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On the distributivity of T-power based implications

Z. Peng[1] and J. Pan[2]

[1,2] *College of Mathematics and Statistics, Yangtze Normal University, Chongqing, PR China*
[2] *Department of Mathematics and Statistics, University of Strathclyde, Glasgow, UK*

364886982@qq.com, jiazhu.pan@strath.ac.uk

Abstract

Due to the fact that Zadeh's quantifiers constitute the usual method to modify fuzzy propositions, the so-called family of T-power based implications was proposed. In this paper, the four basic distributive laws related to T-power based fuzzy implications and fuzzy logic operations (t-norms and t-conorms) are deeply studied. This study shows that two of the four distributive laws of the T-power based implications have a unique solution, while the other two have multiple solutions.

Keywords: T-power based implications, distributivity, t-norms, t-conorms.

1 Introduction

Due to fuzzy implications are the main operations in fuzzy logic, various fuzzy implications have been proposed. For example, the (S, N)-, R- and QL-implications are built by translating different classical logical formulae to the fuzzy context [4, 5]. The f- and g-implications are built from continuous additive generators of continuous Archimedean t-norms or t-conorms, respectively [21]. The probabilistic implications and probabilistic S-implications are built from copula functions [10]. The semicopula based implications are built from initial fuzzy implications and semicopula functions [2]. The fuzzy negation based implications are built from negation functions [15], etc.

In 2017, Massanet et al. noticed that a special property called invariance is required on a fuzzy implication when it is used in approximate reasoning. However, as most of the known fuzzy implications do not have this property, the so-called family of T-power based implications was proposed [13]. Most of the T-power based implications were found to satisfy the invariant property [14]. Nevertheless, there are no corresponding discussions on the distributive laws for the T-power based implications, although the distributive laws play a critical role in both theoretical and practical fields for fuzzy implications [7, 9]. On the other hand, there are many discussions on the distributive equations of fuzzy implications (detail see for [1, 3, 6, 8, 12, 16, 17, 18, 19, 20]). Therefore, as a supplement of this research topic from the theoretical point of view, it is necessary to investigate the distributive laws for the T-power implications.

The paper is organized as follows. In Section 2, some concepts and results are recalled. In Section 3, four distributive equations involving T-power based implications are analyzed. Finally, the paper ends with a section devoted to the conclusions.

2 Preliminaries

For convenience, in this section, the definitions and results to be used in the rest of the paper are outlined.

Definition 2.1. [4] *A function* $I : [0, 1]^2 \to [0, 1]$ *is called a fuzzy implication if it satisfies, for all* x, x_1, x_2, y, y_1, $y_2 \in [0, 1]$, *the following conditions:*

if $x_1 < x_2$, *then* $I(x_1, y) \geq I(x_2, y)$, *i.e.,* $I(\cdot, y)$ *is decreasing,* *(I1)*

if $y_1 < y_2$, *then* $I(x, y_1) \leq I(x, y_2)$, *i.e.,* $I(x, \cdot)$ *is increasing,* *(I2)*

Corresponding Author: Z. Peng

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$I(0,0) = 1,\ I(1,1) = 1,\ I(1,0) = 0.$ *(I3)*

The set of all fuzzy implications will be denoted by FI.

Definition 2.2. [4] *An operator $I : [0,1]^2 \to [0,1]$ is said to satisfy the ordering property, if $I(x,y) = 1 \Leftrightarrow x \le y$ for all $x, y \in [0,1]$.* *(OP)*

Definition 2.3. [11] *An associative, commutative and increasing function $T : [0,1]^2 \to [0,1]$ is called a t-norm if it satisfies $T(x,1) = x$ for all $x \in [0,1]$.*

Example 2.4. [11] *The following are the three basic t-norms T_M, T_P, T_{LK}, given by, respectively:*

$$T_M(x,y) = min(x,y), \qquad T_P(x,y) = xy, \qquad T_{LK}(x,y) = max(x+y-1,0).$$

Definition 2.5. [4] *A t-norm T is called*
- *continuous if it is continuous in both the arguments;*
- *strict, if it is continuous and strictly monotone;*
- *Archimedean, if for all $x, y \in (0,1)$ there exists an $n \in N$ such that $x_T^{(n)} < y$, where*

$$x_T^{(0)} = 1,\ \ x_T^{(1)} = x,\ \ x_T^{(n)} = T(x, x_T^{(n-1)})\ \ for\ \ all\ \ n \ge 2.$$

- *nilpotent, if it is continuous and if each $x \in (0,1)$ is a nilpotent element of T, i.e., if there exists an $n \in N$ such that $x_T^{(n)} = 0$.*

Remark 2.6. [4] *If a t-norm T is strict or nilpotent, then it is Archimedean. Conversely, every continuous and Archimedean t-norm is strict or nilpotent.*

Theorem 2.7. [4] *For a function $T : [0,1]^2 \to [0,1]$ the following statements are equivalent:*
(i) T is a continuous Archimedean t-norm.
(ii) T has a continuous additive generator, i.e., there exists a continuous, strictly decreasing function $t : [0,1] \to [0,\infty]$ with $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that

$$T(x,y) = t^{-1}(min(t(x)+t(y), t(0))),\ \ x,\ y \in [0,1].$$

Remark 2.8. [4] *(i) T is a strict t-norm if and only if each continuous additive generator t of T satisfies $t(0) = \infty$.*
(ii) T is a nilpotent t-norm if and only if each continuous additive generator t of T satisfies $t(0) < \infty$.

Theorem 2.9. [11] *Let A be an index set and $(T_i)_{i\in A}$ a family of t-norms, let $\{(a_i,b_i)\}_{i\in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Then the following function $T : [0,1]^2 \to [0,1]$ is a t-norm:*

$$T(x,y) = \begin{cases} a_i + (b_i - a_i) \cdot T_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}), & if\ x,y \in [a_i,b_i], \\ min(x,y), & otherwise. \end{cases} \tag{1}$$

Definition 2.10. [11] *(i) A t-norm T is called an ordinal sum of t-norms, also known as the summands $< a_i, b_i, T_i >$, $i \in A$, if it is defined as (1). In this case we write $T = (< a_i, b_i, T_i >)_{i\in A}$, where A is an index set, $(T_i)_{i\in A}$ a family of t-norms, and $\{(a_i,b_i)\}_{i\in A}$ is a family of non-empty, pairwise disjoint open subintervals of $[0,1]$.*
(ii) $T = (< a_i, b_i, T_i >)_{i\in A}$ is trivial if $A = \{1\}$, $a_1 = 0$ and $b_1 = 1$.

Theorem 2.11. [4] *For a function $T : [0,1]^2 \to [0,1]$ the following statements are equivalent:*
(i) T is a continuous t-norm.
(ii) T is uniquely representable as an ordinal sum of continuous Archimedean t-norms, i.e, there exist a uniquely determined (finite or countably infinite) index set A, a family of uniquely determined pairwise disjoint open subintervals $\{(a_i,b_i)\}_{i\in A}$ of $[0,1]$ and a family of uniquely determined continuous Archimedean t-norms $(T_i)_{i\in A}$ such that

$$T(x,y) = \begin{cases} a_i + (b_i - a_i) \cdot T_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}), & if\ x,y \in [a_i,b_i], \\ min(x,y), & otherwise. \end{cases}$$

Remark 2.12. *For a continuous t-norm T, if $T \ne T_M$, then it is either a continuous Archimedean t-norm or a non-trivial ordinal sum of continuous Archimedean t-norms.*

Definition 2.13. [4, 11] *(i) An associative, commutative and increasing function $S : [0,1]^2 \to [0,1]$ is called a t-conorm if it satisfies $S(x,0) = x$ for all $x \in [0,1]$.*

(ii) A t-conorm S is idempotent, if $S(x,x) = x$ for all $\in [0,1]$;

Example 2.14. *The following are four basic t-conorms S_M, S_{LK}, S_D, S_{nM} given by, respectively:*

$$S_M(x,y) = max(x,y), \qquad S_{LK}(x,y) = min(x+y,1),$$

$$S_D(x,y) = \begin{cases} 1, & if\ x, y \in (0,1], \\ max(x,y), & otherwise, \end{cases} \qquad S_{nM}(x,y) = \begin{cases} 1, & if\ x+y \geq 1, \\ max(x,y), & otherwise. \end{cases}$$

Definition 2.15. [11, 13] *Let T be a continuous t-norm. For each $x \in [0,1]$, n-th roots and rational powers of x with respect to T are defined by*

$$x_T^{(\frac{1}{n})} = sup\{z \in [0,1] | z_T^{(n)} \leq x\}, \quad x_T^{(\frac{m}{n})} = \left(x_T^{(\frac{1}{n})}\right)_T^{(m)},$$

where m, n are positive integers.

