On the delayed worse-than-minimal repair model and its application to preventive replacement

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Abstract. Minimal repair and other imperfect repair models have been intensively studied in the literature. Much less attention has been payed to the "worse than minimal" repair problem, although it often occurs in practice due to the adverse effects of previous repairs, environmental and internal shocks, etc. To model this type of repair, we define a new point process that behaves as the nonhomogeneous Poisson process up to a certain event or time (minimal repairs) and then it becomes the generalized Polya process of repairs (worse than minimal repairs). The corresponding replacement policy is defined and the optimal solutions that minimize the long run expected cost rate are analyzed. The replacement can be executed univariately either after the given time T or the given number of repairs (on the k-th failure). Moreover, the system can be also replaced by implementing the bivariate strategy, that is, after the time T or on the k-th failure, whichever comes first. The detailed numerical examples illustrate our findings. It is shown that the k-strategy outperforms the T-strategy (lower cost rates), whereas the bivariate strategy is not worse than the best univariate strategy.

Keywords: Minimal repair; worse than minimal repair, generalized Polya process, optimal replacement; long-run expected cost rate

1. Introduction

In the classical renewal setting (Barlow and Proschan, 1975), the failed item is instantaneously replaced by the "as good as new" and therefore, the corresponding process of repairs can be described by the renewal process. However, in practice, there is no perfect repair as such that brings an item to the "as good as new" state. For instance, even the spares that are stored in warehouses and used for replacement are not new as they also age during storage. Of course,

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at many instances, we can assume the corresponding perfect repair as a reasonable approximation. However, it is not always the case. Therefore, during the last several decades, a considerable attention was given in reliability literature to developing and applications of various imperfect repair models.

One of the most popular imperfect repair models (age reduction) is based on the notion of *virtual age* (Kijima, 1989; Liu et al., 1995; Jack, 1998; Ansell et al., 2004; Wang and Pham, 2006; Doyen and Gaudoin, 2004; Finklestein, 2008; Badia and Berrade, 2009; Finkelstein and Cha, 2013; Tanwar et al., 2014; Dijoux et al., 2016; Nguyen et al., 2017; Zhao et al., 2019; to name a few). There are numerous other models as well (Brown and Proschan, 1983; deToledo et al., 2015; Levitin and Lisniansky, 2000; Marais and Saleh, 2009; Kahle, 2019). Imperfect repair models studied in the literature can be considered as a 'natural generalization' of the minimal repair (Barlow and Proschan, 1975; Aven and Jensen, 2000) that restores an item to the "as bad as old" state. It is well-known (Rausand and Høyland, 2003) that this type of repair occurs often in practice when, e.g., a small part of a large failed system is repaired/replaced. A relevant example is a series system of n (large) components. Another possibility is to replace the failed system by the statistically identical one that was operating for the same time but did not fail ("hot" standby). However, this type of minimal repair requires usually a large number of items in the hot standby and, therefore, is not often cost-wise effective. As the failure process for consequent minimal repairs follows the nonhomogeneous Poisson process (NHPP), it usually allows for the closed-form analytical results (e.g., for the long-run expected cost rates) and nice mathematical properties for the optimal solutions in maintenance optimization.

Much less attention in the literature has been devoted to the case when the repair is worse than minimal and, therefore, for justification, we will provide further a list of situations when it can occur in practice (that can be, obviously, extended further). Most often, it is an aftermath of the preceding failure(s) or of the repair action itself. Indeed, at many instances, the failure of a component in a system can result in adverse effects on other components e.g., through the increased stress, temperature, humidity, etc. The latter can increase the overall failure rate of a system, as compared with that just before the failure. (see, e.g., El-Damcese, 1997; Rausand and Hoyland, 2003; Jeong, 2012). The following specific examples (Lee and Cha, 2016) illustrate this possibility:

-a failure of a still wire cable in a bridge or in an elevator instantaneously increases the stress on the remaining cables and leads to some damages that result in the increased failure rate even after the repair.
- a failure of one airplane engine during a flight increases the stresses on other engines, therefore, increasing their failure rates.
-a failure of a pump in a multi pump hydraulic control system instantly increases the pressure for each non-failed pump until the repair of the failed one. This can lead to additional accumulation of wear resulting in the increased failure rate of a system after the repair.
-when an electric device fails effected by an external shock (electric or mechanical shock), the non-failed components also experience this external shock and their reliability performances can be worse than before. Moreover, as a result of a failure, the additional electrical load is distributed among other components that can also result in the increased failure rate (Lee and Cha, 2016).

Thus, for instance, in any load sharing system, the loading (and the failure rates) of components increase after the failure of one of them. Note also that in some standby systems, the components with better reliability characteristics are activated first. Therefore, after the failure of the first activated component, reliability characteristics after the 'standby repair' will be worse. We can also think about situations when the repair itself brings some damage to other parts that were not affected by the failure or 'incepts' the possible causes of the future failure as in the case of imperfect debugging of software when the new bugs can be planted.

All foregoing examples describe the repair that is worse than minimal. To deal with this type of repairs, the generalized Polya process (GPP) was introduced and its mathematical properties were described in Cha (2014), whereas some applications to a preventive maintenance, an optimal replacement under a general failure and repair model and shocks modeling were discussed in Lee and Cha (2016), Badía et al. (2018) and Cha and Finkelstein (2018), respectively. Moreover, Babykina and Couallier (2014) and Le Gat (2014) have shown that the GPP can be better fitted to some real field failure data sets than the traditional NHPP model. This justifies the GPP approach in various practical situations when there is evidence that repair can be worse than minimal. Note that Wu and Scarf (2017) and Nafisah et al. (2019) also considered a repair process that is the superposition of a renewal process and a Poisson process, where the repair effect may be negative, when the intensity of the part that is a renewal process is a decreasing function.

In view of the foregoing considerations, the main goal of the current paper is to extend the capability of the GPP by introducing the combined NHPP+GPP model for describing the combined repair process. For this and for further applications to the problem of optimal maintenance, the corresponding methodology has to be developed. We call the suggested process (as in the title), the GPP process with delay. Indeed, technical systems are often resilient to the first shocks, impacts of failures of some components on the others, etc. However, as the number of these impacts increases, the system's 'strength' in this respect decreases due to deterioration. This means that at the initial stage of operation, the components have sufficiently large strength and they are not directly negatively affected, e.g., by the previous failures of other components. Thus, if the failed component is minimally repaired, the system is *minimally repaired* as well, which is modeled by the corresponding NHPP. However, with the increase of the number of repairs, as described above, the repair starts to be worse than minimal that can be modeled by the corresponding GPP. Therefore, the new, combined process that starts as the NHPP (minimal repairs) and then proceeds as the GPP (worse than minimal) should be introduced and described on the level sufficient for considering the corresponding optimal replacement problems.

The paper is organized as follows. In section 2 we define the combined process for some relevant settings. Section 3 deals with the optimal replacement problems and presents derivation of the corresponding cost rates. Numerical illustration is performed in Section 4, whereas concluding remarks are given in Section 5.

2. Description of the models

2.1. The GPP process

We will define further the combined stochastic point process that starts as the NHPP and then after some delay proceeds as the generalized Polya process (GPP). As the properties of the NHPP that describes the process of minimal repairs are well-known, we start with some relevant definitions and the formal description of the GPP process (Cha, 2014). It should be noted also that the corresponding mathematical description of the combined process should be performed carefully, as it will be used further in deriving the properties of the model suggested in this paper.

