On properties of the phase-type mixed Poisson process and its applications to reliability shock modeling

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Abstract

Although Poisson processes are widely used in various applications for modeling of recurrent point events, there exist obvious limitations. Several specific mixed Poisson processes (which are formally not Poisson processes any more) that were recently introduced in the literature overcome some of these limitations. In this paper, we define a general mixed Poisson process with the phase-type (PH) distribution as the mixing one. As the PH distribution is dense in the set of lifetime distributions, the new process can be used to approximate any mixed Poisson process. We study some basic stochastic properties of the new process and discuss relevant applications by considering the extreme shock model, the stochastic failure rate model and the δ-shock model.

Keywords: Mixed Poisson process; non-homogeneous Poisson process; phase-type distribution; shock models.

1 Introduction

The non-homogeneous Poisson process (NHPP) and its specific version, the homogeneous Poisson process (HPP), are widely used in various applications for modeling of recurrent point

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events. These processes are mathematically tractable and hence, the corresponding explicit results, e.g., in the area of reliability can be effectively obtained at many instances (see Ross [35], Grandell [15], Beichelt [4], Cha and Finkelstein [7, 8], Lam and Zhang [22], to name a few). However, they have important limitations in modeling of the real world phenomena. For example, these processes are characterized by the independent increments, which is often not realistic in practice. Moreover, the mean and the variance for the number of events in $(0, t]$ are equal, which is also restrictive.

Some generalizations for overcoming these limitations were suggested in the literature in the form of various point processes, namely, the compound Poisson process, the filtered Poisson process, two dimensional Poisson process, marked Poisson process, etc. (see Kao [20], and Milne [27]). Apart from these, there are other important generalizations of point processes, such as semi-Markov processes that generalize both the Markov processes as well as the renewal processes (see Limnios and Oprişan [25], and Barbu and Limnios [3]). Neuts [30] discussed a versatile class of point processes which is closely related to finite Markov processes. Several important point processes (namely, renewal processes of phase-type, Markov-modulated Poisson processes, and specific semi-Markov point processes), appear as the particular cases of this class. The switched Poisson processes and the rational arrival processes are two other important class of point processes. The first ones deal with the alternating (stepwise) constant rate switched at random times governed by the alternating renewal process (see, e.g., Bhat [6]) whereas the second ones are used to model in a specific way (using matrix exponential distributions) the dependence between the arrival times in queuing models (see, e.g., Asmussen and Bladt [2]).

From the brief literature review, it follows that the above mentioned models are based on the Markovian (or the semi-Markovian) assumptions, which are restrictive in some applications. For example, the shock models, considered in this paper, are described by the point processes having dependent increments property and the history that takes into account the number of shocks that were previously survived by a system. The mixed Poisson processes are often used in such scenarios as they can posses the desired properties. Note that, although the conditional intensities of the mixed Poisson processes are fixed, their intensity processes (distinct from the HPP and the NHPP) are random. Moreover, the structures of these intensity processes depend, as already mentioned (amid other parameters), on the number of events in the preceding intervals of time. This type of dependency on the past (which is very important for reliability applications), as far as we know, was not considered in the literature apart from the mixed Poisson processes. Recently, Konno [21] have introduced the generalized Pólya process (GPP), characterized by Cha [9] that contains both the HPP and the NHPP as the particular cases. Later, Cha [10] have defined the Poisson-Lindley process (PLP), which is the mixed NHPP with the Lindley mixing distribution. For the PLP, the variance of the number of events in $(0, t]$ is strictly greater than the corresponding mean. In addition, this process has positively dependent increments (i.e., the larger number of events in the past yields the same in the future). However, the Lindley distribution with one parameter is very specific and, therefore, non-flexible enough.

By considering the generalized gamma distribution as a mixing one, Cha and Mercier [11] have recently introduced a new point process, which they have called the Poisson-generalized gamma process (PGGP). This process contains the HPP, the NHPP, the GPP and the PLP as the particular cases. Although the generalized gamma distribution is rather flexible, the corresponding mixed Poisson process is still based on the specific mixing distribution and, therefore, can be not suitable in some applications. This motivates us to consider a more general set up based on a mixing distribution that can approximate any continuous lifetime distribution.

In this paper, we introduce a new counting process to be called the Poisson phase-type process (PPHP), which is the mixed NHPP with the continuous phase-type (PH) mixing distribution. It is well known that "*The set of PH distributions is dense in the set of probability distributions on the non-negative half-line*" (see He [19]). It means that every lifetime distribution can be approximated by the PH distribution and consequently, any mixed Poisson process can also be approximated by the PPHP. This unique property of the PPHP makes it special among other mixed Poisson processes considered in the literature. Furthermore, the PH distribution functions and other statistical measures (e.g., mean, variance, etc.) can be written in matrix forms that are convenient in computations when using the relevant mathematical packages (see, e.g., Eryilmaz [13]). Thus, the PH distribution is mathematically tractable and consequently, the PPHP, as it will be shown, presents a convenient and efficient general tool in various real-life applications. To show this, the corresponding methodology has to be developed in our paper.

As a meaningful, practically sound application that, as we believe, has also its own merit, we consider in detail generalizations of the popular in the literature shock models to the case when the shock arrival process is PPHP. These generalizations are not straightforward and require proper stochastic analysis. Note that, shock models play an important role in describing the lifetime behavior of systems operating in a random environment. There are numerous papers on this topic (see Gut and Hüsler [17, 18], Shanthikumar and Sumita [38, 39], A-Hameed and Proschan [1], Gut [16], Mallor and Omey [26], Li et al. [23], Li and Kong [24], Eryilmaz and Tekin [14], Montoro-Cazorla et al. [28], to name a few). Most of the studies consider shocks arriving in accordance with the the HPP or the NHPP.

Our approach results in a more adequate and precise stochastic description for the real world settings providing, for example, more trustworthy reliability characteristics. Moreover, it can be applied to any system subject to shocks (external or internal). It should be noted that shocks can be understood generally as some jumps in load or stress as, e.g., voltage surges in power generation systems, wind gusts for wind turbines, earthquakes for various structures (for example, bridges), failures of cooling systems that result in a sharp rise of temperature of the main system etc.

The rest of the paper is organized as follows. In Section 2, we provide notations, definitions and also some new results with respect to the PH distributions that will be used in the main part of the paper. In Section 3, we define the PPHP and derive some of its important properties. In

Section 4, we present some applications of the PPHP describing different shock models, namely, the extreme shock model, the stochastic failure rate model and the δ-shock model. Finally, the concluding remarks are given in Section 5.

2 Preliminaries: definitions and new properties of PH distributions

We start with notations and definitions for the PH distributions followed by some new results on stochastic comparisons and aging properties of these distributions to be used in the main part of the paper.

For any random variable U, we denote the cumulative distribution function by $F_U(\cdot)$, the survival function by $\bar{F}_U(\cdot)$, the probability density function by $f_U(\cdot)$, the failure rate function by $r_U(\cdot)$ and the mean residual life function by $m_U(\cdot)$; here $\bar{F}_U(\cdot) = 1 - F_U(\cdot)$, $r_U(\cdot) = f_U(\cdot)/F_U(\cdot)$ and $m_U(t) = E(U - t|U > t)$. We write a matrix A as $A = [A_{ij}]$, where A_{ij} represents the ij-th element of A. For any two matrices $A = [A_{ij}]$ and $B = [B_{ij}]$, $A \otimes B$ is defined as $[A_{ij}B]$, where "$\otimes$" stands for the Kronecker product.

Neuts [29] have defined the set of PH-distributions as a generalization of the exponential distribution. These distributions have a large number of applications in various stochastic modellings (see Neuts [31], Neuts et al. [32], Pérez-Ocón and Montoro-Cazorla [33], to name a few). Below we give the formal definitions (see He ([19], pp. 10, 77), and Neuts ([31], p. 46)).

Definition 2.1 *A non-negative random variable X is said to have a phase-type (PH) distribution if*

$$F_X(x) = 1 - \boldsymbol{\alpha} \exp\{Tx\}\boldsymbol{e} = 1 - \boldsymbol{\alpha}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} T^n\right)\boldsymbol{e}, \quad x \geq 0, \tag{2.1}$$

where

- (i) $\boldsymbol{e}$ *is the column vector with all elements being one;*
- (ii) $\boldsymbol{\alpha}$ *is a substochastic vector of order m, i.e., $\boldsymbol{\alpha}$ is a row vector, all elements of $\boldsymbol{\alpha}$ are nonnegative, and $\boldsymbol{\alpha e} \leq 1$, where m is a positive integer; and*
- (iii) T *is a subgenerator of order m, i.e., T is an $m \times m$ matrix such that: (a) all diagonal elements are negative; (b) all off-diagonal elements are nonnegative; (c) all row sums are non-positive; and (d) T is invertible.* □

We call T and the pair $(\boldsymbol{\alpha}, T)$ the PH generator and the phase-type (PH) representation of order m, respectively. We write $X \sim PH(\boldsymbol{\alpha}, T)$ to indicate that X follows the PH distribution with the PH-representation $(\boldsymbol{\alpha}, T)$. Further, the the corresponding pdf is given by

$$f_X(x) = \boldsymbol{\alpha} \exp\{Tx\}\boldsymbol{T^0}, \quad x \geq 0, \tag{2.2}$$

where $\boldsymbol{T^0} = -T\boldsymbol{e}$.

Definition 2.2 *A random variable X is said to have a discrete PH distribution if its $(m+1)$ state Markov chain P is given by*

$$P = \begin{pmatrix} Q & \boldsymbol{Q^0} \\ 0 & 1 \end{pmatrix},$$

where Q is a sub-stochastic matrix such that $I - Q$ is nonsingular, and $(\boldsymbol{\alpha}, 1 - \boldsymbol{\alpha e})$ is the initial probability vector. Further, the probability mass function of X is given by

$$P(X = n) = \begin{cases} 1 - \boldsymbol{\alpha e}, & \text{if } n = 0 \\ \boldsymbol{\alpha} Q^{n-1} \boldsymbol{Q^0}, & \text{if } n = 1, 2, \ldots, \end{cases}$$

where $\boldsymbol{Q^0} = (I - Q)\boldsymbol{e}$, and $\boldsymbol{\alpha}$ and $\boldsymbol{e}$ are the same as in Definition 2.1. □

We write $X \sim DPH(\boldsymbol{\alpha}, Q)$ to indicate that X follows a discrete PH distribution with PH representation $(\boldsymbol{\alpha}, Q)$.

Remark 2.1 *The following helpful observations can be made (He [19], p. 16).*

(i) A PH distribution with the PH representation $(\boldsymbol{\alpha}, T)$, where $\boldsymbol{\alpha} = 1$ and $T = -\theta$, is the exponential distribution with parameter θ;

(ii) A PH distribution with the PH representation $(\boldsymbol{\alpha}, T)$, where

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{1\times m} \text{ and } T = \begin{pmatrix} -\theta & 0 & \cdots & 0 & 0 \\ \theta & -\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\theta & 0 \\ 0 & 0 & \cdots & \theta & -\theta \end{pmatrix}_{m\times m},$$

is the Erlang distribution with the set of parameters $\{m, \theta\}$. □

Some useful for our further discussion properties of the PH distributions are given in the following lemma (see He [19], pp. 18, 21-22, 25).

Lemma 2.1 *Let X follow the PH distribution with the PH representation $(\boldsymbol{\alpha}, T)$. Then*

(i) $\int_0^\infty \exp\{Tt\}dt = -T^{-1}$;

(ii) The moment generating function of X is given by

$$M_X(s) = (1 - \boldsymbol{\alpha e}) - \boldsymbol{\alpha}\,(sI + T)^{-1}\,\boldsymbol{T^0}, \;\; \boldsymbol{T^0} = -T\boldsymbol{e};$$

(iii) $E(X^n) = (-1)^n n! \boldsymbol{\alpha} T^{-n} \boldsymbol{e}$, $n = 1, 2, 3, \ldots$;

(iv) *If C and D are two PH generators of X, then $C \otimes I + I \otimes D$ is also a PH-generator of X; here the size of the identity matrix I depends on the context.*

(v) *$X - t | X \geq t$ has a PH distribution with the PH representation $(\boldsymbol{\alpha} \exp\{Tt\}/(\boldsymbol{\alpha} \exp\{Tt\}\boldsymbol{e}), T)$, for $t \geq 0$.* □

We will also need the following definitions on ordering and aging notions.

