Integrable Hamiltonian systems defined on the Lie groups $SO(3)$ and $SU(2)$: an application to the attitude control of a spacecraft

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***Abstract*— This paper considers left-invariant control systems defined on the Lie groups $SU(2)$ and $SO(3)$. Such systems have a number of applications in both classical and quantum control problems. The purpose of this paper is two-fold. Firstly, the optimal control problem for a system varying on these Lie Groups, with cost that is quadratic in control is lifted to their Hamiltonian vector fields through the Maximum principle of optimal control and explicitly solved. Secondly, the control systems are integrated down to the level of the group to give the solutions for the optimal paths corresponding to the optimal controls. In addition it is shown here that integrating these equations on the Lie algebra $\mathfrak{su}(2)$ gives simpler solutions than when these are integrated on the Lie algebra $\mathfrak{so}(3)$.**

I. INTRODUCTION

The motivation for studying affine control systems on the Lie Groups $SO(3)$ and $SU(2)$ come from a wealth of applications in both classical and quantum control problems, see [1], [2],[3], [4] and [5]. As the Lie algebras of the Lie Groups $SO(3)$ and $SU(2)$ are isomorphic their symplectic topology is identical and therefore their Hamiltonian lift yield the same vector fields. In the first part of the paper we lift the affine control system with quadratic cost function to its Hamiltonian vector fields through the Maximum Principle of Optimal Control and solve for the optimal controls explicitly. The equations of motion can then be expressed conveniently in Lax Pair Form, see [6].

In the second part of the paper we integrate the Lax Pair equations derived via the Maximum Principle of optimal control, to obtain the corresponding optimal paths in $g(t) \in G$. It was Felix Klein who discovered that in the case of Lagrange's top, simpler solutions were obtained when the Special Unitary group $SU(2)$ is used as the configuration space as opposed to the Special Orthogonal group $SO(3)$, see [7]. Recent work [8] used $SU(2)$ to describe the configuration of the mechanical top instead of $SO(3)$ or Euler angles to represent the moving frame. In this paper we use the more general setting of affine control systems defined on Lie groups to illustrate this.

Here the problem is defined abstractly as a left-invariant control system defined on either the Lie Group $SO(3)$ or $SU(2)$. The general problem is defined as a left-invariant differential systems of the form:

$$\frac{dg(t)}{dt} = g(t)(\sum_{1}^{3} u_i A_i) \tag{1}$$

where the u_i's are the control functions, $g(t) \in SO(3)$ or $g(t) \in SU(2)$, and the A_i's are the standard basis of the Lie algebra of $SO(3)$ or $SU(2)$ respectively. The basis for $\mathfrak{so}(3)$

TABLE I

LIE BRACKET TABLE

[,]	A_1	A_2	A_3
A_1	0	A_3	-A_2
A_2	-A_3	0	A_1
A_3	A_2	-A_1	0

is:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2}$$

and the choice of basis for $\mathfrak{su}(2)$ is:

$$A_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; A_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; A_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{3}$$

It is well known that the Lie algebra $\mathfrak{so}(3)$ is isomorphic to $\mathfrak{su}(2)$, see [9]. The Lie Bracket is defined as $[X, Y] = XY - YX$ for $X, Y \in \mathfrak{g}$. As the two Lie algebras are isomorphic they commute in the same way as is shown in the Lie bracket Table I. The isomorphism between the vector spaces $\mathbb{R}^3 \to \mathfrak{so}(3) \to \mathfrak{su}(2)$ is given explicitly (see [6] for a derivation):

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \to \begin{pmatrix} \frac{i}{2}x_1 & \frac{1}{2}(x_2 + ix_3) \\ -\frac{1}{2}(x_2 - ix_3) & -\frac{i}{2}x_1 \end{pmatrix} \tag{4}$$

where $x_1, x_2, x_3 \in \mathbb{R}$. In this paper we are concerned with the optimal control of systems of the form (1) with the problem of minimizing the cost function quadratic in control:

$$\frac{1}{2}\int_0^T \sum_{i=1}^{3} c_i u_i^2 dt \tag{5}$$

subject to the given boundary conditions $g(0) = g_0$ and $g(T) = g_T$. The general setting of left-invariant control systems on Lie groups can also accommodate vector fields that are not controlled i.e underactuated and systems with drifting vector fields. In equations (1) and (5) a control is set to zero if the vector field is not actuated and a constant if it is drifting. Through the Maximum principal of optimal control a left-invariant maximised Hamiltonian can be constructed from equations (1) and (5), see [6]. In turn the Hamiltonian function is then used along with the Poisson bracket to calculate the corresponding non-canonical Hamiltonian vector fields. It is well known that the Hamiltonian vector fields of any 3 dimensional Lie group are completely integrable in the Liouville sense, see [10]. The solutions to these integrable Hamiltonian vector fields are called extremals. The projected extremal solutions down to the level of the group are called optimal paths.

