SOLAR SAIL ORBITS AT THE EARTH-MOON LIBRATION POINTS

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Solar sail technology offer new capabilities for the analysis and design of space missions. This new concept promises to be useful in overcoming the challenges of moving throughout the solar system. In this paper, novel families of highly non-Keplerian orbits for solar sail spacecraft at linear order are investigated in the Earth-Moon circular restricted three body problem, where the third body is a solar sail. In particular, periodic orbits near the collinear libration points in the Earth-Moon system will be explored along with their applications. The dynamics are completely different from the Earth-Sun system in that the sun line direction constantly changes in the rotating frame but rotates once per synodic lunar month. Using an approximate, first order analytical solution to the nonlinear nonautonomous ordinary differential equations, periodic orbits can be constructed that are displaced above the plane of the restricted three-body system. This new family of orbits have the property of ensuring visibility of both the lunar far-side and the equatorial regions of the Earth, and can enable new ways of performing lunar telecommunications.

Keywords: periodic orbit, solar sail, circular restricted three-body problem

1. Introduction

Solar sailing technology appears as a promising form of advanced spacecraft propulsion, which can enable exciting new space-science mission concepts such as solar system exploration and deep space observation. Although solar sailing has been considered as a practical means of spacecraft propulsion only relatively recently, the fundamental ideas are by no means new (see McInnes[1] for a detailed description).

Solar sails can also be utilised for highly non-Keplerian orbits, such as closed orbits displaced high above the ecliptic plane (see Waters and McInnes[2]). Solar sails are especially suited for such non-Keplerian orbits,

since they can apply a propulsive force continuously. This allows some exciting unique trajectories. In such trajectories, a sail will be used as communication satellites for high latitudes. For example, the orbital plane of the sail can be displaced above the orbital plane of the Earth, so that the sail can stay fixed above the Earth at some distance, if the orbital periods are equal. Orbits around the collinear points of the Earth-Moon system are of great interest because their unique positions are advantageous for several important applications in space mission design (see e.g. Szebehely[3], Roy[4]) and Thurman et al.[5]).
In the last decades several authors have tried to determine more accurate approximations (quasi-Halo orbits) of such equilibrium orbits[6]. The orbits were first studied by Farquhar[7], Farquhar and Kamel[6], Breakwell and Brown[8], Richardson[9], Howell[10]. Halo orbits near the collinear libration points in the Earth-Moon system are of great interest. If the orbit maintains visibility from Earth, a spacecraft on it can be used to assure communications between the equatorial regions of the Earth and the lunar far-side. Thus, the establishment of a bridge for radio communications is a crucial problem for incoming space missions, which plan to use the lunar far-side as a powerful observation point. McInnes[11] investigated a new family of displaced solar sail orbits near the Earth-Moon libration points. In Baoying and McInnes[12–14] and McInnes[11,15], the authors describe the new orbits which are associated with artificial lagrange points. These artificial equilibria have potential applications for future space physics and Earth observation missions. Most work has been done in the Sun-Earth system. In McInnes and Simmons[16], the authors investigate large new families of solar sail orbits, such as Sun-centered halo-type trajectories, with the sail executing a circular orbit of a chosen period above the ecliptic plane. In our study, we will demonstrate the possibility of such trajectories in the Earth-Moon system.

Briefly, in the present study a new family of solar sail periodic orbits have been investigated in the Earth-Moon restricted three-body problem. The first-order approximation is introduced for the linearized system of equations.

The Laplace transform is used to produce the first-order analytic solution of the out-of plane motion. We find families of periodic orbits above the ecliptic plane at linear order. It will be shown for example that, with a suitable sail attitude control program, a 3.5×10^3 km displaced, out-of-plane trajectory around the L_2 point may be executed with a sail acceleration of only 0.2 mms^{-2} (see Figure 2).

This paper is organized as follow: Section 2 provides the mathematical expressions describing the motion of the sail in the circular restricted three-body problem.