Let $\{N(t), t \geq 0\}$ be an *orderly* point process with history (internal filtration) $H_{t-} \equiv \{N(u), 0 \leq u < t\}$ in $[0,t)$, i.e., the set of all point events in $[0,t)$. It can be described via the concept of the stochastic intensity (the intensity process) λ_t, $t \ge 0$ (Aven and Jensen, 2000), which is defined as the following limit

$$\lambda_t = \lim_{\Delta t \to 0} \frac{\Pr[N(t, t+\Delta t) = 1 \mid H_{t-}]}{\Delta t} = \lim_{\Delta t \to 0} \frac{E[N(t, t+\Delta t) \mid H_{t-}]}{\Delta t}, \tag{1}$$

where $N(t_1, t_2)$, $t_1 < t_2$, is the number of events $[t_1, t_2)$. For the NHPP, the stochastic intensity reduces to the rate of the NHPP, i.e., $\lambda_t = \lambda(t), t \ge 0$. It can be shown (Lee and Cha, 2016) that the general 3-parameter definition of the GPP (Cha, 2014), without loss of generality can be reparametrized using two parameters as

Definition 1. Generalized Polya Process (GPP)

A counting process $\{N(t),\ t \ge 0\}$ *is called the Generalized Polya Process (GPP) with the set of parameters* $(\lambda(t), \alpha)$, $\alpha \ge 0$, *if*

(i) $N(0) = 0$;

(ii) $\lambda_t = (\alpha N(t-) + 1)\lambda(t)$. (2)

Thus, the GPP with $(\lambda(t), \alpha = 0)$ reduces to the NHPP with the intensity function $\lambda(t)$ and, accordingly, the GPP can be understood as a generalized version of the NHPP. We see that the history of this process is defined by the number of prior events and the corresponding probability of occurrence of an event in the infinitesimal interval of time is increasing accordingly with this number.

The corresponding repair/failure process is called the GPP repair process (Lee and Cha, 2016). Assume that the duration of a repair is negligible. Define $\{N(t),\ t \ge 0\}$ as the stochastic repair/failure process of the system with its baseline failure rate $\lambda(t)$ that is described by (2) with parameter $\alpha > 0$. Thus, with each repair, the stochastic intensity increases accordingly (and, therefore, the failure rate on the corresponding cycle) which undergoes a type of repair on each failure, where $N(t)$ is the total number of repairs/failures in $(0, t]$.

In what follows, we will consider two main models for the delay in 'inception' of the GPP repair after the process of minimal repairs, whereas before that it is the process of minimal repairs (NHPP).

2.2. GPP starts after *m* minimal repairs (Model 1)

From the start of the operation of the system, until the m-th failure, the repair is minimal and just after the *m*-th failure, the repair starts to be worth than minimal. Thus, the system is 'resilient' up to the *m*-th minimal repair. The resulting combined process can be formally defined as

Definition 2. Combined Repair Process of Type I

(i) $N(0) = 0$;

(ii) $\lambda_t = \lambda(t)$, $N(t-) = 0, 1, \ldots, m$

(iii) $\lambda_t = (\alpha(N(t-) - m) + 1)\lambda(t)$, $N(t-) = m+1, m+2, m+3, \ldots$;

As this process is not pure NHPP, nor pure GPP, in order to proceed, we must characterize it probabilistically.

Derivation of $P(N(t)=n)$
(i)First, let $n=0,1,\ldots,m$.
When we consider the event $\{N(t)=n\}$ for $n=0,1,\ldots,m-1$, we have $s_n<t\le s_{n+1}\le s_m$, where s_n is the n-th arrival time. Thus, the time instant of our interest t is before the change point s_m and, the process until time t is the NHPP. Therefore, obviously, in this case, $P(N(t)=n)=(\Lambda(t)^n\exp\{-\Lambda(t)\})/n!$, where $\Lambda(t)\equiv\int_0^t\lambda(u)du$. Suppose now that $n=m$. Then, for the event $\{N(t)=m\}$, the time instant of our interest t satisfies $s_m<t\le s_{m+1}$. Thus, in the interval $(s_m,t]$, the process is already the GPP. However, due to property (iii) in Definition 2, the stochastic intensity in the interval $(s_m,t]$ is still given by $\lambda_t=\lambda(t)$, which, also results in $P(N(t)=m)=(\Lambda(t)^m\exp\{-\Lambda(t)\})/m!$.

(ii) Let $n=m+1,m+2,\ldots$.
Denote by S_m the mth failure/repair time. Observe that

$$P(N(t)=n)=\int_0^t P(N(t)=n\mid S_m=u)f_{S_m}(u)du,$$

where $f_{S_m}(u)$ is the pdf of S_m

$$\begin{aligned}P(N(t)=n\mid S_m=u)&=P(N(t)=n\mid N(s)<m,0\le s<u,N(u)=m)\\&=P(N(t)-N(u)=n-m\mid N(s)<m,0\le s<u,N(u)=m).\end{aligned}$$

Note that, given $N(s)<m,0\le s<u,N(u)=m$, $\{N(s+u)-N(u),s\ge 0\}$ is the GPP with the parameter set $(\lambda(u+s),\alpha,1)$. From Cha (2014), we have the following relationship for the probability of a conditional increment, i.e.,

$$\begin{aligned}&P(N(t)-N(u)=n-m\mid N(s)<m,0\le s<u,N(u)=m)\\&=\frac{\Gamma(1/\alpha+n-m)}{\Gamma(1/\alpha)(n-m)!}\left(\exp\{-\alpha\Lambda(t,u)\}\right)^{1/\alpha}\left(1-\exp\{-\alpha\Lambda(t,u)\}\right)^{n-m},\end{aligned}$$

where $\Lambda(t,u)\equiv\int_0^{t-u}\lambda(u+s)ds=\int_u^t\lambda(s)ds$.

On the other hand, as $f_{S_m}(t)=\lambda(t)\dfrac{(\Lambda(t))^{m-1}}{(m-1)!}\exp\{-\Lambda(t)\}$, we have:

$$P(N(t)=n)=\int_0^t P(N(t)=n\mid S_m=u)f_{S_m}(u)du$$

$$=\int_0^t\frac{\Gamma(1/\alpha+n-m)}{\Gamma(1/\alpha)(n-m)!}\left(\exp\{-\alpha\Lambda(t,u)\}\right)^{1/\alpha}\left(1-\exp\{-\alpha\Lambda(t,u)\}\right)^{n-m}\lambda(u)\frac{(\Lambda(u))^{m-1}}{(m-1)!}\exp\{-\Lambda(u)\}du,$$

$$n=m+1,m+2,\ldots \quad (3)$$

■

From (3), we can obtain the mean number of failures/repairs in $[0,t)$ as

$$\begin{aligned} E[N(t)] &= \sum_{n=1}^{m-1} n \frac{\Lambda(t)^n}{n!} \exp\{-\Lambda(t)\} \\ &+ \int_0^t \left(m + \frac{1}{\alpha}(\exp\{\alpha\Lambda(t,u)\} - 1) \right) \lambda(u) \frac{(\Lambda(u))^{m-1}}{(m-1)!} \exp\{-\Lambda(u)\} du \end{aligned} \tag{4}$$

The corresponding proof is deferred to the Appendix I.

Remark 1. Note that when $\alpha \to 0$, $\frac{1}{\alpha}(\exp\{\alpha\Lambda(t,u)\}-1) \to (\Lambda(t)-\Lambda(u))$. Thus, in this case, $E[N(t)]$ converges to

$$\sum_{n=1}^{m-1} n \frac{\Lambda(t)^n}{n!} \exp\{-\Lambda(t)\} + \int_0^t \left(m + \Lambda(t) - \Lambda(u)\right) f_{S_m}(u) du$$

$$= E[N^*(t) \mid S_m > t] P(S_m > t) + \int_0^t E[N^*(t) \mid S_m = u] f_{S_m}(u) du$$

$$= E[N^*(t) \mid S_m > t] P(S_m > t) + E[N^*(t) \mid S_m \le t] \cdot P(S_m \le t) = E[N^*(t)] = \Lambda(t),$$

where $\{N^*(t), t \ge 0\}$ is the NHPP with intensity $\lambda(t)$.

