|^2_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}. \tag{77}$$

For $\overline{I}_{43}(t)$, we observe that

$$\begin{aligned}
\left\|\overline{I}_{43}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} &\lesssim \sum_{n\ge1}\int_0^t \left\|\int_s^t \tilde{\mathcal{E}}_{\alpha,2}(t,\tilde{s},s)Q^{\frac12}e_n\left(\frac{s}{\tilde{s}}\right)^{\frac12-H}(\tilde{s}-s)^{H-\frac32}d\tilde{s}\right\|^2_{\dot{H}^{\nu_1}} ds\\
&\le \sum_{n\ge1}\int_0^t \lambda_1^{-2}\left\|\int_s^t \tilde{\mathcal{E}}_{\alpha,2}(t,\tilde{s},s)Q^{\frac12}e_n\left(\frac{s}{\tilde{s}}\right)^{\frac12-H}(\tilde{s}-s)^{H-\frac32}d\tilde{s}\right\|^2_{\dot{H}^{\nu_1+1}} ds\\
&\lesssim \sum_{n\ge1}\int_0^t \left(\int_s^t \left\|\tilde{\mathcal{E}}_{\alpha,2}(t,\tilde{s},s)Q^{\frac12}e_n\right\|_{\dot{H}^{\nu_1+1}}(\tilde{s}-s)^{H-\frac32}d\tilde{s}\right)^2 ds.
\end{aligned}$$

On the other hand, in a similar way as in (67)-(69), we find that

$$\begin{aligned}
\left\|\tilde{\mathcal{E}}_{\alpha,2}(t,\tilde{s},s)Q^{\frac12}e_n\right\|_{\dot{H}^{\nu_1+1}} &\lesssim (t-s)^{-\alpha[1-(\nu_3-\nu_1)]}\left\|\left(\sigma(\tilde{s})-\sigma(s)\right)Q^{\frac12}e_n\right\|_{\dot{H}^{\nu_3}}\\
&\quad + (\tilde{s}-s)^{\alpha[1-(\nu_3-\nu_1)]}(t-\tilde{s})^{-2\alpha[1-(\nu_3-\nu_1)]}\left\|\sigma(s)Q^{\frac12}e_n\right\|_{\dot{H}^{\nu_3}}.
\end{aligned} \tag{78}$$

Hence

$$\left\|\overline{I}_{43}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})}$$

$$\lesssim \sum_{n\geq 1}\int_0^t \left(\int_s^t (t-s)^{-\alpha[1-(\nu_3-\nu_1)]} \left\| (\sigma(\tilde{s})-\sigma(s))\, Q^{\frac{1}{2}} e_n \right\|_{\dot{H}^{\nu_3}} (\tilde{s}-s)^{H-\frac{3}{2}} d\tilde{s} \right)^2 ds$$
$$+ \sum_{n\geq 1}\int_0^t \left(\int_s^t (\tilde{s}-s)^{\alpha[1-(\nu_3-\nu_1)]-(\frac{3}{2}-H)} (t-\tilde{s})^{-2\alpha[1-(\nu_3-\nu_1)]} \left\| \sigma(s) Q^{\frac{1}{2}} e_n \right\|_{\dot{H}^{\nu_3}} d\tilde{s} \right)^2 ds. \tag{79}$$

By exactly the same way as in (70)-(73), one arrives at

$$\left\| \overline{I}_{43}(t) \right\|_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim \mathcal{C}_\sigma + \|\sigma\|_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}. \tag{80}$$

From (76), (77), (80), we now deduce that

$$\left\| \overline{I}_{4}(t) \right\|_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim \mathcal{C}_\sigma + \|\sigma\|_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}. \tag{81}$$