Section 3 is devoted to the study of the periodic orbits around the collinear libration points in the Earth-Moon system. The periodic solutions to the linearized equations of motion (quasi-periodic orbits) are derived analytically. Section 4 is concerned with the numerical computation around the collinear libration points L_1 and L_2 in the Earth-Moon system. Finally some numerical results are presented.

2. Solar sail in the Earth-Moon restricted three-body problem

2.1. *Qualitative approach*

In context of this work, we will assume that m_1 represents the larger primary, m_2 the smaller primary ($m_1 > m_2$) and we will be concerned with the motion of the sail that has negligible mass. It is always assumed that the two more massive bodies (primaries) are moving in circular orbits about their common center of mass and the mass of the third body is too small to affect the motion of the two more massive bodies. The problem of the motion of the third body is called the circular restricted three-body problem (CRTBP).

In order to develop any mathematical model without loss of the generality, it is useful to introduce some parameters that are characteristics of each particular three-body system. This set of parameters is used to normalize the equations of motion. Also the unit mass is taken to be the total mass of the system ($m_1 + m_2$) and the unit of length is chosen to be the constant separation between m_1 and m_2.

Under these considerations the masses of the primaries in the normalized system of units are $m_1 = 1 - \mu$ and $m_2 = \mu$, with $\mu = m_2/(m_1 + m_2)$ (see Figure 1).

2.2. *Equations of motion in presence of solar sail*

The vector dynamical equation for the solar sail in a rotating frame of reference is described by

$$\frac{d^2\boldsymbol{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\boldsymbol{r}}{dt} + \nabla U(\boldsymbol{r}) = \boldsymbol{a}, \tag{1}$$

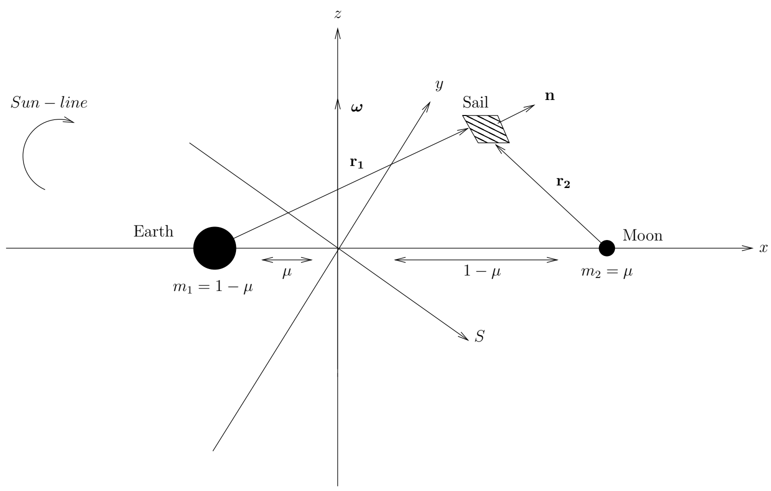

Fig. 1. Schematic geometry of the Earth-Moon restricted three-body problem.

where $\boldsymbol{\omega} = \omega\hat{\boldsymbol{z}}$ ($\hat{\boldsymbol{z}}$ is a unit vector pointing in the direction of $\boldsymbol{z}$) is the angular velocity vector of the rotating frame and $\boldsymbol{r}$ is the position vector of the solar sail relative to the center of mass of the two primaries. The three-body gravitational potential $U(\boldsymbol{r})$ and the solar radiation pressure acceleration $\boldsymbol{a}$ are defined by

$$U(\boldsymbol{r}) = -\left[\frac{1}{2}|\boldsymbol{\omega}\times\boldsymbol{r}|^2 + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right], \qquad \boldsymbol{a} = a_0(\boldsymbol{S}\cdot\boldsymbol{n})^2\boldsymbol{n}, \tag{2}$$

where $\mu = 0.1215$ is the mass ratio for the Earth-Moon system, the sail position vectors w.r.t. the origin, m_1 and m_2 respectively, are defined as $\boldsymbol{r}_1 = [x+\mu, y, z\ ]^T$ and $\boldsymbol{r}_2 = [x-(1-\mu), y, z]^T$, and a_0 is the magnitude of the solar radition pressure force exerted on the sail. The unit normal to the sail $\boldsymbol{n}$ and the Sun line direction are given by

