IAC-21-C1.7.13

Hamiltonian model of a low-thrust spacecraft's capture into 1:1 resonance around Vesta

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Abstract

Vesta is the second largest celestial object of the main asteroid belt and it was visited and investigated by the DAWN mission in 2011. The spacecraft used solar-electric propulsion that generates continuous low-thrust. As the spacecraft slowly descends from high altitude mission orbit (HAMO) to low altitude mission orbit (LAMO), it crosses the 1:1 resonance, putting the spacecraft at risk of being permanently trapped at this altitude. The objective of this paper is to develop a hamiltonian model that represents the phenomenon, which is used as a bases for estimating the probability of capture using the adiabatic invariant theory. Firstly, we define the hamiltonian considering the irregular gravitational field up to the second order and degree and the thrust constant in magnitude and opposite to the velocity direction of the spacecraft. Then, we expand the model around the resonance which results the hamiltonian to be reduced in a pendulum-like expression. The reduced model is validated against numerical simulations and is proven to be a good approximation of the dynamics

Keywords: Vesta, Capture into Resonance, Low-Thrust, Adiabatic Invariant Theory, Hamiltonian Dynamics

1 Introduction

The DAWN mission [1] was one of the first missions to use electric propulsion during the cruise phase and the approach to the asteroid. It demonstrates the possibility of relying on electric propulsion for the majority of the mission duration. As the spacecraft slowly approaches the asteroid, there is a possibility that it is captured by the 1:1 resonance and being permanently trapped at this altitude. Since the application of the electric propulsion is the future tendency, the study of the probability of capture into resonance of a spacecraft around a celestial body needs to be investigated. Resonance orbit is defined as the trajectory for which the ratio of the revolution period of the spacecraft around the asteroid and the rotation period of the asteroid around itself is equal to an integer number, e.g., 1:1 resonance orbit for which the spacecraft does one orbit revolution with the same period with that the asteroid rotates around itself. The spacecraft at each revolution encounters the same gravitational configuration, the effect of which sum up over the revolutions and change noticeably the orbit eccentricity and inclination. Delsante [2] defined the autonomous hamiltonian which describes the 1:1 resonance around Vesta for both circular polar orbits and circular equatorial orbits and, through numerical simulations, analyzed the spacecraft's separatrix crossing.

The adiabatic invariant theory is a useful approach to estimate the probability of capture into resonance of a dynamical system if the time-dependent hamiltonian which describes the problem is dependent on slowly changing parameters, for example a pendulum which length slowly changes with time. The only information required a priori is the initial state of the system and an assumption on the amount of variation of the parameter. Firstly Henrard [3] [4] and later on Artemyev [5] did a review of the adiabatic invariant theory presenting application of it in different fields: the former in celestial mechanics to the problem of capture into resonance of Titan and Hyperion, the latter in space plasma systems. Boccaletti and Pucacco [6] dedicate a chapter of their book to this methodology, applying it to the case of one degree of freedom systems and many degree of freedom system.

With this work, we extend the above research and advance the knowledge about this phenomenon by developing a new hamiltonian model without restrictions on the inclination and eccentricity of the orbit and process it in order to be used in the context of the adiabatic invariant theory. This paper is structured as follows. In Section 2, we define the equations of motion describing the dynamics of the spacecraft's motion around Vesta. Section 3 outlines the estimation process of the adiabatic invariant theory and develops the full hamiltonian model and its expansion around the resonance, for the application of the adiabatic invariant theory. Lastly in Section 4 and Section 5, we use the developed hamiltonian to analyse the cases in which the spacecraft is trapped inside the resonance and the case in which the spacecraft is able to immediately escape the resonance location, continuing its slow descent. Section 6 concludes this study.

2 Numerical model

In this section we define the equations of motion of a spacecraft moving around a uniformly rotating asteroid. The model considered is the two-body problem with perturbations from the irregular gravitational field of the asteroid and the low-thrust to which the spacecraft is subject to. The gravitational field is represented by the spherical harmonics model and is truncated to the fourth order and degree, the low-thrust is constant in magnitude and it always directs to the opposite direction of the spacecraft's velocity.

