Discrete Applied Mathematics 314 (2022) 284–303

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Word-representability of split graphs generated by morphisms

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ARTICLE INFO

Article history:
Received 4 March 2021
Received in revised form 29 January 2022
Accepted 22 February 2022
Available online 29 March 2022

Keywords:
Word-representable graphs
Morphisms
Semi-transitive orientation

ABSTRACT

A graph $G = (V, E)$ is word-representable if and only if there exists a word w over the alphabet V such that letters x and y, $x \neq y$, alternate in w if and only if $xy \in E$. A split graph is a graph in which the vertices can be partitioned into a clique and an independent set. There is a long line of research on word-representable graphs in the literature, and recently, word-representability of split graphs has attracted interest.

In this paper, we first give a characterization of word-representable split graphs in terms of permutations of columns of the adjacency matrices. Then, we focus on the study of word-representability of split graphs obtained by iterations of a morphism, the notion coming from combinatorics on words. We prove a number of general theorems and provide a complete classification in the case of morphisms defined by 2×2 matrices.

1. Introduction

A graph $G = (V, E)$ is *word-representable* if and only if there exists a word w over the alphabet V such that letters x and y, $x \neq y$, alternate in w if and only if $xy \in E$. In this definition, letters x and y *alternate* in w if after removing all letters but x and y, we would either get a word of the form $xyxy\cdots$, or of the form $yxyx\cdots$, of even or odd length. Also, by definition, w must contain each letter in V at least once. For example, the cycle graph on four vertices labeled 1, 2, 3, 4 in clockwise direction is word-representable because it can be represented by the word 14213243.

There is a long line of research on word-representable graphs in the literature that is summarized in [5]. Word-representable graphs are important as they generalize several well-known and well-studied classes of graphs such as 3-colorable graphs, comparability graphs and circle graphs [7]. Note that the class of word-representable graphs is hereditary, that is, removing a vertex in a word-representable graph results in a word-representable graph. The wheel graph W_5 is the smallest non-word-representable graph. One of the key results in the area is the following theorem, where an orientation of a graph is *semi-transitive* if it is acyclic, and for any directed path $v_0 \to v_1 \to \cdots \to v_k$ either there is no edge between v_0 and v_k, or $v_i \to v_j$ is an edge for all $0 \leq i < j \leq k$.

Theorem 1 ([4])**.** *A graph is word-representable if and only if it admits a semi-transitive orientation.*

The following simple lemma will also be of use to us in this paper.

Lemma 2 ([6])**.** *Let K_m be a clique in a graph G. Then any acyclic orientation of G induces a transitive orientation on K_m (where the presence of edges $u \to v$ and $v \to z$ implies the presence of the edge $u \to z$). In particular, any semi-transitive orientation of G induces a transitive orientation on K_m. In either case, the orientation induced on K_m contains a single source and a single sink.*

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https://doi.org/10.1016/j.dam.2022.02.023
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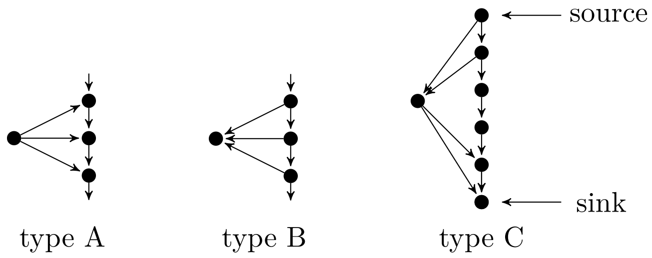

Fig. 1. Three types of vertices in E_{n-m} in a semi-transitive orientation of (E_{n-m}, K_m). The vertical oriented paths are a schematic way to show (parts of) $\vec{P}$.

1.1. Split graphs

A *split graph* is a graph in which the vertices can be partitioned into a clique and an independent set [3]. The paper [6] initiated a systematic study of word-representability of split graphs, which was extended in a follow up paper [1]. In particular, characterizations of split graphs in terms of forbidden induced subgraphs were obtained in [1,6] for cliques of sizes 4 and 5, respectively. Also, a characterization of semi-transitive orientations of split graphs was obtained in [6] (see below), and split graphs were used to solve a long standing problem in the theory of word-representation in [1]. We note though that currently a complete characterization of split graphs (e.g. in terms of forbidden subgraphs) seems to be a non-feasible problem, so a natural research direction is in understanding (non-)word-representable subclasses of split graphs.

Let $S_n = (E_{n-m}, K_m)$ be a word-representable split graph, where K_m is the maximal clique, and E_{n-m} is the independent set. Then, by Theorem 1, S_n admits a semi-transitive orientation. Further, by Lemma 2 we know that any such orientation induces a transitive orientation on K_m with the longest directed path $\vec{P}$. Theorem 5 characterizes semi-transitive orientations of split graphs.

Theorem 3 (*[6]*)**.** *Any semi-transitive orientation of a split graph $S_n = (E_{n-m}, K_m)$ subdivides the set of all vertices in E_{n-m} into three, possibly empty, groups corresponding to each of the following types (also shown schematically in Fig. 1), where $\vec{P} = p_1 \to \cdots \to p_m$ is the longest directed path in K_m:*

- *A vertex in E_{n-m} is of type A if it is a source and is connected to all vertices in $\{p_i, p_{i+1}, \ldots, p_j\}$ for some $1 \le i \le j \le m$;*
- *A vertex in E_{n-m} is of type B if it is a sink and is connected to all vertices in $\{p_i, p_{i+1}, \ldots, p_j\}$ for some $1 \le i \le j \le m$;*
- *A vertex $v \in E_{n-m}$ is of type C if there is an edge $x \to v$ for each $x \in I_v = \{p_1, p_2, \ldots, p_i\}$ and there is an edge $v \to y$ for each $y \in O_v = \{p_j, p_{j+1}, \ldots, p_m\}$ for some $1 \le i < j \le m$.*

There are additional restrictions, given by the next theorem, on relative positions of the neighbors of vertices of types A, B and C.

Theorem 4 (*[6]*)**.** *Let $S_n = (E_{n-m}, K_m)$ be oriented semi-transitively with $\vec{P} = p_1 \to \cdots \to p_m$. For a vertex $x \in E_{n-m}$ of type C, there is no vertex $y \in E_{n-m}$ of type A or B, which is connected to both $p_{|I_x|}$ and $p_{m-|O_x|+1}$. Also, there is no vertex $y \in E_{n-m}$ of type C such that either I_y, or O_y contains both $p_{|I_x|}$ and $p_{m-|O_x|+1}$.*

One can now classify semi-transitive orientations on split graphs.

Theorem 5 (*[6]*)**.** *An orientation of a split graph $S_n = (E_{n-m}, K_m)$ is semi-transitive if and only if*

(i) K_m is oriented transitively;
(ii) each vertex in E_{n-m} is of one of the three types in Theorem 3;
(iii) the restrictions in Theorem 4 are satisfied.

1.2. Split graphs generated by morphisms

Let A and B be alphabets (possibly $A = B$). A map $\varphi : A^* \to B^*$ is called a *morphism*, if we have $\varphi(uv) = \varphi(u)\varphi(v)$ for any $u, v \in A^*$. A morphism φ can be defined by defining $\varphi(a)$ for each $a \in A$. A particular property of a morphism φ is that $\varphi(\varepsilon) = \varepsilon$, where ε is the empty word. Morphisms are a central object in the area of combinatorics on words [8], and there is a natural extension of the notion to two, or more, dimensions. Indeed, one can begin with a matrix M whose entries are elements of A, and then obtain $\varphi(M)$ by substituting each element in M by matrices having the same dimensions and given by some substitution rules.

Relevance of (2-dimensional) morphisms to split graphs is coming through the ideas communicated in [2], where patterns in adjacency matrices are considered to study word-representability of graphs, and the notion of an infinite

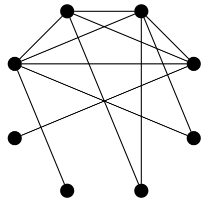

Fig. 2. The split graph $G(M)$ given by $S(M)$ in Example 7.

word-representable graph is introduced. In this paper, we study word-representability of families of split graphs defined by iteration of morphisms, and in particular, we give a complete classification in the case of 2×2 matrices. However, a key theorem we prove that characterizes word-representability of split graphs in terms of permutations of columns in the adjacency matrix is applicable to *any* split graph (see Theorem 15).

1.3. Organization of the paper

After introducing a number of preliminary results and examples in Section 2, that includes our key result (Theorem 15), in Section 3 we present a number of general results on split graphs generated by morphisms. In Section 4 we provide a complete classification of word-representable split graphs defined by iteration of morphisms using two 2×2 matrices; our results in this section are summarized in Tables 1 and 2. Finally, in Section 5 we state a number of open research questions.

2. Preliminaries

Definition 6. Let M be a binary $m \times n$ matrix. Define $S(M)$ to be the matrix

$$\begin{bmatrix} L_n & M^T \\ M & O_m \end{bmatrix}$$

where O_m is the $m \times m$ zero matrix and L_n is the $n \times n$ matrix such that all diagonal entries are 0's and all other entries are 1's.

It is easy to see that for any binary $m \times n$ matrix M, $S(M)$ is the adjacency matrix of a split graph with the maximal clique of order n or $n+1$. We denote the split graph by $G(M)$. Clearly, M gives edges between the clique and independent set in $G(M)$, and the order of the maximal clique depends on the existence of a $11\cdots1$ row in M.

Example 7. If $M = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ then

$$S(M) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is the adjacency matrix of the graph shown in Fig. 2.

Remark 8. If M is a zero matrix, then $G(M)$ is a disjoint union of a clique and isolated vertices and is word-representable, because the clique is semi-transitively (in fact, transitively) orientable and there are no other edges in $G(M)$.

The following lemma is Lemma 8 in [6].

Lemma 9 ([6]). *Let $S_n = (E_{n-m}, K_m)$ be a split graph with the maximum clique K_m, and a spit graph S_{n+1} is obtained from S_n by either adding a vertex of degree 0 (to E_{n-m}), or adding a vertex of degree 1 (to E_{n-m}), or by "coping" a vertex (either in E_{n-m} or in K_m), that is, by adding a vertex whose neighborhood is identical to the neighborhood of a vertex in S_n. Then S_n is word-representable if and only if S_{n+1} is word-representable.*

Table 1
The index of word-representability of infinite split graphs $G(A, B)$ for 2×2 matrices A and B.