$$\boldsymbol{n} = \begin{bmatrix}\cos(\gamma)\cos(\omega_\star t) & -\cos(\gamma)\sin(\omega_\star t) & \sin(\gamma)\end{bmatrix}^T, \tag{3}$$

$$\boldsymbol{S} = \begin{bmatrix}\cos(\omega_\star t) & -\sin(\omega_\star t) & 0\end{bmatrix}^T, \tag{4}$$

where $\omega_\star = 0.923$ is the angular rate of the Sun line in the corotating frame in dimensionless synodic coordinate system. The dynamics of the sail in the neighborhood of the collinear libration points at $\boldsymbol{r}_L$ will be now investigated. Let a small displacement in $\boldsymbol{r}_L$ be $\delta\boldsymbol{r}$ such that $\boldsymbol{r}\rightarrow\boldsymbol{r}_L+\delta\boldsymbol{r}$. We will not consider the small annual changes in the inclination of the Sun line with respect to the plane of the system.

Also, the linear equations for the solar sail can be written

$$\frac{d^2\delta\boldsymbol{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\delta\boldsymbol{r}}{dt} + \nabla U(\boldsymbol{r}_L + \delta\boldsymbol{r}) = \boldsymbol{a}(\boldsymbol{r}_L + \delta\boldsymbol{r}), \tag{5}$$

and retaining only the first-order term in $\delta\boldsymbol{r} = [\delta x, \delta y, \delta y]^T$ in a Taylor-series expansion, the gradient of the potential and the acceleration can be expressed as

$$\nabla U(\boldsymbol{r}_L + \delta\boldsymbol{r}) = \nabla U(\boldsymbol{r}_L) + \left.\frac{\partial \nabla U(\boldsymbol{r})}{\partial \boldsymbol{r}}\right|_{r=r_L} \delta\boldsymbol{r} + O(\delta\boldsymbol{r}^2), \tag{6}$$

$$\boldsymbol{a}(\boldsymbol{r}_L + \delta\boldsymbol{r}) = \boldsymbol{a}(\boldsymbol{r}_L) + \left.\frac{\partial \boldsymbol{a}(\boldsymbol{r})}{\partial \boldsymbol{r}}\right|_{r=r_L} \delta\boldsymbol{r} + O(\delta\boldsymbol{r}^2). \tag{7}$$

It is assumed $\nabla U(\boldsymbol{r}_L) = 0$, and the acceleration is constant with respect to the small displacement $\delta\boldsymbol{r}$, then

$$\left.\frac{\partial \boldsymbol{a}(\boldsymbol{r})}{\partial \boldsymbol{r}}\right|_{r=r_L} = 0. \tag{8}$$

The linear variational system associated with the collinear libration points at $\boldsymbol{r}_L$ can be determined through a Taylor polynomial by substituting Eqs. (6) and (7) into (5)

$$\frac{d^2\delta\boldsymbol{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\delta\boldsymbol{r}}{dt} - K\delta\boldsymbol{r} = \boldsymbol{a}(\boldsymbol{r}_L), \tag{9}$$

where the matrix K is defined as

$$K = -\left[\left.\frac{\partial \nabla U(\boldsymbol{r})}{\partial \boldsymbol{r}}\right|_{r=r_L}\right]. \tag{10}$$

Using the matrix notation the linearized equation about the libration point (Equation (9)) can be represented by the inhomogeneous linear system $\dot{\mathbf{X}} = A\mathbf{X} + \boldsymbol{b}(t)$, where the state vector $\mathbf{X} = (\delta\boldsymbol{r}, \delta\dot{\boldsymbol{r}})^T$, and $\boldsymbol{b}(t)$ is a 6×1 vector, which represents the solar sail acceleration.