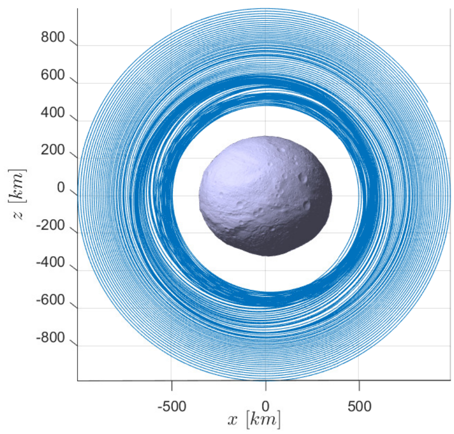

Figure 1: DAWN descent from HAMO to LAMO

Following Kaula [7] , the potential of the gravitational field of an asteroid V in spherical coordinates can be expressed as the sum of the keplerian component and a spherical harmonic expansion up to the degree n and order m

$$V_{r\lambda\phi} = \frac{\mu}{r} + \sum_{n=2}^{\infty}\sum_{m=0}^{n} \frac{\mu}{r}\left(\frac{R_e}{r}\right)^n P_{n,m}(\sin\phi) \\ (C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda) \quad (1)$$

where μ is the gravitational constant of Vesta, R_e is the reference radius of the asteroid, $P_{n,m}(\sin\phi)$ are the associated legendre polynomials, r is the radial distance, ϕ is the geocentric latitude, λ is the longitude, $C_{n,m}$ and $S_{n,m}$ are the spherical harmonics coefficients and n, m are integers.

By transforming in the potential in cartesian coordinate and taking the gradient of the potential and adding the low thrust component to the acceleration, we can define the equations of motion which describe the absolute spacecraft's motion in the asteroid centered inertial frame as Eq.2 where V_{xyz} represents the potential expanded in spherical harmonics as a function of the cartesian coordinates (x, y, z), T is the thrust, m is the spacecraft's mass and $\hat{\boldsymbol{v}}$ is the spacecraft's velocity unit vector. To that, we add the differential equation describing the rate of change of the spacecraft's mass over time as Eq.3 where I_{sp} and g_0 represent the specific impulse and Earth's gravitational acceleration respectively.

$$\ddot{\boldsymbol{x}} = \boldsymbol{\nabla} V_{xyz} - \frac{T}{m(t)}\hat{\boldsymbol{v}} \quad (2)$$

$$\dot{m} = -\frac{T}{I_{sp} g_0} \quad (3)$$

We consider the initial conditions in Table 1.

Table 1: DAWN spacecraft initial conditions at its arrival at Vesta.

Mass (m_0)	1000 kg
Thrust (T)	20 mN
Specific Impulse (I_{sp})	3000 s
SMA (a_0)	1000 km
eccentricity (e_0)	0
Inclination (i_0)	90°
Longitude of the ascending node (Ω_0)	0°
Argument of periapsis (ω_0)	0°

We focus on the 1:1 resonance and we numerically estimate the probability of capture considering 1000 different values of initial true anomalies (θ_0). The outcome of the simulations indicates that if the descent starts at 1000 km and the low-thrust magnitude is 20 mN, the probability of capture of the spacecraft into 1:1 resonance around Vesta is estimated to be $\sim$ 8.6%. The value is coherent with what was previously found by Delsante [2] and Tricarico [8], thus validating our model.

The capture into resonance is a phenomenon that depends on the initial phase angle of the descent. For the

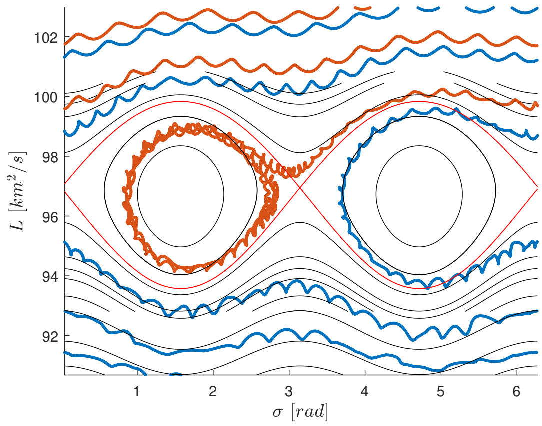

Figure 2: Spacecraft trajectory in the (σ, L) plane. The red line represents the capture case and the blue line represents the escape case.