Case	A	B	$\mathrm{IWR}(A, B)$	Ref.	Case	A	B	$\mathrm{IWR}(A, B)$	Ref.
1	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}*&*\\*&*\end{bmatrix}$	∞	$A = 0$					
2	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	∞	Proposition 24	18	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	∞	Proposition 24
3		$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	∞	Corollary 22	19		$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	3	Remark 32
4		$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	3	Proposition 31	20		$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	∞	Corollary 22
5		$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	3	Remark 32	21		$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	4	Remark 34
6		$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	4	Proposition 33	22		$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	3	Remark 32
7		$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	∞	Remark 43	23		$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	∞	Remark 43
8		$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	∞	Remark 43	24		$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	3	Remark 32
9		$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	∞	Proposition 44	25		$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	3	Remark 32
10		$\begin{bmatrix}0&1\\1&0\end{bmatrix}$	3	Remark 32	26		$\begin{bmatrix}0&1\\1&0\end{bmatrix}$	∞	Remark 45
11		$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	3	Remark 32	27		$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	∞	Remark 43
12		$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	3	Remark 32	28		$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	3	Remark 32
13		$\begin{bmatrix}1&1\\1&0\end{bmatrix}$	∞	Remark 43	29		$\begin{bmatrix}1&1\\1&0\end{bmatrix}$	3	Remark 32
14		$\begin{bmatrix}1&1\\0&1\end{bmatrix}$	3	Remark 32	30		$\begin{bmatrix}1&1\\0&1\end{bmatrix}$	∞	Remark 43
15		$\begin{bmatrix}1&0\\1&1\end{bmatrix}$	3	Remark 32	31		$\begin{bmatrix}1&0\\1&1\end{bmatrix}$	2	Remark 30
16		$\begin{bmatrix}0&1\\1&1\end{bmatrix}$	2	Proposition 29	32		$\begin{bmatrix}0&1\\1&1\end{bmatrix}$	3	Remark 32
17		$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	3	Remark 32	33		$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	3	Remark 32
34	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	∞	Proposition 24	50	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	∞	Proposition 24
35		$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	4	Remark 34	51		$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	∞	Remark 43
36		$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	3	Remark 32	52		$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	∞	Remark 43
37		$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	3	Remark 32	53		$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	4	Remark 34
38		$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	∞	Corollary 22	54		$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	4	Remark 34

(*continued on next page*)

Table 1 (*continued*).

Case	A	B	IWR(A, B)	Ref.	Case	A	B	IWR(A, B)	Ref.
39		$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	3	Remark 32	55		$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	∞	Corollary 22
40		$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	3	Remark 32	56		$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	∞	Remark 43
41		$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	∞	Remark 45	57		$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	3	Remark 32
42		$\begin{bmatrix}0&1\\1&0\end{bmatrix}$	3	Remark 32	58		$\begin{bmatrix}0&1\\1&0\end{bmatrix}$	3	Remark 32
43		$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	∞	Remark 43	59		$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	∞	Remark 43
44		$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	∞	Remark 43	60		$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	∞	Proposition 24
45		$\begin{bmatrix}1&1\\1&0\end{bmatrix}$	2	Remark 30	61		$\begin{bmatrix}1&1\\1&0\end{bmatrix}$	∞	Remark 43
46		$\begin{bmatrix}1&1\\0&1\end{bmatrix}$	3	Remark 32	62		$\begin{bmatrix}1&1\\0&1\end{bmatrix}$	∞	Remark 43
47		$\begin{bmatrix}1&0\\1&1\end{bmatrix}$	3	Remark 32	63		$\begin{bmatrix}1&0\\1&1\end{bmatrix}$	4	Remark 34
48		$\begin{bmatrix}0&1\\1&1\end{bmatrix}$	∞	Remark 43	64		$\begin{bmatrix}0&1\\1&1\end{bmatrix}$	4	Remark 34
49		$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	3	Remark 32	65		$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	∞	Proposition 24

To analyze word-representability of a given split graph, by Lemma 9 we can delete vertices of degree 0 and vertices of degree 1, as well as delete all but one vertex having the same neighborhood.

Proposition 10. *Let M be an $m \times n$ matrix. If every row, or every column of M is of the form $00\cdots0$ or $11\cdots1$, then $G(M)$ is word-representable.*

Proof. If every row of M consists of all 0's or all 1's then in $G(M)$, each vertex in the independent set is an isolated vertex, or is connected to every vertex in the clique. By Lemma 9, word-representability of $G(M)$ is equivalent to word-representability of either the clique K_{n+1}, or the disjoint union of a clique (K_n or K_{n+1}) and an isolated vertex, which are clearly semi-transitively orientable, and thus, by Theorem 1, word-representable.

On the other hand, if every column of M consists of all 0's or all 1's then the neighborhood of each vertex in the independent set is the same and, by Lemma 9, word-representability of $G(M)$ is equivalent to word-representability of the clique K_n with a vertex x connected to some, maybe none or all of clique's vertices. If w.l.o.g. x is connected to vertices $1, 2, \ldots, p$, $0 \le p \le n$, in K_n formed by the vertices $1, 2, \ldots, n$, then the word $x12\cdots px(p+1)(p+2)\cdots n$ represents the graph. □

It is obvious that if M^* is a matrix obtained by a row or column permutation of a matrix M, then $G(M^*)$ is a split graph obtained by relabeling the vertices of the graph $G(M)$. Hence we get the following lemma.

Lemma 11. *Let M be an $m\times n$ binary matrix and M^* is a matrix obtained from a sequence of row and/or column permutations of M. Then, $G(M)$ is word-representable if and only if $G(M^*)$ is word-representable.*

Lemma 12. *Let $M := [m_{ij}]_{m\times n}$ be an $m\times n$ binary matrix such that $m_{k1} = m_{k2} = \cdots = m_{kn} = 1$ for some $k \in \{1, 2, \ldots, m\}$. If*

$$N = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} & 0 \\ m_{21} & m_{22} & \cdots & m_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ m_{(k-1)1} & m_{(k-1)2} & \cdots & m_{(k-1)n} & 0 \\ m_{(k+1)1} & m_{(k+1)2} & \cdots & m_{(k+1)n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ m_{m1} & m_{m2} & \cdots & m_{mn} & 0 \end{bmatrix},$$

is an $(m-1)\times(n+1)$ binary matrix, then $G(M)$ is isomorphic to $G(N)$.

Table 2
The remaining cases of the index of word-representability of infinite split graphs $G(A, B)$ for 2×2 matrices A and B.

Case	A	B	IWR(A, B)	Ref.	Case	A	B	IWR(A, B)	Ref.
66	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	3	Remark 32	82	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	∞	Proposition 24
67		$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	3	Remark 32	83		$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	4	Remark 34
68		$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	3	Remark 32	84		$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	∞	Remark 43
69		$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	3	Remark 32	85		$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	4	Remark 34
70		$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	3	Remark 32	86		$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	∞	Remark 43
71		$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	3	Remark 32	87		$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	4	Remark 34
72		$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	3	Remark 32	88		$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	∞	Proposition 24
73		$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	∞	Corollary 22	89		$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	3	Remark 32
74		$\begin{bmatrix}0&1\\1&0\end{bmatrix}$	∞	Proposition 46	90		$\begin{bmatrix}0&1\\1&0\end{bmatrix}$	3	Remark 32
75		$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	3	Remark 32	91		$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	∞	Corollary 22
76		$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	3	Remark 32	92		$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	4	Remark 34
77		$\begin{bmatrix}1&1\\1&0\end{bmatrix}$	3	Remark 32	93		$\begin{bmatrix}1&1\\1&0\end{bmatrix}$	4	Remark 34
78		$\begin{bmatrix}1&1\\0&1\end{bmatrix}$	3	Remark 32	94		$\begin{bmatrix}1&1\\0&1\end{bmatrix}$	∞	Remark 43
79		$\begin{bmatrix}1&0\\1&1\end{bmatrix}$	3	Remark 32	95		$\begin{bmatrix}1&0\\1&1\end{bmatrix}$	4	Remark 34
80		$\begin{bmatrix}0&1\\1&1\end{bmatrix}$	3	Remark 32	96		$\begin{bmatrix}0&1\\1&1\end{bmatrix}$	∞	Remark 43
81		$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	2	Remark 30	97		$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	∞	Proposition 24
98	$\begin{bmatrix}1&1\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	4	Remark 34	114	$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	∞	Proposition 24
99		$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	3	Remark 32	115		$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	5	Remark 51
100		$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	∞	Remark 43	116		$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	5	Remark 51
101		$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	2	Remark 30	117		$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	5	Proposition 50
102		$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	3	Remark 32	118		$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	5	Remark 51
103		$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	∞	Remark 43	119		$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	∞	Proposition 24

(*continued on next page*)

Table 2 (*continued*).

Case	A	B	IWR(A, B)	Ref.	Case	A	B	IWR(A, B)	Ref.
104		$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	3	Remark 32	120		$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	∞	Proposition 24
105		$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	4	Remark 34	121		$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	3	Proposition 48
106		$\begin{bmatrix}0&1\\1&0\end{bmatrix}$	3	Remark 32	122		$\begin{bmatrix}0&1\\1&0\end{bmatrix}$	3	Remark 49
107		$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	∞	Remark 43	123		$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	∞	Proposition 24
108		$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	3	Remark 32	124		$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	∞	Proposition 24
109		$\begin{bmatrix}1&1\\1&0\end{bmatrix}$	3	Remark 32	125		$\begin{bmatrix}1&1\\1&0\end{bmatrix}$	∞	Theorem 42
110		$\begin{bmatrix}1&1\\0&1\end{bmatrix}$	∞	Corollary 22	126		$\begin{bmatrix}1&1\\0&1\end{bmatrix}$	∞	Theorem 42
111		$\begin{bmatrix}1&0\\1&1\end{bmatrix}$	3	Remark 32	127		$\begin{bmatrix}1&0\\1&1\end{bmatrix}$	∞	Theorem 42
112		$\begin{bmatrix}0&1\\1&1\end{bmatrix}$	3	Remark 32	128		$\begin{bmatrix}0&1\\1&1\end{bmatrix}$	∞	Theorem 42
113		$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	∞	Remark 43	129		$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	∞	Proposition 24

Proof. Let M^* be the matrix obtained from M by making the kth row be the first row. That is,

$$M^* = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{(k-1)1} & m_{(k-1)2} & \cdots & m_{(k-1)n} \\ m_{(k+1)1} & m_{(k+1)2} & \cdots & m_{(k+1)n} \\ \vdots & \vdots & & \vdots \\ m_{m1} & m_{m2} & \cdots & m_{mn} \end{bmatrix}.$$

Since M^* is obtained by reordering rows of M, $G(M^*)$ is obtained by relabeling the vertices of $G(M)$, and thus $G(M^*)$ is isomorphic to $G(M)$. Note that $S(M^*) = S(N)$, and so $G(M^*)$ and $G(N)$ are the same graph. Hence $G(M)$ is isomorphic to $G(N)$. □

It is known [5] that there are no non-word-representable graphs of order less than 6 and the only non-word-representable graph on 6 vertices is the wheel graph W_5, which is not a split graph. Thus, $G(M)$ is word-representable if M is an $m \times n$ matrix and $m+n \leq 6$. In [6], it is shown that any split graph S with maximum clique K_4 is word-representable if and only if S does not contain the graphs T_1, T_2, T_3 and T_4 shown in Fig. 3 as induced subgraphs. As a corollary to this result, we have the following theorem.