In summary the first part of the paper lifts the affine control system (1) using the Maximum principle to its corresponding Hamiltonian Vector fields and the optimal controls solved. The system can then be conveniently expressed in Lax pair form, see [2]:

$$\begin{aligned} \frac{dg(t)}{dt} &= g(t)dH(t), \\ \dot{L}(t) &= [L(t), dH(t)] \end{aligned} \tag{6}$$

on the Lie group $SO(3)$ where $dH(t)$ is defined by the matrix:

$$dH(t) = \begin{pmatrix} 0 & -\frac{\partial H}{\partial M_3} & \frac{\partial H}{\partial M_2} \\ \frac{\partial H}{\partial M_3} & 0 & -\frac{\partial H}{\partial M_1} \\ -\frac{\partial H}{\partial M_2} & \frac{\partial H}{\partial M_1} & 0 \end{pmatrix} \tag{7}$$

where H is the maximized Hamiltonian and the components M_i are the extremal solutions and $L(t)$ is described by the matrix

$$L(t) = \begin{pmatrix} 0 & -M_3 & M_2 \\ M_3 & 0 & -M_1 \\ -M_2 & M_1 & 0 \end{pmatrix} \tag{8}$$

It is assumed in this paper that the extremal solutions M_i and the functions $\frac{\partial H}{\partial M_i}$ are meromorphic functions of time. In the analogy to the spacecraft attitude problem the extremal solutions are the components of angular momentum. Using the isomorphism (4) the Lax Pair equations (6) describe the equations of motion on the Lie Group $SU(2)$, where:

$$\begin{aligned} dH(t) &= \frac{1}{2}\begin{pmatrix} i\frac{\partial H}{\partial M_1} & \frac{\partial H}{\partial M_2} + i\frac{\partial H}{\partial M_3} \\ -(\frac{\partial H}{\partial M_2} - i\frac{\partial H}{\partial M_3}) & -i\frac{\partial H}{\partial M_1} \end{pmatrix} \\ L(t) &= \frac{1}{2}\begin{pmatrix} iM_1 & M_2 + iM_3 \\ -(M_2 - iM_3) & -iM_1 \end{pmatrix}, \end{aligned} \tag{9}$$

In the first part of this paper we derive these equations using the maximum principle of optimal control and the second part of the paper involves integrating these Lax Pair Equations (6) to obtain the corresponding optimal paths in G. Finally, the theory is applied to the under-actuated spacecraft attitude control problem. In [3] it is shown that the rotational kinematics of a spacecraft or rigid body can be described by equation (1) on the Lie Group $SO(3)$.

II. Deriving the Lax Pair Equations via the Maximum Principle

This paper is concerned with the solutions of the equations (6) derived by lifting the state space equation (1) to its corresponding Hamiltonian vector fields via the Maximum principle of optimal control. Here we briefly recall the Maximum Principle and define the lift to the Hamiltonian vector fields (for a detailed description see [6]:

Definition 1 *The Hamiltonian H associated with a vector field X on a manifold M is a function on T^*M defined by $H(\xi) = \xi(X(x))$ for each $\xi \in T^*_x M$. The Hamiltonian vector field $\vec{H}$ is called the Hamiltonian lift of X.*

The control Hamiltonian corresponding to the state space (1) while minimizing the function (5) is written as:

$$H(\xi, u, g) = \sum_{i=1}^{3} u_i \xi(gA_i) - \rho_0 \sum_{i=1}^{3} c_i u_i^2 \tag{10}$$

where $\xi \in T^*_g G$ and $\rho_0 = 1$ for regular extremals and $\rho_0 = 0$ for abnormal extremals. In this paper we shall only consider the regular extremals. As the vector fields are left invariant they can be pulled back by the left group action. The pull-back in this case is explicitly stated as $\xi(\cdot) = \hat{p}(g^{-1}(\cdot))$. i.e $\xi \in T^*G$ is pulled back to give a function $\hat{p} \in \mathfrak{g}^*$. The control Hamiltonian can then be written as

$$H(\hat{p}, u) = \sum_{i=1}^{3} u_i \hat{p}(A_i) - \sum_{i=1}^{3} c_i u_i^2 \tag{11}$$