Definition 2.3 *Let $\boldsymbol{u} = (u_i)$ and $\boldsymbol{v} = (v_i)$ be two real vectors of the same dimension. Then, $\boldsymbol{u}$ is said to be greater (resp. less) than $\boldsymbol{v}$ in the component order, denoted by $\boldsymbol{u} \geq_{comp}$ (resp. $\leq_{comp}$) $\boldsymbol{v}$, if $u_i \geq$ (resp. $\leq$) v_i for all i. We say that $\boldsymbol{u}$ is equal to $\boldsymbol{v}$ in the component order, denoted by $\boldsymbol{u} =_{comp} \boldsymbol{v}$, if both $\boldsymbol{u} \geq_{comp} \boldsymbol{v}$ and $\boldsymbol{u} \leq_{comp} \boldsymbol{v}$ hold.*

Definition 2.4 *Let Y_1 and Y_2 be two absolutely continuous non-negative random variables. Then Y_1 is said to be grater than Y_2 in the*

(i) *mean residual life order, denoted by $Y_1 \geq_{mrl} Y_2$, if $m_{Y_1}(x) \geq m_{Y_2}(x)$ for all $x > 0$;*

(ii) *hazard rate order, denoted by $Y_1 \geq_{hr} Y_2$, if $\bar{F}_{Y_1}(x)/\bar{F}_{Y_2}(x)$ is increasing in $x > 0$;*

(iii) *likelihood ratio order, denoted by $Y_1 \geq_{lr} Y_2$, if $f_{Y_1}(x)/f_{Y_2}(x)$ is increasing in $x > 0$.*

Definition 2.5 *A non-negative random variable Y is said to have the*

(i) *increasing (resp. decreasing) likelihood ratio (ILR) (resp. DLR) property if $f_X(x)$ is log-concave (resp. log-convex);*

(ii) *increasing (resp. decreasing) failure rate (IFR) (resp. DFR) property if $r_Y(x)$ is increasing (resp. decreasing) in $x \geq 0$;*

(iii) *decreasing (resp. increasing) mean residual life (DMRL) (resp. IMRL) property if $m_Y(x)$ is decreasing (resp. increasing) in $x \geq 0$.* □

In the following proposition, we discuss the likelihood ratio ordering, the hazard rate ordering and the mean residual life ordering for two PH-distributed random variables with different PH representations.

Proposition 2.1 *Let X_1 and X_2 be two PH distributed random variables with the PH-representations $(\boldsymbol{\alpha}_1, T_1)$ and $(\boldsymbol{\alpha}_2, T_2)$, respectively.*

(i) *If $(T_2 \otimes T_1^2)\boldsymbol{e} \geq_{comp} (T_2^2 \otimes T_1)\boldsymbol{e}$, then $X_1 \geq_{lr} X_2$;*

(ii) *If $(I \otimes T_1)\boldsymbol{e} \geq_{comp} (T_2 \otimes I)\boldsymbol{e}$, then $X_1 \geq_{hr} X_2$;*

(iii) *If $(I \otimes T_2^{-1})\boldsymbol{e} \geq_{comp} (T_1^{-1} \otimes I)\boldsymbol{e}$, then $X_1 \geq_{mrl} X_2$.*

Proof: We have

$$\frac{f_{X_1}(x)}{f_{X_2}(x)} = \frac{\boldsymbol{\alpha}_1 \exp\{T_1 x\}\boldsymbol{T_1^0}}{\boldsymbol{\alpha}_2 \exp\{T_2 x\}\boldsymbol{T_2^0}}, \quad x > 0,$$

where $\boldsymbol{T_1^0} = -T_1\boldsymbol{e}$ and $\boldsymbol{T_2^0} = -T_2\boldsymbol{e}$. Differentiating the above, we get

$$\left(\frac{f_{X_1}(x)}{f_{X_2}(x)}\right)' = \frac{(\boldsymbol{\alpha}_2 \exp\{T_2 x\}\boldsymbol{T_2^0})(\boldsymbol{\alpha}_1 \exp\{T_1 x\}T_1\boldsymbol{T_1^0}) - (\boldsymbol{\alpha}_1 \exp\{T_1 x\}\boldsymbol{T_1^0})(\boldsymbol{\alpha}_2 \exp\{T_2 x\}T_2\boldsymbol{T_2^0})}{(\boldsymbol{\alpha}_2 \exp\{T_2 x\}\boldsymbol{T_2^0})^2},$$

which is nonnegative if and only if

$$(\boldsymbol{\alpha}_2 \exp\{T_2 x\}\boldsymbol{T_2^0})(\boldsymbol{\alpha}_1 \exp\{T_1 x\}T_1\boldsymbol{T_1^0}) - (\boldsymbol{\alpha}_1 \exp\{T_1 x\}\boldsymbol{T_1^0})(\boldsymbol{\alpha}_2 \exp\{T_2 x\}T_2\boldsymbol{T_2^0}) \geq 0.$$

Now, consider

$$\begin{aligned}
&(\boldsymbol{\alpha}_2 \exp\{T_2 x\}\boldsymbol{T_2^0})(\boldsymbol{\alpha}_1 \exp\{T_1 x\}T_1\boldsymbol{T_1^0}) - (\boldsymbol{\alpha}_1 \exp\{T_1 x\}\boldsymbol{T_1^0})(\boldsymbol{\alpha}_2 \exp\{T_2 x\}T_2\boldsymbol{T_2^0}) \\
&= (\boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_1)(\exp\{T_2 x\} \otimes \exp\{T_1 x\})(T_2 \otimes T_1^2)(\boldsymbol{e} \otimes \boldsymbol{e}) \\
&\quad - (\boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_1)(\exp\{T_2 x\} \otimes \exp\{T_1 x\})(T_2^2 \otimes T_1)(\boldsymbol{e} \otimes \boldsymbol{e}) \\
&= (\boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_1)(\exp\{(T_2 \otimes I + I \otimes T_1)x\})(T_2 \otimes T_1^2 - T_2^2 \otimes T_1)\boldsymbol{e} \\
&= \exp\{-cx\}(\boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_1)(\exp\{(cI + T_2 \otimes I + I \otimes T_1)x\})(T_2 \otimes T_1^2 - T_2^2 \otimes T_1)\boldsymbol{e},
\end{aligned}$$

where c is greater than or equal to the absolute value of any diagonal element of the matrix $T_2 \otimes I + I \otimes T_1$. Thus, $cI + T_2 \otimes I + I \otimes T_1$ and $\exp\{(cI + T_2 \otimes I + I \otimes T_1)x\}$ are nonnegative matrices. Again, from the assumption, we have that $(T_2 \otimes T_1^2) \geq_{comp} (T_2^2 \otimes T_1)\boldsymbol{e}$. On combining these, we get

$$(\boldsymbol{\alpha}_2 \exp\{T_2 x\}\boldsymbol{T_2^0})(\boldsymbol{\alpha}_1 \exp\{T_1 x\}T_1\boldsymbol{T_1^0}) - (\boldsymbol{\alpha}_1 \exp\{T_1 x\}\boldsymbol{T_1^0})(\boldsymbol{\alpha}_2 \exp\{T_2 x\}T_2\boldsymbol{T_2^0}) \geq 0$$

and hence, the result given in part (i) follows. The proof of part (ii) follows in the same line as in part (i) and hence, omitted. Now, we prove part (iii). Note that, from Lemma 2.1(iii) and (v), we have

$$m_{X_1}(x) = -\frac{\boldsymbol{\alpha}_1 \exp\{T_1 x\}T_1^{-1}\boldsymbol{e}}{(\boldsymbol{\alpha}_1 \exp\{T_1 x\}\boldsymbol{e})} \text{ and } m_{X_2}(x) = -\frac{\boldsymbol{\alpha}_2 \exp\{T_2 x\}T_2^{-1}\boldsymbol{e}}{(\boldsymbol{\alpha}_2 \exp\{T_2 x\}\boldsymbol{e})}, \quad x > 0.$$

Now, $m_{X_1}(x) \geq m_{X_2}(x)$ holds if and only if

$$\frac{\boldsymbol{\alpha}_1 \exp\{T_1 x\}T_1^{-1}\boldsymbol{e}}{(\boldsymbol{\alpha}_1 \exp\{T_1 x\}\boldsymbol{e})} \leq \frac{\boldsymbol{\alpha}_2 \exp\{T_2 x\}T_2^{-1}\boldsymbol{e}}{(\boldsymbol{\alpha}_2 \exp\{T_2 x\}\boldsymbol{e})},$$

or equivalently,

$$(\boldsymbol{\alpha}_1 \exp\{T_1 x\}T_1^{-1}\boldsymbol{e})(\boldsymbol{\alpha}_2 \exp\{T_2 x\}\boldsymbol{e}) \leq (\boldsymbol{\alpha}_1 \exp\{T_1 x\}\boldsymbol{e})(\boldsymbol{\alpha}_2 \exp\{T_2 x\}T_2^{-1}\boldsymbol{e}).$$

Further, the above inequality can equivalently be written as

$$(\boldsymbol{\alpha}_1 \otimes \boldsymbol{\alpha}_2)(\exp\{T_1 x\} \otimes \exp\{T_2 x\})(T_1^{-1} \otimes I - I \otimes T_2^{-1})\boldsymbol{e} \leq 0,$$

which holds by using the same argument, as in part (i), with the help of $(I \otimes T_2^{-1})\boldsymbol{e} \geq_{comp} (T_1^{-1} \otimes I)\boldsymbol{e}$. Thus, $m_{X_1}(x) \geq m_{X_2}(x)$ holds, for all $x > 0$, and hence the result given in part (iii) follows. □

The following example illustrates the result given in Proposition 2.1.

Example 2.1 *Let X_1 and X_2 be two PH distributed random variables with the PH representations $(\boldsymbol{\alpha}_1, T_1)$ and $(\boldsymbol{\alpha}_2, T_2)$, respectively, where*

$$\boldsymbol{\alpha}_1 = \begin{pmatrix} 0.2 & 0.8 \end{pmatrix}, \; T_1 = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \; \textit{and } \boldsymbol{\alpha}_2 = \begin{pmatrix} 0.3 & 0.7 \end{pmatrix}, \; T_2 = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}.$$

Clearly, $(T_2 \otimes T_1^2)\boldsymbol{e} \geq_{comp} (T_2^2 \otimes T_1)\boldsymbol{e}$, $(I \otimes T_1)\boldsymbol{e} \geq_{comp} (T_2 \otimes I)\boldsymbol{e}$ and $(I \otimes T_2^{-1})\boldsymbol{e} \geq_{comp} (T_1^{-1} \otimes I)\boldsymbol{e}$ hold and consequently, $X_1 \geq_{lr} X_2$, $X_1 \geq_{hr} X_2$ and $X_1 \geq_{mrl} X_2$ follows from Proposition 2.1, respectively.

In the next proposition, we discuss some stochastic aging properties of a PH distribution. We prove only part (i), whereas the rest can be done in a similar way and hence, omitted.