Theorem 13. *Let A be an $m \times 4$ binary matrix without all 1's rows. Then, $G(A)$ is word-representable if and only if the rows and columns of A cannot be permuted to be a matrix containing* $\begin{bmatrix}1&1&0&0\\1&0&1&0\\0&1&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&0&0\\1&0&1&0\\1&0&0&1\end{bmatrix}$, $\begin{bmatrix}1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}$ *or* $\begin{bmatrix}1&1&1&0\\1&1&0&0\\1&0&0&1\\1&0&1&1\end{bmatrix}$ *as a submatrix.*

Let x^n denote $xx\cdots x$ where x is repeated n times. The next theorem gives a sufficient condition for word-representability of a given graph $G(M)$.

Theorem 14. *Let M be an $m \times n$ binary matrix. If there is a permutation of columns of M giving a matrix such that each row of M is of the form $0^r1^s0^t$ for some non-negative integers r, s, t, then $G(M)$ is word-representable.*

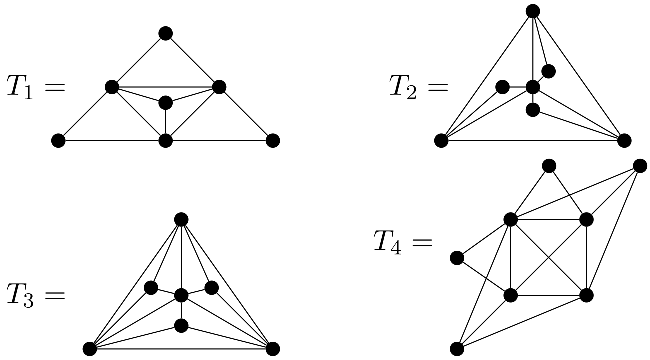

Fig. 3. Non-word-representable split graphs T_1, T_2, T_3, T_4.

Proof. Assume that M^* is the matrix obtained from a sequence of column permutations of M and each row of M^* is of the form $0^r1^s0^t$ where r, s, t are non-negative integers. Let the ith row/column of the adjacency matrix $S(M^*)$ correspond to vertex i in $G(M^*)$. So the clique C in $G(M^*)$ contains vertices $1, 2, \ldots, n$ and the independent set I in $G(M^*)$ contains vertices $n+1, n+2, \ldots, n+m$. Assign the orientation of edges in $G(M^*)$ as $i \to j$ if and only if $i < j$. We have that $1 \to 2 \to \cdots \to n$ is the longest path in the transitively oriented C, and the edges between C and I are oriented from C to I. Thus, each edge in the independent set is of type B, and we are done by Theorems 1 and 5. □

We have the following important generalization of Theorem 14.

Theorem 15. *Let M be an $m \times n$ binary matrix without all 1's rows. The split graph $G(M)$ is word-representable if and only if M satisfies the following conditions:*

(i) there is a sequence of column permutations of M giving a matrix M^* where every row is of the form $0^r1^s0^t$ or $1^r0^s1^t$ for some nonnegative integers r, s, t, and
(ii) for any row of M^* of the form $1^a0^b1^c$ for some positive integers a, b, c, there is no other row having 1's in all positions from a to $a+b+1$.

Proof. "⇐". Assign the orientation of edges in $G(M^*)$ as $i \to j$ if $i < j$ *except* if $j > n$ (i.e. j is a vertex in the independent set) and the row in M^* corresponding to j is of the form $1^r0^s1^t$, in which case we still orient $i \to j$ for $1 \le i \le r$ but $j \to i$ for $r+s+1 \le i \le n$. The vertices in the independent set will then be of types B and C, and taking into account condition (ii), Theorems 1 and 5 can be applied to see that $G(M^*)$ is word-representable, and thus $G(M)$ is word-representable by Lemma 11.

"⇒". By Theorem 1, $G(M)$ admits a semi-transitive orientation. By Theorem 5, under this orientation the clique is oriented transitively, and we can rename the vertices of the clique, if necessary so that the longest path would be formed by $1 \to 2 \to \cdots \to n$. Note that renaming vertices in the clique corresponds to permuting columns in M giving M^*. But then, conditions (ii) and (iii) in Theorem 5 give conditions (i) and (ii) in this theorem. □

Remark 16. If M has an all 1's row, we can see that Theorems 13 and 15 cannot be applied. However, we can use Lemma 12 to change M into an $(m-1) \times (n+1)$ matrix N which does not contain all 1's row. So we can apply the theorems to matrix N instead of M because $G(N)$ is isomorphic to $G(M)$. This observation also applies to Corollary 23.

We can see that Theorem 15 allows us to answer the question on word-representability of $G(M)$ by looking at permutations of columns in M. Let $M = \left[m_{ij}\right]_{m \times n}$ be an $m \times n$ matrix and $\rho = \rho_1\rho_2\cdots\rho_n$ is a permutation of $\{1, 2, \ldots, n\}$ written in one-line notation. We say that

$$M^* = \begin{bmatrix} m_{1\rho_1} & m_{1\rho_2} & \cdots & m_{1\rho_n} \\ m_{2\rho_1} & m_{2\rho_2} & \cdots & m_{2\rho_n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{m\rho_1} & m_{m\rho_2} & \cdots & m_{m\rho_n} \end{bmatrix}$$

is *the matrix obtained from reordering columns of M in the order given by ρ*. The key approach given by Theorem 15 is finding a permutation ρ that turns each row of M^* into the form $0^r1^s0^t$ or $1^r0^s1^t$ (so, all 1's in M^* are cyclically consecutive). Interestingly, to prove word-representability results in this paper, only rows of the form $0^r1^s0^t$ are used, so that condition (ii) in Theorem 15 is not applicable.

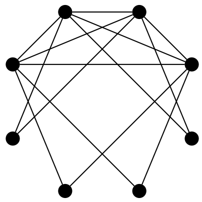

Fig. 4. The split graph $G^2(A, B)$ corresponding to the adjacency matrix $S^2(A, B)$ in Example 19.

Example 17. In the matrix $M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ we can ignore rows 2, 4, 6 and 8 because all entries in these rows are zero. Then we need to find a permutation $\rho = \rho_1\rho_2 \cdots \rho_8$ making

- columns 1, 4, 6 and 7 be (cyclically) consecutive in rows 1;
- columns 1, 3, 5 and 7 be (cyclically) consecutive in row 3 and 7;
- columns 2, 3, 6 and 7 be (cyclically) consecutive in row 5.

It can be implied from the first and the second bullet points that 1 and 7 must be consecutive in ρ and then, w.l.o.g., 4 and 6 are next to the left of these numbers and 3 and 5 are next to the right of them (cyclically). Hence ρ contains

$\{4, 6\}, \{1, 7\}, \{3, 5\}$

where numbers in {} are consecutive in ρ but are in some unknown to us order. But then, we get a contradiction with the second bullet point. Hence, there is no such ρ and $G(M)$ is non-word-representable by Theorem 15.

3. General results on split graphs generated by morphisms

In this section, we discuss rather general results on split graphs generated by morphisms, thus preparing ourselves for a classification of the case of 2×2 matrices coming in the next section.

Definition 18. Let A, B be $m \times n$ binary matrices. The matrix $M^k(A, B)$ is said to be the kth-iteration of the 2-dimensional morphism applied to the 1×1 matrix $[0]$ which maps $[0] \to A$ and $[1] \to B$. Moreover, we write $S^k(A, B)$ for the matrix $S(M^k(A, B))$ and $G^k(A, B)$ for the graph with the adjacency matrix $S^k(A, B)$.

Example 19. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then we have

$$M^0(A, B) = \begin{bmatrix} 0 \end{bmatrix}, M^1(A, B) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M^2(A, B) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Then, $S^2(A, B) = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $G^2(A, B)$ is shown in Fig. 4.

Remark 20. If A is a zero matrix, then $M^k(A, B)$ is always a zero matrix for any $m \times n$ matrix B and positive integer k. So, by Remark 8, $G^k(A, B)$ is word-representable.

3.1. The case of $A = B$

In this case, both [0] and [1] are mapped to the same matrix A, so if A is an $m \times n$ binary matrix, then

$$S^k(A,A) = \begin{bmatrix} L_{n^k} & A_k^T \\ A_k & O_{m^k} \end{bmatrix} \quad \text{where } A_k = \underbrace{\begin{bmatrix} A & A & \cdots & A \\ A & A & \cdots & A \\ \vdots & \vdots & \ddots & \vdots \\ A & A & \cdots & A \end{bmatrix}}_{n^k \text{ columns}}.$$

Clearly, A_k is an $n^k \times m^k$ matrix and $S^k(A,A) = S(A_k)$, so $G^k(A,A)$ is isomorphic to $G(A_k)$.

Theorem 21. *Let A be an $m \times n$ binary matrix. For $k \geq 1$, $G^k(A,A)$ is word-representable if and only if $G(A)$ is word-representable.*

Proof. Firstly, we label a vertex of $G^k(A,A)$ by i if it is represented by the ith column/row in $S^k(A,A)$. Note that rows $i, m+i, 2m+i, \dots, (m^{k-1}-1)m+i$ in A_k are identical for any $i \in \{1,2,\dots,m\}$, and columns $j, n+j, 2n+j, \dots, (n^{k-1}-1)n+j$ in A_k are also identical for any $j \in \{1,2,\dots,n\}$. So, for any $i \in \{1,2,\dots,m\}$, the vertices of $G^k(A,A)$ in $R_i := \{i+n^k, m+i+n^k, 2m+i+n^k, \dots, (m^{k-1}-1)m+i+n^k\}$ have the same neighborhoods. Similarly, any two vertices of $G^k(A,A)$ in $C_j := \{j, n+j, 2n+j, \dots, (n^{k-1}-1)n+j\}$ are connected to the same vertices in the independent set for any $j \in \{1,2,\dots,n\}$. Thus, by Lemma 9, $G^k(A,A)$ is word-representable if and only if the graph G obtained by deleting all vertices but the smallest one in R_i and C_j for all $i,j \in \{1,2,\dots,n\}$ is word-representable. But G is exactly $G(A)$, which complete the proof. □

Corollary 22. *If A is an $m \times n$ binary matrix such that $m+n \leq 6$, then $G^k(A,A)$ is word-representable for any $k \geq 0$.*

Proof. Since the smallest non-word-representable split graph is of order 7, all split graphs of orders less than 7 are word-representable. Hence $G^k(A,A)$ is word-representable for any $m \times n$ matrix A where $m+n \leq 6$. □

Moreover, together with Theorem 13, we have the following result.