The Jacobian matrix A has the general form

$$A = \begin{pmatrix} 0_3 & I_3 \\ K & \Omega \end{pmatrix}, \tag{11}$$

where I_3 is a identity matrix, and

$$\Omega = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{12}$$

For convenience the sail attitude is fixed such that the sail normal vector $\boldsymbol{n}$, which is a unit vector that is perpendicular to the sail surface, points always along the direction of the Sun line with the following constraint $\boldsymbol{S} \cdot \boldsymbol{n} \geq 0$. Its direction is described by the pitch angle γ, which represents the sail attitude.

This yields the linearized nondimensional equations of motion in component form of a solar sail near the collinear libration point

$$\ddot{\xi} - 2\dot{\eta} - U^o_{xx}\xi = a_0 \cos(\omega_\star t)\cos^3(\gamma), \tag{13}$$
$$\ddot{\eta} + 2\dot{\xi} - U^o_{yy}\eta = -a_0 \sin(\omega_\star t)\cos^3(\gamma), \tag{14}$$
$$\ddot{\zeta} - U^o_{zz}\zeta = a_0 \cos^2(\gamma)\sin(\gamma), \tag{15}$$

where U^o_{xx}, U^o_{yy}, and U^o_{zz} are the partial derivatives of the gravitational potential evaluated at the collinear libration points.

3. Solution of the linearized equations of motion

The solution can be made to contain only oscillatory modes with the proper choice of initial conditions. Then, the solar sail will follow a periodic orbit about the libration point.
Clearly, the out-of-plane motion (15) described by a driven harmonic oscillator is decoupled from the in-plane equations of motion ((13)-(14)). These conditions can be met by choosing a particular solution in the plane of the form (see Farquhar[17])

$$\xi(t) = \xi_0 \cos(\omega_\star t), \tag{16}$$
$$\eta(t) = \eta_0 \sin(\omega_\star t). \tag{17}$$

By inserting Equations (16) and (17) in the differential equations, we obtain the linear system in ξ_0 and η_0,

$$\begin{cases} \left(U^o_{xx} - \omega_\star^2\right)\xi_0 - 2\omega_\star\eta_0 = a_0\cos^3(\gamma), \\ -2\omega_\star\xi_0 + \left(U^o_{yy} - \omega_\star^2\right)\eta_0 = -a_0\cos^3(\gamma). \end{cases} \tag{18}$$

Then the amplitudes ξ_0 and η_0 are given by

$$\xi_0 = a_0 \frac{\Big(U^o_{yy} - \omega_\star^2 - 2\omega_\star\Big)\cos^3(\gamma)}{\Big(U^o_{xx} - \omega_\star^2\Big)\Big(U^o_{yy} - \omega_\star^2\Big) - 4\omega_\star^2}, \tag{19}$$

$$\eta_0 = a_0 \frac{\Big(- U^o_{xx} + \omega_\star^2 + 2\omega_\star\Big)\cos^3(\gamma)}{\Big(U^o_{xx} - \omega_\star^2\Big)\Big(U^o_{yy} - \omega_\star^2\Big) - 4\omega_\star^2}, \tag{20}$$

and we have the equality

$$\frac{\xi_0}{\eta_0} = \frac{\omega_\star^2 + 2\omega_\star - U^o_{yy}}{-\omega_\star^2 - 2\omega_\star + U^o_{xx}}. \tag{21}$$

We can find the required radiation pressure be solving the equation (19)

$$a_0 = \cos^{-3}(\gamma)\left[\frac{\omega_\star^4 - \omega_\star^2(U^o_{xx} + U^o_{yy} + 4) + U^o_{xx}U^o_{yy}}{U^o_{yy} - 2\omega_\star - \omega_\star^2}\right]\xi_0. \tag{22}$$

By applying the Laplace transform, the uncoupled out-of-plane ζ-motion defined by the equation (15) can be solved.
The transform version is obtained as

$$s^2 Z - s\xi_0 - \dot{\xi}_0 - U^o_{zz} Z = \frac{a_0\cos^2(\gamma)\sin(\gamma)}{s}, \tag{23}$$

$$(s^2 - U^o_{zz})Z = \dot{\xi}_0 + s\xi_0 + \frac{a_0\cos^2(\gamma)\sin(\gamma)}{s}, \tag{24}$$

also

$$Z(s) = \frac{1}{s^2 - U^o_{zz}}\left(\dot{\xi}_0 + s\xi_0 + \frac{a_0\cos^2(\gamma)\sin(\gamma)}{s}\right). \tag{25}$$

The frequency of the out-of-plane motion is given by solving the equation $s^2 - U^o_{zz} = 0$, also $s_{1,2} = \pm i\sqrt{|U^o_{zz}|} = \pm i\omega_\zeta$.