Proposition 2.2 *Let X be a random variable with the PH representation $(\boldsymbol{\alpha}, T)$.*

- (i) *If $(T^2 \otimes T^2)\boldsymbol{e} \geq_{comp}$ (resp. $\leq_{comp}$) $(T \otimes T^3)\boldsymbol{e}$, then X is ILR (resp. DLR);*
- (ii) *If $(T \otimes T)\boldsymbol{e} \geq_{comp}$ (resp. $\leq_{comp}$) $(I \otimes T^2)\boldsymbol{e}$, then X is IFR (resp. DFR);*
- (iii) *If $\boldsymbol{e} \geq_{comp}$ (resp. $\leq_{comp}$) $(T^{-1} \otimes T)\boldsymbol{e}$, then X is DMRL (resp. IMRL).*

Proof: Not that X has ILR (resp. DLR) property if $-\frac{f_X'(x)}{f_X(x)}$ is increasing (resp. decreasing) in $x > 0$. Now, from (2.2), we have

$$-\frac{f_X'(x)}{f_X(x)} = -\frac{\boldsymbol{\alpha}\exp\{Tx\}T\boldsymbol{T^0}}{\boldsymbol{\alpha}\exp\{Tx\}\boldsymbol{T^0}},$$

where $\boldsymbol{T^0} = -T\boldsymbol{e}$. Differentiating the above, we get

$$\left(-\frac{f_X'(x)}{f_X(x)}\right)' = \frac{-(\boldsymbol{\alpha}\exp\{Tx\}\boldsymbol{T^0})(\boldsymbol{\alpha}\exp\{Tx\}T^2\boldsymbol{T^0}) + (\boldsymbol{\alpha}\exp\{Tx\}T\boldsymbol{T^0})^2}{(\boldsymbol{\alpha}\exp\{Tx\}\boldsymbol{T^0})^2}.$$

Clearly, $\left(-\frac{f_X'(x)}{f_X(x)}\right)' \geq$ (resp. $\leq$) 0 holds if and only if

$$-(\boldsymbol{\alpha}\exp\{Tx\}\boldsymbol{T^0})(\boldsymbol{\alpha}\exp\{Tx\}T^2\boldsymbol{T^0}) + (\boldsymbol{\alpha}\exp\{Tx\}T\boldsymbol{T^0})^2 \geq \text{(resp. } \leq\text{) } 0,$$

or equivalently,

$$(\boldsymbol{\alpha} \otimes \boldsymbol{\alpha})(\exp\{(T \otimes I + I \otimes T)x\})(T^2 \otimes T^2 - T \otimes T^3)\boldsymbol{e} \geq \text{(resp. } \leq\text{) } 0,$$

which holds by using the same argument, as in Proposition 2.1 (i), with the help of $(T^2 \otimes T^2)\boldsymbol{e} \geq_{comp}$ (resp. $\leq_{comp}$) $(T \otimes T^3)\boldsymbol{e}$. Thus, $\left(-\frac{f_X'(x)}{f_X(x)}\right)' \geq$ (resp. $\leq$) 0 holds, for all $x > 0$, and hence the result follows. □

3 Poisson PH process and its properties

Before defining the process, we introduce some additional notations.

For an orderly counting process $\{N(t) : t \geq 0\}$, we write $\{N(t) : t \geq 0\} \sim NHPP(\lambda(t))$ to indicate that $\{N(t) : t \geq 0\}$ is the NHPP with the intensity function $\lambda(\cdot)$. Further, we define

$$\boldsymbol{\tau}(n) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{1\times(n+1)} \text{ and } K(n,l) = \begin{pmatrix} -l & 0 & \cdots & 0 & 0 \\ l & -l & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -l & 0 \\ 0 & 0 & \cdots & l & -l \end{pmatrix}_{(n+1)\times(n+1)}.$$

Throughout the paper we also denote the cumulative intensity function as $\Lambda(t) = \int_0^t \lambda(v)dv$.

Definition 3.1 *A counting process $\{N(t) : t \geq 0\}$ is said to be the Poisson PH process (PPHP) with the set of parameters $\{\lambda(t), \boldsymbol{\alpha}, T\}$ if*

(*i*) $\{N(t) : t \geq 0\}|(X = x) \sim NHPP\,(x\lambda(t))$;

(*ii*) $X \sim PH(\boldsymbol{\alpha}, T)$,

where $\lambda(t) > 0$ and $(\boldsymbol{\alpha}, T)$ is the PH representation of X. □

Below we give a proposition that follows from Remark 2.1.

Proposition 3.1 *The following results hold:*

(*i*) *The PPHP with the set of parameters $\{1, \boldsymbol{\alpha}, T\}$, where $\boldsymbol{\alpha} = 1$ and $T = -1/\lambda$, $\lambda > 0$, is the geometric process with the intensity λ (see Di Crescenzo and Pellerey [12] (p. 204), for the definition of the geometric process);*

(*ii*) *The PPHP with the set of parameters $\{1, \boldsymbol{\alpha}, T\}$, where*

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{1\times m} \text{ and } T = \begin{pmatrix} -\lambda & 0 & \cdots & 0 & 0 \\ \lambda & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & \cdots & \lambda & -\lambda \end{pmatrix}_{m\times m},$$

is the Pólya process with the set of parameters $\{m, \lambda\}$ (see Teugels and Vynckier [40], and Beichelt ([4], p. 133)). □

The proof of the following proposition is obvious and hence, is omitted.

Proposition 3.2 *$\{M(t) : t \geq 0\}$ is the PPHP with the set of parameters $\{1, \boldsymbol{\alpha}, T\}$ if and only if $\{N(t) = M(\Lambda(t)) : t \geq 0\}$is the PPHP with the set of parameters $\{\lambda(t), \boldsymbol{\alpha}, T\}$.* □

In the next *basic theorem*, we derive the distribution of the number of events in a given time interval.

Theorem 3.1 *Let $\{N(t) : t \geq 0\}$ be the PPHP with the set of parameters $\{\lambda(t), \boldsymbol{\alpha}, T\}$. Then, for $t > 0$ and $0 \equiv t_0 < t_1 < \cdots < t_m$, the following results hold.*

(i) $P(N(t) = n) = (1 - \boldsymbol{\alpha e})\mathbf{1_0}(n) - \frac{1}{\Lambda(t)}\left[(\boldsymbol{\beta} \otimes \boldsymbol{\alpha})\,(S \otimes I + I \otimes T)^{-1}\,(S \otimes T)\boldsymbol{e}\right]$, *where* $\boldsymbol{\beta} = \boldsymbol{\tau}(n),\ S = K(n, \Lambda(t)),\ n = 0, 1, 2, 3, \ldots,$ *and*

$$\mathbf{1_0}(n) = \begin{cases} 1, & \textit{for } n = 0, \\ 0, & \textit{for } n = 1, 2, \ldots; \end{cases}$$

(ii) $P(N(t_2) - N(t_1) = n) = (1 - \boldsymbol{\alpha e})\mathbf{1_0}(n) - \frac{1}{\Lambda(t_2) - \Lambda(t_1)}\left[(\boldsymbol{\beta} \otimes \boldsymbol{\alpha})\,(S \otimes I + I \otimes T)^{-1}\,(S \otimes T)\boldsymbol{e}\right]$, *where* $\boldsymbol{\beta} = \boldsymbol{\tau}(n),\ S = K(n, \Lambda(t_2) - \Lambda(t_1)),\ n = 0, 1, 2, 3, \ldots,$ *and* $\mathbf{1_0}(n)$ *is the same as in (i);*

(iii) $$P(N(t_i) - N(t_{i-1}) = n_i, i = 1, \ldots, m) = (1 - \boldsymbol{\alpha e})\mathbf{1_0}(n_i, i = 1, \ldots, m) + (-1)^m \left[\frac{(\boldsymbol{\rho}_m \otimes \boldsymbol{\alpha})(A_m \otimes I + I \otimes T)^{-1}(\phi_m \otimes T)\boldsymbol{e}}{\prod_{i=1}^{m}(\Lambda(t_i) - \Lambda(t_{i-1}))}\right],$$

where

$$A_j = A_{j-1} \otimes I + I \otimes S_j,\ A_1 = S_1,\ j = 2, 3, \ldots, m,$$
$$\boldsymbol{\rho}_m = \boldsymbol{\beta}_1 \otimes \boldsymbol{\beta}_2 \otimes \cdots \otimes \boldsymbol{\beta}_m,\ \phi_m = S_1 \otimes S_2 \otimes \cdots \otimes S_m,\quad \textit{for all } m = 1, 2, \ldots,$$
$$S_i = K(n_i, \Lambda(t_i) - \Lambda(t_{i-1})),\ \boldsymbol{\beta}_i = \boldsymbol{\tau}(n_i),\ n_i = 0, 1, 2, 3, \ldots,\ \textit{for } i = 1, 2, 3, \ldots, m,$$

and

$$\mathbf{1_0}(n_i, i = 1, \ldots, m) = \begin{cases} 1, & \textit{for } n_i = 0,\ i = 1, 2, \ldots, m \\ 0, & \textit{otherwise.} \end{cases}$$

Proof: From Definition 3.1, we have

$$P(N(t) = n) = E(P(N(t) = n|X)) = \int_0^\infty \frac{(x\Lambda(t))^n \exp\{-x\Lambda(t)\}}{n!} dF_X(x). \tag{3.1}$$

Obviously,

$$\frac{(x\Lambda(t))^n \exp\{-x\Lambda(t)\}}{n!} = \frac{\boldsymbol{\beta} \exp\{Sx\}\boldsymbol{S^0}}{\Lambda(t)},$$

where $\boldsymbol{\beta} = \boldsymbol{\tau}(n),\ S = K(n, \Lambda(t))$ and $\boldsymbol{S^0} = -S\boldsymbol{e}$. On using this in (3.1), we get

$$\begin{aligned} P(N(t) = n) &= \frac{1}{\Lambda(t)} \int_0^\infty \boldsymbol{\beta} \exp\{Sx\}\boldsymbol{S^0} dF_X(x) \\ &= \frac{1}{\Lambda(t)} \int_0^{0+} \boldsymbol{\beta S^0} dF_X(x) + \frac{1}{\Lambda(t)} \int_{0+}^\infty \boldsymbol{\beta} \exp\{Sx\}\boldsymbol{S^0} dF_X(x), \end{aligned} \tag{3.2}$$

where the last equality holds because $\exp\{O\} = I$, where the O denotes the zero matrix. Further, note that

$$\frac{\boldsymbol{\beta}\boldsymbol{S^0}}{\Lambda(t)} = \mathbf{1_0}(n) = \begin{cases} 1 & n = 0 \\ 0 & n = 1, 2, \dots \end{cases}$$