Remark 2. The change point *m*, after which the NHPP becomes GPP can be random. Denote the corresponding random variable by M, and its probability mass function by $p(m)$ ($M = 0$ corresponds to the pure GPP and $M = \infty$ corresponds to the pure NHPP). Then, $E[N(t)]$ is given by

$$E[N(t)] = \sum_{m=0}^{\infty} p(m)$$

$$\times \left(\sum_{n=1}^{m-1} n \frac{\Lambda(t)^n}{n!} \exp\{-\Lambda(t)\} + \int_0^t \left(m + \frac{1}{\alpha}(\exp\{\alpha\Lambda(t,u)\} - 1) \right) \lambda(u) \frac{(\Lambda(u))^{m-1}}{(m-1)!} \exp\{-\Lambda(u)\} du \right).$$

See also our discussion in Section 5.

2.3. GPP starts after the fixed time (Model 2)

For $t \le c$, the repair is minimal and, for $t > c$, the repair is GPP repair. This combined repair process can be formally defined as follows:

Definition 3. Combined Repair Process of Type II

(i) $N(0) = 0$;

(ii) $\lambda_t = \lambda(t)$, $t \le c$;

(iii) $\lambda_t = ((N(t-) - N(c))\alpha + 1)\lambda(t)$, $t > c$ (6)

Derivation of $P(N(t) = n)$.

For $t \le c$, as in Model 1, $P(N(t)=n)=(\Lambda(t)^n \exp\{-\Lambda(t)\})/n!$.

When $t>c$, it follows from Definition 3, that the process $\{M_c(s), s \ge 0\}$, where $M_c(s)=N(s+c)-N(c)$, is the GPP with the parameter set $(\lambda(c+s),\alpha)$ and is independent of $N(c)$. Therefore, $(N(t)-N(c)\,|\,N(c)=j)$, $t \ge c$, is stochastically equivalent to $M_c(t-c)$, $t \ge c$, regardless of j. Thus,

$$P(N(t)=n)=\sum_{j=0}^{n} P(N(t)=n\,|\,N(c)=j)P(N(c)=j)$$

$$=\sum_{j=0}^{n} P(N(t)-N(c)=n-j\,|\,N(c)=j)P(N(c)=j)$$

$$=\sum_{j=0}^{n}\frac{1}{j!}\exp\{-\Lambda(c)\}\left(\Lambda(c)\right)^j \frac{\Gamma(1/\alpha+n-j)}{\Gamma(1/\alpha)(n-j)!}\left(\exp\{-\alpha\Lambda(t,c)\}\right)^{1/\alpha}\left(1-\exp\{-\alpha\Lambda(t,c)\}\right)^{n-j}, \quad (7)$$

where $\Lambda(t,c) \equiv \int_0^{t-c} \lambda(c+s)ds = \int_c^t \lambda(s)ds$.

■

The mean number of failures/repairs can be defined as

for $t \le c$, $E[N(t)]=\sum_{n=0}^{\infty} n\frac{\Lambda(t)^n}{n!}\exp\{-\Lambda(t)\}=\Lambda(t)$; (8)

for $t>c$,

$$E[N(t)]=E[N(c)+(N(t)-N(c)]=E[N(c)]+E[N_c(t-c)]$$
$$=\Lambda(c)+\frac{1}{\alpha}(\exp\{\alpha\Lambda(t,c)\}-1). \quad (9)$$

3. Optimal replacement policies

3.1. Model 1

Recall that in this model, the GPP 'starts' after the *m*-th event in the combined process.

(i) Age replacement policy

A system is replaced whenever its age reaches *T*. In-between, the repairs, in accordance with Model 1, are performed. After the replacement, the new cycle starts, etc.

The corresponding long-run expected cost rate (or, equivalently, the expected cost rate on one cycle) for this periodic setting is:

$$c(T)=\frac{c_m E[N(T)]+c_r}{T}, \quad (10)$$

where c_m is the cost of a repair, c_r is the cost for replacement, $c_m < c_r$ and $E[N(T)]$ is given by (4).

Assume that the baseline failure rate $\lambda(t)$ (the failure rate of a system before the first repair) and let, for definiteness, $\lim_{t\to\infty}\lambda(t)=\infty$ be increasing, thus manifesting the system's degradation. This assumption is also usually made in standard optimal replacement problems in the literature (Nakagawa, 2006).

Remark 3. The less 'practical' case of the constant failure rate or even of the decreasing failure rate can be also considered now, as the additional degradation in the process can result from the worse than minimal repairs.

Then it can be easily shown from the properties of the combined process that

$$\lim_{T\to\infty} \frac{E[N(T)]}{T} = \infty . \tag{11}$$

Indeed, under this assumption (i.e., the baseline failure rate $\lambda(t)$ is increasing with $\lim_{t\to\infty} \lambda(t) = \infty$), (11) obviously follows for the NHPP with rate $\lambda(t)$, whereas $E[N(T)]$ for the combined process is larger than that for the NHPP. As

$$\lim_{T\to 0} c(T) = \infty; \lim_{T\to\infty} c(T) = \infty$$

and $c(T)$ is decreasing in the vicinity of $T = 0$, the optimal solution to the problem

$$c(T^*) = \min_{0<T<\infty} c(T)$$

exists. More specifically, differentiating (10) and equating to 0:

$$E'[N(T)]T - E[N(T)] = \frac{c_r}{c_m}, \tag{12}$$

whereas it follows from (4) that (12) can be explicitly written as

$$T\sum_{n=1}^{m-1} n\lambda(T)\frac{\Lambda(T)^{n-1}}{(n-1)!}\exp\{-\Lambda(T)\} - T\sum_{n=1}^{m-1} n\lambda(T)\frac{\Lambda(T)^{n}}{n!}\exp\{-\Lambda(T)\}$$

$$+Tm\lambda(T)\frac{(\Lambda(T))^{m-1}}{(m-1)!}\exp\{-\Lambda(T)\} + T\lambda(T)\int_0^T \exp\{\alpha\Lambda(T,u)\}\lambda(u)\frac{(\Lambda(u))^{m-1}}{(m-1)!}\exp\{-\Lambda(u)\}du$$

$$-\sum_{n=1}^{m-1} n\frac{\Lambda(T)^{n}}{n!}\exp\{-\Lambda(T)\} - \int_0^T \left(m + \frac{1}{\alpha}(\exp\{\alpha\Lambda(T,u)\}-1)\right)\lambda(u)\frac{(\Lambda(u))^{m-1}}{(m-1)!}\exp\{-\Lambda(u)\}du = \frac{c_r}{c_m}$$

This expression is too cumbersome, and it is impossible to analyze the corresponding function (and therefore, the optimal problem) analytically. However, this analysis can be performed from some general consideration. For this, note that

$$\int_0^T tE''[N(t)]dt = E'[N(T)]T - E[N(T)],$$

where $E'[N(t)]$ and $E''[N(t)]$ denote the first and the second derivatives of the function $E[N(t)]$ with respect to *t*, respectively. Thus, we see that the function $E'[N(T)]T - E[N(T)]$ is increasing (from 0). It can be shown, at least, for the considered case $\lim_{t\to\infty} \lambda(t) = \infty$ (then $E''[N^*(t)] > a > 0$ for the NHPP $N^*(t)$ with intensity $\lambda(t)$ and $E''[N^*(t)] = \lambda'(t)$, whereas for the combined process, obviously, $E''[N(t)] \geq \lambda'(t)$) that the left-hand side in (12) is also increasing to ∞ as $T \to \infty$. We will also illustrate this property by numerical examples in the next section. Moreover, it follows from Cha and Finkelstein[27] that $E''[N(t)] > 0$ even for the case when $\lambda(t)=\lambda$. Therefore, depending on parameters in (12), the finite optimal T^* can exist even for this case (see also Remark 3). This dramatically differs from the case of HPP, when no replacements should be obviously made.

(ii)Replacement at the k-th failure

The system is replaced at every kth failure.

The long-run expected cost rate:

$$c(k)=\frac{c_m(k-1)+c_r}{E[S_k]}, \tag{13}$$

where S_k is the kth failure time.