Corollary 23. *Let A be an $m \times 4$ binary matrix with no all 1's row. For any integer k, the graph $G^k(A,A)$ is word-representable if and only if the rows and columns of A cannot be permuted to be the matrix containing $\begin{bmatrix} 1&1&0&0 \\ 1&0&1&0 \\ 0&1&1&0 \end{bmatrix}$, $\begin{bmatrix} 1&1&0&0 \\ 1&0&1&0 \\ 1&0&0&1 \end{bmatrix}$, $\begin{bmatrix} 1&0&1&1 \\ 1&1&0&1 \\ 1&1&1&0 \end{bmatrix}$ or $\begin{bmatrix} 1&1&1&0 \\ 1&1&0&0 \\ 1&0&0&1 \\ 1&0&1&1 \end{bmatrix}$ as a submatrix.*

3.2. The case of $A \neq B$

In what follows, A and B can be distinct.

Proposition 24. *If every row, or every column, in $m \times n$ matrices A and B is either 0^n or 1^n, then $G^k(A,B)$ is word-representable for any $k \geq 0$.*

Proof. If every row (resp., column) in A and B is either 0^n or 1^n, then every row (resp., column) in $M^k(A,B)$ is either 0^{n^k} or 1^{n^k}, so by Proposition 10, $G^k(A,B)$ is word-representable. □

Theorem 25. *Let A and B be $m \times n$ binary matrices. Suppose that A^* and B^* are the matrices obtained from reordering columns of A and B, respectively, in order given by a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$. Then $G^k(A,B)$ is word-representable if and only if $G^k(A^*,B^*)$ is word-representable for any $k \geq 0$.*

Proof. The case $k = 0$ is trivial, so assume that $k \geq 1$. We claim that $M^k(A^*,B^*)$ is obtained from a permutation of columns in $M^k(A,B)$. We will prove the claim by induction on k. Note that $M^1(A,B) = A$ and $M^1(A^*,B^*) = A^*$. So $M^1(A^*,B^*)$ is the matrix obtained from reordering columns of $M^1(A,B)$. Suppose that l is a positive integer and $M^l(A^*,B^*)$ is the matrix obtained from reordering columns of $M^l(A,B)$ in order given by a permutation $\tau = \tau_1\tau_2\cdots\tau_{n^l}$. Let $M^l(A,B) = \begin{bmatrix} C_1 & C_2 & \cdots & C_{n^l} \end{bmatrix}$ where C_i is the ith column of $M^l(A,B)$. Then $M^l(A^*,B^*) = \begin{bmatrix} C_{\tau_1} & C_{\tau_2} & \cdots & C_{\tau_{n^l}} \end{bmatrix}$. For the next iteration of morphism, each column C_i of $M^l(A, B)$ is mapped to n columns $C_{i,1}, C_{i,2}, \ldots, C_{i,n}$, and each column C_{τ_i} of $M^l(A^*, B^*)$ is mapped to n columns $C_{\tau_i,\sigma_1}, C_{\tau_i,\sigma_2}, \ldots, C_{\tau_i,\sigma_n}$. So we have

$$M^{l+1}(A, B) = \left[C_{1,1}\ C_{1,2}\ \cdots C_{1,n}\ \cdots\ C_{n^l,1}\ C_{n^l,2}\ \cdots\ C_{n^l,n}\right]$$

and

$$M^{l+1}(A^*, B^*) = [\ C_{\tau_1,\sigma_1}\ C_{\tau_1,\sigma_2}\ \cdots\ C_{\tau_1,\sigma_n}\ \cdots\ C_{\tau_{n^l},\sigma_1}\ C_{\tau_{n^l},\sigma_2}\ \cdots\ C_{\tau_{n^l},\sigma_n}\ ].$$

A group of columns $C_{i,1}, C_{i,2}, \ldots, C_{i,n}$ is called *block* B_i. Firstly, we can see that reordering the blocks $B_1, B_2, \ldots, B_{n^l}$ of $M^{l+1}(A, B)$ in order given by τ, and then reordering columns in every block B_i in order given by σ, yields the matrix $M^{l+1}(A^*, B^*)$. Thus, $M^{l+1}(A^*, B^*)$ is obtained by a column permutation of $M^{l+1}(A, B)$ and our claim is true. Hence, by Lemma 11, $G^k(A, B)$ is word-representable if and only if $G^k(A^*, B^*)$ is word-representable for any positive integer k. □

Next theorem is a natural extension of Theorem 25 to the case of row permutations, and it can be proved in a similar way to the proof of Theorem 25, so we omit the proof.

Theorem 26. *Let A and B be $m \times n$ binary matrices. Suppose that A^* and B^* are the matrices obtained from reordering rows of A and B, respectively, in order given by the same permutation. Then $G^k(A, B)$ is word-representable if and only if $G^k(A^*, B^*)$ is word-representable for any $k \geq 0$.*

So, we can reorder rows and columns of given matrices A and B while preserving the word-representability of $G^k(A, B)$. If A contains at least one 0, we can reorder rows and columns of A to make the leftmost bottom entry be a 0 (the matrix B will be changed by the same permutation of rows and columns as those applied to A). Thus, in what follows, if A is not all-one matrix, w.l.o.g. we can assume that the leftmost bottom entry of A is always 0. Then, the $m \times n$ leftmost bottom submatrix of $M^2(A, B)$ is A since $M^1(A, B) = A$. Moreover, the $m^{k-1} \times n^{k-1}$ leftmost bottom submatrix of $M^k(A, B)$ is $M^{k-1}(A, B)$. Thus, the limit $\lim_{k\to\infty} M^k(A, B)$, called a *fixed point of the morphism*, is well-defined. So, we have that $G^i(A, B)$ is an induced subgraph of $G^k(A, B)$ if $i \leq k$ and the notion of the infinite split graph $G(A, B)$ is well-defined in the case when A has a 0 as the leftmost bottom entry. So we are interested in the smallest integer l (possibly non-existing) that $G^l(A, B)$ is non-word-representable for given A and B (then $G^i(A, B)$ is non-word-representable for $i \geq l$).

Definition 27. Suppose that a matrix A has a 0 as the leftmost bottom entry. The *index of word-representability* IWR(A, B) of an infinite split graph $G(A, B)$ is the smallest integer l such that $G^l(A, B)$ is non-word-representable. If such l does not exist, that is, if $G^l(A, B)$ is word-representable for all l, then $l := \infty$. If the leftmost bottom entry of A is 1 (so that the $\lim_{k\to\infty} M^k(A, B)$ may not be well-defined as the sequence of graphs $G^k(A, B)$, for $k \geq 0$, may not be a chain of induced subgraphs) then IWR(A, B) is still defined in the same way even though $G(A, B)$ may not be defined.

Note that since $G^0(A, B)$ is a graph with one vertex, we have IWR$(A, B) \geq 1$. Even though Definition 27 is very similar to the respective definition of the index of word-representability of an infinite *Toeplitz graph* in [2] (where the index in our context would be the maximum l such that $G^l(A, B)$ is word-representable), it is more flexible as it makes sense in the situation when the leftmost bottom entry of A is 1.

Theorem 28. *Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ be $m \times n$ binary matrices, and*

$$C = \begin{bmatrix} a_{p_1q_1} & a_{p_1q_2} & \cdots & a_{p_1q_t} \\ a_{p_2q_1} & a_{p_2q_2} & \cdots & a_{p_2q_t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p_sq_1} & a_{p_sq_2} & \cdots & a_{p_sq_t} \end{bmatrix} \text{ and } D = \begin{bmatrix} b_{p_1q_1} & b_{p_1q_2} & \cdots & b_{p_1q_t} \\ b_{p_2q_1} & b_{p_2q_2} & \cdots & b_{p_2q_t} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p_sq_1} & b_{p_sq_2} & \cdots & b_{p_sq_t} \end{bmatrix}$$

be $s \times t$ submatrices of A and B, respectively, where $1 \leq p_1 < p_2 < \cdots < p_s \leq m$ and $1 \leq q_1 < q_2 < \cdots < q_t \leq n$. For any positive integer k, if $G^k(C, D)$ is non-word-representable, then $G^k(A, B)$ is non-word-representable.

Proof. First, we will prove by induction that $M^k(C, D)$ is a submatrix of $M^k(A, B)$ for any positive integer k. It is obvious that $M^1(C, D) = C$ is a submatrix of $M^1(A, B) = A$. Let l be a positive integer such that $M^l(C, D)$ is a submatrix of $M^l(A, B)$ on the columns $c_1, c_2, \ldots, c_{t^l}$ and rows $r_1, r_2, \ldots, r_{s^l}$. For the next iteration of morphism, $M^{l+1}(A, B)$ is formed by replacing each entry of $M^l(A, B)$ with either A or B. So the columns $(c_i - 1)n + q_j$ for $1 \leq i \leq t^l$, $1 \leq j \leq t$, and rows $(r_i - 1)m + p_j$ for $1 \leq i \leq s^l$, $1 \leq j \leq s$, form the matrix $M^{l+1}(C, D)$. Hence $M^k(C, D)$ is a submatrix of $M^k(A, B)$ for any $k \geq 1$. Therefore, $G^k(A, B)$ contains $G^k(C, D)$ as an induced subgraph for $k \geq 1$. As the property of word-representability is hereditary, we have that non-word-representability of $G^k(C, D)$ implies non-word-representability of $G^k(A, B)$. □

Theorem 28 gives a useful tool to study non-word-representability of $G^k(A, B)$ for larger A and B. Indeed, a starting point to justify suspected non-word-representability of $G^k(A, B)$ can be analysis of smaller submatrices of A and B. This is one of our motivation points to conduct a systematic study of IWR(A, B) for 2×2 matrices, to be done in the next section, as they are smallest submatrices that can be used to show non-word-representability of $G^k(A, B)$ for some A, B and k.