On using this in (3.2), we get

$$\begin{aligned} P(N(t) = n) &= [F_X(0+) - F_X(0-)]\mathbf{1_0}(n) + \frac{1}{\Lambda(t)} \int_{0+}^{\infty} \left(\boldsymbol{\beta} \exp\{Sx\}\boldsymbol{S^0}\right) \left(\boldsymbol{\alpha} \exp\{Tx\}\boldsymbol{T^0}\right) dx \\ &= (1 - \boldsymbol{\alpha e})\mathbf{1_0}(n) + \frac{1}{\Lambda(t)} \int_{0+}^{\infty} (\boldsymbol{\beta} \otimes \boldsymbol{\alpha})(\exp\{Sx\} \otimes \exp\{Tx\})(\boldsymbol{S^0} \otimes \boldsymbol{T^0}) dx \\ &= (1 - \boldsymbol{\alpha e})\mathbf{1_0}(n) + \frac{1}{\Lambda(t)} \int_{0+}^{\infty} (\boldsymbol{\beta} \otimes \boldsymbol{\alpha}) \exp\{(S \otimes I + I \otimes T)x\}(\boldsymbol{S^0} \otimes \boldsymbol{T^0}) dx \\ &= (1 - \boldsymbol{\alpha e})\mathbf{1_0}(n) + \frac{1}{\Lambda(t)} \left[(\boldsymbol{\beta} \otimes \boldsymbol{\alpha}) \left(\int_{0+}^{\infty} \exp\{(S \otimes I + I \otimes T)x\} dx\right) \left(\boldsymbol{S^0} \otimes \boldsymbol{T^0}\right)\right] \\ &= (1 - \boldsymbol{\alpha e})\mathbf{1_0}(n) - \frac{1}{\Lambda(t)} \left[(\boldsymbol{\beta} \otimes \boldsymbol{\alpha})(S \otimes I + I \otimes T)^{-1}(S \otimes T)\boldsymbol{e}\right], \end{aligned}$$

where the last equality follows from Lemma 2.1(i) and (iv). Thus, (i) is proved. The proof of (ii) follows in the same line by replacing $\Lambda(t)$ by $\Lambda(t_2) - \Lambda(t_1)$. We will prove ($iii$) now. By using the independent increment property of the NHPP,

$$\begin{aligned} &P(N(t_i) - N(t_{i-1}) = n_i, i = 1, \dots, m | X = x) \\ &= \prod_{i=1}^{m} \frac{(x(\Lambda(t_i) - \Lambda(t_{i-1})))^{n_i} \exp\{-x(\Lambda(t_i) - \Lambda(t_{i-1}))\}}{n_i!} \\ &= \prod_{i=1}^{m} \frac{\left(\boldsymbol{\beta}_i \exp\{S_i x\}\boldsymbol{S_i^0}\right)}{(\Lambda(t_i) - \Lambda(t_{i-1}))} \\ &= \left(\prod_{i=1}^{m} \frac{-1}{(\Lambda(t_i) - \Lambda(t_{i-1})}\right) \boldsymbol{\rho}_m \exp\{A_m x\}\phi_m \boldsymbol{e}, \end{aligned} \tag{3.3}$$

where $\boldsymbol{S_i^0} = -S_i\boldsymbol{e}$, for all $i = 1, \dots, m$. Again, note that

$$P(N(t_i) - N(t_{i-1}) = n_i, i = 1, \dots, m | X = 0) = \mathbf{1_0}(n_i, i = 1, \dots, m). \tag{3.4}$$

Now,

$$\begin{aligned} &P(N(t_i) - N(t_{i-1}) = n_i, i = 1, \dots, m) \\ &= E(P(N(t_i) - N(t_{i-1}) = n_i,\ i = 1, \dots, m | X)) \\ &= \int_0^{\infty} P(N(t_i) - N(t_{i-1}) = n_i, i = 1, \dots, m | X = x) dF_X(x) \\ &= \int_0^{0+} P(N(t_i) - N(t_{i-1}) = n_i, i = 1, \dots, m | X = 0) dF_X(x) \\ &\quad + \int_{0+}^{\infty} P(N(t_i) - N(t_{i-1}) = n_i, i = 1, \dots, m | X = x) dF_X(x) \end{aligned}$$

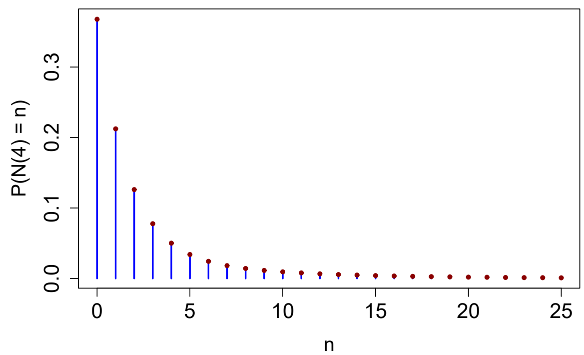

Figure 1: Plot of the probability mass function of $N(4)$ over $n = 0, 1, \ldots, 25$.

$$
\begin{aligned}
&= (1 - \boldsymbol{\alpha}\boldsymbol{e})\mathbf{1_0}(n_i, i = 1, \ldots, m) \\
&\quad + \left(\prod_{i=1}^{m} \frac{-1}{(\Lambda(t_i) - \Lambda(t_{i-1})} \right) \int_{0+}^{\infty} \left(\boldsymbol{\rho}_m \exp\{A_m x\} \phi_m \boldsymbol{e}\right) \left(\boldsymbol{\alpha} \exp\{Tx\} \boldsymbol{T^0}\right) dx \\
&= (1 - \boldsymbol{\alpha}\boldsymbol{e})\mathbf{1_0}(n_i, i = 1, \ldots, m) \\
&\quad + \left(\prod_{i=1}^{m} \frac{-1}{(\Lambda(t_i) - \Lambda(t_{i-1})} \right) \int_{0+}^{\infty} \left(\boldsymbol{\rho}_m \otimes \boldsymbol{\alpha}\right) \left(\exp\{A_m x\} \otimes \exp\{Tx\}\right) \left(\phi_m \boldsymbol{e} \otimes \boldsymbol{T^0}\right) dx \\
&= (1 - \boldsymbol{\alpha}\boldsymbol{e})\mathbf{1_0}(n_i, i = 1, \ldots, m) \\
&\quad + (-1)^m \left(\prod_{i=1}^{m} \frac{1}{(\Lambda(t_i) - \Lambda(t_{i-1})} \right) \left[\left(\boldsymbol{\rho}_m \otimes \boldsymbol{\alpha}\right) \left(A_m \otimes I + I \otimes T\right)^{-1} (\phi_m \otimes T)\boldsymbol{e} \right],
\end{aligned}
$$

where the fourth equality follows from (3.3) and (3.4), and the sixth equality follows from Lemma 2.1(i) and (iv). Thus, (iii) is proved. □

The following example illustrates the result given in Theorem 3.1(i).

Example 3.1 *Let $\{N(t) : t \geq 0\}$ be the PPHP with the set of parameters $\{\lambda(t), \boldsymbol{\alpha}, T\}$, where*

$$
\boldsymbol{\alpha} = \begin{pmatrix} 0.2 & 0.8 \end{pmatrix}, \; T = \begin{pmatrix} -2 & 1 \\ 0.5 & -10 \end{pmatrix} \; \textit{and } \lambda(t) = 3, \quad \textit{for all } t \geq 0.
$$

In Figure 1, we plot the probability mass function for $N(4)$ over $n = 0, 1, \ldots, 25$. □

In the next theorem, we derive the distribution of inter-arrival times for the PPHP.

Theorem 3.2 *Let $T_i, i = 0, 1, 2, \ldots,$ denote the arrival time of the i-th event, and let $X_i = T_i - T_{i-1}, i = 1, 2, \ldots$ denote the inter-arrival time between the i-th and the $(i-1)$-th events, where $T_0 = 0$. Assume that events occur according to the PPHP with the set of parameters $\{\lambda, \boldsymbol{\alpha}, T\}$. Then the cumulative distribution function and the probability density function of X_i are given by*

$$
F_{X_i}(t) = \boldsymbol{\alpha}\boldsymbol{e} + \boldsymbol{\alpha}(-\lambda t I + T)^{-1} \boldsymbol{T^0}
$$

and

$$f_{X_i}(t) = \lambda\boldsymbol{\alpha}(-\lambda t I + T)^{-2}\boldsymbol{T^0},$$

respectively, where $\boldsymbol{T^0} = -T\boldsymbol{e}$.

Proof: Note that, on condition $X = x$, the PPHP with the set of parameters $\{\lambda, \boldsymbol{\alpha}, T\}$ is the same as the HPP with the rate $x\lambda$. Therefore,

$$P(X_i > t | X = x) = \exp\{-\lambda t x\}, \quad i = 1, 2, 3, \ldots .$$

Then

$$\begin{aligned}
\bar{F}_{X_i}(t) &= \int_0^\infty \exp\{-\lambda t x\} dF_X(x) \\
&= \int_0^{0+} \exp\{0\} dF_X(x) + \int_{0+}^\infty \exp\{-\lambda t x\} dF_X(x) \\
&= (1 - \boldsymbol{\alpha e}) + \int_{0+}^\infty \boldsymbol{\alpha} \exp\{(-\lambda t I + T)x\}\boldsymbol{T^0} dx \\
&= (1 - \boldsymbol{\alpha e}) - \boldsymbol{\alpha}(-\lambda t I + T)^{-1}\boldsymbol{T^0},
\end{aligned}$$

and hence, the result follows. □

In the following theorem, we discuss some properties of the PPHP. Note that, the second result gives an alternative representation for the probabilities in Theorem 3.1(i).

Theorem 3.3 *Let $\{N(t) : t \geq 0\}$ be the PPHP with the set of parameters $\{\lambda(t), \boldsymbol{\alpha}, T\}$. Then the following results hold.*

(i) The probability generating function (pgf) of $N(t)$ is given by

$$\phi_{N(t)}(z) = (1 - \boldsymbol{\alpha e}) + \boldsymbol{\alpha}(\Lambda(t)(1 - z)I - T)^{-1}\boldsymbol{T^0}, \quad \boldsymbol{T^0} = -T\boldsymbol{e};$$

(ii) The probability mass function (pmf) of $N(t)$ is given by

$$P(N(t) = n) = \begin{cases} (1 - \boldsymbol{\alpha e}) + \boldsymbol{\alpha}(\Lambda(t)I - T)^{-1}\boldsymbol{T^0}, & n = 0 \\ (\Lambda(t))^n \boldsymbol{\alpha}(\Lambda(t)I - T)^{-(n+1)}\boldsymbol{T^0}, & n = 1, 2, \ldots, \end{cases}$$

where $\boldsymbol{T^0} = -T\boldsymbol{e}$;

(iii) The moment generating function (mgf) of $N(t)$ is given by

$$M_{N(t)}(s) = (1 - \boldsymbol{\alpha e}) - \boldsymbol{\alpha}\left((\Lambda(t)(\exp\{s\} - 1))I + T\right)^{-1}\boldsymbol{T^0},$$

where $\boldsymbol{T^0} = -T\boldsymbol{e}$;

(*iv*) *The n-th order raw moment of* $N(t)$ *is given by*

$$E(N(t)^n) = \sum_{i=1}^{n} (-\Lambda(t))^i (i!) \boldsymbol{\alpha} T^{-n} \mathbf{e} S_{i,n}, \;\; n = 1, 2, 3, \ldots$$

where $S_{i,n}$ *stands for the Stirling number of second kind;*

(*v*) *The mean and the variance of* $N(t)$ *are given by*

$$E(N(t)) = (-\Lambda(t))\boldsymbol{\alpha} T^{-1}\mathbf{e} = \Lambda(t)E(X)$$

and

$$Var(N(t)) = (-\Lambda(t))\boldsymbol{\alpha} T^{-1}\mathbf{e} + (\Lambda(t)^2)\left(2\boldsymbol{\alpha} T^{-2}\mathbf{e} - (\boldsymbol{\alpha} T^{-1}\mathbf{e})^2\right) = E(N(t)) + (\Lambda(t)^2)Var(X),$$

respectively.

Proof:

(*i*) The pgf of $N(t)$ is given by

$$\begin{aligned} \phi_{N(t)}(z) = E(z^{N(t)}) &= \sum_{n=0}^{\infty} z^n P(N(t) = n) \\ &= \sum_{n=0}^{\infty} \int_0^{\infty} \frac{(zx\Lambda(t))^n}{n!} \exp\{-x\Lambda(t)\} dF_X(x) \\ &= \int_0^{\infty} \exp\{-x\Lambda(t)(1-z)\} dF_X(x) \\ &= (1 - \boldsymbol{\alpha e}) + \boldsymbol{\alpha}(\Lambda(t)(1-z)I - T)^{-1}\boldsymbol{T^0}, \end{aligned}$$

where the fourth equality holds due to the Dominated Convergence Theorem.

(*ii*) The proof immediately follows from part (*i*) by using the fact that $P(N(t) = n) = \phi^{(n)}_{N(t)}(0)/n!$, where $\phi^{(n)}_{N(t)}(0)$ represents the n-th order derivative of the pgf of $N(t)$ at point $z = 0$.