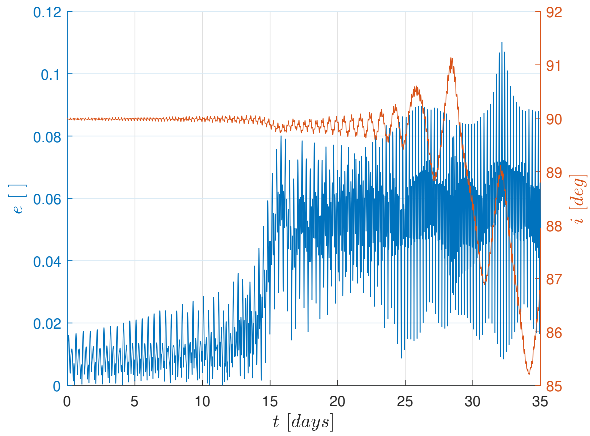

Figure 3: Eccentricity and inclination evolution during DAWN's descent.

analysis, we consider two different values of initial true anomalies to represent the case in which the spacecraft is capture into resonance and the case in which it escapes:

$$\begin{cases} \theta_0 = 30°, & \text{for the capture case} \\ \theta_0 = 50°, & \text{for the escape case} \end{cases} \tag{4}$$

It is possible to represent the results of these two cases in the phase-space plane, in which the x-axis represents the resonance angle $\sigma = \lambda - \vartheta$ defined as the difference between the mean longitude and the sidereal time, while the y-axis represents the angular momentum L.

Notice the different behaviour of the phase space trajectory between the two cases: for the first case in Figure 2, with $\theta_0 = 30°$, the trajectory remains inside the region between the upper and lower separatrices (lines highlighted with the red color), which corresponds to the 1:1 resonance; for the second case, with $\theta_0 = 50°$, the trajectory crosses the upper separatrix and immediately crosses the lower one, thus escaping from the resonance. The results from the numerical simulations are characterized with large oscillations, for this reason we use the MATLAB function `movmean` with a window length of 200 to smooth the results and have a better visualization of the phase space trajectory.

Considering the capture case ($\theta_0 = 30°$), Figure 3 shows the evolution of the eccentricity and inclination of the trajectory as the spacecraft crosses the 1:1 resonance location and the capture happens around the 25^{th} day: after the capture the average value of the eccentricity stays constant as the spacecraft crosses the resonance with Vesta, while the inclination starts oscillating with a larger amplitude and linearly decreases over time. For more details, please refer to our previous paper [9]. These simulations provide us insight on making proper assumptions of applying the AIT.

3 Adiabatic invariant theory

The adiabatic invariant theory (AIT) was initially introduced in the field of quantum mechanics, to give a more solid bases to the quantification rules [6]. Later on, the theory was found useful also in the field of celestial mechanics due to the possibility for many system's hamiltonian to be transformed to a pendulum-like expression [4] and [10] as

$$\mathcal{H} = \frac{1}{2}p^2 - b\cos q \tag{5}$$

The hamiltonian generally has one degree of freedom and depends on slowly varying parameters. By providing realistic assumption on how the parameters change over time, the AIT can provide precise information regarding the system's dynamical evolution.

Formally [11], a variable $I(p, q, \lambda)$ is recognized as an adiabatic invariant if, for every $\epsilon > 0$ there exist a $\delta_\epsilon > 0$ so that, for every $\delta < \delta_\epsilon$ and $t < 1/\delta$

$$|I(p(t), q(t), \delta t) - I(p_0, q_0, 0)| < \epsilon \tag{6}$$

which means that for a time variation of almost $1/\epsilon$, the adiabatic invariant changes in the order of ϵ.

To estimate the probability of capture into resonance using the AIT the following steps are required:

- definition of the hamiltonian model related to the problem which presents phase space trapping;
- expansion of the hamiltonian around the resonance, resulting in a pendulum-like equation;
- determination of the area inside the separatrix and its rate of change as a function of the slowly changing parameter;
- estimation of the probability of capture into 1:1 resonance as a function of the rate of change of the area inside the separatrix.