Using Mathematica, we can find the inverse Laplace transform, which will be the general solution of the out-of-plane component

$$\begin{aligned}\zeta(t) &= \zeta_0\cos(\omega_\zeta t) + \dot{\zeta}_0|U^o_{zz}|^{-1/2}\sin(\omega_\zeta t) \qquad (26)\\ &\quad + a_0\cos^2(\gamma)\sin(\gamma)|U^o_{zz}|^{-1}[U(t) - \cos(\omega_\zeta t)],\\ &= U(t)a_0\cos^2(\gamma)\sin(\gamma)|U^o_{zz}|^{-1} + \dot{\zeta}_0|U^o_{zz}|^{-1/2}\sin(\omega_\zeta t) \qquad (27)\\ &\quad + \cos(\omega_\zeta t)[\zeta_0 - a_0\cos^2(\gamma)\sin(\gamma)|U^o_{zz}|^{-1}],\end{aligned}$$

where the nondimensional frequency is defined as $\omega_\zeta = |U^o_{zz}|^{1/2}$ and $U(t)$ is the unit step function.

Specifically for the choice of the initial data $\dot{\zeta}_0 = 0$, the equation (27) can be more conveniently expressed as

$$\zeta(t) = U(t) a_0 \cos^2(\gamma) \sin(\gamma) |U^o_{zz}|^{-1} + \cos(\omega_\zeta t)[\zeta_0 - a_0 \cos^2(\gamma) \sin(\gamma) |U^o_{zz}|^{-1}]. \tag{28}$$

The solution can be made to contain only the periodic oscillatory modes at an out-of-plane distance

$$\zeta_0 = a_0 \cos^2(\gamma) \sin(\gamma) |U^o_{zz}|^{-1}. \tag{29}$$

Of course, this distance can be maximized by an optimal choice of the sail pitch angle determined by

$$\frac{d}{d\gamma^\star} \cos^2(\gamma^\star) \sin(\gamma^\star) = 0, \tag{30}$$

$$\gamma^\star = \tan^{-1}(2^{-1/2}), \tag{31}$$

$$\gamma^\star = 35^\circ.264. \tag{32}$$

4. Numerical integration of the nonlinear equations of motion

This section is concerned with the computation of periodic orbits around the collinear libration points L_1 and L_2 in the Earth-Moon system.

The linear approximation is only valid at small distances from the Lagrange points at linear order. Then, we should be able to find an out-of-plane halo-type motion around the Earth-Moon L_2 point.

In order to model motion in the nonlinear case the linearized equations of motion are used to find a guess for the appropriate initial conditions to generate out-of-plane periodic (halo-type) trajectory around the collinear libration points. Thus, at the beginning the solutions given by analytical approximations are used as initial guess. These approximate analytical solutions are utilized in a numerical search to determine displaced periodic orbits in the full nonlinear model. However, the initial guess found is not good enough to close the orbit in the nonlinear system.

Our purpose, hereafter, is to apply a robust control approach, namely a time-delayed feedback control[18], which takes into account the solution found from the linearized dynamics to generate a periodic reference trajectory.