Definition 2.16. [13] *A binary operator $I : [0,1]^2 \to [0,1]$ is said to be a T-power based implication (power based implication for short) if there exists a continuous t-norm T such that*

$$I(x,y) = sup\{r \in [0,1] | y_T^{(r)} \geq x\}, \quad for\ all\ x, y \in [0,1]. \tag{2}$$

If I is a T-power based implication, then it will be denoted by I^T.

Proposition 2.17. [13] *Let T be a continuous t-norm and I^T its power based implication defined by (2).*

(i) If $T = T_M$, then $I^T(x,y) = \begin{cases} 1, & if\ x \leq y, \\ 0, & if\ x > y, \end{cases}$ the Rescher implication I_{RS}.

(ii) If T is an Archimedean t-norm with additive generator t, then

$$I^T(x,y) = \begin{cases} 1, & if\ x \leq y, \\ \frac{t(x)}{t(y)}, & if\ x > y, \end{cases}$$

with the convention that $\frac{a}{\infty} = 0$ for all $a \in [0,1]$.

(iii) If T is an ordinal sum of t-norms of the form $T = (< a_i, b_i, T_i >)_{i \in A}$, where T_i is an Archimedean t-norm with additive generator t_i for all $i \in A$, then

$$I^T(x,y) = \begin{cases} 1, & if\ x \leq y, \\ \frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{y-a_i}{b_i-a_i})}, & if\ x > y\ and\ x, y \in [a_i, b_i], \\ 0, & otherwise. \end{cases}$$

3 Distributivity of the T-power based implications

The four distributive laws involving a fuzzy implication I are given as follows:

$$I(S(x,y),z) = T(I(x,z), I(y,z)), \tag{3}$$

$$I(T(x,y),z) = S(I(x,z), I(y,z)), \tag{4}$$

$$I(x, T_1(y,z)) = T_2(I(x,y), I(x,z)), \tag{5}$$

$$I(x, S_1(y,z)) = S_2(I(x,y), I(x,z)), \tag{6}$$

for all x, y, $z \in [0,1]$, where T, T_1, T_2 are t-norms, S, S_1, S_2 are t-conorms [1, 4, 8].

For the power based implication I^{T_M}, it is Rescher implication. The solutions of distributivity equations involving I^{T_M} are shown in Table 1, since its solutions are easily obtained. The complete proof of Table 1 is shown in Appendix A.

In the following, let us study the distributive laws of the T-power based implication I^T, where T is a continuous Archimedean t-norm, or a non-trivial ordinal sum of continuous Archimedean t-norms.

Table 1: Distributivity solutions of fuzzy implication I^{T_M}

Equation	Solution
$I^{T_M}(S(x,y),z) = T(I^{T_M}(x,z), I^{T_M}(y,z))$	$S = S_M$, any t-norm T
$I^{T_M}(T(x,y),z) = S(I^{T_M}(x,z), I^{T_M}(y,z))$	$T = T_M$, any t-conorm S
$I^{T_M}(x, T_1(y,z)) = T_2(I^{T_M}(x,y), I^{T_M}(x,z))$	$T_1 = T_M$, any t-norm T_2
$I^{T_M}(x, S_1(y,z)) = S_2(I^{T_M}(x,y), I^{T_M}(x,z))$	$S_1 = S_M$, any t-conorm S_2

3.1 On the equation $I(S(x,y),z) = T(I(x,z), I(y,z))$

Lemma 3.1. *Let a function $I : [0,1]^2 \to [0,1]$ satisfy (OP), T be a t-norm and S a t-conorm. If the triple (I, S, T) satisfies (3), then $S = S_M$.*

Proof. Assume that the triple (I, S, T) satisfies (3), then $I(S(x,y),z) = T(I(x,z), I(y,z))$ for all $x, y, z \in [0,1]$. Putting $x = y = z$, we get $I(S(x,x),x) = T(I(x,x), I(x,x)) = 1$ for all $x \in [0,1]$. Since I satisfies (OP), then $S(x,x) \le x$. Note that $S(x,x) \ge x$ for all $x \in [0,1]$. Then $S(x,x) = x$ for all $x \in [0,1]$, i.e., $S = S_M$. □

Theorem 3.2. *Let T be a continuous Archimedean t-norm (a non-trivial ordinal sum of continuous Archimedean t-norms, respectively) and I^T its power based implication, let T_1 be a t-norm and S a t-conorm. Then the following statements are equivalent:*

(i) The triple (I^T, S, T_1) satisfies (3).

(ii) $S = S_M$ and $T_1 = T_M$.

Proof. (i⇒ ii) Let the triple (I^T, S, T_1) satisfy (3). Since I^T satisfies (OP) ([13], Proposition 8), then $S = S_M$ by Lemma 3.1. Thus

$$I^T(\max(x,y),z) = T_1(I^T(x,z), I^T(y,z)) \text{ for all } x,\ y,\ z \in [0,1].$$

Let $x = y$. Then $I^T(x,z) = T_1(I^T(x,z), I^T(x,z))$ for all $x,\ z \in [0,1]$.

Case 1: T is a continuous Archimedean t-norm.

Let t be an additive generator of T, and let $x > z > 0$ in above equation, then

$$\frac{t(x)}{t(z)} = T_1\left(\frac{t(x)}{t(z)}, \frac{t(x)}{t(z)}\right).$$

Let $a = \frac{t(x)}{t(z)}$. Then $a \in [0,1)$ and $a = T_1(a,a)$. Hence $T_1 = T_M$.

Case 2: T is a non-trivial ordinal sum of continuous Archimedean t-norms.

Without loss of generality assume that $T = (< a_i, b_i, T_i >)_{i \in A}$, where A is an index set, T_i is a continuous Archimedean t-norm with additive generator t_i for all $i \in A$, and $\{(a_i, b_i)\}_{i \in A}$ is a family of non-empty, pairwise disjoint open subintervals of $[0,1]$.

Let $x,\ z \in [a_i, b_i]$ for some $i \in A$ with $x > z > a_i$. Then

$$\frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{z-a_i}{b_i-a_i})} = T_1\left(\frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{z-a_i}{b_i-a_i})}, \frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{z-a_i}{b_i-a_i})}\right).$$

Let $m = \frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{z-a_i}{b_i-a_i})}$. Then $m \in [0,1)$ and $m = T_1(m,m)$. Hence $T_1 = T_M$.

(ii⇒ i) Obvious. □

3.2 On the equation $I(T(x,y),z) = S(I(x,z), I(y,z))$

Theorem 3.3. *Let T be a continuous Archimedean t-norm (a non-trivial ordinal sum of continuous Archimedean t-norms, respectively) and I^T its power based implication, and let S be a t-conorm. Then the triple (I^T, T, S) satisfies (4) if and only if $S = S_{LK}$.*

Proof. **Case 1:** T is a continuous Archimedean t-norm.