In this paper, we focus on the first two steps of the process.

3.1 Hamiltonian model definition

The hamiltonian of our problem includes a gravitational component represented in Eq.1 and a low-thrust component. We can express the latter as a function of the orbital parameters as

$$\mathcal{H}_{LT} = -\frac{T}{m} a\vartheta \tag{7}$$

where T is the magnitude of the thrust, m is the instantaneous spacecraft's mass, a is the semi-major axis (SMA) and ϑ is the sidereal time. So the complete hamiltonian related to the 1:1 resonance and containing the gravitational term up to the second order is given as

$$\begin{aligned}\mathcal{H}_{1:1} = &-\frac{\mu^2}{2L^2} + R_e^2 \frac{\mu^4}{L^6} F_{201} G_{210} C_{20} + \\ &\frac{3}{4} R_e^2 \frac{\mu^4}{L^6} F_{220} G_{200} C_{22} \cos 2\lambda + \dot{\vartheta}\Lambda - \frac{T}{m}\frac{L^2}{\mu}\vartheta\end{aligned} \tag{8}$$

where L is the angular momentum, Λ is the is the momentum conjugate to the sidereal time ϑ and the inclination and eccentricity functions are

$$\begin{cases} F_{201} = \left(\frac{3}{4}\sin^2 i(t) - \frac{1}{2}\right) \\ G_{210} = \frac{1}{(1-e^2)^{3/2}} \\ F_{220} = \frac{3}{4}\left(1 + \cos i(t)\right)^2 \\ G_{200} = \left(1 - \frac{5}{2}e^2 + \frac{13}{16}e^4\right) \end{cases} \tag{9}$$

Following [2], we define the resonance angle σ related to the 1:1 resonance as

$$\sigma = \lambda - \vartheta \tag{10}$$

where λ is the mean longitude and ϑ is the sidereal time. We consider a symplectic transformation which leads to the new set of canonical variables

$$\sigma \quad , \quad L' = L \quad , \quad \vartheta' = \vartheta \quad , \quad \Lambda' = \Lambda + L \tag{11}$$

and by selecting only the resonant contributions, the new Hamiltonian is defined as

$$\begin{aligned}\mathcal{H}_{1:1} = &-\frac{\mu^2}{2L^2} - R_e^2 \frac{\mu^4}{L^6} F_{201} G_{210} C_{20} - \\ &-\frac{3}{4} R_e^2 \frac{\mu^4}{L^6} F_{220} G_{200} C_{22} \cos 2\sigma - \dot{\vartheta} L + \frac{T}{m}\frac{L^2}{\mu}\sigma\end{aligned} \tag{12}$$

3.2 Hamiltonian model expansion

The second part of the estimation process involves the expansion of the hamiltonian around resonances and its reduction to a pendulum-like expression.

We search for the equilibria (σ_{eq},L_{eq}) as solutions of

$$\frac{\partial \mathcal{H}}{\partial L} = \frac{\partial \mathcal{H}}{\partial \sigma} = 0 \tag{13}$$

around which the expansion is performed. The full hamiltonian model in Eq.12 related to the 1:1 resonance is divided in two parts $\mathcal{H}_{20}$ and $\mathcal{H}_{22}$ defined as

$$\mathcal{H}_{1:1} = \mathcal{H}_{20} + \mathcal{H}_{22} \tag{14}$$

where

$$\mathcal{H}_{20} = -\frac{\mu^2}{2L^2} - R_e^2 \frac{\mu^4}{L^6} F_{201} G_{210} C_{20} - \dot{\vartheta} L \tag{15}$$

$$\mathcal{H}_{22} = -R_e^2 \frac{\mu^4}{L^6} F_{220} G_{200} C_{22} \cos(2\sigma) + \frac{T}{m}\frac{L^2}{\mu}\sigma \tag{16}$$