$E[S_k]$ is derived as follows:

$$P(S_k>t)=P(N(t)<k)=\sum_{n=0}^{k-1} P(N(t)=n).$$

Thus, from (1), for $k=1,2,...,m+1$,

$$P(S_k>t)=\sum_{n=0}^{k-1} P(N(t)=n)=\sum_{n=0}^{k-1} \frac{\Lambda(t)^n}{n!}\exp\{-\Lambda(t)\}. \tag{14}$$

For $k=m+2,m+3,...$,

$$P(S_k>t)=\sum_{n=0}^{k-1} P(N(t)=n)=\sum_{n=0}^{m} \frac{\Lambda(t)^n}{n!}\exp\{-\Lambda(t)\}$$
$$+\sum_{n=m+1}^{k-1}\int_0^t \frac{\Gamma(1/\alpha+n-m)}{\Gamma(1/\alpha)(n-m)!}\left(\exp\{-\alpha\Lambda(t,u)\}\right)^{1/\alpha}\left(1-\exp\{-\alpha\Lambda(t,u)\}\right)^{n-m}\lambda(u)\frac{(\Lambda(u))^{m-1}}{(m-1)!}\exp\{-\Lambda(u)\}du.$$

Then,

$$E[S_k]=\int_0^\infty P(S_k>t)dt. \tag{15}$$

Similar to the previous case, for increasing baseline $\lambda(t)$, when $\lim_{t\to\infty}\lambda(t)=\infty$, the finite optimal solution of the optimization problem

$$c(k^*)=\min_{1\le k<\infty} c(k)$$

should exist. Due to the fact that equation (15) is cumbersome, the analytical analysis of (13) is practically impossible. However, as previously, some general considerations hold. Let us rewrite (13) as

$$c(k)=\frac{c_m(k-1)+c_r}{E[S_k]}=\frac{c_m k}{E[S_k]}+\frac{(c_r-c_m)}{E[S_k]},\ k=1,2,....$$

Consider first, the case $\lambda(t)=\lambda$. We will show that, distinct from the HPP (when there is no need for replacements), in this case, we can have an optimal solution, as ageing is manifested by the GPP with parameters (λ,α).

For $k\le m$: $c(k)=c_m\lambda+\dfrac{(c_r-c_m)\lambda}{k}$ and, obviously, no replacement should be planned. For $k>m$, when the GPP starts, $c_m k/E[S_k]$ is increasing in k. This is because for the HPP this quotient is a constant, whereas $E[S_k]$ is increasing slower than *k* for the GPP. The latter follows from the definition of the GPP, as its stochastic intensity is piecewise increasing (starting with λ), therefore, majorizing the HPP case. On the other hand, $(c_r - c_m)/E[S_k]$ is decreasing in *k* to 0. Thus, similar to the previous case, depending on parameters, it can result in an optimal k^* that minimizes $c(k)$. For the increasing $\lambda(t)$, when $\lim_{t\to\infty}\lambda(t)=\infty$, the function $c_m k / E[S_k]$ is increasing to ∞ as $k \to \infty$ and the *finite* optimal k^* exists. We shall also illustrate this reasoning by numerical examples in the next section.

(iii)Replacement at the kth Failure or at T, whichever occurs first

The long-run expected cost rate in this case is given by:

$$c(k,T) = \frac{c_m E[N_{repair}] + c_r}{E[\min\{S_k, T\}]}, \tag{16}$$

where N_{repair} is the number of minimal repairs in a cycle.

$E[\min\{S_k,T\}]$ is obtained as

$$E[\min\{S_k,T\}] = \int_0^T P(\min\{S_k,T\} > t)\,dt = \int_0^T P(S_k > t)\,dt = \int_0^T P(N(t) \le k-1)\,dt .$$

Thus, for $k = 1,2,...,m+1$,

$$E[\min\{S_k,T\}] = \int_0^T \sum_{n=0}^{k-1} \frac{\Lambda(t)^n}{n!} \exp\{-\Lambda(t)\}\,dt .$$

For $k = m+2, m+3,...$

$$E[\min\{S_k,T\}] = \int_0^T \sum_{n=0}^{m} \frac{\Lambda(t)^n}{n!} \exp\{-\Lambda(t)\}\,dt$$

$$+\int_0^T \sum_{n=m+1}^{k-1} \int_0^t \frac{\Gamma(1/\alpha+n-m)}{\Gamma(1/\alpha)(n-m)!} \left(\exp\{-\alpha\Lambda(t,u)\}\right)^{1/\alpha} \left(1-\exp\{-\alpha\Lambda(t,u)\}\right)^{n-m} \lambda(u) \frac{(\Lambda(u))^{m-1}}{(m-1)!} \exp\{-\Lambda(u)\} du dt$$

The proof of the following relationships is deferred to the Appendix II.

For $k = 1,2,...,m+1$,

$$E(N_{repair}) = (k-1) - \sum_{n=0}^{k-1} (k-1-n) \cdot \frac{\Lambda(T)^n}{n!} \exp\{-\Lambda(T)\}$$

For $k = m+2, m+3,...$,

$$E(N_{repair}) = (k-1) - \sum_{n=0}^{m} (k-1-n) \cdot \frac{\Lambda(T)^n}{n!} \exp\{-\Lambda(T)\} - \sum_{n=m+1}^{k-1} (k-1-n)$$

$$\times \int_0^T \frac{\Gamma(1/\alpha+n-m)}{\Gamma(1/\alpha)(n-m)!} \left(\exp\{-\alpha\Lambda(t,u)\}\right)^{1/\alpha} \left(1-\exp\{-\alpha\Lambda(t,u)\}\right)^{n-m} \lambda(u) \frac{(\Lambda(u))^{m-1}}{(m-1)!} \exp\{-\Lambda(u)\} du .$$

The corresponding bivariate optimization problem can be formulated as

$$c(k^*,T^*) = \min_{1\le k<\infty, 0<T<\infty} c(k,T) \tag{17}$$

and will be analyzed numerically in the next section.

3.2. Model 2

Recall that in this model, for $t \le c$, the repair is minimal and, for $t > c$, it is the GPP repair. There are slight alterations, as compared with Model 1 that are briefly described in what follows. The analysis of optimal procedures is similar; therefore, the numerical illustration in the next section will be provided only for Model 1.

(i)Age Replacement Policy

The system is replaced whenever its age reaches *T.* The long-run expected cost rate is given by (10), where $E[N(T)]$ is obtained in (8) for $T \le c$ and in (9) for $T > c$.

(ii)Replacement at the kth Failure

The system is replaced at every kth failure. The long-run expected cost rate is given by (13), where $E[S_k]$ is derived as.

$$P(S_k > t) = P(N(t) < k) = \sum_{n=0}^{k-1} P(N(t) = n),$$

where $P(N(t) = n)$ is given by (6) for $t \le c$ and (7) for $t > c$. Then,

$$E[S_k] = \int_0^\infty P(S_k > t)dt .$$

(iii)Replacement at the kth Failure or at T, whichever occurs first.

The long-run expected cost rate in this case is given by (16), where $E[\min\{S_k, T\}]$ is obtained now as

$$E[\min\{S_k,T\}] = \int_0^T P(\min\{S_k,T\} > t)\,dt = \int_0^T P(S_k > t)\,dt = \int_0^T P(N(t) \le k-1)\,dt$$

$$= \int_0^T \sum_{n=0}^{k-1} P(N(t) = n)dt ,$$

and $P(N(t) = n)$ is given by (6) for $t \le c$ and (7) for $t > c$.

Similar to Model 1,

$$E(N_{repair}) = (k-1) - \sum_{n=0}^{k-1} (k-1-n) \cdot P(N(T) = n),$$

where $P(N(T) = n)$ is given by (6) for $T \le c$ and (7) for $T > c$.