For the nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \mathbb{R}^n. \tag{33}$$

We want to find a time-delayed feedback control ($\tau > 0$ is a delay-time)

$$\boldsymbol{u}(t) = -K(\mathbf{x}(t) - \boldsymbol{x}(t-\tau)), \tag{34}$$

to be added to $\boldsymbol{f}(\boldsymbol{x}, t)$ such that the controlled system orbit can track the target

$$\lim_{t\to\infty} ||\boldsymbol{x}(t) - \boldsymbol{x}(t-\tau)|| = 0. \tag{35}$$

Thus, the design problem is to determine the control gain matrix K to achieve the goal (equation 35), such that

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t) + K(\boldsymbol{x}(t) - \boldsymbol{x}(t-\tau)). \tag{36}$$

In order to obtain a one-month orbit, the delay-time will be $2\pi/\omega_\star$ and the matrix K is a scalar multiple of the identity matrix $I_{6\times 6}$ which is computed experimentally. The final trajectory that corresponds to the minimum feedback requirement will be used as a reference orbit (see Figure 2).

It will be shown that the sail may execute an out-of-plane distance of 5×10^3 km with the semimajor and minor axes of 1.174×10^4 km ($\eta_0 = 3.051 \times 10^{-2}$) and 1.208×10^3 km ($\xi_0 = 3.140 \times 10^{-3}$) in the neighborhood of the cislunar libration point L_1. In the same way the trajectory arround the L_2 point would be a narrow ellipse with semi-major and minor axes of 1.105×10^4 km ($\eta_0 = 2.876 \times 10^{-2}$) and 5.655×10^2 km ($\xi_0 = 1.471 \times 10^{-3}$) and a period of 28 days (synodic lunar month). Therefore the sail may be placed on such a trajectory by inserting it into a suitable elliptical trajectory about the L_2 point. The lunar far-side and the equatorial regions would be visible with only a sail acceleration of the order of 0.2 mms^{-2}. This small performance solar sail could be used to demonstrate the use of the L_2 point for lunar far-side communications. At the Eart-Moon L_2 point, the same sail could be displaced into a modified equilibrium point 10^4 km following the trajectory with the semimajor and minor axes of 3.116×10^4 ($\eta_0 = 8.21 \times 10^{-2}$) and 1.614×10^3 km ($\xi_0 = 4.197 \times 10^{-3}$).

Because of the instability of the collinear libration points, such orbits cannot be maintained without active control.

Future work will be focussed on linear control techniques to the problem of tracking and maintaining the solar sail on prescribed orbits. Also, we will use a linear feedback regulator (LQR) to track a periodic reference trajectory (Figure 2) based on a time-delayed feedback mechanism.

5. Conclusion

In this study a new family of displaced solar sail orbits near the collinear libration points in the Earth-Moon system have been identified. The Laplace

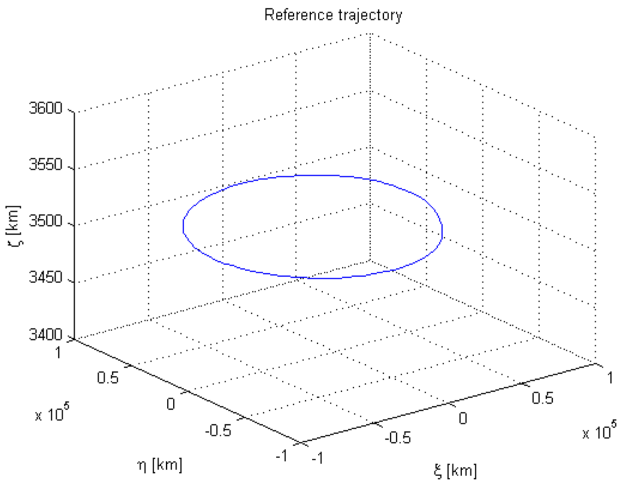

Fig. 2. Out-of-plane solar sail trajectory at the translunar libration point L_2.

transform was used to give the general solution to the uncoupled out-of plane motion. It can be seen from this form of solution that once the sail is pitched from $\gamma = 0$ at $t = 0$, the motion of the sail is of the form of periodic oscillations at an out-of plane given distance. Also, by choosing this initial distance, the sail remains at this fixed distance. It was found that periodic orbits can be developed at linear order, that are displaced above the plane of the restricted three-body problem. These new families of highly non-Keplerian orbits that are unique to solar sails can enable new ways of performing space-science missions. Despite the fact that the accurate results by studying the linearized system may be found, it should be remembered that these solutions are only approximations to the real behaviour.

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