(*iii*) The mgf of $N(t)$ is given by

$$M_{N(t)}(s) = E(\exp\{sN(t)\}) = E(E(\exp\{sN(t)\}|X)),$$

where $[N(t)|X = x]$ is the Poisson distribution with parameter $\Lambda(t)x$. Now, on using the mgf of the Poisson distribution (see Ross [36], p. 63), we get

$$\begin{aligned} M_{N(t)}(s) &= E(\exp\{\Lambda(t)(\exp\{s\} - 1)X\}) = M_X(\Lambda(t)(\exp\{s\} - 1)) \\ &= (1 - \boldsymbol{\alpha e}) - \boldsymbol{\alpha}\left((\Lambda(t)(\exp\{s\} - 1))I + T\right)^{-1} \boldsymbol{T^0}, \end{aligned}$$

where the last equality holds due to Lemma 2.1(*ii*).

(*iv*) The n-th order moment of $N(t)$ about zero is given by

$$\begin{aligned} E(N(t)^n) &= E(E(N(t)^n|X)) \\ &= \sum_{i=1}^{n} E((\Lambda(t)X)^i)S_{i,n} \\ &= \sum_{i=1}^{n}(-\Lambda(t))^i(i!)\boldsymbol{\alpha}T^{-i}\mathbf{e}S_{i,n}, \text{ for } n = 1, 2, \ldots, \end{aligned}$$

where the second equality follows from the expression of moments of the Poisson distribution (see Riordan [34], p. 105), and the last equality follows from Lemma 2.1(*iii*).

(*v*) The proof immediately follows from the definition of variance of a random variable and the fact that $S_{1,1} = S_{1,2} = S_{2,2} = 1$. □

The following corollary immediately follows from Theorem 3.3 (*ii*).

Corollary 3.1 *Let X follow the PH distribution with the PH representation $(\boldsymbol{\alpha}, T)$. If $\boldsymbol{\alpha e} = 1$ then $N(t) + 1 \sim DPH(\alpha, Q)$, where $Q = \left(I - \Lambda(t)^{-1}T\right)^{-1}$.*

In the following theorem, we give a characterization result of the PPHP.

Theorem 3.4 *Let $\{N(t) : t \geq 0\}$ be a PPHP with the set of parameters $\{\lambda(t), \boldsymbol{\alpha}, T\}$. Then the stochastic intensity λ_t of $\{N(t) : t \geq 0\}$ is given by*

$$\lambda_t = \frac{-(\mathbf{g} \otimes \boldsymbol{\alpha})(G \otimes I + I \otimes T)^{-1}(G \otimes T)\boldsymbol{e}}{(\Lambda(t))^2[(1 - \boldsymbol{\alpha e}) + \boldsymbol{\alpha}(\Lambda(t)I - T)^{-1}\boldsymbol{T^0}]}\lambda(t), \text{ for } N(t-) = 0,$$

where $t > 0$, $\mathbf{g} = (0, 1)$, $G = K(1, \Lambda(t))$, $\boldsymbol{T^0} = -T\boldsymbol{e}$ and

$$\lambda_t = \left(\frac{N(t-)+1}{\Lambda(t)}\right)\frac{(\mathbf{g_1} \otimes \boldsymbol{\alpha})(G_1 \otimes I + I \otimes T)^{-1}(G_1 \otimes T)\boldsymbol{e}}{(\mathbf{g_2} \otimes \boldsymbol{\alpha})(G_2 \otimes I + I \otimes T)^{-1}(G_2 \otimes T)\boldsymbol{e}}\lambda(t), \text{ for } N(t-) = 1, 2, 3, \ldots,$$

where $t > 0$, $\mathbf{g_1} = \boldsymbol{\tau}(N(t-) + 1)$, $\mathbf{g_2} = \boldsymbol{\tau}(N(t-))$, $G_1 = K(N(t-) + 1, \Lambda(t))$ and $G_2 = K(N(t-), \Lambda(t))$.

Proof: Let $\{M(t) : t \geq 0\}$ be a PPHP with the set of parameters $\{1, \boldsymbol{\alpha}, T\}$. Now, from Proposition 4.1 of Grandell [15], the stochastic intensity $\hat{\lambda}_t$ of the point process $\{M(t) : t \geq 0\}$ is given by

$$\hat{\lambda}_t = \frac{\int_0^\infty x^{N(t-)+1}\exp\{-xt\}dF_X(x)}{\int_0^\infty x^{N(t-)}\exp\{-xt\}dF_X(x)}. \tag{3.5}$$

By proceeding in the same line as in the proof of Theorem 3.1, for $t > 0$ and $n = 0, 1, 2, \ldots$, we can write

$$\int_0^\infty x^n \exp\{-xt\}dF_X(x) = \frac{n!}{t^n}\int_0^\infty \frac{(xt)^n}{n!}\exp\{-xt\}dF_X(x)$$

$$= \frac{n!}{t^n}\left[(1-\boldsymbol{\alpha e})\mathbf{1_0}(n) - \frac{1}{t}\left\{(\boldsymbol{\beta}\otimes\boldsymbol{\alpha})(S\otimes I + I\otimes T)^{-1}(S\otimes T)\boldsymbol{e}\right\}\right],$$

where $\boldsymbol{\beta} = \boldsymbol{\tau}(n)$, $S = K(n,t)$ and $\mathbf{1_0}(n)$ is the same as in Theorem 3.1. Now, by using the above equality in (3.5), we get

$$\hat{\lambda}_t = \frac{-(\mathbf{g}\otimes\boldsymbol{\alpha})(G\otimes I + I\otimes T)^{-1}(G\otimes T)\boldsymbol{e}}{t^2[(1-\boldsymbol{\alpha e}) + \boldsymbol{\alpha}(tI-T)^{-1}\boldsymbol{T^0}]}, \text{ for } N(t-) = 0,$$

where $t > 0$, $\mathbf{g} = (0,1)$, $G = K(1,t)$, $\boldsymbol{T^0} = -T\boldsymbol{e}$ and

$$\hat{\lambda}_t = \left(\frac{N(t-)+1}{t}\right)\frac{(\mathbf{g_1}\otimes\boldsymbol{\alpha})(G_1\otimes I + I\otimes T)^{-1}(G_1\otimes T)\boldsymbol{e}}{(\mathbf{g_2}\otimes\boldsymbol{\alpha})(G_2\otimes I + I\otimes T)^{-1}(G_2\otimes T)\boldsymbol{e}}, \text{ for } N(t-) = 1,2,3,\ldots,$$

where $t > 0$, $\mathbf{g_1} = \boldsymbol{\tau}(N(t-)+1)$, $\mathbf{g_2} = \boldsymbol{\tau}(N(t-))$, $G_1 = K(N(t-)+1,t)$ and $G_2 = K(N(t-),t)$. From Proposition 3.2, the PPHP with the set of parameters $\{\lambda(t),\boldsymbol{\alpha},T\}$ can be recovered by considering $\{N(t) = M(\Lambda(t)) : t \geq 0\}$. Thus, the stochastic intensity λ_t of $\{N(t) : t \geq 0\}$ can be written as

$$\lambda_t = \hat{\lambda}_{\Lambda(t)}\lambda(t),$$

(see Grandell [15], p. 79) and hence, the result is proved. □

Due to the complicated expression of the stochastic intensity, the monotonic behaviour of λ_t with respect to $N(t-)$ cannot be analytically studied. Below we give a numerical example.

Example 3.2 *Let $\{N(t) : t \geq 0\}$ be a PPHP with the set of parameters $\{\lambda(t),\boldsymbol{\alpha},T\}$, where*

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & 1 \end{pmatrix},\ T = \begin{pmatrix} -2 & 0 \\ 2 & -2 \end{pmatrix} \text{ and } \lambda(t) = 1,\ \ t \geq 0.$$

In Figure 2, we plot the stochastic intensity λ_t against $N(t-)$, for fixed $t = 4$. This shows that λ_t increases as $N(t-)$ increases. □

In the next theorem, we study some stochastic ordering properties of the PPHP.

Theorem 3.5 *Let $\{N_i(t) : t \geq 0\}$ be a PPHP with the set of parameters $(\lambda(t),\boldsymbol{\alpha}_i,T_i)$, for $i = 1,2$. Then the following results hold.*

(i) *If $(T_2\otimes T_1^2)\boldsymbol{e} \geq_{comp} (T_2^2\otimes T_1)\boldsymbol{e}$, then $N_1(t) \geq_{lr} N_2(t)$;*

(ii) *If $(I\otimes T_1)\boldsymbol{e} \geq_{comp} (T_2\otimes I)\boldsymbol{e}$, then $N_1(t) \geq_{hr} N_2(t)$.*

Proof: Let $\{M_i(t), t \geq 0\}$ be a PPHP with the set of parameters $\{1,\boldsymbol{\alpha_i},T_i\}$, for $i = 1,2$. Then

$$\{M_i(t) : t \geq 0\}|(X_i = x) \sim HPP(x), \text{ for } i = 1,2,$$

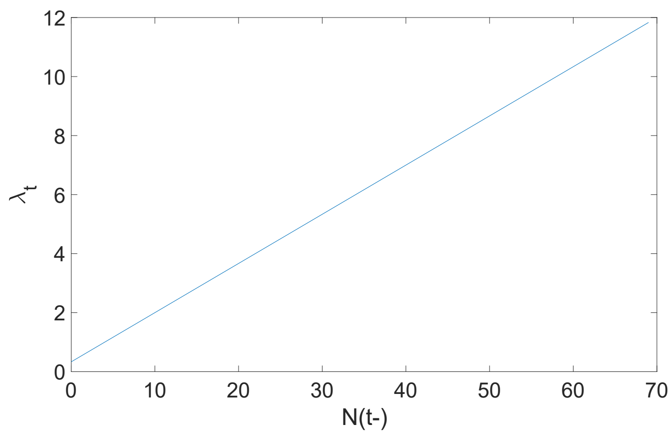

Figure 2: Plot of λ_t against $N(t-)$ for fixed $t = 4$

where $X_i \sim PH(\boldsymbol{\alpha_i}, T_i)$. Consequently, $[M_i(t)|X_i = x]$, for any fixed $t \geq 0$, follows the Poisson distribution with parameter xt, for $i = 1, 2$. Now, by using the result given in Table 2.5 of Belzunce [5], we get

$$[M_1(t)|X_1 = x_1] \geq_{lr} [M_2(t)|X_2 = x_2], \text{ for all } x_1 \geq x_2 \text{ and for fixed } t \geq 0, \quad (3.6)$$

which further gives

$$[M_1(t)|X_1 = x_1] \geq_{hr} [M_2(t)|X_2 = x_2], \text{ for all } x_1 \geq x_2 \text{ and for fixed } t \geq 0. \quad (3.7)$$

Since $(T_2 \otimes T_1^2)\boldsymbol{e} \geq_{comp} (T_2^2 \otimes T_1)\boldsymbol{e}$, we have, from Proposition 2.1 (i), that $X_1 \geq_{lr} X_2$. Further, by using this together with (3.6) in Theorem 1.C.17 of Shaked and Shanthikumar [37], we get that $M_1(t) \geq_{lr} M_2(t)$ for all $t \geq 0$. Again this, in view of Proposition 3.2, implies that $M_1(\Lambda(t)) \geq_{lr} M_2(\Lambda(t))$ or equivalently, $N_1(t) \geq_{lr} N_2(t)$. Hence part (i) is proved. Again, by using Proposition 2.1 (ii), $(I \otimes T_1)\boldsymbol{e} \geq_{comp} (T_2 \otimes I)\boldsymbol{e}$ implies $X_1 \geq_{hr} X_2$. Further, by using this together with (3.7) in Theorem 1.B.14 of Shaked and Shanthikumar [37], we get that $M_1(t) \geq_{hr} M_2(t)$ for all $t \geq 0$. Finally, the result given in part (ii) follows from Proposition 3.2.□

In the following theorem, we discuss an ageing property of the PPHP.