4. Classification of word-representable split graphs defined by iteration of morphisms using two 2 × 2 matrices

A summary of our classification of word-representability of $G^k(A, B)$ for 2×2 matrices A and B can be found in Tables 1 and 2, where the index of word-representability IWR(A, B) is given along with a reference, or a comment to the respective result.

4.1. The case of A is not all-one matrix

For any 2×2 matrices A and B, the graph $G^1(A, B)$ is a split graph of order 4 which is always word-representable. Then, IWR(A, B) ≥ 2. However, 2×2 matrices A and B such that $G^2(A, B)$ is non-word-representable can be found.

Proposition 29. *For* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, *IWR*$(A, B) = 2$.

Proof. We have $M^2(A, B) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Reordering columns of $M^2(A, B)$ in order given by the permutation 2314 yields the matrix $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So, by Theorem 13, $G^2(A, B)$ is non-word-representable ($G^2(A, B)$ contains T_1 as an induced subgraph). □

Remark 30. Permuting rows or/and columns in A and B similarly to Proposition 29, we see that IWR(A, B) $= 2$ for A and B in Cases 31, 45, 81 and 101 in Tables 1 and 2 (in Case 81 $G^2(A, B)$ contains T_3, and in the other cases $G^2(A, B)$ contains T_1).

Proposition 31. *For* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, *IWR*$(A, B) = 3$.

Proof. We have

$$M^2(A, B) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } M^3(A, B) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Reordering columns of $M^2(A, B)$ in order given by the permutation 4231 yields the matrix $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. By Theorem 14, we have $G^2(A, B)$ is word-representable. However, we have shown in Example 17 that $G^3(A, B)$ is non-word-representable. So IWR(A, B) $= 3$. □

Remark 32. Proposition 31 gives Case 4 in Tables 1. In each of Cases 5, 10, 11, 12, 14, 15, 17, 57, 66, 67, 68, 71, 72, 77, 78, 89, 99, 102, 104, 106, 108, 109, 111 and 112 in Tables 1 and 2, $M^2(A, B)$ has a permutation satisfying the conditions of Theorem 25 and $M^3(A, B)$ does not (similarly to Proposition 31). So IWR(A, B) $= 3$ for A and B in these cases. Moreover, by Theorem 25, column and row permutations of A and B give the same IWR. Consequently, we also have IWR(A, B) $= 3$ for A and B in Cases 19, 22, 24, 25, 28, 29, 32, 33, 36, 37, 39, 40, 42, 46, 47, 49, 58, 69, 70, 75, 76, 79, 80 and 90.

Proposition 33. *For* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, *IWR*$(A, B) = 4$.

Proof. Reordering columns of $M^3(A, B)$ in order given by the permutation 51732648 yields a matrix such that each row is of the form $0^r1^s0^t$. By Theorem 15, we have $G^3(A, B)$ is word-representable. For

$$M^4(A,B) = \begin{bmatrix}
0&0&1&0&0&0&1&0&1&0&1&0&0&0&1&0\\
0&1&0&0&0&1&0&0&0&0&0&0&0&1&0&0\\
1&0&1&0&1&0&1&0&1&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&1&0&1&0&1&0&0&0&1&0&0&0&1&0\\
0&1&0&0&0&0&0&0&0&1&0&0&0&1&0&0\\
1&0&1&0&1&0&0&0&1&0&1&0&1&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&1&0&1&0&1&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0\\
1&0&0&0&1&0&1&0&1&0&0&0&1&0&1&0\\
0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0\\
0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix},$$

suppose that a reordering of columns $\rho = \rho_1\rho_2\cdots\rho_{16}$ exists showing word-representability of $G^4(A, B)$ by Theorem 15. Then,

- from row 3, columns 1, 3, 5, 7, 9, 13 and 15 must be (cyclically) consecutive;
- from row 5, columns 3, 5, 7, 11 and 15 must be (cyclically) consecutive;
- from row 7, columns 1, 3, 5, 9, 11, 13 and 15 must be (cyclically) consecutive.

But then, from the first and the third bullet points, columns 1, 3, 5, 9, 13 and 15 must be consecutive and then column 7 or 11 is next to the left of them and the other one is next to the right of them. This contradicts to the second bullet point. So there is no such ρ and $G^4(A, B)$ is non-word-representable. Therefore, $\mathrm{IWR}(A, B) = 4$. □

Remark 34. Proposition 33 gives Case 6 in Table 1. In each of Cases 53, 63, 83, 87, 93, 98 and 105 in Tables 1 and 2, $M^3(A, B)$ has a permutations satisfying condition in Theorem 25 but $M^4(A, B)$ does not (similarly to Proposition 33). So $\mathrm{IWR}(A, B) = 4$ for A and B in these cases. Moreover, by Theorem 25, we also have that $\mathrm{IWR}(A, B) = 3$ for A and B in Cases 21, 35, 54, 64, 85, 92 and 95.

By Remarks 30, 32 and 34, we can see that in many cases the index of word-representability is 2, 3 or 4. Next, we will introduce certain definitions and theorems to present the cases where the index of word-representability is infinity.

Let M be an $m \times n$ binary matrix. For convenience, we will represent rows of M by binary strings of length n. For example, we will represent three rows of $\begin{bmatrix}1&1&0&1\\0&1&0&0\\0&0&0&1\end{bmatrix}$ by 1101, 0100 and 0001.

Definition 35. Let A and B be $m \times n$ binary matrices. Define $R^k(A, B)$ to be the set of binary strings representing rows of $M^k(A, B)$. So every element of $R^k(A, B)$ is a binary string of length n^k. Each element of $R^k(A, B)$ is called a *row pattern of* $M^k(A, B)$.

Definition 36. Let $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$ and $B = \begin{bmatrix}e&f\\g&h\end{bmatrix}$ be 2×2 binary matrices and B^n be the set of binary strings of length n. We define functions $u_{A,B} : \{0, 1\} \to B^2$ and $l_{A,B} : \{0, 1\} \to B^2$ by

$$u_{A,B}(0) = ab,\ l_{A,B}(0) = cd,\ u_{A,B}(1) = ef \quad\text{and}\quad l_{A,B}(1) = gh.$$

Moreover, if $v = v_1v_2\cdots v_k \in B^k$, $k \geq 2$, we extend the definition of the functions $u_{A,B}$ and $l_{A,B}$ to the case of $B^k \to B^{2k}$ by

$$u_{A,B}(v) = u_{A,B}(v_1)u_{A,B}(v_2)\cdots u_{A,B}(v_k)$$

and

$$l_{A,B}(v) = l_{A,B}(v_1)l_{A,B}(v_2)\cdots l_{A,B}(v_k).$$

When A and B are clear from the context, we can omit the subscript and write u and l instead of $u_{A,B}$ and $l_{A,B}$, respectively.

Example 37. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Then we have

$$M^3(A,B) = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $R^4(A,B) = \{10111011, 10001000, 10101010, 11111111, 00000000\}$. In fact, we can find $R^4(A,B)$ by using the functions $u_{(A,B)}$ and $l_{A,B}$. As we start with $M^0(A,B) = \begin{bmatrix} 0 \end{bmatrix}$, we have $R^0(A,B) = \{0\}$. Then we apply the functions $u_{A,B}$ and $l_{A,B}$ to all elements in $R^0(A,B)$ to get all elements in $R^1(A,B)$:

$$u_{A,B}(0) = 11 \text{ and } l_{A,B}(0) = 00.$$

So, $R^1(A,B) = \{11, 00\}$. Now,

$$u_{A,B}(11) = 1010 \text{ and } l_{A,B}(11) = 1010$$
$$u_{A,B}(00) = 1111 \text{ and } l_{A,B}(00) = 0000$$

so $R^2(A,B) = \{1010, 1111, 0000\}$. Repeating the procedure one more time yields

$$u_{A,B}(1010) = 10111011 \text{ and } l_{A,B}(1010) = 10001000$$
$$u_{A,B}(1111) = 10101010 \text{ and } l_{A,B}(1111) = 10101010$$
$$u_{A,B}(0000) = 11111111 \text{ and } l_{A,B}(0000) = 00000000,$$

and so we can see that $R^3(A,B)$ is the same set as we get above.

So, all elements in $R^k(A,B)$ are obtained by applying $u_{A,B}$ and $l_{A,B}$ to every element in $R^{k-1}(A,B)$. The next theorem generalizes this observation, and it can be proved easily by induction.

Theorem 38. *Let A and B be 2 × 2 binary matrices. Then*

$$R^k(A,B) = \{f_k(\cdots f_2(f_1(0))\cdots) | f_i \in \{u_{A,B}, l_{A,B}\}\} \text{ for any } k \geq 1.$$

Definition 39. Let $v = v_1v_2\cdots v_k \in B^k$. Then, $\Gamma(v) := \{m \in \{1,2,\ldots,k\} | v_m = 1\}$.

In order to study row patterns, we introduce a relation $\leq$ on B^n. Let $x = x_1x_2\cdots x_k$ and $y = y_1y_2\cdots y_k$ be in B^k. We say that $x \leq y$ if and only if $x_i = 1$ implies $y_i = 1$ for every $i \in \{1,2,\ldots,k\}$. In other words, $x \leq y$ if and only if $\Gamma(x) \subseteq \Gamma(y)$. It is easy see that $\leq$ is reflexive, antisymmetric and transitive, and thus $\leq$ is a partial order.