4. Illustrative examples and discussion

4.1. Age replacement

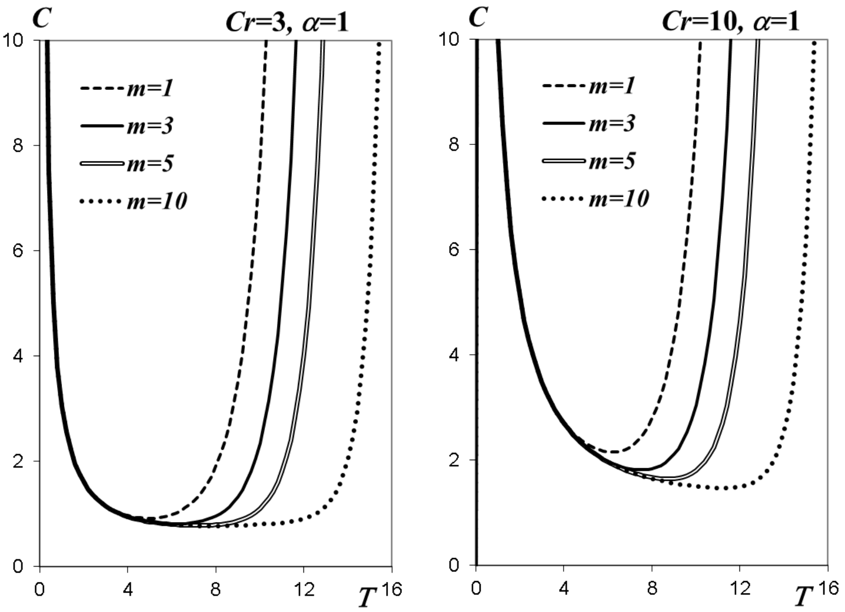
Consider the NHPP with the intensity modeled by the power function, which corresponds to the Weibull baseline distribution of a system (before the first repair). Let, specifically,

$\lambda(t) = \lambda t, \lambda = 0.1; c_m = 1.$ The corresponding cost rate (10) for different values of *m* is presented in Fig.1 by using (4), where

$$\Lambda(t,c) \equiv \int_0^{t-c} \lambda(c+s)ds = \int_c^t \lambda(s)ds = \frac{\lambda}{2}(t^2 - c^2); \Lambda(t,0) \equiv \Lambda(t) = \frac{\lambda}{2}t^2$$

Thus we can see that the optimal problem $c(T^*) = \min_{0<T<\infty} c(T)$ has a clear solution and the optimal T^* is increasing as the number of minimal repairs *m* increases*,* as the larger *m* means that for a longer time the failure rate is not increasing with repairs. This is similar to the classical maintenance strategies when the larger failure rate (with all other parameters the same) results in the smaller optimal time of replacement (Finkelstein *et al.*, 2016). Note that, *m* is not optimized here as it is given as the inherent property of the repair process.

Comparing curves for $c_r = 3$, $c_r = 10$, it can be observed that the optimal replacement times are larger for the second case (see the more detailed explanation of a similar effect in Section 4.2). Furthermore, these values decrease with increase of the GGP parameter α, as can be clearly seen by comparing curves for α =1, α =2. The same result should hold when increasing the shape parameter of Weibull distribution. This effect also follows from general considerations (see, e.g., Finkelstein et al (2016)), i.e., increasing these parameters increases deterioration (aging) in the model that leads to the more frequent PM actions (smaller values of optimal *T* or optimal *k*) as this increases deterioration, that has to be dealt with more frequent PM actions.



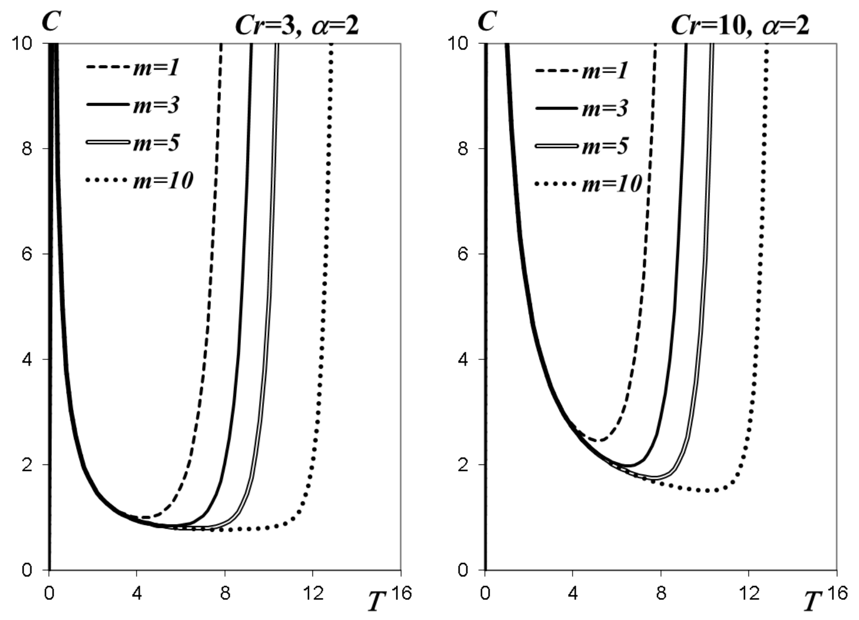


Fig.1. The cost rates for α =1, α =2, λ=0.1, Cm=1, $c_r = 3$, $c_r = 10$.

In Section 3, we have discussed why the function $E'[N(T)]T - E[N(T)]$ is increasing to ∞ as $T \to \infty$. Fig. 2 illustrates it for the specific case under consideration.

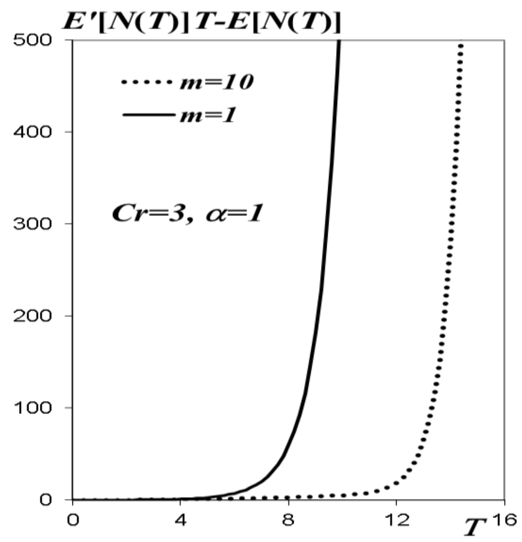


Fig.2. The function $E'[N(T)]T - E[N(T)]$ for α =1, λ=0.1.

4.2. Replacement at the k-th failure

For the same values of parameters as in the previous case and using relationship (15), the cost rate (13) is given in Fig. 3. This is done for different values of the cost of replacement, which affects the result (see our forthcoming reasoning).

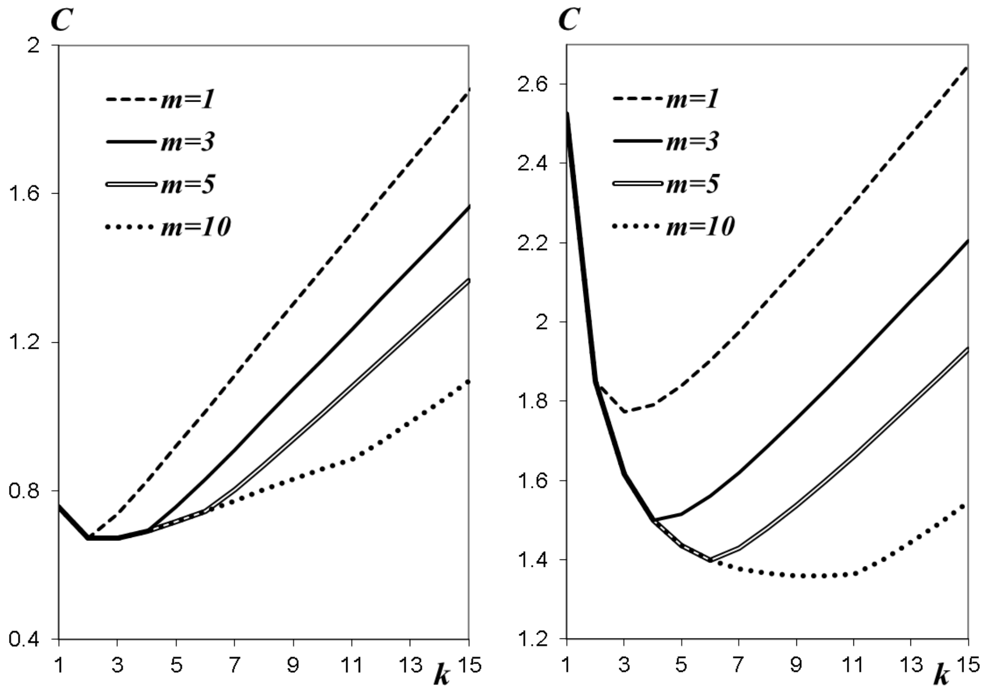


Fig.3. The cost rate for $\alpha=1$, λ=0.1, $c_m=1$, $c_r=3$ (left) and $c_r=10$ (right)