(Necessity) Let the triple (I^T, T, S) satisfy (4). Suppose that $S \neq S_{LK}$, then there exist a, $b \in (0,1)$ such that

$$S(a,b) \neq \min(a+b,1). \tag{7}$$

Assume that t is an additive generator of T, then t is continuous, strictly decreasing ([4], Theorem 2.1.5). Thus there exist x_0, y_0, $z_0 \in (0,1)$ with $x_0 > z_0$, $y_0 > z_0$ such that

$$\frac{t(x_0)}{t(z_0)} = a \text{ and } \frac{t(y_0)}{t(z_0)} = b, \tag{8}$$

i.e., $I^T(x_0, z_0) = a$, $I^T(y_0, z_0) = b$.

If $a + b < 1$, i.e., $t(x_0) + t(y_0) < t(z_0)$, by (7) and (8) we get

$$S\left(I^T(x_0,z_0), I^T(y_0,z_0)\right) = S(a,b) \neq a+b = \frac{t(x_0)}{t(z_0)} + \frac{t(y_0)}{t(z_0)}. \tag{9}$$

However, by $t(z_0) < t(0)$, we get $t(x_0) + t(y_0) < t(0)$. Then

$$T(x_0,y_0) = t^{-1}(\min(t(x_0)+t(y_0), t(0))) = t^{-1}(t(x_0)+t(y_0)) > z_0.$$

Hence

$$I^T(T(x_0,y_0),z_0) = \frac{t(x_0)+t(y_0)}{t(z_0)} = a+b. \tag{10}$$

From (9), (10) we get $I^T(T(x_0,y_0),z_0) \neq S(I^T(x_0,z_0), I^T(y_0,z_0))$, this contradicts the fact that the triple (I^T, T, S) satisfies (4).

If $a + b \geq 1$, i.e., $t(x_0) + t(y_0) \geq t(z_0)$, by (7) we get

$$S\left(\frac{t(x_0)}{t(z_0)}, \frac{t(y_0)}{t(z_0)}\right) = S(a,b) \neq 1,$$

i.e., $S\left(I^T(x_0,z_0), I^T(y_0,z_0)\right) \neq 1$.

However, since $t^{-1}(t(0)) = 0 < z_0$, then $t^{-1}(\min(t(x_0)+t(y_0), t(0))) \leq z_0$, i.e., $T(x_0,y_0) \leq z_0$. Hence $I^T(T(x_0,y_0),z_0) = 1$. Thus $I^T(T(x_0,y_0),z_0) > S(I^T(x_0,z_0), I^T(y_0,z_0))$. A contradiction to the fact that the triple (I^T, T, S) satisfies (4).

(Sufficiency) Let $S = S_{LK}$. It suffices to prove that the triple (I^T, T, S) satisfies (4) for all x, y, $z \in [0,1]$ with $x > z$ and $y > z$.

If $T(x,y) > z$, i.e., $t^{-1}(\min(t(x)+t(y), t(0))) > z$, then $\min(t(x)+t(y), t(0)) < t(z)$. Note that $t(z) \leq t(0)$, then $t(x) + t(y) < t(z) \leq t(0)$. Thus

$$I^T(T(x,y),z) = \frac{t(T(x,y))}{t(z)} = \frac{\min(t(x)+t(y), t(0))}{t(z)} = \frac{t(x)+t(y)}{t(z)} = S_{LK}(I^T(x,z), I^T(y,z)).$$

If $T(x,y) \leq z$, i.e., $t^{-1}(\min(t(x)+t(y), t(0))) \leq z$, then

$$I^T(T(x,y),z) = 1 \text{ and } \min(t(x)+t(y), t(0)) \geq t(z).$$

Since $t(0) \geq t(z)$, then $t(x) + t(y) \geq t(z)$. Thus $\frac{t(x)}{t(z)} + \frac{t(y)}{t(z)} \geq 1$. Therefore,

$$S_{LK}(I^T(x,z), I^T(y,z)) = \min\left(\frac{t(x)}{t(z)} + \frac{t(y)}{t(z)}, 1\right) = 1.$$

Hence $I^T(T(x,y),z) = S_{LK}(I^T(x,z), I^T(y,z))$.

Thus we complete the proof in the case that T is a continuous Archimedean t-norm.

Case 2: T is a non-trivial ordinal sum of continuous Archimedean t-norms.

Without loss of generality assume that $T = (< a_i, b_i, T_i >)_{i \in A}$, where A is an index set, T_i is a continuous Archimedean t-norm for all $i \in A$, and $\{(a_i, b_i)\}_{i \in A}$ is a family of non-empty, pairwise disjoint open subintervals of $[0,1]$.

Let $x,\ y,\ z \in [0,1]$ with $x > z,\ y > z$. If there is not an $i \in A$ such that $x,\ y,\ z \in [a_i, b_i]$, then equation $I^T(T(x,y),z) = S(I^T(x,z), I^T(y,z))$ holds for any t-conorm S.

In fact, consider the following cases.

Case 2.1: for all $i \in A$, $z \notin [a_i, b_i]$. Obviously, $I^T(x,z) = 0$, and $I^T(y,z) = 0$.

If there exists a $k \in A$ such that $x,\ y \in [a_k, b_k]$, then

$$T(x,y) = a_k + (b_k - a_k) \cdot T_i(\frac{x - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k}) \in [a_k, b_k].$$

Since $x > z,\ y > z$, then $z < a_k$. Thus $I^T(T(x,y),z) = 0$. If there is not a $k \in A$ such that $x,\ y \in [a_k, b_k]$, obviously, $T(x,y) = \min(x,y) > z$. Thus $I^T(T(x,y),z) = 0$. Hence, $I^T(T(x,y),z) = S(I^T(x,z), I^T(y,z))$ holds for any t-conorm S.

Case 2.2: there exists an $i \in A$ such that $z \in [a_i, b_i]$, $x \notin [a_i, b_i]$, $y \notin [a_i, b_i]$, and there is not a $k \in A$ such that x, $y \in [a_k, b_k]$. Then $T(x,y) = \min(x,y) > z$, and $T(x,y) \notin [a_i, b_i]$. Thus

$$I^T(T(x,y),z) = 0,\ \ I^T(x,z) = 0,\ \text{ and }\ I^T(y,z) = 0.$$

Hence $I^T(T(x,y),z) = S(I^T(x,z), I^T(y,z))$ holds for any t-conorm S.

Case 2.3: there exists an $i \in A$ such that $z \in [a_i, b_i]$, $x \notin [a_i, b_i]$, $y \notin [a_i, b_i]$, and there exists a $k \in A$ such that x, $y \in [a_k, b_k]$. Then

$$I^T(x,z) = 0,\ \ I^T(y,z) = 0,\ \text{ and } T(x,y) = a_k + (b_k - a_k) \cdot T_k(\frac{x - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k}).$$

Since $x > z,\ y > z$, then $b_i \leq a_k$.

If $b_i < a_k$, then $T(x,y) \notin [a_i, b_i]$. Thus, $I^T(T(x,y),z) = 0$.

If $b_i = a_k$, then $z < b_i$, since $z \in [a_i, b_i]$ and $z \notin [a_k, b_k]$. Note that $T(x,y) \geq a_k = b_i$. Obviously, $I^T(T(x,y),z) = 0$. The reason is that $T(x,y) \notin [a_i, b_i]$ when $T(x,y) > b_i$, and $I^T(T(x,y),z) = \frac{t_i(\frac{b_i - a_i}{b_i - a_i})}{t_i(\frac{z - a_i}{b_i - a_i})} = 0$ when $T(x,y) = b_i$.

Hence, equation $I^T(T(x,y),z) = S(I^T(x,z), I^T(y,z))$ holds for any t-conorm S.