The expansion is implemented at two different orders: we expand $\mathcal{H}_{20}$ up to the second order with respect to L_{eq}, while $\mathcal{H}_{22}$ is expanded up to the zero order with respect to L_{eq}. This procedure leads to the following expressions

$$\mathcal{H}_{20} = \frac{1}{2}\alpha(L - L_{eq})^2 + \gamma(L - L_{eq}) + const \tag{17}$$

$$\mathcal{H}_{22} = \beta \cos(2\sigma) + \tau\sigma \tag{18}$$

where

$$\begin{cases} \alpha = 2\left(\frac{3\mu^2}{2L_{eq}^4} + 21 C_{20} F_{201} G_{210} R_e^2 \frac{\mu^4}{L_{eq}^8}\right) \\ \beta = -R_e^2 \frac{\mu^4}{L_{eq}^6} F_{220} G_{200} C_{22} \\ \gamma = \left(\frac{\mu^2}{L_{eq}^3} - \dot{\vartheta} + 6 C_{20} F_{201} G_{210} R_e^2 \frac{\mu^4}{L_{eq}^7}\right) \\ \tau = \frac{T}{m}\frac{L_{eq}^2}{\mu} \end{cases} \tag{19}$$

The reason for this division is to avoid the dependency of the parameters α, β, γ and τ on the coordinate σ. As the simulation starts near the resonance the parameter γ related to the first order term drops to zero and the constant term in $\mathcal{H}_{20}$ does not contribute to the dynamics.

In this scenario the slowly varying parameter is related to the inclination and follows the linear relation

$$i(t) = i_0 - s\Delta t \tag{20}$$

where s is the slope of the linear relation. More details about this assumption in Section 3.2.1. The expression can be manipulated as

$$i(t) = i_0(1 - \lambda) \tag{21}$$

where

$$\lambda = \frac{s}{i_0}\Delta t \tag{22}$$

By defining $p = (L - L_{eq})$, the 1:1 resonance hamiltonian can be written as

$$\mathcal{H}_{1:1} = \frac{1}{2}\alpha(\lambda)p^2 + \beta(\lambda)\cos 2\sigma + \tau\sigma \tag{23}$$

This expression is the one related to a forced pendulum. This problem was previously studied in [12]. We adopt the following change of variables

$$\begin{cases} x = \sqrt{2|p|}\cos(2\sigma) \\ y = \sqrt{2|p|}\sin(2\sigma) \end{cases} \tag{24}$$

leading to the final expression of the fully expanded hamiltonian related to the 1:1 resonance to be used in the probability of capture estimation process

$$\mathcal{H}_{1:1} = \frac{1}{2}\alpha(\lambda)\frac{(x^2+y^2)^2}{4} + \beta(\lambda)\frac{x}{\sqrt{x^2+y^2}} + \\ + \frac{1}{2}\tau\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right) \tag{25}$$

3.2.1 Hybrid approach

The dynamics of the system can be also simulated using Hamilton's equations. Taking into consideration the differential equation describing the rate of change of the spacecraft's mass over time, for a hamiltonian $\mathcal{H} = \mathcal{H}(\sigma, L)$ the Hamilton's equations can be defined as

$$\dot{\sigma} = \frac{\partial \mathcal{H}}{\partial L} \tag{26}$$

$$\dot{L} = -\frac{\partial \mathcal{H}}{\partial \sigma} \tag{27}$$

$$\dot{m} = -\frac{T}{I_{sp}g_0} \tag{28}$$

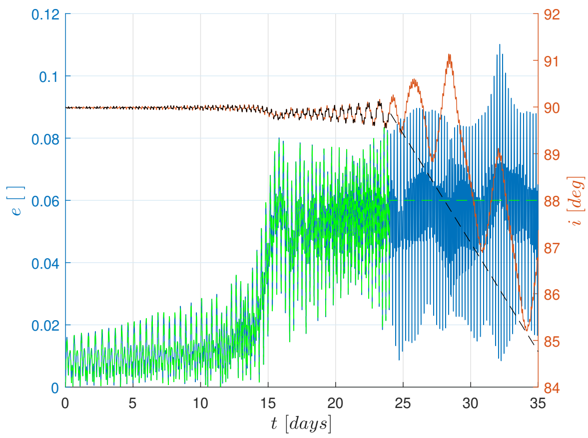

Figure 4: Eccentricity and inclination assumptions for the hybrid approach against the numerical results obtained in the previous sections during DAWN's descent.

