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A control volume based formulation of the discrete Kirchhoff triangular thin plate bending element

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ABSTRACT

A control volume method is presented for predicting the displacement and rotation of thin transversely loaded flat plates. The new procedure uses discrete Kirchhoff triangle (DKT) elements but introduces a dual mesh of interconnected control volumes (CVs) centred on the finite element (FE) vertices. Discrete equations for the unknown degrees of freedom are subsequently derived by enforcing equilibrium on these CVs; as such this implementation is a quadrature free routine. To allow a comparison, a quadrature free implementation of the DKT element, using the standard finite element procedure, was developed using symbolic mathematics. The CV based procedure is validated by patch tests for a state of pure bending and twist. Convergence tests for various loading types show enhanced performance for coarse meshes over the equivalent FE method.

1 INTRODUCTION

Plate bending elements remain an active research area, with work focusing upon the selection of deformation theory, application of suitable boundary conditions and the avoidance of element shear locking. Recent developments in finite volume (FV) methods have identified two different approaches, a cell centred method that is a geometrically versatile formulation with multifaceted control volumes [1] and vertex centred methods [2]. Both formulations are presented as locking free for both thick and thin Mindlin plates. This paper presents a vertex centred FV thin plate formulation based upon the discrete Kirchhoff triangle element [3]. The DKT element is based upon Mindlin plate theory but has Kirchhoff constraints, that is transverse shear is zero, applied at each node giving rise to an element that converges to the Kirchhoff thin plate solution. The solution convergence is compared against the existing finite element DKT. Both the FE-DKT and CV-DKT formulations where built upon the same moment curvature matrix and both where solved without quadrature in order to have the best comparison of the numerical procedures. The FV method differs from the FE by introducing a dual mesh of interconnecting control volumes over a standard finite element mesh. The element stress resultants are then integrated around the control volume faces and equilibrium is imposed on that CV. The resulting equilibrium equations then relate the control volume centre unknown displacements to those at neighbouring centres, in a manner equivalent to the relationships between nodal displacements characteristic of the FE method. A quadrature free implementation is achieved using the symbolic maths toolbox of MATLAB which is built upon the Maple kernel. Symbolic integration (SI) of the moment curvature matrix is carried out in both the FE-DKT and CV-DKT codes to obtain the element stiffness matrix.

2 Element Stiffness Matrix

As already stated both the FE and CVFE methods are founded upon the same moment curvature relationships, but the formulations differ with regards to the element stiffness matrix. In the FE-DKT element the stiffness matrix is derived using the principle of minimum potential energy, equation (1). The CV-DKT differs from this because the stiffness matrix is composed of a set of discrete equilibrium equations. In the finite element method the stiffness matrix for the DKT element is:

$$\mathbf{K_{FEM}} = 2A \int_0^1 \int_0^{1-\eta} \mathbf{B}^T \mathbf{D}_b \mathbf{B} d\xi d\eta \tag{1}$$

where $\mathbf{B}$ [3] is the moment curvature matrix, $\mathbf{D_b}$ is the constitutive matrix in bending, A is the element area and ξ, η are the area coordinates of the traingular element.

To determine the stiffness matrix for the CV-DKT a dual mesh of interconnecting control volumes is set up, with each control volume centred upon a node of the finite element mesh. The control volume faces are constructed by connecting a point mid way along the finite element mesh edge to the centre of area of that corresponding finite element.

The stress resultants per unit length are integrated along each face of the control volume with respect to the line coordinate r, anti-clockwise around the CV node. This integration gives rise to the stress resultants acting on each face. For a face i these are $\mathbf{T}_z^i$, $\mathbf{M}_x^i$ and $\mathbf{M}_y^i$, representing the total transverse force and total moments about the x and y axes respectively. Equations (2), (3) and (4) are thus functions to determine the internal actions upon each face of the control volume.

Transverse Force:

$$\mathbf{T}_z^i = \int T_x cos\theta dr + \int T_y sin\theta dr \tag{2}$$

Total Moment about the x-axis:

$$\mathbf{M}_x^i = \int M_y sin\theta dr + \int M_{xy} cos\theta dr - \int (y_r - y_i)(T_x cos\theta dr + T_y sin\theta dr) \tag{3}$$

Total Moment about the y-axis:

$$\mathbf{M}_y^i = \int M_x cos\theta dr + \int M_{xy} sin\theta dr - \int (x_r - x_i)(T_x cos\theta dr + T_y sin\theta dr) \tag{4}$$

where (x_r, y_r) are the coordinates of the moment arm along the differential line, (x_i, y_i) are the coordinates of the centre of the control volume, M_n are the bending moment resultants, T_n are the shear stress resultants and θ denoted the inclination of the control volume face. M_n and T_n are determined from the product of the moment curvature, $\mathbf{B}$, and constitutive, $\mathbf{D_b}$, matices