Theorem 40. *Let A and B be m × n binary matrices. For any $k > 1$, if $(R^k(A,B), \leq)$ is a total order, then $G^k(A,B)$ is word-representable.*

Proof. Let $R^k(A,B) = \{x_1, x_2, \ldots, x_l\}$ where $l \geq 1$ and $x_1, x_2, \ldots, x_l$ are binary strings of length n^k. Since $(R^k(A,B), \leq)$ is a total order, w.l.o.g., we assume that $x_1 \leq x_2 \leq \cdots \leq x_l$. That is $\Gamma(x_1) \subseteq \Gamma(x_2) \subseteq \cdots \subseteq \Gamma(x_l)$. Let

$$\begin{aligned} D_1 &:= \Gamma(x_1), \\ D_2 &:= \Gamma(x_2) \setminus \Gamma(x_1), \\ D_3 &:= \Gamma(x_3) \setminus \Gamma(x_2), \\ &\vdots \\ D_l &:= \Gamma(x_l) \setminus \Gamma(x_{l-1}) \quad \text{and} \\ D_{l+1} &:= \{1,2,\ldots,n^k\} \setminus \Gamma(x_l). \end{aligned}$$

If $D_j = \{i_{j,1}, i_{j,2}, \ldots, i_{j,|D_j|}\}$ for $i_{j,1} < i_{j,2} < \cdots < i_{j,|D_j|}$, then

$$\rho = i_{1,1}i_{1,2}\cdots i_{1,|D_1|}i_{2,1}i_{2,2}\cdots i_{2,|D_2|}\cdots i_{l,1}i_{l,2}\cdots i_{l,|D_l|}$$

is a n^k-permutation. Let M^* be the matrix obtained by reordering columns of $M^k(A,B)$ according to the order given by ρ. Then we want to prove that every row of $M^*(A,B)$ is of the form 1^s0^t for some $s, t \geq 0$. Let $y = y_1y_2\cdots y_{n^k}$ be a row pattern of $M^k(A, B)$ and $y^* = y_1^* y_2^* \cdots y_{n^k}^*$ be the row pattern after reordering y. Since $y \in R^k(A, B)$, then $y = x_q$ for some $q \in \{1, 2, \ldots, l\}$. So $y_i = 1$ for all $i \in \Gamma(x_q)$ and $y_i = 0$ for all $i \notin \Gamma(x_q)$. That is, $\Gamma(y)$ is

$$\{i_{1,1}, i_{1,2}, \ldots, i_{1,|D_1|}, i_{2,1}, i_{2,2}, \ldots, i_{2,|D_2|}, \ldots, i_{q,1}, i_{q,2}, \ldots, i_{l,|D_q|}\}.$$

Hence $y^* = 1^s0^t$ where $s = |D_1| + |D_2| + \cdots + |D_q|$ and $t = n^k - |D_1| - |D_2| - \cdots - |D_q|$. Therefore every row of $M^*(A, B)$ is of the form 1^s0^t. By Theorem 14, we have that $G^k(A, B)$ is word-representable. □

Theorem 41. *Let A and B be 2 × 2 binary matrices. If $x \leq y$ implies $\{u_{A,B}(x), l_{A,B}(x), u_{A,B}(y), l_{A,B}(y)\}$ is comparable under $\leq$ for binary strings x and y of the same length, then $G^k(A, B)$ is word-representable for any $k \geq 0$.*

Proof. We will prove by induction on k that $R^k(A, B)$ is comparable under $\leq$. Because $0 \leq 0$, we have $\{u_{A,B}(0), l_{A,B}(0)\} = R^1(A, B)$ is comparable under $\leq$. Suppose that $R^l(A, B)$ is comparable under $\leq$ for some $l \geq 1$. Let $v, w \in R^{l+1}(A, B)$. Then

$$v = u_{A,B}(x) \text{ or } v = l_{A,B}(x) \text{ for some } x \in R^l(A, B)$$

and

$$w = u_{A,B}(y) \text{ or } w = l_{A,B}(y) \text{ for some } y \in R^l(A, B).$$

By induction hypothesis, w.l.o.g., we can assume that $x \leq y$. So, $\{u_{A,B}(x), l_{A,B}(x), u_{A,B}(y), l_{A,B}(y)\}$ is comparable and v and w belong to this set. Thus we have that v and w are comparable. Hence $R^k(A, B)$ is comparable under $\leq$ for any $k \geq 0$. Hence, by Theorem 40, $G^k(A, B)$ is word-representable for any $k \geq 0$. □

Theorem 42. *Let A and B be 2 × 2 binary matrices. If $\{u_{A,B}(100), l_{A,B}(100), u_{A,B}(101), l_{A,B}(101)\}$ is comparable under $\leq$, then $G^k(A, B)$ is word-representable for any $k \geq 0$.*

Proof. For convenience, we write u and l instead of $u_{A,B}$ and $l_{A,B}$, respectively. Suppose that $\{u(100), l(100), u(101), l(101)\}$ is comparable under $\leq$. Let $x = x_1x_2 \cdots x_m$ and $y = y_1y_2 \cdots y_m$ be binary strings of length m such that $x \leq y$. Also, let

$$R := \{i | x_i = 0, y_i = 0\},$$
$$S := \{i | x_i = 1, y_i = 1\} \text{ and}$$
$$T := \{i | x_i = 0, y_i = 1\}.$$

Note that R, S, T partition the set $\{1, 2, \ldots, m\}$. Since $u(100)$ and $l(100)$ are comparable, we have two cases to consider.

If $u(100) \leq l(100)$, then $u(1) \leq l(1)$ and $u(0) \leq l(0)$. So we have $u(101) \leq l(101)$ and it is impossible that $l(101) \leq u(100)$ and $l(100) \leq u(101)$. So there are four possible cases here, which are

$$u(100) \leq u(101) \leq l(100) \leq l(101),$$
$$u(100) \leq u(101) \leq l(101) \leq l(100),$$
$$u(101) \leq u(100) \leq l(101) \leq l(100) \text{ and}$$
$$u(101) \leq u(100) \leq l(100) \leq l(101).$$

Similarly, in the case of $l(100) \leq u(100)$, we have $l(1) \leq u(1)$ and $l(0) \leq u(0)$. So $l(101) \leq u(101)$ and $u(100) \nleq l(100)$ and $u(101) \nleq l(100)$. So we have four more cases, which are

$$l(100) \leq l(101) \leq u(100) \leq u(101),$$
$$l(100) \leq l(101) \leq u(101) \leq u(100),$$
$$l(101) \leq l(100) \leq u(101) \leq u(100) \text{ and}$$
$$l(101) \leq l(100) \leq u(100) \leq u(101).$$

Next, we will consider comparability of the set $\{u(x), l(x), u(y), l(y)\}$ in each case.

- $u(100) \leq u(101) \leq l(100) \leq l(101)$. So we have $u(0) \leq u(1) \leq l(0) \leq l(1)$. Note that $u(x_i) \leq u(y_i)$ and $l(x_i) \leq l(y_i)$ for any $i \in T$. Then $u(x) \leq u(y)$ and $l(x) \leq l(y)$. Since $u(y_i) \leq l(x_i)$ where i belongs to R, S or T, so $u(y) \leq l(x)$. Hence, $u(x) \leq u(y) \leq l(x) \leq l(y)$.
- $u(100) \leq u(101) \leq l(101) \leq l(100)$. So we have $u(0) \leq u(1) \leq l(1) \leq l(0)$. Note that $u(x_i) \leq u(y_i)$ and $l(y_i) \leq l(x_i)$ for any $i \in T$. Then $u(x) \leq u(y)$ and $l(y) \leq l(x)$. Since $u(y_i) \leq l(y_i)$ where i belongs to R, S or T, so $u(y) \leq l(y)$. Hence, $u(x) \leq u(y) \leq l(y) \leq l(x)$.
- $u(101) \leq u(100) \leq l(101) \leq l(100)$. So we have $u(1) \leq u(0) \leq l(1) \leq l(0)$. Note that $u(y_i) \leq u(x_i)$ and $l(y_i) \leq l(x_i)$ for any $i \in T$. Then $u(y) \leq u(x)$ and $l(y) \leq l(x)$. Since $u(x_i) \leq l(y_i)$ where i belongs to R, S or T, so $u(x) \leq l(y)$. Hence, $u(y) \leq u(x) \leq l(y) \leq l(x)$.
- $u(101) \leq u(100) \leq l(100) \leq l(101)$. So we have $u(1) \leq u(0) \leq l(0) \leq l(1)$. Note that $u(y_i) \leq u(x_i)$ and $l(x_i) \leq l(y_i)$ for any $i \in T$. Then $u(y) \leq u(x)$ and $l(x) \leq l(y)$. Since $u(x_i) \leq l(x_i)$ where i belongs to R, S or T, so $u(x) \leq l(x)$. Hence, $u(y) \leq u(x) \leq l(x) \leq l(y)$.

- $l(100) \le l(101) \le u(100) \le u(101)$. So we have $l(0) \le l(1) \le u(0) \le u(1)$. Note that $l(x_i) \le l(y_i)$ and $u(x_i) \le u(y_i)$ for any $i \in T$. Then $l(x) \le l(y)$ and $u(x) \le u(y)$. Since $l(y_i) \le u(x_i)$ where i belongs to R, S or T, so $l(y) \le u(x)$. Hence, $l(x) \le l(y) \le u(x) \le u(y)$.
- $l(100) \le l(101) \le u(101) \le u(100)$. So we have $l(0) \le l(1) \le u(1) \le u(0)$. Note that $l(x_i) \le l(y_i)$ and $u(y_i) \le u(x_i)$ for any $i \in T$. Then $l(x) \le l(y)$ and $u(y) \le u(x)$. Since $l(y_i) \le u(y_i)$ where i belongs to R, S or T, so $l(y) \le u(y)$. Hence, $l(x) \le l(y) \le u(y) \le u(x)$.
- $l(101) \le l(100) \le u(101) \le u(100)$. So we have $l(1) \le l(0) \le u(1) \le u(0)$. Note that $l(y_i) \le l(x_i)$ and $u(y_i) \le u(x_i)$ for any $i \in T$. Then $l(y) \le l(x)$ and $u(y) \le u(x)$. Since $l(x_i) \le u(y_i)$ where i belongs to R, S or T, so $l(x) \le u(y)$. Hence, $l(y) \le l(x) \le u(y) \le u(x)$.
- $l(101) \le l(100) \le u(100) \le u(101)$. So we have $l(1) \le l(0) \le u(0) \le u(1)$. Note that $l(y_i) \le l(x_i)$ and $u(x_i) \le u(y_i)$ for any $i \in T$. Then $l(y) \le l(x)$ and $u(x) \le u(y)$. Since $l(x_i) \le u(x_i)$ where i belongs to R, S or T, so $l(x) \le u(x)$. Hence, $l(y) \le l(x) \le u(x) \le u(y)$.

We can see that $\{u(x), l(x), u(y), l(y)\}$ is comparable in every case. By Theorem 41, $G^k(A, B)$ is word-representable for any $k \ge 0$. □

Remark 43. Theorem 42 can be applied to check word-representability of $G^k(A, B)$. We can see that $\{u_{A,B}(100), l_{A,B}(100), u_{A,B}(101), l_{A,B}(101)\}$ is comparable under $\le$ in Cases 2, 7, 8, 13, 18, 23, 27, 30, 34, 43, 44, 48, 51, 52, 56, 59, 61, 62, 84, 86, 94, 96, 100, 103, 107 and 113 in Tables 1 and 2. Then $\mathrm{IWR}(A, B) = \infty$ for A and B in these cases.

Proposition 44. *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. *Then* $G^k(A, B)$ *is word-representable for any* $k \ge 0$.