Case 2.4: there exists an $i \in A$ such that $z,\ x \in [a_i, b_i]$, $y \notin [a_i, b_i]$. Then $I^T(y,z) = 0$. Since $y > z$, then $y > b_i \geq x$. Thus $T(x,y) = \min(x,y) = x$. Therefore,

$$I^T(T(x,y),z) = I^T(x,z).$$

Hence, equation $I^T(T(x,y),z) = S(I^T(x,z), I^T(y,z))$ holds for any t-conorm S.

Case 2.5: there exists an $i \in A$ such that $z,\ y \in [a_i, b_i]$, $x \notin [a_i, b_i]$. Similarly to Case 2.4, equation $I^T(T(x,y),z) = S(I^T(x,z), I^T(y,z))$ holds for any t-conorm S.

Hence, it suffices to consider $x,\ y,\ z \in [a_i, b_i]$ for some $i \in A$. The rest proof is similar to the proof of Case 1. □

To show the application of Theorem 3.3, an example is given.

Example 3.4. *Let T be a continuous Archimedean t-norm with additive generator $t(x) = 1 - x$, $x \in [0,1]$, then*

$$T = T_{LK},\ \text{and}\ \ I^T(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{1-x}{1-y}, & \text{if } x > y. \end{cases}$$

If $x \leq z$ or $y \leq z$, then $I^T(T(x,y),z) = 1 = S_{LK}(I^T(x,z), I^T(y,z))$.

If $x > z$ and $y > z$, then

$$I^T(T(x,y),z) = \begin{cases} 1, & \text{if } x + y - 1 \leq z, \\ \frac{2-(x+y)}{1-z}, & \text{if } x + y - 1 > z, \end{cases} = min\left(\frac{2 - (x+y)}{1 - z}, 1\right),$$

$$S_{LK}(I^T(x,z), I^T(y,z)) = S_{LK}\left(\frac{t(x)}{t(z)}, \frac{t(y)}{t(z)}\right) = min\left(\frac{2 - (x+y)}{1 - z}, 1\right).$$

Thus $I^T(T(x,y),z) = S_{LK}(I^T(x,z), I^T(y,z))$ for all $x,\ y,\ z \in [0,1]$. Hence the triple (I^T, T, S_{LK}) satisfies (4).

Remark 3.5. *Note that the triple (I, T_M, S_M) satisfies (4) for any fuzzy implication I. Therefore, equation (4) is also satisfied by the triple (I^T, T_M, S_M). This result indicates that there exist a t-norm T_1 different from T and a t-conorm S different from S_{LK}, such that the triple (I^T, T_1, S) satisfies (4).*

In the following, we study the t-norm T_1 different from T and the t-conorm S different from S_{LK} such that the triple (I^T, T_1, S) satisfies (4).

Lemma 3.6. *Let $\alpha \in (0, \infty)$ and $S : [0,1]^2 \to [0,1]$ be a function defined as*

$$S(x,y) = min\left((x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}})^{\alpha}, 1\right), \ \ x,\ y \in [0,1],$$

then S is φ-conjugate with S_{LK}, i.e., S is a t-conorm.

Proof. Let $\varphi : [0,1] \to [0,1]$ be a function defined by

$$\varphi(x) = x^{\frac{1}{\alpha}},\ x \in [0,1],\ \alpha > 0.$$

Obviously, φ is an automorphism. Consider the Lukasiewicz t-conorm S_{LK}, i.e.,

$$S_{LK}(x,y) = \min(x+y, 1),\ \ x, y \in [0,1].$$

Then, for all $x,\ y \in [0,1]$, we have

$$\varphi^{-1}(S_{LK}(\varphi(x), \varphi(y))) = \left(\min(x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}}, 1)\right)^{\alpha} = \min\left((x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}})^{\alpha}, 1\right) = S(x,y),$$

that is, S is φ-conjugate with S_{LK}. Therefore, S is a t-conorm. □

Proposition 3.7. *Let T be a continuous Archimedean t-norm with additive generator t and I^T its power based implication. Let T_1 be a continuous Archimedean t-norm with additive generator t_1 defined by*

$$t_1(x) = (k \cdot t(x))^{\frac{1}{\alpha}},\ \ x \in [0,1],$$

and S be a t-conorm defined by $S(x,y) = min\left((x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}})^{\alpha}, 1\right)$. Then the triple (I^T, T_1, S) satisfies (4), where k, α are constants, and $\alpha > 0$, $k > 0$.

Proof. Let $x,\ y,\ z \in [0,1]$. It suffices to prove that the triple (I^T, T_1, S) satisfies (4) for $x > z$ and $y > z$.

Since t_1 is an additive generator of T_1, then

$$T_1(x,y) = t_1^{-1}(\min(t_1(x) + t_1(y), t_1(0))),\ \ x, y \in [0,1].$$

If $T_1(x,y) \le z$, then $t_1(x) + t_1(y) \ge t_1(z)$, and $I^T(T_1(x,y), z) = 1$. From $t_1(x) + t_1(y) \ge t_1(z)$ we get

$$\frac{t_1(x)}{t_1(z)} + \frac{t_1(y)}{t_1(z)} \ge 1,$$

that is

$$\frac{t_1(t^{-1}(t(x)))}{t_1(t^{-1}(t(z)))} + \frac{t_1(t^{-1}(t(y)))}{t_1(t^{-1}(t(z)))} \ge 1. \tag{11}$$

From $t_1(x) = (k \cdot t(x))^{\frac{1}{\alpha}}$ we get $t_1(t^{-1}(x)) = (kx)^{\frac{1}{\alpha}}$, $x \in [0, t(0)]$. Then from (11) we have

$$\left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}} + \left(\frac{t(y)}{t(z)}\right)^{\frac{1}{\alpha}} \ge 1.$$

Thus

$$S(I^T(x,z), I^T(y,z)) = S\left(\frac{t(x)}{t(z)}, \frac{t(y)}{t(z)}\right) = 1.$$

Therefore, $I^T(T_1(x,y), z) = 1 = S(I^T(x,z), I^T(y,z))$.

If $T_1(x,y) > z$, similarly, we obtain $\left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}} + \left(\frac{t(y)}{t(z)}\right)^{\frac{1}{\alpha}} < 1$, then

$$S(I^T(x,z), I^T(y,z)) = S\left(\frac{t(x)}{t(z)}, \frac{t(y)}{t(z)}\right) = \left(\left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}} + \left(\frac{t(y)}{t(z)}\right)^{\frac{1}{\alpha}}\right)^{\alpha}.$$

On the other hand, from $T_1(x,y) > z$ we obtain $\min(t_1(x)+t_1(y), t_1(0)) < t_1(z)$. Since $t(z) \le t_1(0)$, then $t_1(x)+t_1(y) < t_1(z) \le t_1(0)$. Thus

$$I^T(T_1(x,y),z) = \frac{t(T_1(x,y))}{t(z)} = \frac{t(t_1^{-1}(t_1(x)+t_1(y)))}{t(z)} = \frac{1}{k}\cdot\frac{(t_1(x)+t_1(y))^\alpha}{t(z)}$$
$$= \frac{1}{k}\cdot\left(\frac{(k\cdot t(x))^{\frac{1}{\alpha}}+(k\cdot t(y))^{\frac{1}{\alpha}}}{t(z)^{\frac{1}{\alpha}}}\right)^\alpha = \left(\frac{t(x)^{\frac{1}{\alpha}}+t(y)^{\frac{1}{\alpha}}}{t(z)^{\frac{1}{\alpha}}}\right)^\alpha = \left(\left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}}+\left(\frac{t(y)}{t(z)}\right)^{\frac{1}{\alpha}}\right)^\alpha.$$

Thus $I^T(T_1(x,y),z) = S(I^T(x,z), I^T(y,z))$. From the above discussion it is easy to see that the triple (I^T, T_1, S) satisfies (4). □

Similarly, we have the following result for the case that T is a non-trivial ordinal sum of continuous Archimedean t-norms.