Through the maximum principle of optimal control and the fact that the control Hamiltonian is a quadratic function of the control functions u_i and $\frac{d^2H}{du_i^2} < 0$ implies that there exists exactly one global maximum at each point. Then calculating $\frac{\partial H}{\partial u_i} = 0$ gives the optimal controls in feedback form as:

$$u_i^* = \frac{1}{c_i}\hat{p}(A_i) \tag{12}$$

where $i = 1, 2, 3$. Then substituting (12) back into (11) gives the optimal Hamiltonian $H(\hat{p}, u^*)$ which will be denoted as H for simplicity. Define the extremal solutions $M_i = \hat{p}(A_i)$. From this the Hamiltonian vector fields can be calculated using the Poisson bracket. The Poisson bracket is a Lie algebra homomorphism

$$\{M_i, M_j\} = -\hat{p}([A_i, A_j]) \tag{13}$$

Then let $l \in \mathfrak{g}^*$ where the coordinates of l are M_1, M_2, M_3 then the Hamiltonian vector fields can be written in compact form as:

$$\frac{dl}{dt} = \{l, H\} \tag{14}$$

on semi-simple Lie groups each element in $\mathfrak{g}^*$ can be uniquely identified with an element in $\mathfrak{g}$, implies that the element $l \in \mathfrak{g}^*$ can be identified with an element $L(t) \in \mathfrak{g}$ where

$$L(t) = M_1 A_1 + M_2 A_2 + M_3 A_3 \tag{15}$$

then the equation (14) can be expressed in the well known dual form as:

$$\dot{L}(t) = [L(t), dH(t)] \tag{16}$$

In addition to this equation, substituting the optimal controls (12) into (1) gives

$$\frac{dg(t)}{dt} = g(t)dH(t) \tag{17}$$

Then equations (16) and (17) give the equations in Lax pair form. These equations will be integrated on the Lie algebras of $SO(3)$ and $SU(2)$.

III. The integration procedure; solving the optimal paths

In this section we derive a conserved quantity from the Lax Pair equations and also identify a particular orbit, (assuming an initial $g_0 \in G$) that greatly simplifies the integration procedure. Integrating equation (17) with respect to this particular orbit enables us to compute explicit formulae for the optimal paths $g(t) \in G$. Firstly, recall the Lax Pair equations describing the optimal solutions derived in the previous section:

$$\frac{dg(t)}{dt} = g(t)dH \tag{18}$$

$$\dot{L}(t) = [L(t), dH(t)] \tag{19}$$

in order to solve for $g(t)$ we use equation (18) and the general solution of (19):

$$L(t) = g(t)^{-1}L(0)g(t) \tag{20}$$

Indeed (20) can be shown to be the general solution of (19) by differentiation:

$$\begin{aligned} \frac{dL(t)}{dt} &= \frac{dg(t)^{-1}}{dt}L(0)g(t) + g(t)^{-1}L(0)\frac{dg(t)}{dt} \\ &= -dHg(t)^{-1}L(0)g(t) + g(t)^{-1}L(0)g(t)dH \\ &= L(t)dH - dHL(t) \\ &= [L(t), dH(t)] \end{aligned} \tag{21}$$

Here $L(0)$ is the $L(t)$ matrix at $t = 0$ and is therefore a matrix with constant entries. As $g(t)$ varies, $g(t)L(t)g(t)^{-1}$ describes the conjugacy class of $L(t)$. Therefore, the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ of $L(t)$ (equation (8)) are constant along each orbit:

$$\begin{aligned} \lambda_1 = 0, \lambda_2 &= -\sqrt{-M_1^2 - M_2^2 - M_3^2} \\ \lambda_3 &= \sqrt{-M_1^2 - M_2^2 - M_3^2} \end{aligned} \tag{22}$$

Let us denote $I_2 = M_1^2 + M_2^2 + M_3^2$, therefore it follows from (22) that I_2 is constant along the Hamiltonian flow. This conserved quantity will be used to derive explicit expressions for $g(t) \in G$ later in the paper. Through the same argument the eigenvalues of $L(t) \in \mathfrak{su}(2)$ imply that $M_1^2 + M_2^2 + M_3^2$ is also constant for the system on $SU(2)$.

To derive equations for the optimal curves $g(t) \in G$ in its most explicit form, it is useful to consider a particular solution of $g(t)L(t)g(t)^{-1} = L(0)$, from equation (20). It is shown in [6], that as $SO(3)$ acts transitively on the sphere, there always exists an initial $g_0 = g(0) \in SO(3)$ such that $L(0) \in \mathfrak{so}(3)$ can be conjugated to $g_0 L(0) g_0^{-1} = \sqrt{I_2}A_1$, where A_1 is the basis vector in (2). This corollary extends to $g(t) \in SU(2)$ for $L(0) \in \mathfrak{su}(2)$ where A_1 is the basis (3). Thus, for simplicity and to obtain more explicit solutions it suffices to integrate the particular orbit

$$g(t)L(t)g(t)^{-1} = \sqrt{I_2}A_1 \tag{23}$$

In summary to obtain formulae for the optimal curves $g(t) \in G$ we integrate:

$$g(t)^{-1}\frac{dg(t)}{dt} = dH(t) \tag{24}$$

with respect to the particular orbit:

$$g(t)L(t)g(t)^{-1} = \sqrt{I_2}A_1 \tag{25}$$

where

$$I_2 = M_1^2 + M_2^2 + M_3^2 \tag{26}$$

is an integral of motion. I_2 is constant along the Hamiltonian flow and it is assumed in the remainder of the paper that $M_2^2 + M_3^2 \neq 0$.