Proof. The case of $k = 0$ is trivial. For $k \ge 1$, we let $S_0^k := \emptyset$,

$$S_j^k := \{s \in \{1, 2, \ldots, 2^k\} | s \equiv 2^{j-1} \pmod{2^j}\} \text{ for } j \in \{1, 2, \ldots, k\}$$

and

$$T^k := \{x_1 x_2 \cdots x_{2^k} | x_i = \begin{cases} 0 \text{ if } i \notin S_j^k, \\ 1 \text{ if } i \in S_j^k \end{cases} \quad \text{for some } j \in \{0, 1, \ldots, 2^k\}\}.$$

We claim that $R^k(A, B) = T^k$ for any $k \ge 1$ and prove it by induction on k. It is obvious in the case of $k = 1$ because $R^1(A, B) = \{00, 10\}$ while $S_0^1 = \emptyset$ and $S_1^1 = \{1\}$. Suppose l is a positive integer such that $R^l(A, B) = T^l$. Let $y \in R^{l+1}(A, B)$, then $y = u_{A,B}(z)$ or $y = l_{A,B}(z)$ for some $z \in R^l(A, B)$. Since $R^l(A, B) = T^l$, we have, for some $r \in \{0, 1, \ldots, l\}$,

$$z = z_1 z_2 \cdots z_{2^l} \text{ where } z_i = \begin{cases} 0 \text{ if } i \notin S_r^l, \\ 1 \text{ if } i \in S_r^l. \end{cases}$$

If $y = u_{A,B}(z)$, then $y = 1010 \cdots 10$. That is, $y_i = \begin{cases} 0 \text{ if } i \notin S_1^{l+1}, \\ 1 \text{ if } i \in S_1^{l+1} \end{cases}$, and so $y \in T^{l+1}$. If $y = l_{A,B}(z)$, the number of 1's in y and z is identical because 0 is mapped to 00 and 1 is mapped to 01. We can see that $y_{2i} = 1$ if and only if $z_i = 1$. Hence,

$$y_i = \begin{cases} 0 \text{ if } i \notin S_{r+1}^{l+1}, \\ 1 \text{ if } i \in S_{r+1}^{l+1}. \end{cases}$$

So $R^{l+1}(A, B) \subseteq T^{l+1}$. Conversely, let $v = v_1 v_2 \cdots v_{2^{l+1}}$ where

$$v_i = \begin{cases} 0 \text{ if } i \notin S_t^{l+1}, \\ 1 \text{ if } i \in S_t^{l+1} \end{cases}$$

for some $t \in \{0, 1, \ldots, l+1\}$. If $t = 0$, note that $v = 000 \cdots 0 = l_{A,B}(000 \cdots 0)$ and $000 \cdots 0 \in R^l(A, B)$, and so $v \in R^{l+1}(A, B)$. If $t = 1$, we have $v = 1010 \cdots 10 = u_{A,B}(\bar{v})$ for any $\bar{v} \in R^l(A, B)$. That is $v \in R^{l+1}(A, B)$. Suppose $t \in \{2, 3, \ldots, l+1\}$ and $w = w_1 w_2 \cdots w_{2^l}$ be an element of T^l such that

$$w_i = \begin{cases} 0 \text{ if } i \notin S_{t-1}^l, \\ 1 \text{ if } i \in S_{t-1}^l. \end{cases}$$

By induction hypothesis, $w \in R^l(A, B)$. Note that $v = l_{A,B}(w)$ and then $v \in R^{l+1}(A, B)$. So we have $T^{l+1} \subseteq R^{l+1}(A, B)$. Hence we have already proved the claim.

Note that $\{1, 2, \ldots, 2^k - 1\}$ is a disjoint union of $S_1, S_2, \ldots, S_k$. If $S_j^k = \{i_{j,1}, i_{j,2}, \ldots, i_{j,|S_j^k|}\}$ for $i_{j,1} < i_{l,2} < \cdots < i_{j,|S_j^k|}$, then we can let ρ be the permutation

$$i_{1,1} i_{1,2} \cdots i_{1,|S_1^k|} i_{2,1} i_{2,2} \cdots i_{2,|S_2^k|} \cdots i_{k,1} i_{k,2} \cdots i_{k,|S_k^k|} 2^k.$$

Since $R^k(A, B) = T^k$, then all 1's in each row pattern in $R^k(A, B)$ is in columns $i_{j,1}, i_{j,2}, \ldots, i_{1,|S_j^k|}$ for some $1 < j < k$. Therefore we can see that every row of the matrix obtained by reordering columns of $M^k(A, B)$ according to the order given by ρ is of the form $0^a1^b0^c$ for some non-negative integers a, b and c. By Theorem 14, $G^k(A, B)$ is word-representable for ant $k \geq 1$. □

Remark 45. Proposition 44 gives $\mathrm{IWR}(A, B) = \infty$ for A and B in Case 9 in Tables 1. By Theorem 25, we also have $\mathrm{IWR}(A, B) = \infty$ for A and B in Cases 26 and 41.

Proposition 46. *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$*. Then* $G^k(A, B)$ *is word-representable for any* $k \geq 0$.

Proof. The case of $k = 0$ is trivial. For any binary string x, we define $\bar{x}$ to be the binary string obtained by changing digits of x from 0 to 1 and from 1 to 0. We claim that, for $k \geq 1$, $R^k(A, B) = \{x, \bar{x}\}$ for some $x \in B^{2^k}$ and prove it by induction on k. As $R^1(A, B) = \{10, 01\}$ and $01 = \overline{10}$, the case of $k = 1$ is done. Suppose that $k > 1$ is a positive integer such that $R^k(A, B) = \{x, \bar{x}\}$ for some $x \in B^{2^k}$. Then $R^{k+1}(A, B) = \{u_{A,B}(x), l_{A,B}(x), u_{A,B}(\bar{x}), l_{A,B}(\bar{x})\}$. Note that $u_{A,B}(0) = l_{A,B}(1)$ and $u_{A,B}(1) = l_{A,B}(0)$. Then $u_{A,B}(x) = l_{A,B}(\bar{x})$ and $l_{A,B}(x) = u_{A,B}(\bar{x})$. So we have $R^{k+1}(A, B) = \{u_{A,B}(x), l_{A,B}(x)\}$. Since $u_{A,B}(0) = \overline{l_{A,B}(0)}$ and $u_{A,B}(1) = \overline{l_{A,B}(1)}$, we have $u_{A,B}(x) = \overline{l_{A,B}(x)}$. That is, $R^{k+1}(A, B) = \{u_{A,B}(x), \overline{u_{A,B}(x)}\}$. Hence we have proved the claim by induction.

So, for any $k \geq 1$, $R^k(A, B) = \{x, \bar{x}\}$ for some binary string x. Suppose that there are s 1's in x. Let ρ be a permutation such that reordering $\begin{bmatrix} x \end{bmatrix}$ according to the order giving by ρ make all 1's in x be together in the first s columns. Then reordering $\begin{bmatrix} \bar{x} \end{bmatrix}$ according to the order giving by ρ makes all 1's in x be together in the last $2^k - s$ columns. Hence each row of the matrix obtained by reordering $M^k(A, B)$ according to the order giving by ρ is $1^s0^{2^k-s}$ or $0^s1^{2^k-s}$. By Theorem 14, $G^k(A, B)$ is word-representable for any $k \geq 0$. □

4.2. The case when A is the all-one matrix

There are only 16 cases when $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. If $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, we have $M^0(A, B) = \begin{bmatrix} 0 \end{bmatrix}$ and $M^k(A, B)$ is all-one matrix for any $k \geq 1$ which is word-representable by Theorem 14. The rest of this paper, except Theorem 47, deals with the cases when B has at least one 0.

Theorem 47. *Let* $A = \begin{bmatrix} 1 \end{bmatrix}_{m\times n}$ *and B be an* $m \times n$ *binary matrix. Then,* $M^k(A, B)$ *is a submatrix of* $M^{k+2}(A, B)$*, and* $G^k(A, B)$ *is an induced subgraph of* $G^{k+2}(A, B)$*, for any* $k \geq 0$.

Proof. Since the case of B being an all-one matrix is trivial, we assume that B is not an all-one matrix. Let $B = \{b_{ij}\}_{m\times n}$ and $b_{rs} = 0$ for some $1 \leq r \leq m$ and $1 \leq s \leq n$. We will prove by induction on k that $M^k(A, B)$ is contained in $M^{k+2}(A, B)$ as a submatrix by rows $(r-1)m^k + 1, (r-1)m^k + 2, \ldots, rm^k$ and columns $(s-1)n^k + 1, (s-1)n^k + 2, \ldots, sn^k$ for any $k \geq 0$.

Note that $M^0(A, B) = \begin{bmatrix} 0 \end{bmatrix}$, $M^1(A, B) = \begin{bmatrix} 1 \end{bmatrix}_{m\times n}$ and

$$M^2(A, B) = \begin{bmatrix} B & B & \cdots & B \\ B & B & \cdots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \cdots & B \end{bmatrix}.$$

Let $M^2(A, B) = \begin{bmatrix} m_{ij} \end{bmatrix}$, then $m_{rs} = 0$. So $M^0(A, B)$ is a submatrix of $M^2(A, B)$ by row $(r-1)m^0 + 1$ and column $(s-1)n^0 + 1$.

Let $l \geq 1$ be an integer such that $M^l(A, B)$ is a submatrix of $M^{l+2}(A, B)$ by rows $(r-1)m^k + 1, (r-1)m^k + 2, \ldots, rm^k$ and columns $(s-1)n^k + 1, (s-1)n^k + 2, \ldots, sn^k$. For the next iteration of morphism applied to $M^{l+2}(A, B)$, it is easy to see that $M^l(A, B)$ in the rows $(r-1)m^k + 1, (r-1)m^k + 2, \ldots, rm^k$ and the columns $(s-1)n^k + 1, (s-1)n^k + 2, \ldots, sn^k$ $M^{l+2}(A, B)$ is mapped to $M^{l+1}(A, B)$ in the rows $(r-1)m^{k+1} + 1, (r-1)m^{k+1} + 2, \ldots, rm^{k+1}$ and columns $(s-1)n^{k+1} + 1, (s-1)n^{k+1} + 2, \ldots, sn^{k+1}$ of $M^{l+3}(A, B)$. Hence, $M^k(A, B)$ is a submatrix of $M^{k+2}(A, B)$ for any $k \geq 0$. Consequently, $G^k(A, B)$ is an induced subgraph of $G^{k+2}(A, B)$ for any $k \geq 0$. □

We know from Theorem 47 that $G^0(A, B) \preceq G^2(A, B) \preceq G^4(A, B) \preceq \cdots$ and $G^1(A, B) \preceq G^3(A, B) \preceq G^5(A, B) \preceq \cdots$ where $G \preceq H$ means G is an induced subgraph of H. So we are interested in investigating the smallest integer l such that $G^l(A, B)$ is non-word-representable in both cases of l being even and l being odd. The cases 114, 119, 120, 123, 124 and 129 in Table 2 are given by using Proposition 24. Theorem 42 can be applied to Cases 125, 126, 127 and 128, and the index of word-representability in these cases is infinity. The following propositions discuss the remaining cases.