For the hamiltonian model defined in Eq.12, the eccentricity is assumed to be constant, while the inclination is assumed to be decreasing linearly over time as in Figure 4. In particular

$$e = 0.06 \tag{29}$$

$$i(t) = i_0 - s\Delta t \tag{30}$$

where s is the slope that better approximates the inclination behaviour after the capture as in Figure 4. The simulation using Hamilton's equations is more precise if it starts near the resonance region. The reason is that the value of eccentricity and inclination change consistently over time and the hamiltonian we defined is incapable of producing very accurate results for long time before the resonance crossing. To address this problem, we adopt a hybrid approach to simulate the phenomenon: far from the resonance location, at 1000 km, the numerical approach described in Section 2 is used to simulate the first part of the descent; as the spacecraft is closer to the 1:1 resonance location the simulation switches using the Hamilton's equations.

4 Capture Case

We simulate the slow descent of Dawn from a high altitude orbit to a low altitude one, using a thrust of 20 mN. An initial polar circular orbit and a continuous thrusting are assumed, with the direction of the thrust opposite to that of the relative velocity of Vesta. Considering the case in which the capture happens ($\theta_0 = 30°$), in Figure 5 we

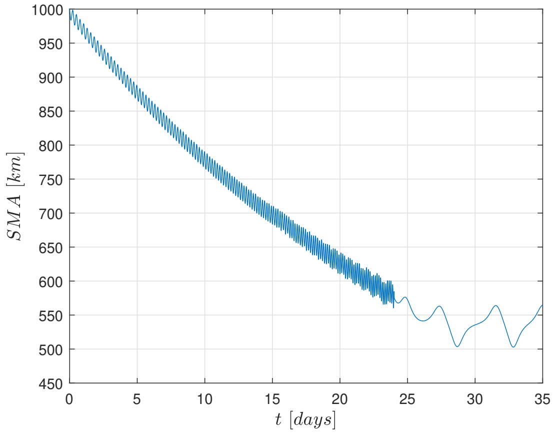

Figure 5: Capture case: semi-major axis (SMA) evolution during descent using the hybrid approach.

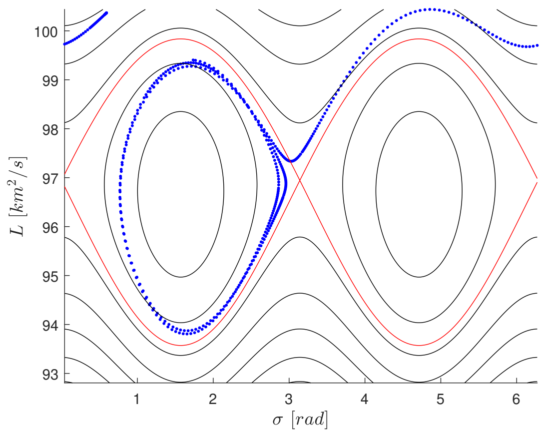

Figure 6: Capture case: Spacecraft trajectory in the (σ, L) plane.

demonstrate the evolution of the semi-major axis (SMA) over time using the hybrid approach.

The capture happens after 25 days. In Figure 6, we demonstrate the result of the last part of the descent simulation that is obtained from Hamilton's equations, in phase space.

The simulation starts in the upper circulation region and, as it crosses the separatrix highlighted in red, the spacecraft is restricted within the rotation region between the separatrix lines. The simulation shows phase space trapping, thus satisfying the first requirement for the estimation of the probability of capture.

The expansion of the hamiltonian leads the phase space to preserve its configuration and it translates the rotation region to the origin. As we change variables from (σ, p) to (x, y), the new phase space is given in 7.