We can see that the optimal problem $c(k^*)=\min_{1\le k<\infty} c(k)$ has a clear solution and the optimal k^* is increasing with increase in the number of minimal repairs *m,* as the larger *m* means that for a longer time the failure rate is not increasing with repairs, which is similar to the previous case. It can be also observed that for the smaller values of the cost of repair ($c_r=3$) irrespectively of *m*, the minimal cost rate is achieved at $k^*=2$ (the system should be replaced more often, i.e., at every second failure). However, when this cost is relatively large $(c_r=10)$, more repairs are allowed before replacement that is delayed due to the larger cost. The minimal cost rate in this case is decreasing when *m* is increasing, which means that the larger *m* results in a smaller failure rate, and therefore, in a smaller cost rate (Finkelstein *et al.*, 2016). Specifically, for $m=10$, the replacement should be performed after the 11-th failure.

In Section 3, we have discussed why the function $\dfrac{k}{E[S_k]}$ is increasing to ∞ as $k\to\infty$. Fig. 4 illustrates it for the specific case under consideration.

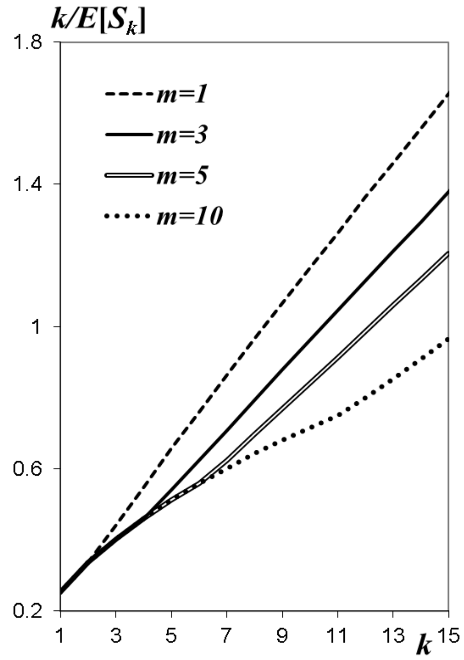


Fig.4.The function $\dfrac{k}{E[S_k]}$ for α =1, λ=0.1.

4.3. Replacement at the kth failure or at *T*, whichever occurs first

For the same values of parameters as in the previous case (for $c_r = 3$) and using relationship (15), the cost rate (13) is given in Fig. 5. We see that for all considered values of *m*, the case $k = 2$ results in minimal values of the cost rate for all values of *T* and achieves

$$c(2,T^*) = \min_{0<T<\infty} c(2,T)$$

for all $T > 10$. This is in agreement with Fig.3 (left). Thus, the bivariate optimization, actually reduces to the univariate with respect to *k.* The possible intuitive explanation of this fact is that for the considered problems, there is more information in the event that we observe the *i*th failure/repair than in the fixed observation time of some event (without its number). Another interesting observation is that if *k* is fixed and is relatively large, then the corresponding minimum with respect to *T* exists. For instance, when $m = 1$, this minimum can be observed for $k \geq 5$, meaning that information on time starts to be more valuable as the corresponding likelihoods for the large values of *k* decrease.

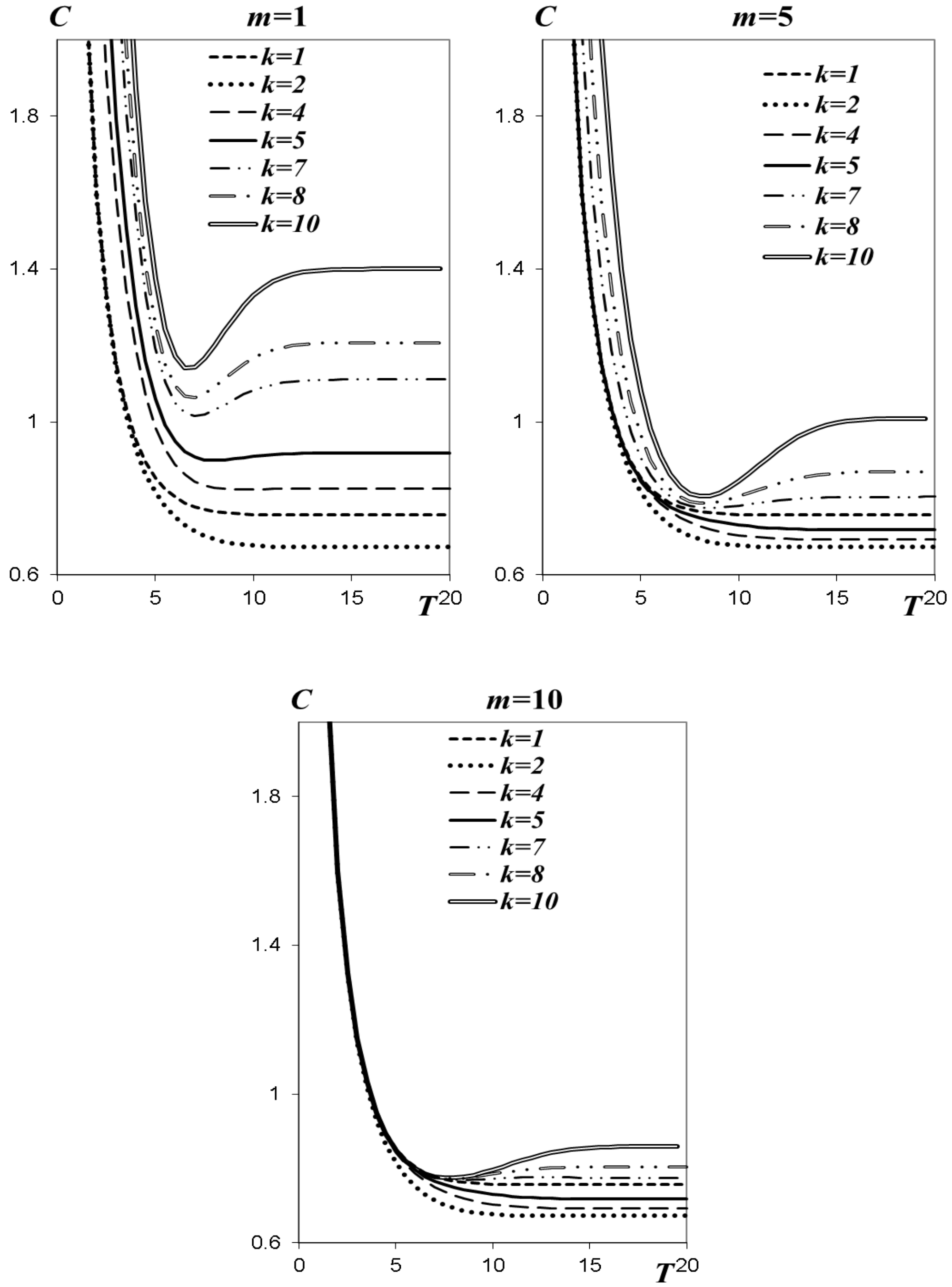


Fig.5. The cost rate for α =1, λ=0.1, $c_m = 1$, $c_r = 3$ for different m

When the cost of replacement is larger ($c_r = 10$), the situation can be different (Fig. 6). For example, when $m = 10$, it can be seen that the cost rate is no longer minimal for the case k=2 (it practically does not change for $k > 5$, which is in agreement with Fig.3 (right), where $k = 11$ results in the minimum cost rate for this univariate optimization). This is the effect of the 'late switch' of the GPP that still had to be explained theoretically. On the other hand, for smaller values of *m,* the case $k = 2$ is still optimal. Thus, the 'interaction' between variables c_r and m is complicated and needs further study.