Proposition 3.8. *Let A be an index set and $\{(a_i,b_i)\}_{i\in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Let $T = (< a_i, b_i, T_i >)_{i\in A}$ be a non-trivial ordinal sum of Archimedean t-norms and I^T its power based implication, where T_i is a continuous Archimedean t-norm with additive generator t_i for all $i \in A$. Let $T_1 = (< a_i, b_i, T_{1i} >)_{i\in A}$ be an ordinal sum of Archimedean t-norms, where T_{1i} is a continuous Archimedean t-norm with additive generator t_{1i} defined as*

$$t_{1i}(x) = (k\cdot t_i(x))^{\frac{1}{\alpha}},\ \ x\in[0,1],\ i\in A.$$

Let S be a t-conorm defined as

$$S(x,y) = min\left((x^{\frac{1}{\alpha}}+y^{\frac{1}{\alpha}})^\alpha, 1\right).$$

Then the triple (I^T, T_1, S) satisfies (4), where k, α are constants with $\alpha > 0$, $k > 0$.

Proof. Let $x, y, z \in [0,1]$ with $x > z$, $y > z$. Analogues to the proof in case 2 of Theorem 3.3, if there is not an $i \in A$ such that $x, y, z \in [a_i, b_i]$, then $I^T(T_1(x,y),z) = S(I^T(x,z), I^T(y,z))$ holds for any t-conorm S.

Hence, it suffices to consider $x, y, z \in [a_i, b_i]$ for some $i \in A$. The rest proof is similar to the proof of Proposition 3.7. □

3.3 On the equation $I(x, T_1(y,z)) = T_2(I(x,y), I(x,z))$

Lemma 3.9. *Let a function $I : [0,1]^2 \to [0,1]$ satisfy (OP), and let T_1, T_2 be t-norms. If the triple (I, T_1, T_2) satisfies (5), then $T_1 = T_M$.*

Proof. Assume that the triple (I, T_1, T_2) satisfies (5), i.e.,

$$I(x, T_1(y,z)) = T_2(I(x,y), I(x,z)) \text{ for all } x,y,z\in[0,1].$$

Taking $x = y = z$, then

$$I(x, T_1(x,x)) = T_2(I(x,x), I(x,x)) \text{ for all } x\in[0,1].$$

Since I satisfies (OP), then $I(x, T_1(x,x)) = 1$. Hence $x \le T_1(x,x)$ for all $x \in [0,1]$. As $T_1(x,x) \le x$ for all $x \in [0,1]$, then $T_1(x,x) = x$ for all $x \in [0,1]$. Thus $T = T_M$. □

Theorem 3.10. *Let T be a continuous Archimedean t-norm (a non-trivial ordinal sum of continuous Archimedean t-norms, respectively) and I^T its power based implication, and let T_1, T_2 be t-norms. Then the following statements are equivalent:*

(i) The triple (I^T, T_1, T_2) satisfies (5).

(ii) $T_1 = T_2 = T_M$.

Proof. (i ⇒ ii) Let the triple (I^T, T_1, T_2) satisfy (5). Since I^T satisfies (OP), then $T_1 = T_M$ by Lemma 3.9. Thus, for all $x, y, z \in [0,1]$, we get

$$I^T(x, \min(y,z)) = T_2(I^T(x,y), I^T(x,z)).$$

Taking $y = z$, then

$$I^T(x,y) = T_2(I^T(x,y), I^T(x,y)).$$

Case 1: T is a continuous Archimedean t-norm.

Consider $x > y > 0$. Let t be an additive generator of T, and let $I^T(x,y) = a$, then $a = \frac{t(x)}{t(y)}$. Thus $a \in [0,1)$ by the continuity of T. Therefore,

$$a = T_2(a,a) \text{ for all } a \in [0,1),$$

i.e., $T_2 = T_M$.

Case 2: T is a non-trivial ordinal sum of continuous Archimedean t-norms.

Without loss of generality assume that $T = (< a_i, b_i, T_i >)_{i \in A}$, where A is an index set and T_i is a continuous Archimedean t-norm with additive generator t_i for all $i \in A$, and $\{(a_i,b_i)\}_{i\in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$.

Let $x, y \in [a_i, b_i]$ for some $i \in A$ with $x > y > a_i$. Then

$$\frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{y-a_i}{b_i-a_i})} = T_2\left(\frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{y-a_i}{b_i-a_i})}, \frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{y-a_i}{b_i-a_i})}\right).$$

Let $m = \frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{y-a_i}{b_i-a_i})}$. Then $m \in [0,1)$ and $m = T_2(m,m)$. Hence $T_2 = T_M$.

(ii $\Rightarrow$ i) Obvious. □

3.4 On the equation $I(x, S_1(y,z)) = S_2(I(x,y), I(x,z))$

Lemma 3.11. [4] *For a function $I : [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) I is increasing in the second variable, i.e., I satisfies (I2).

(ii) I satisfies $I(x, \max(y,z)) = \max(I(x,y), I(x,z))$ for all $x,y,z \in [0,1]$, i.e., the triple (I, S_M, S_M) satisfies (6).

Remark 3.12. (i) The t-conorm S_2 such that the triple (I, S_M, S_2) satisfies (6) may not be unique. To see this consider the Rescher implication I_{RS}, i.e., I^{T_M}. It is easy to see that the triple (I_{RS}, S_M, S_2) satisfies (6) for any t-conorm S_2 from Table 1.

(ii) It is easy to see that the pair (S_M, S_M) is a solution of equation (6) involving I_T.

Lemma 3.13. *Let $I \in FI$ satisfy one of the following conditions:*

(i) For some x, the function $I_x(y)$ defined by $I_x(y) = I(x,y)$, $y \in [0,1]$ is onto [0,1].

(ii) For some y, the function $I_y(x)$ defined by $I_y(x) = I(x,y)$, $x \in [0,1]$ is onto [0,1].

If the triple (I, S_M, S_2) satisfies (6), then $S_2 = S_M$,

Proof. Assume that the triple (I, S_M, S_2) satisfies (6), i.e.,

$$I(x, \max(y,z)) = S_2(I(x,y), I(x,z)) \text{ for all } x,y,z \in [0,1].$$

Taking $y = z$, then

$$I(x,y) = S_2(I(x,y), I(x,y)) \text{ for all } x,y \in [0,1]. \tag{12}$$

For condition (i): the function $I_x(y)$ defined by $I_x(y) = I(x,y)$, $y \in [0,1]$ is onto [0,1] for some x. Taking $p = I_x(y)$, then $p = S_2(p,p)$ for all $p \in [0,1]$. Therefore, $S_2 = S_M$.

For the condition (ii): for some y, the function $I_y(x)$ defined by $I_y(x) = I(x,y), x \in [0,1]$ is onto [0,1]. Similarly, taking $p = I_y(x)$ in (12), then $p = S_2(p,p)$ for all $p \in [0,1]$, thus $S_2 = S_M$. □

Lemma 3.14. *Let $I \in FI$ satisfy one of the following conditions:*

(i) For some x, the function $I_x(y)$ defined by $I_x(y) = I(x,y)$ is a strictly increasing function.

(ii) I satisfies (OP).

If the triple (I, S_1, S_M) satisfies (6), then $S_1 = S_M$.

Proof. Assume that the triple (I, S_1, S_M) satisfies (6), i.e.,

$$I(x, S_1(y,z)) = \max(I(x,y), I(x,z)) \text{ for all } x,y,z \in [0,1].$$

Taking $y = z$, then $I(x, S_1(y,y)) = I(x,y)$ for all $x,y \in [0,1]$, i.e.,

$$I_x(S_1(y,y)) = I_x(y) \text{ for all } y \in [0,1].$$

For condition (i): for some x, the function $I_x(y)$ is a strictly increasing function. Then $S_1(y,y)) = y$ for all $y \in [0,1]$. Therefore $S_1 = S_M$.