Equilibrium is imposed on the control volume by summing all the internal actions on each face for the control volume. The equilibrium equations can be expressed as:

$$\sum_{i=1}^{n} \mathbf{T}_z^i + \mathbf{T}_z^E = 0 \qquad \sum_{i=1}^{n} \mathbf{M}_x^i + \mathbf{M}_x^E = 0 \qquad \sum_{i=1}^{n} \mathbf{M}_y^i + \mathbf{M}_y^E = 0 \tag{5}$$

where $\mathbf{T}_z^E$, $\mathbf{M}_x^E$ and $\mathbf{M}_y^E$ are any externally applied forces or moments on the control volume and n is the number of faces for a given control volume. This is carried out for each element in the mesh and assembled into the global stiffness matrix in a manner analogous to the standard finite element procedure [4].

$$\begin{bmatrix} \mathbf{K}_{zw} & \mathbf{K}_{zw,y} & \mathbf{K}_{zw,x} \\ \mathbf{K}_{m_x w} & \mathbf{K}_{m_x w,y} & \mathbf{K}_{m_x w,x} \\ \mathbf{K}_{m_y w} & \mathbf{K}_{m_y w,y} & \mathbf{K}_{m_y w,x} \end{bmatrix} \begin{Bmatrix} w \\ w_{,y} \\ -w_{,x} \end{Bmatrix} = \begin{Bmatrix} -\mathbf{T}_z^E \\ -\mathbf{M}_x^E \\ -\mathbf{M}_y^E \end{Bmatrix} \tag{6}$$

Now the structural equations are in a form equivalent to the finite element procedure (7),

$$[\mathbf{K}]\{\mathbf{U}\} = \{\mathbf{P}\} \tag{7}$$

where $\{\mathbf{U}\}$ is the vector of nodal displacements, $\{\mathbf{P}\}$is the vector of externally applied force and moments and $[\mathbf{K}]$ is the stiffness matrix (6). The load vector can now be modified to include applied loads and the appropriate boundary constraints applied to the stiffness matrix as in the finite element method. The stiffness matrix can then be solved by either a direct or iterative solution strategy. In the presented work the MATLAB matrix left division routine was used.

3 Implementation

In the formulation of the FE-DKT (1) and CV-DKT (6) stiffness matrices, symbolic integration, using the Maple kernel of MATLAB, was employed. The advantage of using symbolics is that an explicit solution to the stiffness matrix is achieved. The symbolic toolbox is capable of integrating the moment curvature matrix in a relatively quick time, with the solution extractable to form conventional code. This was validated by comparing the solution to the FE-DKT element stiffness matrix against the explicit FORTRAN code of Jeychandrabose et. al. [5]. Both methods gave an identical solution. Results from the quadrature free FE-DKT and CV-DKT elements were compared against the shell63 element of ANSYS, a quadratic element composed of 4 DKT elements [6] where 3 point quadrature is employed in evaluating the stiffness matrix.

4 Results

The CV-DKT has a proper rank to its stiffness matrix and passes the patch test for states of pure bending and twist. Convergence tests of maximum displacement against increasing discretisation, for various loading types and boundary conditions where used to asses the performance of the CV-DKT element against the FE-DKT, quadrature free DKT, and the quadrature based ANSYS-DKT. Shown here are the normalised central displacements for square plates of thickness $h = 0.05$ with clamped boundaries, loaded with a uniform pressure load of $10N/m^2$ (Figure 1(a)) and a centrally applied point load of $1N$ (Figure 1(b)). For the uniform pressure load case it can be seen that convergence to the exact solution is more rapid than in the FE equivalents. For the point load case it is noted that the CV-DKT method predicts the central displacement as accurately as the quadrature free FE formulation at a given mesh refinement.

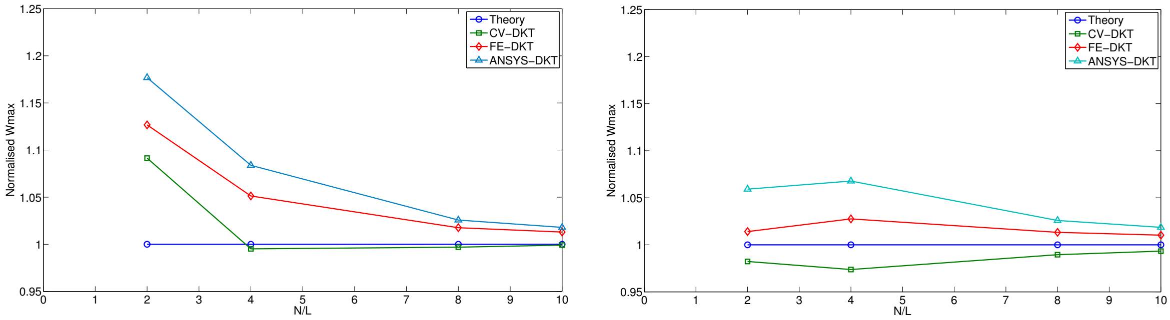

Figure 1: Convergence of maximum transverse displacement to thin plate theory. (a) Clamped plate with uniform pressure load, $a/b = 1$ and $h = 0.05$. (b) Clamped plate with centrally applied point load, $a/b = 1$ and $h = 0.05$.

5 Conclusion

A control volume based finite element method is presented for the prediction of bending deformations in thin plates. The method is a direct equivalent to the existing discrete Kirchhoff triangular element and displays equivalent or better displacement convergence under various loads. The method is quadrature free, utilising the symbolic integration tools of the Maple kernel of MATLAB. This work is presented to show the promise of the CVFEM formulation in plate bending problems.

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