Theorem 3.6 *Let* $\{N(t) : t \geq 0\}$ *be a PPHP with the set of parameters* $\{\lambda(t), \boldsymbol{\alpha}, T\}$*. If* $(T \otimes T)\boldsymbol{e} \geq_{comp}$ *(resp.* $\leq_{comp}$*)* $(I \otimes T^2)\boldsymbol{e}$ *then* $N(t)$ *is IFR (resp. DFR).*

Proof: From Proposition 7.2 of Grandell [15], we have that $N(t)$ is IFR (resp. DFR) if and only if X is IFR (resp. DFR). Hence, the result follows from Proposition 2.2 (i).

4 Application: Shock models

In this section, we apply our results obtained in the previous sections for describing the popular extreme shock model, the stochastic failure rate model and the δ-shock model when the shock process is the PPHP.

4.1 Extreme shock model

Let L be the lifetime of a system subject to external shocks that occur according to the PPHP with the set of parameters $\{\lambda(t), \boldsymbol{\alpha}, T\}$. Let $0 = T_0 < T_1 < T_2 < \cdots < T_n$ be the sequence of random variables representing the arrival times of n shocks. Assume that the system survives from a shock, occurred at time t, with probability $q(t)$ and fails with probability $p(t) = 1 - q(t)$. Further, assume that the system is absolutely reliable in the absence of shocks.

Theorem 4.1 *The survival function for the defined extreme shock model is given by*

$$P(L > t) = (1 - \boldsymbol{\alpha e}) + \boldsymbol{\alpha} \left(\left(\int_0^t p(x)\lambda(x)dx \right) I - T \right)^{-1} \boldsymbol{T^0}, \tag{4.1}$$

where $\boldsymbol{T^0} = -T\boldsymbol{e}$.

Proof: We have

$$P(L > t | T_1, T_2, \ldots, T_{N(t)}, N(t)) = \prod_{i=1}^{N(t)} q(T_i),$$

where $\prod_{i=1}^{0}(\cdot) \equiv 1$. Now,

$$P(L > t) = \sum_{n=0}^{\infty} P(L > t, N(t) = n) = \sum_{n=0}^{\infty} P(N(t) = n) P(L > t | N(t) = n). \tag{4.2}$$

Note that, for any mixed Poisson process with arrival rate $\lambda(\cdot)$, the conditional joint distribution of arrival times $(T_1, T_2, \ldots, T_{N(t)})$, given that $N(t) = n$, is (see, e.g.,Cha and Mercier [11])

$$f_{T_1, T_2, \ldots, T_{N(t)} | N(t)}(t_1, t_2, \ldots, t_n | n) = (n!) \prod_{i=1}^{n} \left(\frac{\lambda(t_i)}{\Lambda(t)} \right), \;\; 0 \le t_1 \le t_2 \le \cdots \le t_n,$$

where $\Lambda(t) = \int_0^t \lambda(v)dv$.
Using this general result we can write

$$\begin{aligned} P(L > t | N(t) = n) &= \frac{n!}{\Lambda(t)^n} \int_0^t \int_0^{t_n} \cdots \int_0^{t_n} \prod_{i=1}^{n} q(t_i)\lambda(t_i) dt_1 dt_2 \ldots dt_n \\ &= \frac{1}{\Lambda(t)^n} \left(\int_0^t q(v)\lambda(v)dv \right)^n . \end{aligned}$$

By using the above equality along with (3.1) in (4.2), we get

$$\begin{aligned} P(L > t) &= \sum_{n=0}^{\infty} \int_0^{\infty} \frac{x^n \exp\{-x\Lambda(t)\}}{n!} \left(\int_0^t q(v)\lambda(v)dv \right)^n dF_X(x) \\ &= \int_0^{\infty} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \left(\int_0^t q(v)\lambda(v)dv \right)^n \right) \exp\{-x\Lambda(t)\} dF_X(x) \\ &= \int_0^{\infty} \exp\left\{ -x \int_0^t p(v)\lambda(v)dv \right\} dF_X(x) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{0+} \exp\{0\} dF(x) + \int_{0+}^{\infty} \exp\left\{-x\int_0^t p(v)\lambda(v)dv\right\} \boldsymbol{\alpha} \exp\{Tx\}\boldsymbol{T^0} dx \\
&= (1-\boldsymbol{\alpha e}) + \int_{0+}^{\infty} \boldsymbol{\alpha} \exp\left\{x\left(-I\int_0^t p(v)\lambda(v)dv + T\right)\right\}\boldsymbol{T^0} dx \\
&= (1-\boldsymbol{\alpha e}) + \boldsymbol{\alpha}\left(\left(\int_0^t p(v)\lambda(v)dv\right) I - T\right)^{-1}\boldsymbol{T^0},
\end{aligned}
$$

where the second equality holds due to the Dominated Convergence Theorem, and the last equality follows from Lemma 2.1 (i) and the fact that $-sI+T$ is invertible for all $s \geq 0$. Hence, the result is proved. □

The next corollary, presents a simple practical example and follows from Theorem 4.1.

Corollary 4.1 *If* $T=-1$, $\boldsymbol{\alpha e}=1$ *and* $p(t)=p$ *(independent of time). Then*

$$
P(L>t) = \frac{1}{1+\Lambda(t)p},
$$

where $\Lambda(t) = \int_0^t \lambda(t)$. □

Another corollary of Theorem 4.1 is stated as follows.

Corollary 4.2 *If* T *is a diagonalizable matrix of order* m *then*

$$
P(L>t) = (1-\boldsymbol{\alpha e}) + \boldsymbol{\alpha} P\hat{T}P^{-1}\boldsymbol{e},
$$

where

$$
\hat{T} = \begin{pmatrix}
\frac{-\lambda_1}{\int_0^t p(x)\lambda(x)dx-\lambda_1} & 0 & \cdots & 0 \\
0 & \frac{-\lambda_2}{\int_0^t p(x)\lambda(x)dx-\lambda_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{-\lambda_m}{\int_0^t p(x)\lambda(x)dx-\lambda_m}
\end{pmatrix}_{m\times m},
$$

and λ_i *is the* i*-th eigenvalue of* T, $i=1,2,\ldots,m$, *and* P *is a non-singular matrix whose* i*-th column is the eigenvector corresponding to the eigenvalue* λ_i, *for* $i=1,2,\ldots,m$.

Proof: Since T is a diagonalizable matrix, we can write $T=PDP^{-1}$, where

$$
D = \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_m
\end{pmatrix}_{m\times m},
$$

and λ_i is the i-th eigenvalue of T, for $i=1,2,\ldots,m$, and P is a non-singular matrix whose i-th column is the eigenvector corresponding to the eigenvalue λ_i, for $i=1,2,\ldots,m$. Now, by replacing $T=PDP^{-1}$ in (4.1), we get

$$
P(L>t) = (1-\boldsymbol{\alpha e}) - \boldsymbol{\alpha}\left(\left(\int_0^t p(x)\lambda(x)dx\right) PP^{-1} - PDP^{-1}\right)^{-1} PDP^{-1}\boldsymbol{e}
$$

$$
\begin{aligned}
&= (1-\boldsymbol{\alpha e}) - \boldsymbol{\alpha} P \left(\left(\int_0^t p(x)\lambda(x)dx \right) I - D \right)^{-1} P^{-1}PDP^{-1}\boldsymbol{e} \\
&= (1-\boldsymbol{\alpha e}) - \boldsymbol{\alpha} P \left(\left(\int_0^t p(x)\lambda(x)dx \right) I - D \right)^{-1} DP^{-1}\boldsymbol{e}.
\end{aligned}
$$

As D is a diagonal matrix, $\left(\left(\int_0^t p(x)\lambda(x)dx \right) I - D \right)$ is a diagonal matrix with the i-th diagonal entry $1/\left(\int_0^t p(x)\lambda(x)dx - \lambda_i \right)$ and hence, $\hat{T} = -\left(\left(\int_0^t p(x)\lambda(x)dx \right) I - D \right)^{-1} D$ is also a diagonal matrix with the i-th diagonal entry $-\lambda_i/\left(\int_0^t p(x)\lambda(x)dx - \lambda_i \right)$. Thus, the result is proved. □

In the following theorem, we derive the failure rate function for the defined model. The proof is trivial and hence, omitted.

Theorem 4.2 *The failure rate function for the defined extreme shock model is given by*

$$
r_L(t) = \frac{p(t)\lambda(t)\boldsymbol{\alpha} \left(\left(\int_0^t p(x)\lambda(x)dx \right) I - T \right)^{-2} \boldsymbol{T^0}}{(1-\boldsymbol{\alpha e}) + \boldsymbol{\alpha} \left(\left(\int_0^t p(x)\lambda(x)dx \right) I - T \right)^{-1} \boldsymbol{T^0}}, \quad t \geq 0,
$$

where $\boldsymbol{T^0} = -T\boldsymbol{e}$. □

For illustration of Theorems 4.1 and 4.2, let

$$
\boldsymbol{\alpha} = \begin{pmatrix} 0.2 & 0.8 \end{pmatrix}, \; T = \begin{pmatrix} -2 & 1 \\ 0.5 & -10 \end{pmatrix}, \; q(t) = \exp\{-t\} \text{ and } \lambda = 2.
$$

In Figure 3, we plot the system's survival function over $t \in [0, 45]$ and the system's failure rate function over $t \in [0, 200]$. We see that as q(t) asymptotically tends to 0 and $\lambda = 2$, the failure rate, obviously, tends to the same value.

We will now discuss some relevant stochastic comparisons. But first, we formulate the following lemma.

Lemma 4.1 *Let D be any diagonal matrix of order m with non-negative diagonal entries d_i, $i = 1, 2, \ldots, m$. Then, for any non-singular matrix P, we have $PDP^{-1}\boldsymbol{e} \geq_{comp} d_{\min}\boldsymbol{e}$, where $d_{\min} = min\{d_i | i = 1, 2, \ldots, m\}$.*

Proof: For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m$, let the ij-th entry of the matrices P and P^{-1} be denoted by P_{ij} and P^{-1}_{ij}, respectively. Clearly, $PDP^{-1}\boldsymbol{e} \geq_{comp} d_{min}\boldsymbol{e}$ holds if and only if each row sum of the matrix PDP^{-1} is greater than or equal to d_{min}. Note that the i-th row sum of PDP^{-1} is equal to $\sum_{k=1}^m \sum_{j=1}^m d_j P_{ij} P^{-1}_{jk}$, for $i = 1, 2, \ldots, m$. Now,

$$
\sum_{k=1}^m \sum_{j=1}^m d_j P_{ij} P^{-1}_{jk} \quad \geq \quad \min\{d_j : j = 1, 2, \ldots, m\} \sum_{k=1}^m \sum_{j=1}^m P_{ij} P^{-1}_{jk}
$$

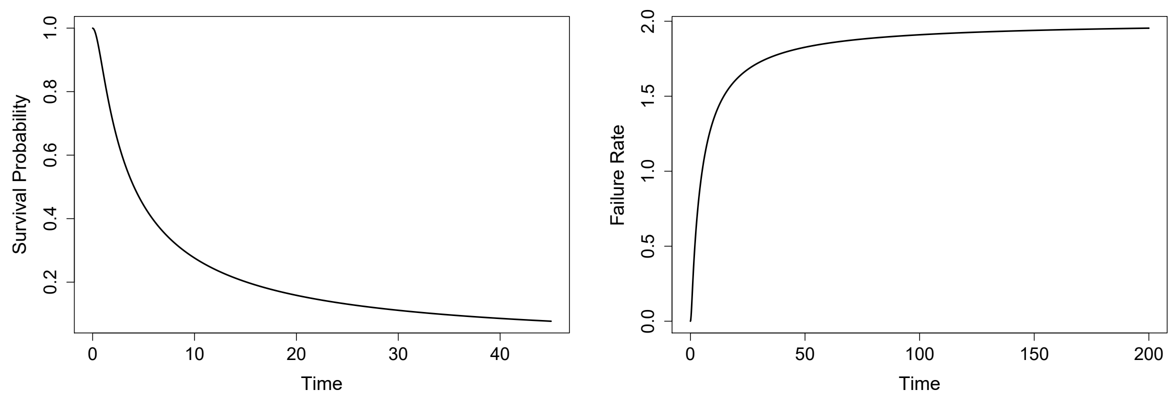

Figure 3: Plot of the system's survival function over $t \in [0, 45]$ and the system's failure rate function over $t \in [0, 200]$, respectively.

$$
\begin{aligned}
&= d_{min} \sum_{k=1}^{m} \sum_{j=1}^{m} P_{ij} P_{jk}^{-1} \\
&= d_{\min},
\end{aligned}
$$

where the last equality holds because the i-th row sum of PP^{-1} is equal to 1, i.e., $\sum_{k=1}^{m} \sum_{j=1}^{m} P_{ij} P_{jk}^{-1} = 1$. Hence, the result is proved. $\square$

In the following theorem, we compare the lifetimes of two systems operating under different random environments.