The proof is complete. □

Proof of Lemma 3.5. Estimating $\overline{I}_5(t)$. Let us begin with the estimate for $\overline{I}_{51}(t)$ as

$$\begin{aligned} &\left\| \overline{I}_{51}(t) \right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \\ &\lesssim \sum_{n\geq 1}\int_0^T \left\| \left(\frac{T}{s}\right)^{H-\frac{1}{2}} (T-s)^{H-\frac{1}{2}} \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)Q^{\frac{1}{2}}e_n \right\|^2_{\dot{H}^{\nu_1+r_1}} ds \\ &\lesssim \sum_{n\geq 1}\int_0^T (T-s)^{2H-1} \left\| \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s) \right\|^2_{\mathcal{L}(\dot{H}^{\nu_3},\dot{H}^{\nu_1+r_1})} \left\| \sigma(s)Q^{\frac{1}{2}}e_n \right\|^2_{\dot{H}^{\nu_3}} ds. \end{aligned}$$

Applying part d) of Lemma 2.3 with $\mu_1 = \nu_1 + r_1$, $\mu_2 = \nu_1$ and $\mu_4 = \nu_3$, we have

$$\left\| \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s) \right\|_{\mathcal{L}(\dot{H}^{\nu_3},\dot{H}^{\nu_1+r_1})} \lesssim t^{-\alpha r_1}(T-s)^{-\alpha(\nu_1+1-\nu_3)}. \tag{82}$$

The above property and the observation (63) lead to

$$\begin{aligned} \left\| \overline{I}_{51}(t) \right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} &\lesssim t^{-2\alpha r_1}\int_0^T (T-s)^{-[1-2H+\alpha(\nu_1+1-\nu_3)]} \|\sigma(s)\|^2_{L_0^2(X,\dot{H}^{\nu_3})} ds \\ &\lesssim t^{-2\alpha r_1} \|\sigma\|^2_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}. \end{aligned} \tag{83}$$

In the same way as in the estimate for $\overline{I}_{52}(t)$ in the proof of Lemma 3.4, but the property (82) is used instead of (62), one can easily get

$$\left\| \overline{I}_{52}(t) \right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \lesssim t^{-2\alpha r_1} \|\sigma\|^2_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}. \tag{84}$$

To estimate $\overline{I}_{53}(t)$, we need to bound the term $\left\|\mathcal{E}_{\alpha,1}(t)\tilde{\mathcal{E}}_{\alpha,2}(T,\tilde{s},s)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1+r_1}}$. Applying part b) of Lemma 2.3 with $\mu_1=\nu_1+r_1$, $\mu_2=\nu_1$ and using the estimate (69), one has

$$\begin{aligned}
&\left\|\mathcal{E}_{\alpha,1}(t)\tilde{\mathcal{E}}_{\alpha,2}(T,\tilde{s},s)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1+r_1}}\\
&\qquad\lesssim \|\mathcal{E}_{\alpha,1}(t)\|_{\mathcal{L}(\dot{H}^{\nu_1+1},\dot{H}^{\nu_1+r_1})}\left\|\tilde{\mathcal{E}}_{\alpha,2}(T,\tilde{s},s)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1+1}}\\
&\qquad\lesssim t^{-\alpha r_1}\left\|\tilde{\mathcal{E}}_{\alpha,2}(T,\tilde{s},s)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1+1}}\\
&\qquad\lesssim t^{-\alpha r_1}(T-s)^{-\alpha[1-(\nu_3-\nu_1)]}\left\|\left(\sigma(\tilde{s})-\sigma(s)\right)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_3}}\\
&\qquad+t^{-\alpha r_1}(\tilde{s}-s)^{\alpha[1-(\nu_3-\nu_1)]}(T-\tilde{s})^{-2\alpha[1-(\nu_3-\nu_1)]}\left\|\sigma(s)Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_3}}. \qquad (85)
\end{aligned}$$

In the same way as in the estimate for $\overline{I}_{53}(t)$ in the proof of Lemma 3.4, but the property (85) is used instead of (69), one arrives at

$$\left\|\overline{I}_{53}(t)\right\|_{L^2(\Omega,\dot{H}^{\nu_1+r_1})}\lesssim t^{-\alpha r_1}\left(\mathcal{C}_\sigma+\|\sigma\|_{C((0,T];L^2_0(X,\dot{H}^{\nu_3}))}\right). \qquad (86)$$