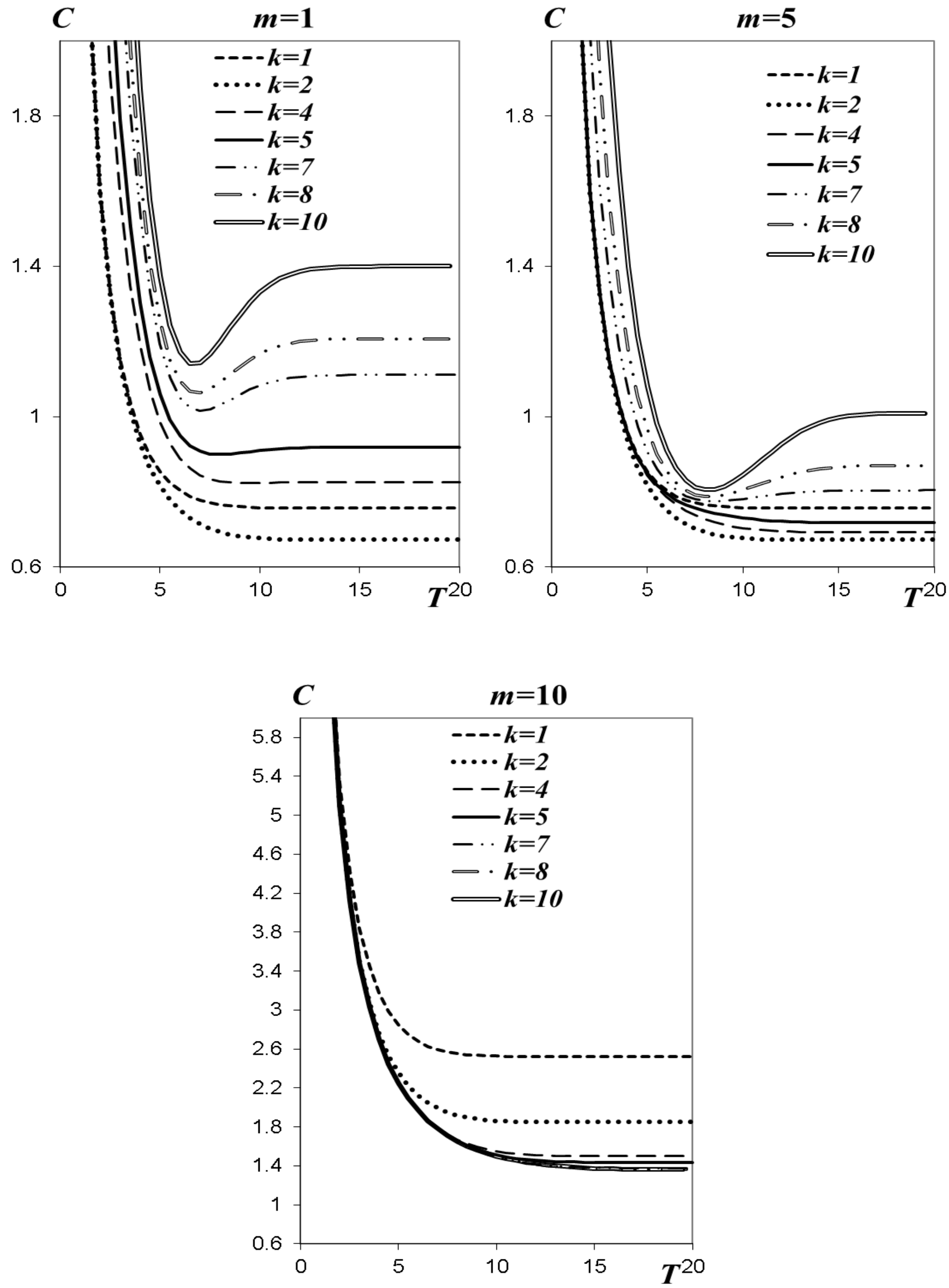


Fig.6. The cost rate for $\alpha=1$, λ=0.1, $c_m=1$, $c_r=10$ for different *m*

It follows from the considered examples that the *k*-strategy of PM outperforms the T-strategy (lower cost rates), whereas the bivariate strategy (by definition) is not worse than the best univariate strategy.

5. Concluding remarks

Perfect repair brings an item to ‘as good as new’ state, whereas minimal repair does this to the ‘bad as old’ state. However, at many practical instances, the repair can be even worse than minimal. Most often, it is an aftermath of the preceding failure(s) or the repair itself. Indeed, the failure of a component in a system can results in adverse effects on other components, e.g., through the increased stress, temperature, humidity, etc. This setting, mathematically described by the GPP, was discussed only in a few publications.

In this paper, we go further and define and characterize the combined NHPP+GPP process and the corresponding repair model. Technical systems are often resilient to the first shocks, impacts of failures of some components on the other components, etc. However, as the number of these impacts increases, the system's 'strength' in this respect decreases due to deterioration. This means that in the initial stage of operation, a system is *minimally repaired*, which is modeled by the corresponding NHPP. However, with the increase of the number of repairs, the repair starts to be *worse than minimal* that can be modeled by the corresponding GPP.

We discuss the relevant properties of the new process and derive expressions that are necessary for considering the corresponding PM problems. Specifically, the replacement can be made either after the given time or the given number of repairs. The existence of the optimal solution is discussed, and the detailed numerical examples are given for illustration of our findings. Moreover, the relevant illustrations are provided for the corresponding bivariate optimization problem when a system is replaced after the time T or on the k-th failure, whichever comes first. Specifically, it is shown that the bivariate PM strategy reduces to the univariate k-strategy, at least, for the relatively small values of the cost of replacement, c_r. The case of larger values of this parameter needs further study.

There can be different directions of further research based on the proposed model. In the foregoing, the number of minimal repairs (delay) before the onset of the GPP, m was set. However, obviously, it can be considered as a random variable. In this case, an appropriate distribution should be assumed and stochastic properties of this more complex model can be considered (e.g., relevant stochastic comparisons for different delay distributions). Similarly, in Model 2, the case of random change time C instead of a constant c could also be considered.

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References

Ansell, J. I., Archibald, T. W., & Thomas, L. C. (2004). The elixir of life: using a maintenance, repair and replacement model based on virtual and operating age in the water industry. *IMA Journal of Management Mathematics*, **15**, 151-160.

Badía, F. G., Berrade, M. D., Cha, J. H., & Lee, H. (2018). Optimal replacement policy under a general failure and repair model: Minimal versus worse than old repair. *Reliability Engineering & System Safety*, **180**, 362-372.

Aven, T. & Jensen, U. (2000) A general minimal repair model. *Journal of Applied Probability*, **37**, 187–197 .

Babykina, G. & Couallier, V. (2014) Modelling pipe failures in water distribution systems: Accounting for harmful repairs and a time-dependent covariate. *International Journal of Performability Engineering*, **10**, 31–42.

Badia, F. G. & Berrade, M. D. (2009) Optimum maintenance policy of a periodically inspected system under imperfect repair. *Advances in Operations Research*: Article ID 691203.