A. Integrating down to $SO(3)$

The optimal control problem defined on the Lie algebra of $SO(3)$ is now integrated down to the level of the group. $L(t)$ and $dH(t)$ are defined as (8) and (7) respectively. For convenience define a constant $K^2 = I_2$. K is constant along the Hamiltonian flow. Let ϕ_1, ϕ_2, ϕ_3 denote the coordinates of a point in $SO(3)$ according to the formula:

$$g(t) = \exp(\phi_1 A_1)\exp(\phi_2 A_2)\exp(\phi_3 A_1) \tag{27}$$

where A_1 and A_2 are as in (2). ϕ_1, ϕ_2, ϕ_3 are known as Euler angles (see [6]). Using the equation (25) write:

$$L(t) = Kg(t)^{-1}A_1 g(t) \tag{28}$$

and therefore,

$$L(t) = K\exp(-A_1\phi_3)\exp(-A_2\phi_2)A_1\exp(A_2\phi_2)\exp(A_1\phi_3) \tag{29}$$

It follows that

$$L(t) = K\begin{pmatrix} 0 & -\cos\phi_3\sin\phi_2 & \sin\phi_2\sin\phi_3 \\ \cos\phi_3\sin\phi_2 & 0 & -\cos\phi_2 \\ -\sin\phi_2\sin\phi_3 & \cos\phi_2 & 0 \end{pmatrix} \tag{30}$$

then equating the $L(t)$ matrix (30) to the $L(t)$ matrix in (8) gives:

$$M_1 = K\cos\phi_2 \tag{31}$$

and therefore

$$\sin\phi_2 = \pm\sqrt{1 - \frac{M_1^2}{K^2}} = \pm\frac{\sqrt{M_2^2 + M_3^2}}{K} \tag{32}$$

furthermore

$$\begin{aligned} M_2 &= K\sin\phi_2\sin\phi_3 \\ M_3 &= K\sin\phi_2\cos\phi_3 \end{aligned} \tag{33}$$

Then in (33) dividing M_2 by M_3 gives ϕ_3 in terms of the extremal solutions:

$$\frac{M_2}{M_3} = \tan\phi_3 \tag{34}$$

Therefore

$$\sin\phi_3 = \pm\frac{M_2}{\sqrt{M_2^2 + M_3^2}}; \cos\phi_3 = \pm\frac{M_3}{\sqrt{M_2^2 + M_3^2}} \tag{35}$$

In order to obtain an expression for ϕ_1 we use the coordinate representation of $g(t)$ (equation (27)) and substitute this equation in (24) to yield:

$$g(t)^{-1}\frac{dg(t)}{dt} = \dot{\phi}_1\begin{pmatrix} 0 & -\cos\phi_3\sin\phi_2 & \sin\phi_2\sin\phi_3 \\ \cos\phi_3\sin\phi_2 & 0 & -\cos\phi_2 \\ -\sin\phi_2\sin\phi_3 & \cos\phi_2 & 0 \end{pmatrix} + \dot{\phi}_2\begin{pmatrix} 0 & \sin\phi_3 & \cos\phi_3 \\ -\sin\phi_3 & 0 & 0 \\ -\cos\phi_3 & 0 & 0 \end{pmatrix} + \dot{\phi}_3\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{36}$$

then equating (36) to $dH(t)$ in (7) yields:

$$\frac{\partial H}{\partial M_2} = \dot{\phi}_1\sin\phi_2\sin\phi_3 + \dot{\phi}_2\cos\phi_3 \tag{37}$$

$$\frac{\partial H}{\partial M_3} = \dot{\phi}_1\sin\phi_2\cos\phi_3 - \dot{\phi}_2\sin\phi_3 \tag{38}$$

dividing (37) by $\cos\phi_3$ and (38) by $\sin\phi_3$, adding the two equations and rearranging gives:

$$\dot{\phi}_1 = \frac{\frac{\partial H}{\partial M_3}\cos\phi_3 + \frac{\partial H}{\partial M_2}\sin\phi_3}{\sin\phi_2} \tag{39}$$

then substituting equations (32) and (35) into (39) and simplifying yields:

$$\dot{\phi}_1 = K\left(\frac{\frac{\partial H}{\partial M_2}M_2 + \frac{\partial H}{\partial M_3}M_3}{M_2^2 + M_3^2}\right) \tag{40}$$