Proposition 48. *For* $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$*, IWR(A, B) = 3. Moreover,* $G^k(A, B)$ *is not word-representable for* $k \geq 3$.

Proof. Note that $M^2(A,B) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ and

$$M^3(A,B) = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Reordering columns of $M^2(A,B)$ in the order given by the permutation 1324, and applying Theorem 14, yields that $G^2(A,B)$ is word-representable.

Now, let M^* be a matrix obtained from reordering columns of $M^3(A,B)$ and every row of M^* is of the form $0^r1^s0^t$ or $1^r0^s1^t$. Then, the set of row patterns of M^* is

$$\{00111111, 11001111, 11110011, 11111100\}$$

or

$$\{01111110, 10011111, 11100111, 11111001\}.$$

If the set of row patterns of M^* is {00111111, 11001111, 11110011, 11111100}, then the occurrences of 11001111 and 11111100 do not satisfy the condition (ii) in Theorem 15. So, $G^3(A,B)$ is non-word-representable. Similarly, if the set of row patterns of M^* is {01111110, 10011111, 11100111, 11111001}, then the presence of 10011111 and 11111001 implies $G^3(A,B)$ is non-word-representable.

Now we consider the word-representability of $G^4(A,B)$. We have

$$\begin{aligned} R^4(A,B) = \{&1011101010111010, 0111010101110101, 1110101011101010,\\ &1101010111010101, 1010101110101011, 0101011101010111,\\ &1010111010101110, 0101110101011101\}. \end{aligned}$$

In order to apply Theorem 15, we assume the existence of an order $\rho = \rho_1\rho_2\cdots\rho_{16}$ with the following properties:

- from $1011101010111010 \in R^4(A,B)$, columns 2, 6, 8, 10, 14 and 16 must be cyclically consecutive;
- from $1110101011101010 \in R^4(A,B)$, columns 4, 6, 8, 12, 14 and 16 must be cyclically consecutive;
- from $1010101110101011 \in R^4(A,B)$, columns 2, 4, 6, 10, 12 and 14 must be cyclically consecutive.

It follows from the first and the second bullet points that 6, 8, 14, and 16 must be consecutive and then, w.l.o.g., 2 and 10 is next to the left of them and then 4 and 12 is next to the right them. That means that 2 and 4 cannot be cyclically consecutive, which contradicts the second bullet point. So there is no such ρ and $G^4(A,B)$ is non-word-representable. As $G^3(A,B)$ is non-word-representable, and the class of word-representable graphs is hereditary, by Theorem 47, $G^{2k+1}(A,B)$ is non-word-representable for any $k \geq 1$. Similarly, $G^{2k}(A,B)$ is non-word-representable for any $k \geq 2$ because $G^4(A,B)$ is non-word-representable. Therefore, $G^k(A,B)$ is not word-representable for $k \geq 3$. □

Remark 49. Proposition 48 gives Case 121 in Tables 1, and we obtain IWR$(A,B) = 3$ in Case 122 by a column and row permutation of A and B.

Proposition 50. *For* $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, *IWR*$(A,B) = 5$. *Moreover,* $G^k(A,B)$ *is not word-representable for* $k \geq 5$.

Proof. It is easy to see that $G^0(A,B)$, $G^1(A,B)$ and $G^2(A,B)$ are word-representable. Since reordering columns of $M^3(A,B)$ in order given by the permutation 26153748 yields a matrix satisfying the condition in Theorem 14, $G^3(A,B)$ is word-representable. Further, reordering columns of $M^4(A,B)$ in order given by the permutation

$$2(10)4(12)3(11)195(13)7(15)6(14)(16)$$

also yields a matrix satisfying the condition in Theorem 14, so $G^4(A,B)$ is word-representable.

Next, we consider $G^5(A,B)$. We use u and l instead of $u_{A,B}$ and $l_{A,B}$, respectively. In order to apply Theorem 15, we assume the existence of an order $\rho = \rho_1\rho_2\cdots\rho_{32}$ proving word-representability of $G^5(A,B)$.

- $l(l(l(u(u(0))))) = 10111011101110111011101110111011 \in R^5(A,B)$ so columns 2, 6, 10, 14, 18, 22, 26 and 30 must be cyclically consecutive in ρ.

- $l(u(u(l(l(0))))) = 10101010111111111010101011111111 \in R^5(A, B)$ so columns 2, 4, 6, 8, 18, 20, 22 and 24 must be cyclically consecutive in ρ.
- $l(u(l(l(u(0))))) = 11111010111111111111101011111111 \in R^5(A, B)$ so columns 6, 8, 22 and 24 must be cyclically consecutive in ρ.
- $u(u(l(l(u(0))))) = 11110000111111111111000011111111 \in R^5(A, B)$ so columns 5, 6, 7, 8, 21, 22, 23 and 24 must be cyclically consecutive in ρ.

It follows from the first and the second bullet points that 2, 6, 18, and 22 must be consecutive and then, w.l.o.g., 10, 14, 26 and 30 are next to the left of these numbers and 4, 8, 20 and 24 are next to the right them. Hence ρ contains

$$\{10, 14, 26, 30\}, \{2, 6, 18, 22\}, \{4, 8, 20, 24\}$$

where numbers in {} are consecutive in ρ but are in some unknown to us order. From the third bullet point, ρ contains

$$\{10, 14, 26, 30\}, \{2, 18\}, \{6, 22\}, \{8, 24\}, \{4, 20\}.$$

We obtain a contradiction with the fourth bullet point. So there is no such ρ and $G^5(A, B)$ is non-word-representable.

Next, we consider word-representability of $G^6(A, B)$. In order to apply Theorem 15, we assume that there exist a permutation $\tau = \tau_1\tau_2 \cdots \tau_{64}$ proving word-representability of $G^6(A, B)$.

- $l(u(u(u(u(u(0)))))) \in R^6(A, B)$ is 1010101010101010101010101010101010101010101010101010101010101010. So, all odd columns must be cyclically consecutive in τ.
- $u(u(u(u(l(l(0)))))) \in R^6(A, B)$ is 1111111111111111000000000000000011111111111111110000000000000000. So, columns in $\{1, 2, \ldots, 16, 33, 34, \ldots, 48\}$ must be cyclically consecutive in τ.
- $u(l(l(u(u(u(0)))))) \in R^6(A, B)$ is 0011000000110000001100000011000000110000001100000011000000110000. So, columns in $\{3, 4, 11, 12, 19, 20, 27, 28, 35, 36, 43, 44, 51, 52, 59, 60\}$ must be cyclically consecutive in τ.

Since all odd columns must be cyclically consecutive in τ, there is a unique $s \in \{1, 2, \ldots, 64\}$ such that τ_s is odd and τ_{s+1} is even, where for $s = 64$, $s + 1 := 1$. From the second bullet point, we have

$$\{\tau_s, \tau_{s-1}, \ldots, \tau_{s-15}\} = \{1, 3, 5, \ldots, 15, 33, 35, 37, \ldots, 47\}$$
$$\text{and } \{\tau_{s+1}, \tau_{s+2}, \ldots, \tau_{s+16}\} = \{2, 4, 6, \ldots, 16, 34, 36, 38, \ldots, 48\}$$

where for $i \geq 0$, $\tau_{-i} := \tau_{64-i}$ and for $s+i \geq 65$, $s+i := s+i-64$. On the other hand, from the third bullet point, we have

$$\{\tau_s, \tau_{s-1}, \ldots, \tau_{s-7}\} = \{3, 11, 19, 27, 35, 43, 51, 59\}$$
$$\text{and } \{\tau_{s+1}, \tau_{s+2}, \ldots, \tau_{s+8}\} = \{4, 12, 20, 28, 36, 44, 52, 60\}$$

where the indices less than 1 and larger than 64 are treated as above. We obtain a contradiction because $19 \notin \{\tau_s, \tau_{s-1}, \ldots, \tau_{s-15}\}$ but $19 \in \{\tau_s, \tau_{s-1}, \ldots, \tau_{s-7}\}$. Hence, there is no such τ and $G^6(A, B)$ is non-word-representable. Since the graphs $G^5(A, B)$ and $G^6(A, B)$ are not word-representable and the class of word-representable graphs is hereditary, by Theorem 47, $G^k(A, B)$ is non-word-representable for any $k \geq 5$. □

Remark 51. Proposition 50 is Case 117 in Table 1. By Theorem 25, column and row permutations of A and B give the same IWR. Consequently, we also have $\text{IWR}(A, B) = 5$ for A and B in Cases 115, 116 and 117.

5. Concluding remarks

This paper is a major contribution to the study of word-representability of split graphs. Two key achievements in the paper are as follows:

- Theorem 15 can be used to study word-representability of split graphs via adjacency matrices, which is a novel approach.
- Necessary conditions for word-representability of split graphs obtained by iteration of morphism can be checked in polynomial time. Indeed, for a given such graph defined by matrices A and B, we can go through all of 2×2 submatrices in A, and the respective submatrices in B, and then use our classification results in Tables 1 and 2 to detect non-word-representability.

As for open directions of research, it would be useful to provide a classification, similar to that we provided for 2×2 matrices in Tables 1 and 2, for larger matrices, for example, for 2×3 matrices, or 3×3 matrices, etc. This would enlarge our knowledge of word-representable split graphs obtained by iteration of morphisms, and hopefully, will eventually lead to a complete classification of such graphs.

For another research question, we noted in Tables 1 and 2 that if $G^5(A, B)$ is word-representable, then $IWR(A, B) = \infty$. In other words, the largest finite IWR in the case of 2×2 matrices is 5. Is there a reason for that? Does there exist a positive integer t (a constant, or a function of n and m) making the following statement true "If A, B are $n \times m$ matrices and $G^t(A, B)$ is word-representable, then $IWR(A, B) = \infty$"?

Finally, recall that if the leftmost bottom entry of A is 1 then $\lim_{k\to\infty} M^k(A, B)$ may not be well-defined as the sequence of graphs $G^k(A, B)$, for $k \geq 0$, may not be a chain of induced subgraphs. In all such cases, for 2×2 matrices, we still have that non-word-representability of $G^k(A, B)$ implies non-word-representability of $G^{k+1}(A, B)$. Is it always the case for $m \times n$ matrices A and B? If not, then how do we characterize the situations when it is the case?

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