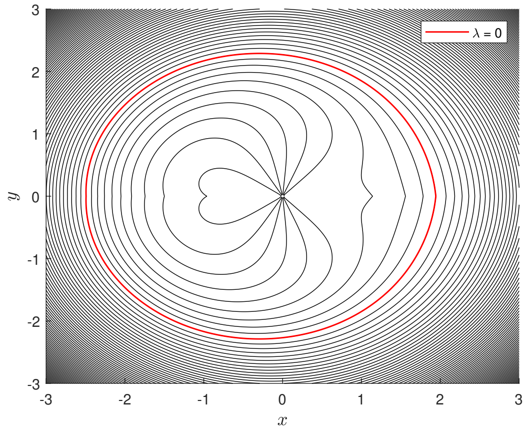

Figure 7: Phase space configuration in the coordinate set (x, y)

The separatrix encloses a single region of space and the area inside the separatrix is the resonance domain that depends on λ as defined in Eq.22. As λ increases, the area enclosed by the separatrix increases as well. Figure 8 shows the dependency of the resonance domain with respect to λ.

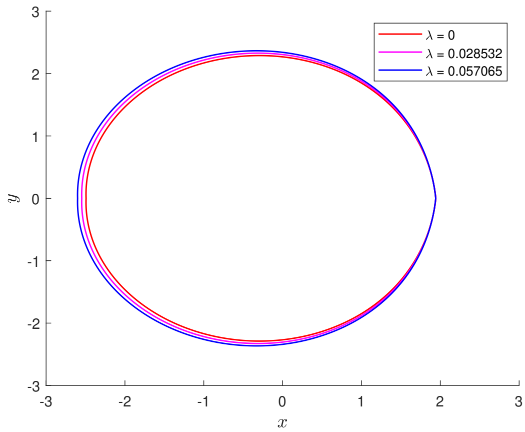

Figure 8: Separatrix evolution as a function of

We project the simulations in Figure 6 in the new phase space (x, y) and obtain the new trajectory represented by the black dashed line in Figure 9. It is noticed that as the area inside the resonance region increases the trajectory remains restricted inside of it.

By assuming that both inclination and eccentricity re-

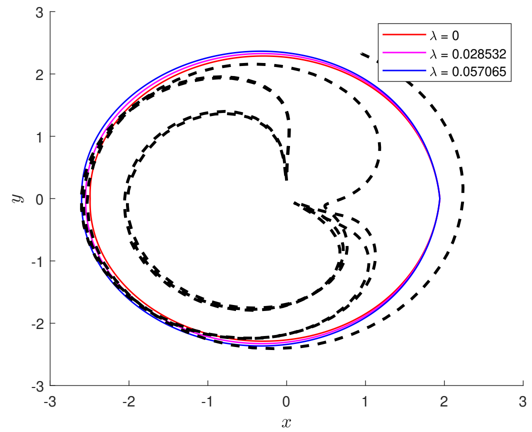

Figure 9: Capture case: Spacecraft trajectory in the (x, y) plane.

main constant,

$$e = 0.06 \tag{31}$$
$$i(t) = i_0 \tag{32}$$

we perform the same simulation again with the same initial conditions and demonstrate the result in Figure 10. The simulations show that the capture does not happen neither $\theta = 30°$ nor for any initial value of true anomaly. The capture is proven to be caused by the decrease of the inclination.

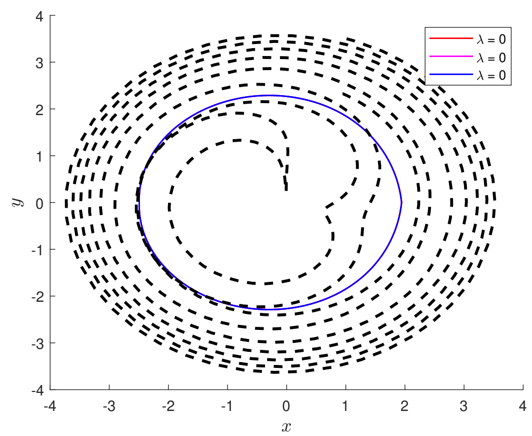

Figure 10: Capture case: Spacecraft trajectory in the (x, y) plane assuming inclination stays constant.