For condition (ii): I satisfies (OP). Suppose that $S_1 \neq S_M$, then there exists a $y \in (0,1)$ such that $S_1(y,y) > y$. Hence, there exists an $x \in (0,1)$ such that $S_1(y,y) > x > y$, then $I(x, S_1(y,y)) = 1 > I(x,y)$ by (OP). A contradiction to $I(x, S_1(y,y)) = I(x,y)$. □

Proposition 3.15. *Let T be a continuous Archimedean t-norm (a non-trivial ordinal sum of continuous Archimedean t-norms, respectively) and I^T its power based implication. If the triple (I^T, S_1, S_2) satisfies (6), then $S_1 = S_M \Leftrightarrow S_2 = S_M$.*

Proof. $(S_1 = S_M \Rightarrow S_2 = S_M)$

Case 1: T is a continuous Archimedean t-norm. Suppose that t is an additive generator of T. Let $x \geq y$, fix $y \in (0,1)$. Since t is a continuous function with $t(1) = 0$, then $I_y(x) = \frac{t(x)}{t(y)}$ is onto $[0,1]$. Hence $S_2 = S_M$ by Lemma 3.13.

Case 2: T is a non-trivial ordinal sum of continuous Archimedean t-norms.

Without loss of generality assume that $T = (< a_i, b_i, T_i >)_{i \in A}$, where A is an index set, T_i is a continuous Archimedean t-norm with additive generator t_i for all $i \in A$, and $\{(a_i, b_i)\}_{i \in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$.

Taking x, $y \in [a_i, b_i]$ with $x \geq y > a_i$. Fix y, then the following function

$$I_y(x) = \frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{y-a_i}{b_i-a_i})},\ x \in [y, b_i],$$

is onto $[0,1]$. Therefore $S_2 = S_M$ by Lemma 3.13.

$(S_2 = S_M \Rightarrow S_1 = S_M)$ Since I^T satisfies (OP), then $S_2 = S_M \Rightarrow S_1 = S_M$ by Lemma 3.14. □

Theorem 3.16. *Let T be a nilpotent, continuous t-norm and I^T its power based implication, then the triple (I^T, S_1, S_2) satisfies (6) if and only if $S_1 = S_M$, $S_2 = S_M$.*

Proof. (Necessity) Let the triple (I^T, S_1, S_2) satisfy (6), i.e,

$$I^T(x, S_1(y,z)) = S_2(I^T(x,y), I^T(x,z)). \tag{13}$$

for all x, y, $z \in [0,1]$.

Suppose that t is an additive generator of T. Taking $y = 0$, $z = 0$ in (13), then

$$\frac{t(x)}{t(0)} = S_2\left(\frac{t(x)}{t(0)}, \frac{t(x)}{t(0)}\right) \text{ for all } x \in [0,1].$$

Let $p = \frac{t(x)}{t(0)}$, then $p = S_2(p,p)$ for all $p \in [0,1]$. Hence $S_2 = S_M$. Therefore, $S_1 = S_M$ by Lemma 3.14 (ii).

(Sufficiency) Obvious. □

Proposition 3.17. *Let A be an index set and $(T_i)_{i \in A}$ a family of continuous Archimedean t-norms, let $(a_i, b_i)_{i \in A}$ be a family of non-empty, pairwise disjoint open subintervals of [0,1]. Let T be a non-trivial ordinal sum of continuous Archimedean t-norms with the form $(< a_i, b_i, T_i >)_{i \in A}$ and I^T its power based implication, let S_1, S_2 be t-conorms. If there exists an $i \in A$ such that $a_i = 0$ and T_i is a nilpotent t-norm, or a_i is an idempotent point of S_1 and T_i is a nilpotent t-norm, then the following statements are equivalent:*

(i) The triple (I^T, S_1, S_2) satisfies (6).

(ii) $S_1 = S_M$, $S_2 = S_M$.

Proof. Taking $y = z = a_i$, and $x \in [a_i, b_i]$. The rest proof is similar to the proof of Theorem 3.16. □

Problem 3.18. For the power based implication I^T generated from a strict t-norm T, does the fact that the triple (I^T, S_1, S_2) satisfies (6) if and only if $S_1 = S_2 = S_M$ is true ?

Unfortunately, the answer is negative. To see this consider the following example.

Example 3.19. *Let T be a strict t-norm with additive generator $t(x)=\frac{1}{x}-1$, $x\in[0,1]$ and I^T its power based implication, i.e.,*

$$I^T(x,y)=\begin{cases}1, & \text{if } x\le y,\\ \frac{y(1-x)}{x(1-y)}, & \text{otherwise}\end{cases}$$

with the understanding $\frac{0}{0}=1$. Let S_1 be a t-conorm defined as following:

$$S_1(x,y)=\frac{x+y-2xy}{1-xy},\ \ x,y\in[0,1],$$

with the understanding $\frac{0}{0}=1$. Let S_2 be the t-conorm S_{LK}, i.e.,

$$S_2(x,y)=min(x+y,1),\ \ x,y\in[0,1].$$

For x, y, $z\in[0,1]$ with $x>y$, $x>z$.
Case 1: $x=1$. Obviously, $I^T(x,S_1(y,z))=0=S_2(0,0)=S_2(I^T(x,y),I^T(x,z))$.
Case 2: $y=0$ or $z=0$. Obviously, $I^T(x,S_1(y,z))=S_2(I^T(x,y),I^T(x,z))$.
Case 3: x, y, $z\in(0,1)$. If $x>S_1(y,z)$, i.e., $x>\frac{y+z-2yz}{1-yz}$, then

$$\begin{aligned}I^T(x,S_1(y,z))&=\frac{t(x)}{t(S_1(y,z))}=t(x)\cdot\frac{S_1(y,z)}{1-S_1(y,z)}=t(x)\cdot\frac{y+z-2yz}{1-y-z+yz}\\&=t(x)\cdot\frac{(y-yz)+(z-yz)}{(1-y)(1-z)}=t(x)\cdot\left(\frac{y}{1-y}+\frac{z}{1-z}\right)=t(x)\cdot\left(\frac{1}{t(y)}+\frac{1}{t(z)}\right).\end{aligned}$$

On the other hand, since

$$\begin{aligned}x>\frac{y+z-2yz}{1-yz}&\Leftrightarrow\frac{1}{x}<\frac{1-yz}{y+z-2yz}\\&\Leftrightarrow\frac{1}{x}-1<\frac{1-y-z+yz}{y+z-2yz}\\&\Leftrightarrow(\frac{1}{x}-1)\frac{y+z-2yz}{1-y-z+yz}<1\\&\Leftrightarrow(\frac{1}{x}-1)\frac{(y-yz)+(z-yz)}{(1-y)(1-z)}<1\\&\Leftrightarrow(\frac{1}{x}-1)(\frac{y}{1-y}+\frac{z}{1-z})<1\\&\Leftrightarrow(\frac{1}{x}-1)\left(\frac{1}{\frac{1}{y}-1}+\frac{1}{\frac{1}{z}-1}\right)<1\\&\Leftrightarrow\frac{\frac{1}{x}-1}{\frac{1}{y}-1}+\frac{\frac{1}{x}-1}{\frac{1}{z}-1}<1\\&\Leftrightarrow\frac{t(x)}{t(y)}+\frac{t(x)}{t(z)}<1.\end{aligned}$$

Then

$$S_2(I^T(x,y),I^T(x,z))=min\left(\frac{t(x)}{t(y)}+\frac{t(x)}{t(z)},1\right)=\frac{t(x)}{t(y)}+\frac{t(x)}{t(z)}.$$

Hence $I^T(x,S_1(y,z))=S_2(I^T(x,y),I^T(x,z))$.
If $x\le S_1(y,z)$, i.e., $x\le\frac{y+z-2yz}{1-yz}$, then $I^T(x,S_1(y,z))=1$. Note that

$$x\le\frac{y+z-2yz}{1-yz}\Leftrightarrow\frac{t(x)}{t(y)}+\frac{t(x)}{t(z)}\ge 1.$$

Then $S_2(I^T(x,y),I^T(x,z))=1$. Thus $I^T(x,S_1(y,z))=S_2(I^T(x,y),I^T(x,z))$.
From the above discussion, we get that the triple (I^T,S_1,S_2) satisfies (6).