As the right hand side of equation (40) is a meromorphic function, as M_i and $\frac{\partial H}{\partial M_i}$ are meromorphic functions of time, implies that $\dot{\varphi}_1$ is also a meromorphic function. Therefore, equation (40) can be integrated to obtain ϕ_1. From this $\sin\phi_1$ and $\cos\phi_1$ are easily found and can be substituted directly into (27). Therefore, the coordinates on $g(t) \in SO(3)$ have been solved in terms of the elements in the dual of the Lie Algebra. Calculating (27) explicitly and substituting in these coordinates gives an explicit expression for the optimal curves $g(t) \in SO(3)$. See [6] for the explicit form of equation (27).

B. Integrating down to $SU(2)$

The optimal control problem defined on the Lie algebra of $SU(2)$ is now integrated down to the level of the group. $L(t)$ and $dH(t)$ are defined as (9). For convenience define a constant $K^2 = I_2$. K is constant along the Hamiltonian flow. Define $\varphi_1, \varphi_2, \varphi_3$ to denote the coordinates of a point in $SU(2)$ according to the formula:

$$g_1(t) = \exp(\varphi_1 A_1)\exp(\varphi_2 A_2)\exp(\varphi_3 A_1) \tag{41}$$

where A_1 and A_2 are as in (3). Assume now that K is non-zero. Then from equation (25):

$$L(t) = K g_1(t)^{-1} A_1 g_1(t) \tag{42}$$

and therefore,

$$L(t) = K\exp(-A_1\varphi_3)\exp(-A_2\varphi_2)A_1\exp(A_2\varphi_2)\exp(A_1\varphi_3) \tag{43}$$

It follows that

$$L(t) = \frac{iK}{2}\begin{pmatrix} \cos\varphi_2 & e^{-i\varphi_3}\sin\varphi_2 \\ e^{i\varphi_3}\sin\varphi_2 & -\cos\varphi_2 \end{pmatrix} \tag{44}$$

Then equating the $L(t)$ matrix in (9) to (44) gives

$$M_1 = K\cos\varphi_2 \tag{45}$$

and furthermore

$$\begin{aligned} M_2 + iM_3 &= iKe^{-i\varphi_3}\sin\varphi_2 \\ iM_3 - M_2 &= iKe^{i\varphi_3}\sin\varphi_2 \end{aligned} \tag{46}$$

from (45) it is easily shown that:

$$\sin\varphi_2 = \pm\frac{\sqrt{M_2^2 + M_3^2}}{K} \tag{47}$$

substituting equation (47) into the equations (46) then adding the two equations and simplifying gives:

$$\cos\varphi_3 = \pm\frac{M_3}{\sqrt{M_2^2 + M_3^2}} \tag{48}$$

following the same procedure but subtracting one equation from another in (46) yields:

$$\sin\varphi_3 = \pm\frac{M_2}{\sqrt{M_2^2 + M_3^2}} \tag{49}$$

It remains to solve for φ_1. Using the coordinate representation of $g(t)$ (41) and substituting into the equation (24) yields:

$$g_1(t)^{-1}\frac{dg_1(t)}{dt} = \frac{\dot{\varphi}_1}{2}\begin{pmatrix} i\cos\varphi_2 & ie^{-i\varphi_3}\sin\varphi_2 \\ ie^{i\varphi_3}\sin\varphi_2 & -i\cos\varphi_2 \end{pmatrix} + \frac{\dot{\varphi}_2}{2}\begin{pmatrix} 0 & e^{-i\varphi_3} \\ -e^{i\varphi_3} & 0 \end{pmatrix} + \frac{\dot{\varphi}_3}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{50}$$

then equating (50) to $dH(t)$ in (9) yields:

$$\frac{\partial H}{\partial M_1} = \dot{\varphi}_1\cos\varphi_2 + \dot{\varphi}_3 \tag{51}$$

and

$$\begin{aligned} \frac{\partial H}{\partial M_2} + i\frac{\partial H}{\partial M_3} &= \dot{\varphi}_1 ie^{-i\varphi_3}\sin\varphi_2 + \dot{\varphi}_2 e^{-i\varphi_3} \\ -\frac{\partial H}{\partial M_2} + i\frac{\partial H}{\partial M_3} &= \dot{\varphi}_1 ie^{i\varphi_3}\sin\varphi_2 - \dot{\varphi}_2 e^{i\varphi_3} \end{aligned} \tag{52}$$

the two equations in (52) can be rearranged to give:

$$\begin{aligned} \frac{\frac{\partial H}{\partial M_2}}{e^{-i\varphi_3}} + \frac{i\frac{\partial H}{\partial M_3}}{e^{-i\varphi_3}} &= \dot{\varphi}_1 i\sin\varphi + \dot{\varphi}_2 \\ -\frac{\frac{\partial H}{\partial M_2}}{e^{i\varphi_3}} + \frac{i\frac{\partial H}{\partial M_2}}{e^{i\varphi_3}} &= \dot{\varphi}_1 i\sin\varphi - \dot{\varphi}_2 \end{aligned} \tag{53}$$

then adding the two equations in (53) and rearranging:

$$\frac{\frac{\partial H}{\partial M_2}}{e^{-i\varphi_3}} - \frac{\frac{\partial H}{\partial M_2}}{e^{i\varphi_3}} + \frac{i\frac{\partial H}{\partial M_3}}{e^{i\varphi_3}} + \frac{i\frac{\partial H}{\partial M_3}}{e^{-i\varphi_3}} = 2\dot{\varphi}_1 i\sin\varphi_2 \tag{54}$$

on substituting the expressions (46) into (54) and simplifying obtain:

$$\dot{\varphi}_1 = K\left(\frac{\frac{\partial H}{\partial M_2}M_2 + \frac{\partial H}{\partial M_3}M_3}{M_2^2 + M_3^2}\right) \tag{55}$$

As the right hand side of equation (55) is a meromorphic function, as M_i and $\frac{\partial H}{\partial M_i}$ are meromorphic functions of time, implies that $\dot{\varphi}_1$ is also a meromorphic function. Therefore, equation (55) can be integrated to obtain φ_1. This illustrates that integrating on $SU(2)$ gives exactly the same expressions for optimal paths in local coordinates or Euler angles as integrating on $SO(3)$ i.e $\phi_i = \varphi_i$. However, the solutions $g(t)$ are expressed much more compactly on $SU(2)$. Calculating (41) explicitly yields:

$$g_1(t) = \begin{pmatrix} e^{\frac{1}{2}i\varphi_1} e^{\frac{1}{2}i\varphi_3} \cos\frac{\varphi_2}{2} & e^{\frac{1}{2}i\varphi_1} e^{-\frac{1}{2}i\varphi_3} \sin\frac{\varphi_2}{2} \\ -e^{-\frac{1}{2}i\varphi_1} e^{\frac{1}{2}i\varphi_3} \sin\frac{\varphi_2}{2} & e^{-\frac{1}{2}i\varphi_1} e^{-\frac{1}{2}i\varphi_3} \cos\frac{\varphi_2}{2} \end{pmatrix} \tag{56}$$

with

$$\begin{aligned} \cos\frac{\varphi_2}{2} &= \sqrt{\frac{1+\cos\varphi_2}{2}} = \sqrt{\frac{K+M_1}{2K}} \\ \sin\frac{\varphi_2}{2} &= \sqrt{\frac{1-\cos\varphi_2}{2}} = \sqrt{\frac{K-M_1}{2K}} \\ e^{\frac{1}{2}i\varphi_3} &= \cos\frac{\varphi_3}{2} + i\sin\frac{\varphi_3}{2} \\ &= \left(\frac{\sqrt{M_2^2+M_3^2}+M_3}{2\sqrt{M_2^2+M_3^2}}\right)^{1/2} + i\left(\frac{\sqrt{M_2^2+M_3^2}-M_3}{2\sqrt{M_2^2+M_3^2}}\right)^{1/2} \end{aligned} \tag{57}$$

and as $\dot{\varphi}_1$ is a meromorphic function, (55) can be integrated and it then follows that $e^{\frac{1}{2}i\varphi_1}$ and $e^{-\frac{1}{2}i\varphi_1}$ are easily calculated and substituted into (56). Therefore, for any left-invariant Hamiltonian system the corresponding solutions on $SU(2)$ can be expressed much more compactly and simply than when expressed on $SO(3)$. This generalizes the findings of Felix Klein who discovered simpler solutions on $SU(2)$ for Lagrange's top, a subsystem of this problem. Having solved for the optimal curves in the group it is also of interest to study the projections of the element $g(t) \in G$ onto the base space, in this case $\mathbb{S}^2$. The projections onto $\mathbb{S}^2$ will be the same for $SO(3)$ and $SU(2)$. For $SO(3)$, this projection is done by multiplying the matrix (27) on the right hand side by the vector $\vec{e}_1 = [1,0,0]^T$, then the vector $\vec{x} = g(t)\vec{e}_1 = [x,y,z]^T$ is:

$$\begin{aligned} x &= \cos\varphi_2 \\ y &= \sin\varphi_1 \sin\varphi_2 \\ z &= -\cos\varphi_1 \sin\varphi_2 \end{aligned} \tag{58}$$