Combining (83), (84), (86), we deduce that

$$\left\|\overline{I}_{5}(t)\right\|_{L^2(\Omega,\dot{H}^{\nu_1+r_1})}\lesssim t^{-\alpha r_1}\left(\mathcal{C}_\sigma+\|\sigma\|_{C((0,T];L^2_0(X,\dot{H}^{\nu_3}))}\right). \qquad (87)$$

• <u>*Estimating $\overline{I}_4(t)$*</u>. Applying part b) of Lemma 2.3 with $\mu_1=\nu_1+1$ and $\mu_2=\nu_3$, one has

$$\|\mathcal{E}_{\alpha,2}(t-s)\|_{\mathcal{L}(\dot{H}^{\nu_3},\dot{H}^{\nu_1+1})}\lesssim (t-s)^{-\alpha(1-(\nu_3-\nu_1))}. \qquad (88)$$

Hence, for $\overline{I}_{41}(t)$, we can see

$$\begin{aligned}
&\left\|\overline{I}_{41}(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_1})}\\
&\lesssim \sum_{n\geq 1}\int_0^t\left(\frac{t}{s}\right)^{2H-1}(t-s)^{2H-1}\left\|\mathcal{E}_{\alpha,2}(t-s)\sigma(s)Q^{\frac{1}{2}}e_n\right\|^2_{\dot{H}^{\nu_1+r_1}}ds\\
&\leq \sum_{n\geq 1}\int_0^t\left(\frac{t}{s}\right)^{2H-1}(t-s)^{2H-1}\lambda_1^{-2(1-r_1)}\left\|\mathcal{E}_{\alpha,2}(t-s)\sigma(s)Q^{\frac{1}{2}}e_n\right\|^2_{\dot{H}^{\nu_1+1}}ds\\
&\lesssim \int_0^t(t-s)^{2H-1}\left\|\mathcal{E}_{\alpha,2}(t-s)\right\|^2_{\mathcal{L}(\dot{H}^{\nu_3},\dot{H}^{\nu_1+1})}\left\|\sigma(s)\right\|^2_{L^2_0(X,\dot{H}^{\nu_3})}ds
\end{aligned}$$

$$\lesssim \int_0^t (t-s)^{-[1-2H+2\alpha(1-(\nu_3-\nu_1))]} \|\sigma(s)\|^2_{L_0^2(X,\dot{H}^{\nu_3})} ds.$$

By the observation (63), we deduce that

$$\|\overline{I}_{41}(t)\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \lesssim \|\sigma\|^2_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}. \tag{89}$$

Similarly, by using (88) and a similar way as in the estimate for $\overline{I}_{42}(t)$ in the proof of Lemma 3.4, one can easily get

$$\|\overline{I}_{42}(t)\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \lesssim \|\sigma\|^2_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}. \tag{90}$$

For $\overline{I}_{43}(t)$, we observe that

$$\begin{aligned}
&\|\overline{I}_{43}(t)\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \\
&\lesssim \sum_{n\ge 1} \int_0^t \left\| \int_s^t \tilde{\mathcal{E}}_{\alpha,2}(t,\tilde{s},s) Q^{\frac{1}{2}} e_n \left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H} (\tilde{s}-s)^{H-\frac{3}{2}} d\tilde{s} \right\|^2_{\dot{H}^{\nu_1+r_1}} ds \\
&\le \sum_{n\ge 1} \int_0^t \lambda_1^{-2(1-r_1)} \left\| \int_s^t \tilde{\mathcal{E}}_{\alpha,2}(t,\tilde{s},s) Q^{\frac{1}{2}} e_n \left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H} (\tilde{s}-s)^{H-\frac{3}{2}} d\tilde{s} \right\|^2_{\dot{H}^{\nu_1+1}} ds \\
&\lesssim \sum_{n\ge 1} \int_0^t \left(\int_s^t \left\| \tilde{\mathcal{E}}_{\alpha,2}(t,\tilde{s},s) Q^{\frac{1}{2}} e_n \right\|_{\dot{H}^{\nu_1+1}} (\tilde{s}-s)^{H-\frac{3}{2}} d\tilde{s} \right)^2 ds.
\end{aligned}$$