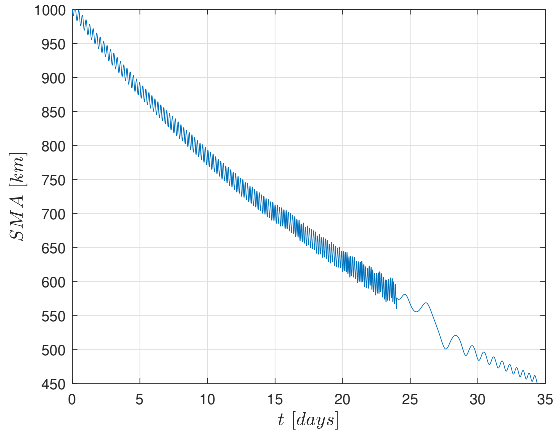

Figure 11: Escape case: SMA evolution during descent using the hybrid approach.

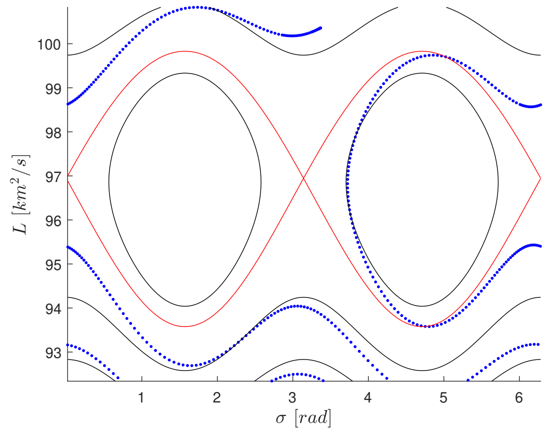

Figure 12: Escape case: Spacecraft trajectory in the (σ, L) plane.

5 Escape case

Considering the case in which the capture does not happen ($\theta = 50°$), in Figure 11 we demonstrate the evolution of the semi-major axis over time using the hybrid approach.

In Figure 12, we demonstrate in phase space the result of the last part of the descent simulation, obtained from Hamilton's equations. It can be noticed that this simulation and the one in Figure 6 are identical to the numerical results shown previously in Figure 2.

We project the simulations in Figure 12 in the new phase space (x, y) and obtain the new trajectory represented by the black dashed line in Figure 13. In this last case, the trajectory crosses and escapes the resonance immediately.

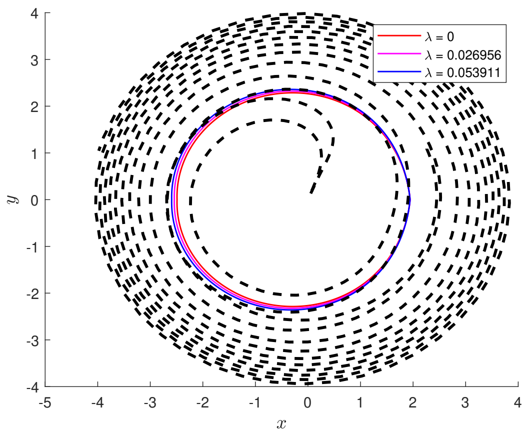

Figure 13: Escape case: Spacecraft trajectory in the (x, y) plane.

6 Conclusion

In this paper the hamiltonian model of a low-thrust spacecraft around an asteroid with irregular gravitational field has been developed. The estimation process of a low-thrust spacecraft's probability of capture into 1:1 resonance has been investigated and the first two steps of the AIT have been performed. The hamiltonian model is proved to be capable of approximating the phase space trapping. Then, we proceeded expanding the hamiltonian model around the resonance and obtain a pendulum-like expression. We represented the trajectory evolution in the phase space of the pendulum-like hamiltonian and found that the capture into 1:1 resonance is due to the slowly varying parameter related to the inclination. Lastly, we analyzed the simulation results for the case of capture into 1:1 resonance and the case in which the spacecraft escapes the resonance.

For future work, we will focus on the last two steps of the estimation process: analytically determine the rate of change of the area inside the separatrix and use this last quantity to estimate the probability of capture.

Acknowledgements

We thank Anton V. Artemyev for his expertise and helpful discussions for certain aspects of our study. This work is funded by ESA OSIP with project title "Resonance Capture of Low-Thrust Spacecraft Around a Small Body" and by the John Anderson Research Award Studentship.

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