In the case of $SU(2)$ with $g_1(t)$ defined as (56) the equivalent projection:

$$g_1(t) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} g_1(t)^{-1} \to \mathbb{S}^2 \tag{59}$$

gives an element of $SU(2)$ isomorphic to $\mathbb{S}^2$ through equation (4). Therefore substituting equations (45), (47) and (55) into (58) gives the optimal curves $\vec{x} \in \mathbb{S}^2 \subset \mathbb{R}^3$ in terms of the extremal solutions:

$$\begin{aligned} x &= \frac{M_1}{K} \\ y &= \pm\frac{\sqrt{M_2^2+M_3^2}}{K} \sin\left(K\int_0^t \left(\frac{\frac{\partial H}{\partial M_2}M_2 + \frac{\partial H}{\partial M_3}M_3}{M_2^2+M_3^2}\right)dt\right) \\ z &= \mp\frac{\sqrt{M_2^2+M_3^2}}{K} \cos\left(K\int_0^t \left(\frac{\frac{\partial H}{\partial M_2}M_2 + \frac{\partial H}{\partial M_3}M_3}{M_2^2+M_3^2}\right)dt\right) \end{aligned} \tag{60}$$

Clearly, this projection is onto the unit sphere as $\|\vec{x}\|^{1/2} = 1$.

IV. Integrable Hamiltonian Control Systems Example

The attitude control of a Spacecraft has been modelled as a left-invariant control system defined on the Lie group $SO(3)$, see [3]. In this section we solve for the optimal controls explicitly for an under-actuated left invariant control system. The methods used here to derive the extremal solutions are outlined in [11]. Once the extremal solutions have been solved explicitly they are substituted into the equations derived in the previous section to yield the optimal curves $g(t) \in G$ and the base space $\hat{x} \in \mathbb{S}^2$. The Spacecraft can only be controlled about two axis (in the general equations (1) set $u_3 = 0$) and therefore the differential equation describing the spacecraft is:

$$\frac{dg(t)}{dt} = g(t)\left(u_1 A_1 + u_2 A_2\right) \tag{61}$$

which is a controllable single bracket system since $A_3 = [A_1, A_2]$, see [3] for detail. In the spacecraft attitude control problem we wish to minimize some energy type cost function quadratic in control, in the under-actuated case because $u_3 = 0$ this function is:

$$\frac{1}{2}\int_0^t c_1 u_1^2 + c_2 u_2^2 dt \tag{62}$$

where $c_1, c_2 \in \mathbb{R}$ then from equation (11) the regular Hamiltonian is:

$$H = u_1 M_1 + u_2 M_2 - \frac{1}{2}(c_1 u_1^2 + c_2 u_2^2) \tag{63}$$

from equation (12) the optimal controls are expressed in feedback form as:

$$\begin{aligned} u_1 &= \frac{M_1}{c_1} \\ u_2 &= \frac{M_2}{c_2} \end{aligned} \tag{64}$$

substituting (64) into (63) gives the maximized Hamiltonian:

$$H = \frac{1}{2}\left(\frac{M_1^2}{c_1} + \frac{M_2^2}{c_2}\right) \tag{65}$$

using the Poisson bracket (13) and the Lie bracket Table I, the Hamiltonian vector fields can be calculated as:

$$\begin{aligned} \dot{M}_1 &= \{M_1, H\} = \frac{M_2}{c_2}\{M_1, M_2\} = -\frac{M_2 M_3}{c_2} \\ \dot{M}_2 &= \frac{M_1 M_3}{c_1} \\ \dot{M}_3 &= \left(\frac{c_1 - c_2}{c_1 c_2}\right) M_1 M_2 \end{aligned} \tag{66}$$

these equations can then be solved in terms of Weierstrass elliptic functions. However, for simplicity of exposition assume azimuthal symmetry of the spacecraft $c_1 = c_2 = c$, and that M_3 is some constant, call $\sqrt{s}$. Then the Casimir function (26) which in this problem is analogous to the conservation of angular momentum gives, $I_2 - s = M_1^2 + M_2^2$ then parameterizing, using polar coordinates the extremal solutions are:

$$\begin{aligned} M_1 &= r\sin\theta \\ M_2 &= r\cos\theta \\ M_3 &= \sqrt{s} \end{aligned} \tag{67}$$

where $r = \sqrt{I_2 - s}$ and θ is calculated in the same manner as [11], to give $\dot{\theta} = -\frac{\sqrt{s}}{c}$. $\dot{\theta}$ is constant and therefore $\theta = (-\frac{\sqrt{s}}{c})t + D$ where D is a constant of integration. Notice that the condition $M_2^2 + M_3^2 \neq 0$ holds. Finally the constant s is calculated by equating the Hamiltonian H (equation (65)) to the Casimir function I_2 (equation (26) to yield:

$$s = I_2 - 2cH \tag{68}$$

In addition from the Hamiltonian (65) the partial derivatives with respect to the extremal solutions are:

$$\begin{aligned} \frac{\partial H}{\partial M_2} &= \frac{M_2}{c_2} = \frac{r\cos\theta}{c_2} \\ \frac{\partial H}{\partial M_3} &= 0 \end{aligned} \tag{69}$$

then substituting (69) and the extremal solutions (67) into the equations (47), (48) and (55) to obtain the expressions for the optimal paths in local coordinates (or Euler angles), recall $\phi_i = \varphi_i$:

$$\begin{aligned} \varphi_1 &= K\int \frac{r^2\cos^2\theta}{c_2(r^2\cos^2\theta + s)}dt \\ \varphi_2 &= \arcsin\left(\frac{\sqrt{r^2\cos^2\theta + s}}{K}\right) \\ \varphi_3 &= \arccos\left(\sqrt{\frac{s}{r^2\cos^2\theta + s}}\right) \end{aligned} \tag{70}$$

where $K, c_2, r, s \in \mathbb{R}$ and θ is linear in t. In addition the relations:

$$\begin{aligned} &\cos\frac{\varphi_2}{2} = \sqrt{\frac{K + r\sin\theta}{2K}}; \sin\frac{\varphi_2}{2} = \sqrt{\frac{K - r\sin\theta}{2K}} \\ &e^{\pm\frac{1}{2}\varphi_3} = \left(\frac{\sqrt{r^2\cos^2\theta + s} + \sqrt{s}}{2\sqrt{r^2\cos^2\theta + s}}\right)^{1/2} \\ &\pm i\left(\frac{\sqrt{r^2\cos^2\theta + s} - \sqrt{s}}{2\sqrt{r^2\cos^2\theta + s}}\right)^{1/2} \\ &\varphi_1 = K\int \frac{r^2\cos^2\theta}{c_2(r^2\cos^2\theta + s)}dt \end{aligned} \tag{71}$$

can be substituted directly into (56) to yield the optimal paths $g(t) \in SU(2)$. In that same manner for $g(t) \in SO(3)$ the equations:

$$\begin{aligned} &\cos\phi_2 = \frac{r\sin\theta}{K}; \sin\phi_2 = \frac{\sqrt{r^2\cos^2\theta + s}}{K} \\ &\cos\phi_3 = \frac{\sqrt{s}}{\sqrt{r^2\cos^2\theta + s}}; \sin\phi_3 = \frac{r\cos\theta}{\sqrt{r^2\cos^2\theta + s}} \\ &\phi_1 = K\int \frac{r^2\cos^2\theta}{c_2(r^2\cos^2\theta + s)}dt \end{aligned} \tag{72}$$

can be substituted directly into (27) to obtain the optimal paths $g(t) \in SO(3)$. Furthermore, it is of interest to project the optimal curves onto the base space $\mathbb{S}^2$. Substituting (67) and (69) into (60) yields the optimal paths on $\mathbb{S}^2$:

$$\begin{aligned} x &= \frac{r}{K}\sin\theta \\ y &= \pm\frac{\sqrt{r^2\cos^2\theta + s}}{K}\sin\left(K\int \frac{r^2\cos^2\theta}{c_2(r^2\cos^2\theta + s)}dt\right) \\ z &= \mp\frac{\sqrt{r^2\cos^2\theta + s}}{K}\cos\left(K\int \frac{r^2\cos^2\theta}{c_2(r^2\cos^2\theta + s)}dt\right) \end{aligned} \tag{73}$$

Therefore, the explicit expressions have been derived for the optimal paths (minimizing control energy) of an under-actuated spacecraft, in local coordinates, as a path $g(t) \in G$ and as a path in the base space $\mathbb{S}^2$.

V. CONCLUSION

In this paper it is shown how to derive the Lax Pair equations for an affine control system defined on the semi-simple Lie groups $SU(2)$ and $SO(3)$. In addition the Lax Pair equations are integrated to derive explicit expressions for the corresponding optimal paths in $SU(2)$ and $SO(3)$. As Felix Klein discovered when integrating the Lagrange top, it is illustrated here in the more general setting of left-invariant systems on Lie groups that integrating on $SU(2)$ yields simpler equations than integrating on $SO(3)$. In addition the equations derived in this paper are applied to the optimal control of an under-actuated spacecraft, giving the optimal curves in Euler angles, as a curve $g(t) \in G$ and also as a curve in the base space $\hat{x} \in \mathbb{S}^2$.

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