Obviously, the solution (S_1,S_2) of equation (6) involving I^T may not be unique when T is a strict t-norm. Moreover, we can be sure that $S_2\neq S_D$ (S_{nM}, respectively). See the following remark.

Remark 3.20. *(i) Let T be a continuous Archimedean t-norm. If the triple (I^T, S_1, S_2) satisfies (6), then $S_2 \neq S_D$.*

Actually, suppose that $S_2 = S_D$, then $S_1 \neq S_M$ by Proposition 3.15. Hence there exists a $y_0 \in (0,1)$ such that $1 > S_1(y_0, y_0) > y_0$.

Consider an $x_0 \in [0,1]$ such that $1 > x_0 > S_1(y_0, y_0)$, we get

$$I^T(x_0, S_1(y_0, y_0)) < 1, \;\; I^T(x_0, y_0) \in (0,1).$$

Hence $S_2(I^T(x_0, y_0), I^T(x_0, y_0)) = 1$, a contradiction to

$$I^T(x_0, S_1(y_0, y_0)) = S_2(I^T(x_0, y_0), I^T(x_0, y_0)).$$

(ii) For a power based implication I^T ($T \neq T_M$), if the triple (I^T, S_1, S_2) satisfies (6), then $S_2 \neq S_{nM}$.

Actually, suppose that $S_2 = S_{nM}$, then $S_1 \neq S_M$ by Proposition 3.15. Hence, there exists a $y_0 \in (0,1)$ such that $1 > S_1(y_0, y_0) > y_0$.

Case 1: *T is a continuous Archimedean t-norm.*

Assume that t is an additive generator of T. Consider an $x_0 \in (0,1)$ such that

$$1 > x_0 > max\left(S_1(y_0, y_0), t^{-1}(\frac{1}{2}t(y_0))\right),$$

then $\frac{t(x_0)}{t(y_0)} < \frac{1}{2}$. Thus

$$I^T(x_0, S_1(y_0, y_0)) = \frac{t(x_0)}{t(S_1(y_0, y_0))} > \frac{t(x_0)}{t(y_0)} = S_2(I^T(x_0, y_0), I^T(x_0, y_0)),$$

a contradiction to $I^T(x_0, S_1(y_0, y_0)) = S_2(I^T(x_0, y_0), I^T(x_0, y_0))$.

Case 2: *T is a non-trivial ordinal sum t-norms.*

Without loss of generality assume that $T = (< a_i, b_i, T_i >)_{i \in A}$, where A is an index set, T_i is a continuous Archimedean t-norm with additive generator t_i for all $i \in A$, and $(a_i, b_i)_{i \in A}$ be a family of non-empty, pairwise disjoint open subintervals of [0,1].

Case 2.1: *$y_0 \notin [a_i, b_i]$ for all $i \in A$. Consider an $x_0 \in (y_0, S_1(y_0, y_0))$, then*

$$I^T(x_0, S_1(y_0, y_0)) = 1 > 0 = S_2(I^T(x_0, y_0), I^T(x_0, y_0)),$$

a contradiction to $I^T(x_0, S_1(y_0, y_0)) = S_2(I^T(x_0, y_0), I^T(x_0, y_0))$.

Case 2.2: *$y_0 \in [a_i, b_i]$ for an $i \in A$.*

If $S_1(y_0, y_0) > b_i$, consider an $x_0 \in (b_i, S_1(y_0, y_0))$, then

$$I^T(x_0, S_1(y_0, y_0)) = 1 > 0 = S_2(I^T(x_0, y_0), I^T(x_0, y_0)),$$

a contradiction to $I^T(x_0, S_1(y_0, y_0)) = S_2(I^T(x_0, y_0), I^T(x_0, y_0))$.

If $S_1(y_0, y_0) = b_i$, consider an $x_0 \in [a_i, b_i]$ such that

$$b_i > x_0 > a_i + (b_i - a_i) \cdot t^{-1}\left(\frac{1}{2}t(\frac{y_0 - a_i}{b_i - a_i})\right),$$

then $\frac{t(\frac{x_0 - a_i}{b_i - a_i})}{t(\frac{y_0 - a_i}{b_i - a_i})} < \frac{1}{2}$. Thus

$$I^T(x_0, S_1(y_0, y_0)) = 1 > \frac{t(\frac{x_0 - a_i}{b_i - a_i})}{t(\frac{y_0 - a_i}{b_i - a_i})} = S_2\left(I^T(x_0, y_0), I^T(x_0, y_0)\right),$$

a contradiction to $I^T(x_0, S_1(y_0, y_0)) = S_2(I^T(x_0, y_0), I^T(x_0, y_0))$.

If $S_1(y_0, y_0) < b_i$, consider an $x_0 \in [a_i, b_i]$ such that

$$b_i > x_0 > max\left(S_1(y_0, y_0), a_i + (b_i - a_i) \cdot t^{-1}\left(\frac{1}{2}t(\frac{y_0 - a_i}{b_i - a_i})\right)\right),$$

then $\frac{t(\frac{x_0-a_i}{b_i-a_i})}{t(\frac{y_0-a_i}{b_i-a_i})} < \frac{1}{2}$. *Thus*

$$I^T(x_0, S_1(y_0, y_0)) = \frac{t(\frac{x_0-a_i}{b_i-a_i})}{t(\frac{S_1(y_0,y_0)-a_i}{b_i-a_i})} > \frac{t(\frac{x_0-a_i}{b_i-a_i})}{t(\frac{y_0-a_i}{b_i-a_i})} = S_2\left(I^T(x_0, y_0), I^T(x_0, y_0)\right),$$

a contradiction to $I^T(x_0, S_1(y_0, y_0)) = S_2(I^T(x_0, y_0), I^T(x_0, y_0))$.

Therefore, $S_2 \neq S_{nM}$.

In the following, we give a result on the solution of equation (6) involving I^T when T is a strict t-norm.

Proposition 3.21. *Let T be a strict t-norm and I^T its power based implication, and let S_1, S_2 be t-conorms. If the triple (I^T, S_1, S_2) satisfies (6), then S_1 is either idempotent or $S_1(y, y) > y$ for all $y \in (0, 1)$.*

Proof. Let t be an additive generator of T. If there exists a $y_0 \in (0, 1)$ such that $S_1(y_0, y_0) = y_0$, then from the triple (I^T, S_1, S_2) satisfies (6) we get that for all $x \in [y_0, 1]$,

$$\frac{t(x)}{t(y_0)} = S_2\left(\frac{t(x)}{t(y_0)}, \frac{t(x)}{t(y_0)}\right).$$

Let $p = \frac{t(x)}{t(y_0)}$, $x \in [y_0, 1]$. Then $S_2(p, p) = p$ for all $p \in [0, 1]$. Hence $S_2 = S_M$, thus $S_1 = S_M$ by Proposition 3.15.

If there is not a $y_0 \in (0, 1)$ such that $S_1(y_0, y_0) = y_0$, obviously, $S_1(y, y) > y$ for all $y \in (0, 1)$. □

Proposition 3.22. *Let T be a strict t-norm with additive generator t and I^T its power based implication, let S_2 be the following t-conorm:*

$$S_2(x, y) = min\left((x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}})^\alpha, 1\right), \ \ x, y \in [0, 1], \ \ \alpha > 0.$$

Then there exists a t-conorm S_1 with the following additive generator

$$s_1(x) = t(x)^{-\frac{1}{\alpha}}, \ \ x \in [0, 1], \ \ \alpha > 0,$$

such that the triple (I^T, S_1, S_2) satisfies (6).