Theorem 4.3 *Let L_1 and L_2 be the lifetimes of two systems subject to random shocks that occur according to the PPHP with the sets of parameters $\{\lambda_1(t), \boldsymbol{\alpha}, T\}$ and $\{\lambda_2(t), \boldsymbol{\alpha}, T\}$, respectively. Further, let $q_1(t) = 1 - p_1(t)$ and $q_2(t) = 1 - p_2(t)$ be the survival probabilities of the first and the second systems under a shock, occurred at time t, respectively. Let $H_i(t) = \int_0^t p_i(x)\lambda_i(x)dx$, $i = 1, 2$. If T is a diagonalizable matrix with real eigenvalues and $H_1(t) \leq H_2(t)$, for all t, then $L_2 \leq_{st} L_1$.*

Proof: From Corollary 4.2, we have

$$
\bar{F}_{L_i}(t) = (1 - \boldsymbol{\alpha e}) + \boldsymbol{\alpha} P \hat{T}_i P^{-1} \boldsymbol{e},
$$

where

$$
\hat{T}_i = \begin{pmatrix} \frac{-\lambda_1}{H_i(t)-\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{-\lambda_2}{H_i(t)-\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{-\lambda_m}{H_i(t)-\lambda_m} \end{pmatrix}_{m \times m}, \quad i = 1, 2.
$$

Since T is a PH-generator, all eigenvalues, λ_j, $j = 1, 2, \ldots, m$, of T are negative. Then, from the assumption '$H_1(t) \leq H_2(t)$ for all $t \geq 0$', we get $-\lambda_j/(H_2(t) - \lambda_j) \leq -\lambda_j/(H_1(t) - \lambda_j)$, for $j = 1, 2, \ldots, m$ and for all $t \geq 0$. Further, this implies that $\hat{T}_1 - \hat{T}_2$ is a diagonal matrix with non-negative entries and hence, $\boldsymbol{\alpha} P(\hat{T}_1 - \hat{T}_2)P^{-1}\boldsymbol{e} \geq 0$ follows from Lemma 4.1. Thus, the result is proved. □

The following two corollaries immediately follow from Theorem 4.3.

Corollary 4.3 *Suppose that* $\lambda_1(t) = \lambda_2(t)$ *for all* t*. If* T *is a diagonalizable matrix with real eigenvalues and* $p_1(t) \leq p_2(t)$ *for all* t*, then* $L_2 \leq_{st} L_1$*.*

Corollary 4.4 *Suppose that* $p_1(t) = p_2(t)$ *for all* t*. If* T *is a diagonalizable matrix with real eigenvalues and* $\lambda_1(t) \leq \lambda_2(t)$ *for all* t*, then* $L_2 \leq_{st} L_1$*.*

4.2 Stochastic failure rate model

Let L_0 be the lifetime of a system in a baseline (without shocks) environment, and let $\bar{F}_0(\cdot)$ and $r_0(\cdot)$ be the corresponding survival and failure rate functions, respectively. Further, let L be the lifetime of an identical system under the influence of random shocks that occur according to the PPHP with the set of parameters $\{\lambda(t), \boldsymbol{\alpha}, T\}$. As earlier, let $0 = T_0 \leq T_1 \leq T_2 \leq \cdots \leq T_n \leq \ldots$ be the arrival times of shocks. Assume that each shock increases the system's failure rate by the fixed value $\mu > 0$. Then

$$P(L > t | T_1, T_2, \ldots, T_{N(t)}, N(t)) = \exp\left\{ - \left[\int_0^t r_0(x) + \sum_{i=1}^{N(t)} \mu \mathbf{1}_{[T_i, \infty)}(x) \right] dx \right\}, \quad t \geq 0,$$

where

$$\mathbf{1}_{[T_i,\infty)}(x) = \begin{cases} 1 & \text{if } x \in [T_i, \infty), \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots$.

Theorem 4.4 *The survival function of a system for the defined stochastic failure rate model is given by*

$$P(L > t) = \bar{F}_0(t) \left[(1 - \boldsymbol{\alpha e}) + \boldsymbol{\alpha} \left(M(t) I - T \right)^{-1} \boldsymbol{T^0} \right],$$

where $M(t) = \Lambda(t) - \exp\{-\mu t\} \int_0^t \exp\{\mu x\} \lambda(x) dx$ *and* $\boldsymbol{T^0} = -T\boldsymbol{e}$*.*

Proof: Note that,

$$\mu \sum_{i=0}^{N(t)} \int_0^t \mathbf{1}_{[T_i,\infty)}(x) dx = \mu t N(t) - \mu \sum_{i=1}^{N(t)} T_i,$$

we can write

$$P(L > t | T_1, T_2, \ldots, T_{N(t)}, N(t)) = \bar{F}_0(t) \exp\{-\mu t N(t)\} \left(\prod_{i=1}^{N(t)} \exp\{\mu T_i\} \right).$$

On using this and similar arguments as in the proof of Theorem 4.1,

$$\begin{aligned}
P(L>t|N(t)=n) &= \frac{n!\bar{F}_0(t)\exp\{-\mu t n\}}{\Lambda(t)^n}\int_0^t\int_0^{t_n}\cdots\int_0^{t_3}\int_0^{t_2}\left(\prod_{i=1}^{n}\exp\{\mu t_i\}\lambda(t_i)\right)dt_1dt_2\ldots dt_n \\
&= \frac{n!\bar{F}_0(t)\exp\{-\mu t n\}}{\Lambda(t)^n}\frac{\left(\int_0^t \exp\{uv\}\lambda(v)dv\right)^n}{n!} \\
&= \frac{\bar{F}_0(t)\left(\exp\{-\mu t\}\int_0^t \exp\{\mu v\}\lambda(v)dv\right)^n}{\Lambda(t)^n}.
\end{aligned}$$

Further, by using this along with (3.1), we get

$$\begin{aligned}
P(L>t) &= E(P(L>t|N(t))) \\
&= \bar{F}_0(t)\sum_{n=0}^{\infty}\int_0^{\infty}\frac{x^n\exp\{-x\Lambda(t)\}}{n!}\left(\exp\{-\mu t\}\int_0^t\exp\{\mu v\}\lambda(v)dv\right)^n dF_X(x) \\
&= \bar{F}_0(t)\int_0^{\infty}\sum_{n=0}^{\infty}\left(\frac{\left(x\exp\{-\mu t\}\int_0^t\exp\{\mu v\}\lambda(v)dv\right)^n}{n!}\right)\exp\{-x\Lambda(t)\}dF_X(x) \\
&= \bar{F}_0(t)\int_0^{\infty}\exp\left\{-x\left(\Lambda(t)-\exp\{-\mu t\}\int_0^t\exp\{\mu v\}\lambda(v)dv\right)\right\}dF_X(x) \\
&= \bar{F}_0(t)\left[(1-\boldsymbol{\alpha e})+\boldsymbol{\alpha}\left(\left(\Lambda(t)-\exp\{-\mu t\}\int_0^t\exp\{\mu x\}\lambda(x)dx\right)I-T\right)^{-1}\boldsymbol{T^0}\right] \\
&= \bar{F}_0(t)\left[(1-\boldsymbol{\alpha e})+\boldsymbol{\alpha}\left(M(t)I-T\right)^{-1}\boldsymbol{T^0}\right],
\end{aligned}$$

where the third equality holds due to the Dominated Convergence Theorem and the last equality holds because $\int_0^t \exp\{\mu x\}\lambda(x)dx \le \exp\{\mu t\}\Lambda(t)$ for all $t \ge 0$. Hence, the result is proved. □

The following corollary immediately follows from Theorem 4.4.

Corollary 4.5 *If T is a diagonalizable matrix of order m, then*

$$P(L>t)=\bar{F}_0(t)\left[(1-\boldsymbol{\alpha e})+\boldsymbol{\alpha}P\hat{T}P^{-1}\boldsymbol{e}\right],$$

where

$$\hat{T}=\begin{pmatrix}\frac{-\lambda_1}{M(t)-\lambda_1} & 0 & \cdots & 0\\ 0 & \frac{-\lambda_2}{M(t)-\lambda_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \frac{-\lambda_m}{M(t)-\lambda_m}\end{pmatrix}_{m\times m},$$

and λ_i is the i-th eigenvalue of T, for $i=1,2,\ldots,m$, and P is a non-singular matrix whose i-th column is the eigenvector corresponding to the eigenvalue λ_i, for $i=1,2,\ldots,m$. □

In the next theorem, we obtain the failure rate function for the defined model. The proof is omitted.

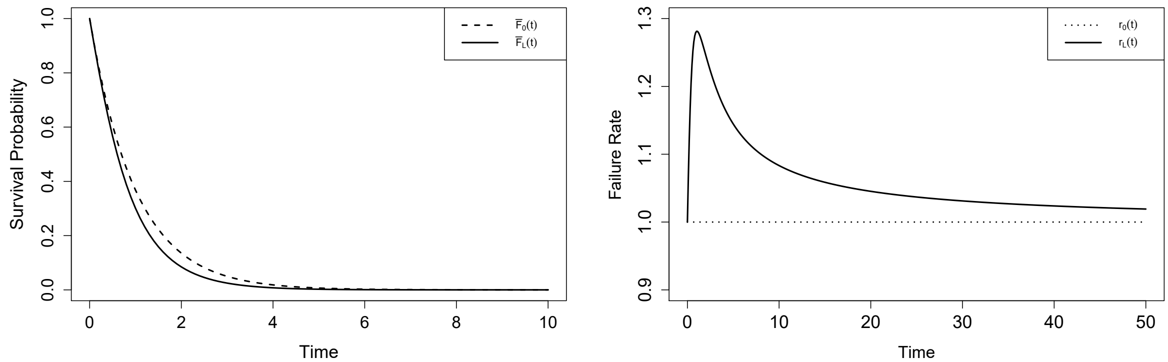

Figure 4: Plot of the system's survival function over $t \in [0, 10]$ and the system's failure rate function over $t \in [0, 50]$, respectively.

Theorem 4.5 *The failure rate function of a system for the defined stochastic failure rate model is given by*

$$r_L(t) = r_0(t) + \frac{\{\boldsymbol{\alpha}\,(M(t)I - T)^{-2}\,\boldsymbol{T^0}\} \times \mu \times (\Lambda(t) - M(t))}{(1 - \boldsymbol{\alpha e}) + \boldsymbol{\alpha}\,(M(t)I - T)^{-1}\,\boldsymbol{T^0}}, \quad t \geq 0,$$

where $M(t)$ and $\boldsymbol{T^0}$ are the same as in Theorem 4.4.

The following example illustrates results of Theorems 4.4 and 4.5. Let

$$\boldsymbol{\alpha} = \begin{pmatrix} 0.2 & 0.8 \end{pmatrix},\ T = \begin{pmatrix} -2 & 1 \\ 0.5 & -10 \end{pmatrix},\ \bar{F}_0(t) = \exp\{-t\},\ \lambda = 3 \text{ and } \mu = 1.$$

In Figure 4, we plot the survival function and the failure rate function of a system both for the normal and the random environments. We can see that the failure rate initially increases and then decreases. The latter behavior is due to 'heterogeneity' in the corresponding sample paths of the stochastic failure rate (the weaker populations are dying out first).