By exactly the same techniques as in (78), (79), one also have

$$\|\overline{I}_{43}(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \lesssim \mathcal{C}_\sigma + \|\sigma\|_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}. \tag{91}$$

From (89), (90), (91), we deduce that

$$\|\overline{I}_{4}(t)\|_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \lesssim \mathcal{C}_\sigma + \|\sigma\|_{C((0,T];L_0^2(X,\dot{H}^{\nu_3}))}. \tag{92}$$

The proof is complete. □

Proof of Lemma 3.6. In the case $H \in (\frac{1}{2}, 1)$, the two terms $\overline{I}_5(t)$, $\overline{I}_4(t)$ have simpler explicit representations as follows

$$\overline{I}_5(t) = \sum_{n\ge 1} \int_0^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\sigma(s)Q^{\frac{1}{2}} e_n d\beta_n^H(s)$$

$$= \sum_{n\geq 1}\int_0^T K^*_{H,T}\left(\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-\cdot)\sigma(\cdot)Q^{\frac{1}{2}}e_n\right)(s)d\beta_n(s)$$

$$= \sum_{n\geq 1}\int_0^T\left(\int_s^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\frac{\partial K^H}{\partial\tilde{s}}(\tilde{s},s)d\tilde{s}\right)\beta_n(s)$$

$$= c_H\sum_{n\geq 1}\int_0^T\left(\int_s^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right)\beta_n(s), \tag{93}$$

and

$$\overline{I}_4(t) = \sum_{n\geq 1}\int_0^t \mathcal{E}_{\alpha,2}(t-s)\sigma(s)Q^{\frac{1}{2}}e_n d\beta_n^H(s) = \sum_{n\geq 1}\int_0^t K^*_{H,t}\left(\mathcal{E}_{\alpha,2}(t-\cdot)\sigma(\cdot)Q^{\frac{1}{2}}e_n\right)(s)d\beta_n(s)$$

$$= \sum_{n\geq 1}\int_0^t\left(\int_s^t \mathcal{E}_{\alpha,2}(t-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\frac{\partial K^H}{\partial\tilde{s}}(\tilde{s},s)d\tilde{s}\right)d\beta_n(s)$$

$$= c_H\sum_{n\geq 1}\int_0^t\left(\int_s^t \mathcal{E}_{\alpha,2}(t-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right)d\beta_n(s). \tag{94}$$

• <u>*Estimating* $\overline{I}_5(t)$</u>. Since $\|\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-\tilde{s})\|_{\mathcal{L}(\dot{H}^{\nu_4},\dot{H}^{\nu_1})} \lesssim (T-\tilde{s})^{-\alpha[1-(\nu_4-\nu_1)]}$, we can see that

$$\left\|\overline{I}_5(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})}$$

$$= \sum_{n\geq 1}\int_0^T\left\|\int_s^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right\|^2_{\dot{H}^{\nu_1}} ds$$

$$\lesssim \sum_{n\geq 1}\int_0^T\left[\int_s^T \left\|\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1}}\left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right]^2 ds$$

$$\lesssim \sum_{n\geq 1}\int_0^T\left[\int_s^T (T-\tilde{s})^{-\alpha[1-(\nu_4-\nu_1)]}\left\|\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_4}}\left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right]^2 ds. \tag{95}$$

Using the Hölder inequality, we obtain the following estimate

$$\left\|\overline{I}_5(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim \int_0^T\left(\int_s^T\left(\frac{s}{\tilde{s}}\right)^{1-2H}(\tilde{s}-s)^{H-\frac{3}{2}}(T-\tilde{s})^{-\alpha[1-(\nu_4-\nu_1)]}d\tilde{s}\right)\times$$

$$\times \left(\int_s^T (\tilde{s}-s)^{H-\frac{3}{2}} (T-\tilde{s})^{-\alpha[1-(\nu_4-\nu_1)]} \|\sigma(\tilde{s})\|^2_{L^2_0(X,\dot{H}^{\nu_4})} d\tilde{s} \right) ds.$$