Barlow, R.E. & Proschan, F. (1975) *Statistical Theory of Reliability and Life Testing*. New York: Holt, Rinehart & Winston.
Brown, M. & Proschan, F. (1983) Imperfect repair. *Journal of Applied Probability*, **20**, 851-862.
Cha, J. H. (2014) Characterization of the generalized Polya process and its applications. *Advances in Applied Probability*, **46**, 1148–1171.
Cha, J. H. & Finkelstein, M. (2016) Justifying the Gompertz curve of mortality via the generalized Polya process of shocks. *Journal of Theoretical Population Biology*, **109**, 54-62.
Cha, J. H. & Finkelstein, M. (2018) *Point Processes for Reliability Analysis. Shocks and Repairable systems*. London: Springer.
deToledo, M. L. G., Freitas, M. A., Colosimo, E. A. & Gilardoni, G. L. (2015) ARA and ARI imperfect repair models: Estimation, goodness-of-fit and reliability prediction. *Reliability Engineering and System Safety*, **140**, 107-115.
Dijoux, Y., Fouladirad, M. & Nguyen, D. T. (2016) Statistical inference for imperfect maintenance models with missing data. *Reliability Engineering and System Safety*, **154**, 84-96.
Doyen, L. & Gaudoin, O. (2004) Classes of imperfect repair models based on reduction of failure intensity or virtual age. *Reliability Engineering and System Safety*, **84**, 45–56.
El-Damcese, M. A. (1997) Suggested procedure for preventive maintenance policy. Microelectronics Reliability, **37**, 1173–1177
Finkelstein, M. (2008) *Failure Rate Modelling for Reliability and Risk*. London: Springer.
Finkelstein, M. & Cha, J. H. (2013) *Stochastic Modelling for Reliability (Shocks, Burn-in and Heterogeneous Populations)*. London: Springer.
Finkelstein, M., Shafiee, M. & Kotchap, A.N. (2016) Classical optimal replacement strategies revisited. *IEEE Transaction on Reliability*, **65**, 540-546.
Jack, N. (1998). Age-reduction models for imperfect maintenance. *IMA Journal of Management Mathematics*, **9**, 347-354.
Jeong, J. S. (2012) Failure mechanism and reliability test method for USB inter- face circuitry on CPUs for mobile devices. *Microelectronics Reliability,* **52**, 2014–2018.
Kahle, W. (2019) Imperfect repair in degradation processes: a Kijima-type approach. *Applied Stochastic Modelling in Business and Industry*, **35**, 211-220.
Kijima, M. (1989) Some results for repairable systems with general repair. *Journal of Applied Probability*, **26**, 89-102.
Le Gat, Y. (2014) Extending the Yule process to model recurrent pipe failures in water supply networks. *Urban Water Journal*, **11**, 617–630.
Lee, H. & Cha, J. H. (2016) New stochastic models for preventive maintenance and maintenance optimization. *European Journal of Operational Research*, **255**, 80-90.
Levitin, G. & Lisniansky, A. (2000) Optimization of imperfect preventive maintenance for multi-state systems. *Reliability Engineering and System Safety*, **67**, 193-203.
Liu, X. G., Makis, V., & Jardine, A. K. (1995). A model of age-based replacement with regular services. *IMA Journal of Management Mathematics*, **6**, 145-154.
Marais, K. B. & Saleh, J. H. (2009) Beyond its cost, the value of maintenance: An analytical framework for capturing its net present value. *Reliability Engineering and System Safety*, **94**, 644–657.
Nakagawa, T. (2006) *Maintenance Theory of Reliability*. London: Springer.
Nguyen, D.T., Dijoux, Y. & Fouladirad, M. (2017) Analytical properties of an imperfect repair model and application in preventive maintenance scheduling. *European Journal of Operational Research*, **256**, 439-453.
Rausand, M. & Hoyland, A. (2003) *System Reliability Theory: Models, Statistical Methods, and Applications*, **396**. New York: John Wiley & Sons

Tanwar, M., Rai, R. N. & Bolia, N. (2014) Imperfect repair modeling using Kijima type generalized renewal process. *Reliability Engineering and System Safety*, **124**, 24-31.
Wang, H. & Pham, H. (2006) *Reliability and Optimal Maintenance*. London: Springer.
Zhao, X., Gaudoin, O., Doyen, L. & Xie, M. (2019) Optimal inspection and replacement policy based on experimental degradation data with covariates. *IISE Transactions*, **5**, 322-336.

Appendix I

Derivation of $E[N(t)]$

$$E[N(t)] = \sum_{n=0}^{m} n \frac{\Lambda(t)^n}{n!} \exp\{-\Lambda(t)\}$$

$$+\sum_{n=m+1}^{\infty} n \int_0^t \frac{\Gamma(1/\alpha+n-m)}{\Gamma(1/\alpha)(n-m)!} \left(\exp\{-\alpha\Lambda(t,u)\}\right)^{1/\alpha} \left(1-\exp\{-\alpha\Lambda(t,u)\}\right)^{n-m} \lambda(u) \frac{(\Lambda(u))^{m-1}}{(m-1)!} \exp\{-\Lambda(u)\} du$$

$$= \sum_{n=0}^{m-1} n \frac{\Lambda(t)^n}{n!} \exp\{-\Lambda(t)\}$$

$$+\sum_{n=m}^{\infty} n \int_0^t \frac{\Gamma(1/\alpha+n-m)}{\Gamma(1/\alpha)(n-m)!} \left(\exp\{-\alpha\Lambda(t,u)\}\right)^{1/\alpha} \left(1-\exp\{-\alpha\Lambda(t,u)\}\right)^{n-m} \lambda(u) \frac{(\Lambda(u))^{m-1}}{(m-1)!} \exp\{-\Lambda(u)\} du$$

$$= \sum_{n=0}^{m-1} n \frac{\Lambda(t)^n}{n!} \exp\{-\Lambda(t)\}$$

$$+\int_0^t \left(\sum_{k=0}^{\infty} (m+k) \frac{\Gamma(1/\alpha+k)}{\Gamma(1/\alpha)k!} \left(\exp\{-\alpha\Lambda(t,u)\}\right)^{1/\alpha} \left(1-\exp\{-\alpha\Lambda(t,u)\}\right)^{k} \right) \lambda(u) \frac{(\Lambda(u))^{m-1}}{(m-1)!} \exp\{-\Lambda(u)\} du$$

$$= \sum_{n=1}^{m-1} n \frac{\Lambda(t)^n}{n!} \exp\{-\Lambda(t)\} + \int_0^t \left(m + \frac{1}{\alpha}(\exp\{\alpha\Lambda(t,u)\}-1) \right) \lambda(u) \frac{(\Lambda(u))^{m-1}}{(m-1)!} \exp\{-\Lambda(u)\} du \, .$$

Appendix II

Derivation of $E[N_{repair}]$

Observe that

$$E[N_{repair}] = E(N_{repair} \mid S_k \le T) P(S_k \le T) + E(N_{repair} \mid S_k > T) P(S_k > T) \, .$$

Obviously, $E(N_{repair} \mid S_k \le T) = k-1$. On the other hand,

$$E(N_{repair} \mid S_k > T) = E(N(T) \mid N(T) \le k-1)$$

$$= \sum_{n=0}^{k-1} n \, P(N(T)=n \mid N(T) \le k-1) = \sum_{n=0}^{k-1} n \frac{P(N(T)=n)}{P(N(T) \le k-1)} \, .$$

Therefore,

$$E(N_{repair}) = (k-1) \cdot P(S_k \le T) + \sum_{n=0}^{k-1} n \frac{P(N(T)=n)}{P(N(T) \le k-1)} P(S_k > T)$$

$$= (k-1)\cdot(1-P(N(T)\le k-1)) + \sum_{n=0}^{n-1} n \frac{P(N(T)=n)}{P(N(T)\le k-1)} P(N(T)\le k-1)$$

$$= (k-1) - \sum_{n=0}^{k-1} (k-1)\cdot P(N(T)=n) + \sum_{n=0}^{k-1} nP(N(T)=n)$$

$$= (k-1) - \sum_{n=0}^{k-1} (k-1-n)\cdot P(N(T)=n)\ .$$