The proof of the next theorem is similar to that in Theorem 4.3 and hence, omitted.

Theorem 4.6 *Let L_1 and L_2 be the lifetimes of two systems subject to random shocks that occur according to the PPHP with the sets of parameters $\{\lambda_1(t), \boldsymbol{\alpha}, T\}$ and $\{\lambda_2(t), \boldsymbol{\alpha}, T\}$, respectively. Assume that each shock increases the failure rates of the first and the second systems by μ_1 and μ_2, respectively. For $i = 1, 2$, let $M_i(t) = \Lambda_i(t) - \exp\{-\mu_i t\} \int_0^t \exp\{\mu_i x\}\lambda_i(x)dx$, where $\Lambda_i(t) = \int_0^t \lambda_i(x)dx$. If T is a diagonalizable matrix with real eigenvalues and $M_1(t) \leq M_2(t)$ for all t, then $L_2 \leq_{st} L_1$.*

The following two corollaries immediately follow from Theorem 4.6.

Corollary 4.6 *Suppose that $\lambda_1(t) = \lambda_2(t)$ for all t. If T is a diagonalizable matrix with real eigenvalues and $\mu_1 \leq \mu_2$, then $L_2 \leq_{st} L_1$.*

Corollary 4.7 *Suppose that $\mu_1 = \mu_2$. If T is a diagonalizable matrix with real eigenvalues and $\lambda_1(t) \leq \lambda_2(t)$ for all t, then $L_2 \leq_{st} L_1$.*

4.3 δ-shock model

Let L be the lifetime of a system subject to external shocks that occur according to a PPHP with the set of parameters $\{\lambda, \boldsymbol{\alpha}, T\}$. Further, let $0 = T_0 \leq T_1 \leq T_2 \leq \cdots \leq T_n \leq \dots$ be the arrival times of shocks, and let $X_i \equiv T_i - T_{i-1}$ be the inter-arrival time between i-th and $(i-1)$-th shocks, $i = 1, 2, \dots$. According to the δ-shock model, a system fails if the time lag between two consecutive shocks is less than a prefixed threshold value δ. Then, we have

$$P(L > t | T_1, T_2, \dots, T_{N(t)}, N(t) = n) = P(X_1 > \delta_0, X_2 > \delta_0, \dots, X_n > \delta_0 | N(t) = n).$$

Theorem 4.7 *The survival function of a system for the defined δ-shock model is given by*

$$P(L > t) = \sum_{n=0}^{\lfloor \frac{t}{\delta} \rfloor} \left(\frac{t - n\delta}{t} \right)^n A_n, \quad t > 0,$$

where $A_n = (1 - \boldsymbol{\alpha e}) \mathbf{1_0}(n) - \frac{1}{\lambda t} \left[(\boldsymbol{\beta} \otimes \boldsymbol{\alpha}) (S \otimes I + I \otimes T)^{-1} (S \otimes T) \boldsymbol{e} \right]; \boldsymbol{\beta} = \boldsymbol{\tau}(n),\ S = K(n, \lambda t)$, $n = 0, 1, 2, 3, \dots,$ *and*

$$\mathbf{1_0}(n) = \begin{cases} 1, & \textit{for } n = 0, \\ 0, & \textit{for } n = 1, 2, \dots. \end{cases}$$

Proof: We have

$$P(L > t) = \sum_{n=0}^{\infty} P(L > t, N(t) = n). \tag{4.3}$$

Note that the system survives n-shocks in $[0, t)$ if $X_1 > \delta, X_2 > \delta, \dots, X_n > \delta$ hold or equivalently, $T_1 > \delta, T_2 > \delta + T_1, \dots, T_n > \delta + T_{n-1}$ hold. Again, this condition can equivalently be written as $t > T_n > \delta + T_{n-1} > 2\delta + T_{n-2} > \cdots > n\delta$. Thus, if $t \leq n\delta$, then the probability of the event 'the system survives n shocks till time t' is zero. Now, for $1 \leq n \leq \lfloor \frac{t}{\delta} \rfloor$, we can write

$$P(L > t, N(t) = n) = P(L > t | N(t) = n) P(N(t) = n). \tag{4.4}$$

Consider

$$\begin{aligned} P(L > t | N(t) = n) &= P(X_1 > \delta, X_2 > \delta, \dots, X_n > \delta | N(t) = n) \\ &= P(T_1 > \delta, T_2 > \delta + T_1, \dots, T_n > \delta + T_{n-1} | N(t) = n) \\ &= \int_{n\delta}^{t} \int_{(n-1)\delta}^{t_n - \delta} \cdots \int_{\delta}^{t_2 - \delta} f_{T_1, T_2, \dots, T_{N(t)} | N(t)}(t_1, t_2, \dots, t_n | n) dt_1 dt_2 \dots dt_n \\ &= \left(\frac{n!}{t^n} \right) \int_{n\delta}^{t} \int_{(n-1)\delta}^{t_n - \delta} \cdots \int_{\delta}^{t_2 - \delta} dt_1 dt_2 \dots dt_n \end{aligned}$$

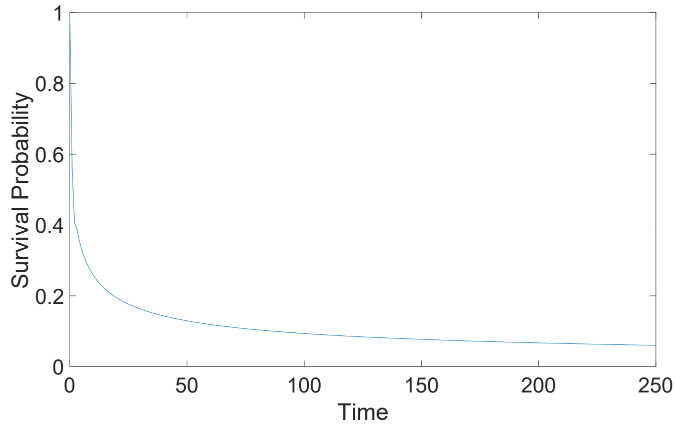

Figure 5: Plot of the system's survival function against $t \in [0, 250]$

$$= \left(\frac{t - n\delta}{t}\right)^n,$$

where the fourth equality holds due to the similar arguments as in the proof of Theorem 4.1. On using the above expression along with (4.4) in (4.3), we get

$$\begin{aligned} P(L > t) &= \sum_{n=0}^{\lfloor \frac{t}{\delta} \rfloor} \left(\frac{(t - n\delta)}{t}\right)^n P(N(t) = n) \\ &= \sum_{n=0}^{\lfloor \frac{t}{\delta} \rfloor} \left(\frac{(t - n\delta)}{t}\right)^n A_n, \end{aligned}$$

where the second equality follows from Theorem 3.1(i). Thus, the result is proved. □

The following example illustatrtes this result. Let

$$\boldsymbol{\alpha} = \begin{pmatrix} 0.2 & 0.8 \end{pmatrix}, \; T = \begin{pmatrix} -2 & 1 \\ 0.5 & -10 \end{pmatrix} \text{ and } \lambda = 5.$$

In Figure 5, we plot the system's survival function over $t \in [0, 250]$, for the fixed $\delta = 2$.

In the next theorem we derive the distribution for the number of a fatal shock that causes the system's failure. Moreover, we obtain the expected number of shocks that the system had experienced till the failure. The proof of the second part of this theorem immediately follows from the first part and hence, is omitted.

Theorem 4.8 *Let shocks occur according to the PPHP with the set of parameters $\{\lambda, \boldsymbol{\alpha}, T\}$. Further, let M be a random variable representing the number of a fatal shock. Then the probability mass function and the mean of M, for the defined δ-shock model, are given by*

$$P(M = m) = \boldsymbol{\alpha} \left((m-1)\delta\lambda I - T\right)^{-1} \boldsymbol{T^0} - \boldsymbol{\alpha} \left(m\delta\lambda I - T\right)^{-1} \boldsymbol{T^0}, \quad m = 1, 2, \ldots,$$

and

$$E(M) = \sum_{m=0}^{\infty} \boldsymbol{\alpha} \left(m\delta\lambda I - T\right)^{-1} \boldsymbol{T^0},$$

respectively, where $\boldsymbol{T^0} = -T\boldsymbol{e}$*.*

Proof: From the definition of the δ-shock model, we have

$$\begin{aligned} P(M = m) &= P(X_1 > \delta, X_2 > \delta, \ldots, X_{m-1} > \delta) \\ &\quad -P(X_1 > \delta, X_2 > \delta, \ldots, X_m > \delta), \quad m = 1, 2, \ldots \end{aligned} \tag{4.5}$$

Note that, on condition $X = x$, the PPHP with the set of parameters $\{\lambda, \boldsymbol{\alpha}, T\}$ is the same as the HPP with the rate λx. Further, we know that the inter-arrival times for the HPP are i.i.d. and follow the exponential distribution. Thus, on condition $X = x$, the inter-arrival times for the PPHP with the set of parameters $\{\lambda, \boldsymbol{\alpha}, T\}$ are also i.i.d. and follow the exponential distribution with the parameter λx. Thus, we can write

$$P(X_1 > \delta, X_2 > \delta, \ldots, X_m > \delta | X = x) = \exp\{-m\delta\lambda x\},$$

which gives

$$\begin{aligned} P(X_1 > \delta, X_2 > \delta, \ldots, X_m > \delta) &= \int_0^{\infty} \exp\{-m\delta\lambda x\} dF_X(x) \\ &= \int_0^{0+} \exp\{-m\delta\lambda x\} dF_X(x) + \int_{0+}^{\infty} \exp\{-m\delta\lambda x\} dF_X(x) \\ &= F_X(0+) - F_X(0-) + \int_{0+}^{\infty} \exp\{-m\delta\lambda x\} \boldsymbol{\alpha} \exp\{Tx\} \boldsymbol{T^0} dx \\ &= (1 - \boldsymbol{\alpha e}) + \boldsymbol{\alpha} \left(m\delta\lambda I - T\right)^{-1} \boldsymbol{T^0}. \end{aligned}$$

On using this in (4.5), we get

$$P(M = m) = \boldsymbol{\alpha} \left((m-1)\delta\lambda I - T\right)^{-1} \boldsymbol{T^0} - \boldsymbol{\alpha} \left(m\delta\lambda I - T\right)^{-1} \boldsymbol{T^0}$$

and hence, the result is proved. □

5 Concluding remarks

We have introduced a new counting process that was called the Poisson phase-type process (PPHP), which is the mixed NHPP with the continuous phase-type mixing distribution. As the set of PH distributions is dense in the set of probability distributions on the non-negative half-line, every lifetime distribution can be approximated by the PH distribution and consequently, any mixed Poisson process can also be *approximated* by the PPHP. This unique property of the PPHP makes it special among other mixed Poisson processes considered in the literature.

The PH distribution functions and other statistical measures (e.g., mean, variance, etc.) can be written in matrix forms that are convenient in computations when using the relevant mathematical packages. Thus, the PPHP presents a convenient and efficient general tool in various real-life applications.

In this paper, we characterize the PPHP by describing the corresponding counting measure and presenting the relationships for the distribution of the number of events in a given interval, distributions of inter-arrival times, etc. Some relevant stochastic comparisons and aging properties are also considered.

As an important, practically sound application that, as we believe, has also its own merit, we consider three different setting for shocks modeling with the PPHP process of shocks. In the future applied research, a combination of these models can be considered, e.g., when each shock in the random failure rate model results not only in the jump in the failure rate but also leads to immediate failure with a given probability (as in the extreme shock model).

Data availability statement Data sharing is not applicable to this article as no data sets were generated or applied during this study.

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