To estimate two integrals of the right-hand side, we prepare a material as follows

$$\int_0^t \tau^{a-1}(t-\tau)^{b-1} d\tau = t^{a+b-1} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad \text{for } a,b>0. \tag{96}$$

By the assumption (σ_3), the above property allows that

$$\begin{aligned}
&\int_0^{T-s} \tau^{-\frac{\rho(3/2-H)}{\rho-2}} (T-s-\tau)^{-\frac{\rho\alpha[1-(\nu_4-\nu_1)]}{\rho-2}} d\tau \\
&= (T-s)^{1-\frac{\rho}{\rho-2}\left[\frac{3}{2}-H+\alpha(1-(\nu_4-\nu_1))\right]} \frac{\Gamma(1-\frac{\rho(3/2-H)}{\rho-2})\Gamma(1-\frac{\rho\alpha[1-(\nu_4-\nu_1)]}{\rho-2})}{\Gamma\Big(2-\frac{\rho}{\rho-2}\left[\frac{3}{2}-H+\alpha(1-(\nu_4-\nu_1))\right]\Big)}.
\end{aligned}$$

Using the Hölder inequality and the above identity, we have

$$\begin{aligned}
&\int_s^T (\tilde{s}-s)^{H-\frac{3}{2}} (T-\tilde{s})^{-\alpha[1-(\nu_4-\nu_1)]} \|\sigma(\tilde{s})\|^2_{L^2_0(X,\dot{H}^{\nu_4})} d\tilde{s} \\
&= \int_0^{T-s} \tau^{H-\frac{3}{2}} (T-s-\tau)^{-\alpha[1-(\nu_4-\nu_1)]} \|\sigma(\tau+s)\|^2_{L^2_0(X,\dot{H}^{\nu_4})} d\tau \\
&\lesssim \left(\int_0^{T-s} \tau^{-\frac{\rho(3/2-H)}{\rho-2}} (T-s-\tau)^{-\frac{\rho\alpha[1-(\nu_4-\nu_1)]}{\rho-2}} d\tau \right)^{\frac{\rho-2}{\rho}} \left(\int_0^{T-s} \|\sigma(\tau+s)\|^\rho_{L^2_0(X,\dot{H}^{\nu_4})} d\tau \right)^{\frac{2}{\rho}} \\
&\lesssim (T-s)^{\frac{\rho-2}{\rho}-\left[\frac{3}{2}-H+\alpha(1-(\nu_4-\nu_1))\right]} \|\sigma\|^2_{L^\rho(0,T;L^2_0(X,\dot{H}^{\nu_4}))}.
\end{aligned} \tag{97}$$

In addition, the property (96) also yields that

$$\begin{aligned}
&\int_s^T \left(\frac{s}{\tilde{s}}\right)^{1-2H} (\tilde{s}-s)^{H-\frac{3}{2}} (T-\tilde{s})^{-\alpha[1-(\nu_4-\nu_1)]} d\tilde{s} \\
&\lesssim s^{1-2H} \int_s^T (\tilde{s}-s)^{H-\frac{3}{2}} (T-\tilde{s})^{-\alpha[1-(\nu_4-\nu_1)]} d\tilde{s} \\
&= s^{1-2H} \int_0^{T-s} \tau^{H-\frac{3}{2}} (T-s-\tau)^{-\alpha[1-(\nu_4-\nu_1)]} d\tilde{s}
\end{aligned}$$

$$\lesssim s^{1-2H}(T-s)^{1-\left[\frac{3}{2}-H+\alpha(1-(\nu_4-\nu_1))\right]}.$$

Hence, we deduce that

$$\begin{aligned}\left\|\overline{I}_5(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} &\lesssim \|\sigma\|^2_{L^\rho(0,T;L^2_0(X,\dot{H}^{\nu_4}))}\int_0^T s^{1-2H}(T-s)^{\frac{\rho-2}{\rho}+1-2\left[\frac{3}{2}-H+\alpha(1-(\nu_4-\nu_1))\right]}ds \\ &\lesssim \|\sigma\|^2_{L^\rho(0,T;L^2_0(X,\dot{H}^{\nu_4}))}T^{\frac{\rho-2}{\rho}-2\alpha(1-(\nu_4-\nu_1))}, \end{aligned} \tag{98}$$

where we note that $1-2H>-1$ and $(\frac{\rho-2}{\rho}+1)-[3-2H+2\alpha(1-(\nu_4-\nu_1))]>0$, which follows from Assumption (σ_3).

• <u>*Estimating* $\overline{I}_4(t)$</u>. Now, we will use a similar technique as in above to estimate $\overline{I}_4(t)$. Using the fact that $\|\mathcal{E}_{\alpha,2}(t-\tilde{s})\|_{\mathcal{L}(\dot{H}^{\nu_4},\dot{H}^{\nu_4})}\lesssim 1$, we can see that

$$\begin{aligned}&\left\|\overline{I}_4(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} \\ &= \sum_{n\ge 1}\int_0^t \left\|\int_s^t \mathcal{E}_{\alpha,2}(t-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right\|^2_{\dot{H}^{\nu_1}} ds \\ &\le \sum_{n\ge 1}\int_0^t\left[\int_s^t \lambda_1^{-(\nu_4-\nu_1)}\left\|\mathcal{E}_{\alpha,2}(t-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_4}}\left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right]^2 ds \\ &\lesssim \sum_{n\ge 1}\int_0^t\left[\int_s^t \left\|\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_4}}\left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right]^2 ds.\end{aligned}$$

Using the Hölder inequality, we obtain

$$\begin{aligned}&\left\|\overline{I}_4(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} \\ &\lesssim \int_0^t\left(\int_s^t\left(\frac{s}{\tilde{s}}\right)^{1-2H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right)\left(\int_s^t(\tilde{s}-s)^{H-\frac{3}{2}}\|\sigma(\tilde{s})\|^2_{L^2_0(X,\dot{H}^{\nu_4})}d\tilde{s}\right)ds,\end{aligned}$$

Since $H-\frac{3}{2}>-1$, one easily observe that $\int_s^t\left(\frac{s}{\tilde{s}}\right)^{1-2H}(\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\lesssim s^{1-2H}$. In addition, the Hölder inequality allows that

$$\begin{aligned}&\int_s^t(\tilde{s}-s)^{H-\frac{3}{2}}\|\sigma(\tilde{s})\|^2_{L^2_0(X,\dot{H}^{\nu_4})}d\tilde{s} \\ &\lesssim \left(\int_s^t(\tilde{s}-s)^{\frac{\rho}{\rho-2}(H-\frac{3}{2})}d\tilde{s}\right)^{\frac{\rho-2}{\rho}}\left(\int_s^t\|\sigma(\tilde{s})\|^p_{L^2_0(X,\dot{H}^{\nu_4})}d\tilde{s}\right)^{\frac{2}{\rho}}\end{aligned}$$

$$\lesssim \|\sigma\|^2_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}.$$

From three latter estimates, we deduce that

$$\left\|\overline{I}_4(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim \|\sigma\|^2_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))} \int_0^t s^{1-2H} ds \lesssim \|\sigma\|^2_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}. \tag{99}$$

The proof is complete. □

Proof of Lemma 3.7. • Estimating $\overline{I}_5(t)$. To estimate $\overline{I}_5(t)$, we will use the property

$$\|\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-s)\|_{\mathcal{L}(\dot{H}^{\nu_4},\dot{H}^{\nu_1+r_1})} \lesssim t^{-\alpha r_1}(T-s)^{-\alpha(\nu_1+1-\nu_4)},$$

which is obtained by applying part d) of Lemma 2.3 with $\mu_1 = \nu_1 + r_1$, $\mu_2 = \nu_1$ and $\mu_4 = \nu_4$. In this way, we arrive at

$$\begin{aligned}
&\left\|\overline{I}_5(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \\
&= \sum_{n\ge 1} \int_0^T \left\| \int_s^T \mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-\tilde{s})\sigma(\tilde{s})Q^{\frac12}e_n \left(\frac{s}{\tilde{s}}\right)^{\frac12-H} (\tilde{s}-s)^{H-\frac32} d\tilde{s} \right\|^2_{\dot{H}^{\nu_1+r_1}} ds \\
&\lesssim \sum_{n\ge 1} \int_0^T \left[\int_s^T \left\|\mathcal{E}_{\alpha,1}(t)\mathcal{E}_{\alpha,2}(T-\tilde{s})\sigma(\tilde{s})Q^{\frac12}e_n\right\|_{\dot{H}^{\nu_1+r_1}} \left(\frac{s}{\tilde{s}}\right)^{\frac12-H} (\tilde{s}-s)^{H-\frac32} d\tilde{s} \right]^2 ds \\
&\lesssim t^{-2\alpha r_1} \sum_{n\ge 1} \int_0^T \left[\int_s^T (T-\tilde{s})^{-\alpha[1-(\nu_4-\nu_1)]} \left\|\sigma(\tilde{s})Q^{\frac12}e_n\right\|_{\dot{H}^{\nu_4}} \left(\frac{s}{\tilde{s}}\right)^{\frac12-H} (\tilde{s}-s)^{H-\frac32} d\tilde{s} \right]^2 ds.
\end{aligned}$$

By using exactly the same way employed to obtain (95)-(98), one easily obtain

$$\left\|\overline{I}_5(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1})} \lesssim t^{-2\alpha r_1} \|\sigma\|^2_{L^\rho(0,T;L_0^2(X,\dot{H}^{\nu_4}))}. \tag{100}$$

• Estimating $\overline{I}_4(t)$. To estimate $\overline{I}_4(t)$, we will use the property

$$\|\mathcal{E}_{\alpha,2}(t-s)\|_{\mathcal{L}(\dot{H}^{\nu_4},\dot{H}^{\nu_1+r_1})} \lesssim (t-s)^{-\alpha(1-(\nu_4-\nu_1))},$$

which is obtained by applying part b) of Lemma 2.3 with $\mu_1 = \nu_1 + r_1$ and $\mu_2 = \nu_4$. In this way, we arrive at

$$\left\|\overline{I}_4(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} = \sum_{n\ge 1} \int_0^t \left\| \int_s^t \mathcal{E}_{\alpha,2}(t-\tilde{s})\sigma(\tilde{s})Q^{\frac12}e_n \left(\frac{s}{\tilde{s}}\right)^{\frac12-H} (\tilde{s}-s)^{H-\frac32} d\tilde{s} \right\|^2_{\dot{H}^{\nu_1+r_1}} ds$$

$$\leq \sum_{n\geq 1}\int_0^t \left[\int_s^t \left\|\mathcal{E}_{\alpha,2}(t-\tilde{s})\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_1+r_1}} \left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H} (\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right]^2 ds$$

$$\lesssim \sum_{n\geq 1}\int_0^t \left[\int_s^t (t-s)^{-\alpha(1-(\nu_4-\nu_1))} \left\|\sigma(\tilde{s})Q^{\frac{1}{2}}e_n\right\|_{\dot{H}^{\nu_4}} \left(\frac{s}{\tilde{s}}\right)^{\frac{1}{2}-H} (\tilde{s}-s)^{H-\frac{3}{2}}d\tilde{s}\right]^2 ds. \tag{101}$$

Noting that the latter series in (101) has a similar form as in (95) (the difference is that T is replaced by t). Hence, one can verify that

$$\left\|\overline{I}_4(t)\right\|^2_{L^2(\Omega,\dot{H}^{\nu_1+r_1})} \lesssim \|\sigma\|^2_{L^\rho(0,T;L^2_0(X,\dot{H}^{\nu_4}))}. \tag{102}$$

The proof is complete. □

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