Proof. Since T is strict, then t is continuous, strictly decreasing, with $t(0) = \infty$ and $t(1) = 0$. Thus the function $s_1 : [0, 1] \to [0, \infty]$ defined by

$$s_1(x) = t(x)^{-\frac{1}{\alpha}}, \ \ x \in [0, 1], \ \ \alpha > 0,$$

is continuous, strictly increasing, with $s_1(0) = 0$ and $s_1(1) = \infty$. Therefore,

$$S_1(x, y) = s_1^{-1}(s_1(x) + s_1(y)) = t^{-1}\left(\left(t(x)^{-\frac{1}{\alpha}} + t(y)^{-\frac{1}{\alpha}}\right)^{-\alpha}\right),$$

is a strict t-conorm by Theorem 2.2.6 in [4].

Let x, y, $z \in [0, 1]$ with $x > y$ and $x > z$.

Case 1: $x = 1$, or $y = 0$, or $z = 0$. Obviously, $I^T(x, S_1(y, z)) = S_2(I^T(x, y), I^T(x, z))$.

Case 2: x, y, $z \in (0, 1)$. If $x > S_1(y, z)$, then

$$I^T(x, S_1(y, z)) = \frac{t(x)}{t(S_1(y, z))} = t(x) \cdot \left(t(y)^{-\frac{1}{\alpha}} + t(z)^{-\frac{1}{\alpha}}\right)^\alpha = \left(\left(\frac{t(x)}{t(y)}\right)^{\frac{1}{\alpha}} + \left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}}\right)^\alpha.$$

On the other hand, note that $x > S_1(y, z) \Leftrightarrow \left(\left(\frac{t(x)}{t(y)}\right)^{\frac{1}{\alpha}} + \left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}}\right)^\alpha < 1$. Then

$$S_2\left(I^T(x, y), I^T(x, z)\right) = \min\left(\left(\left(\frac{t(x)}{t(y)}\right)^{\frac{1}{\alpha}} + \left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}}\right)^\alpha, 1\right) = \left(\left(\frac{t(x)}{t(y)}\right)^{\frac{1}{\alpha}} + \left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}}\right)^\alpha,$$

thus, we get

$$I^T(x, S_1(y, z)) = S_2(I^T(x, y), I^T(x, z)).$$

If $x \leq S_1(y, z)$, note that $x \leq S_1(y, z) \Leftrightarrow \left(\left(\frac{t(x)}{t(y)}\right)^{\frac{1}{\alpha}} + \left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}}\right)^\alpha \geq 1$, then

$$I^T(x, S_1(y, z)) = 1 = S_2(I^T(x, y), I^T(x, z)).$$

From the above discussion, the triple (I^T, S_1, S_2) satisfies (6). □

Next, we give a result on the solution of equation (6) involving I^T when $T = (< a_i, b_i, T_i >)_{i\in A}$, where T_i is a strict t-norm for all $i \in A$.

Proposition 3.23. *Let T be a non-trivial ordinal sum of t-norms with the form $(< a_i, b_i, T_i >)_{i\in A}$ and I^T its power based implication, where A is an index set, $(T_i)_{i\in A}$ is a family of strict t-norms, and $(a_i, b_i)_{i\in A}$ be a family of non-empty, pairwise disjoint open subintervals of [0,1]. Let S_2 be the following t-conorm:*

$$S_2(x, y) = min\left((x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}})^{\alpha}, 1\right),\ \ x, y \in [0, 1],\ \ \alpha > 0.$$

Then there exists a t-conorm S_1 with the following form:

$$S_1 = (< a_i, b_i, S_{1i} >)_{i\in A},$$

such that the triple (I^T, S_1, S_2) satisfies (6), where S_{1i} is a t-conorm with additive generator $s_{1i}(x) = t_i(x)^{-\frac{1}{\alpha}}$, $x \in [0, 1]$, and t_i is an additive generator of T_i for all $i \in A$.

Proof. It is easy to see that, for every $i \in A$, the following function

$$s_{1i}(x) = t_i(x)^{-\frac{1}{\alpha}}, x \in [0, 1]$$

is strictly increasing, continuous, with $s_{1i}(0) = 0$ and $s_{1i}(1) = \infty$. Therefore,

$$S_{1i}(x, y) = s_{1i}^{-1}(s_{1i}(x) + s_{1i}(y)) = t_i^{-1}\left(\left(t_i(x)^{-\frac{1}{\alpha}} + t_i(y)^{-\frac{1}{\alpha}}\right)^{-\alpha}\right),\ \ x, y \in [0, 1]$$

is a t-conorm by Theorem 2.2.6 in [4]. Obviously, for $x < 1$ and $y < 1$, we have

$$S_{1i}(x, y) < 1. \tag{14}$$

In fact, suppose that $x < 1$ and $y < 1$, then $s_{1i}(x) < \infty$, $s_{1i}(y) < \infty$. Thus $s_{1i}(x) + s_{1i}(y) < \infty$. Therefore, $s_{1i}^{-1}(s_{1i}(x) + s_{1i}(y)) < 1$, i.e., $S_{1i}(x, y) < 1$.

Let S_1 be a function defined by

$$S_1(x, y) = \begin{cases} a_i + (b_i - a_i) \cdot S_{1i}(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}), & \text{if } x, y \in [a_i, b_i], \\ \max(x, y), & \text{otherwise.} \end{cases} \tag{15}$$

Then S_1 is a non-trivial ordinal sum of t-conorms by Corollary 3.58 in [11], i.e., $S_1 = (< a_i, b_i, S_{1i} >)_{i\in A}$. Obviously, if $x < a_i$ and $y < a_i$ for some $i \in A$, then we have

$$S_1(x, y) < a_i. \tag{16}$$

In fact, let $x < a_i$, $y < a_i$ for some $i \in A$. If there exists a $k \in A$ such that x, $y \in [a_k, b_k]$ $(k \neq i)$, then $b_k \leq a_i$. For $b_k < a_i$, we get

$$S_1(x, y) = a_k + (b_k - a_k) \cdot S_{1k}(\frac{x - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k}) \leq a_k + (b_k - a_k) = b_k < a_i.$$

For $b_k = a_i$, since $x < a_i$ and $y < a_i$, i.e., $x < b_k$ and $y < b_k$, then

$$\frac{x - a_k}{b_k - a_k} < 1,\ \ \frac{y - a_k}{b_k - a_k} < 1.$$

Hence, by (14) we get

$$S_{1k}(\frac{x - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k}) < 1.$$

Thus,

$$S_1(x, y) = a_k + (b_k - a_k) \cdot S_{1k}(\frac{x - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k}) < a_k + (b_k - a_k) = b_k = a_i.$$

If there is not a $k \in A$ such that x, $y \in [a_k, b_k]$, then $S_1(x, y) = \max(x, y) < a_i$.

In the following, we prove that the triple (I^T, S_1, S_2) satisfies (6).

Let x, y, $z \in [0, 1]$ with $x > y$ and $x > z$.

Case 1: for every $i \in A$, $x \notin [a_i, b_i]$, $y \notin [a_i, b_i]$ and $z \notin [a_i, b_i]$. Then

$$I^T(x, S_1(y,z)) = I^T(x, \max(y,z)) = 0 = S_2(I^T(x,y), I^T(x,z)).$$

Case 2: there exists an $i \in A$, such that $x \notin [a_i, b_i]$, $y \notin [a_i, b_i]$ and $z \in [a_i, b_i]$. If $y \geq z$, then

$$I^T(x, S_1(y,z)) = I^T(x, \max(y,z)) = I^T(x,y) = S_2(I^T(x,y), 0) = S_2(I^T(x,y), I^T(x,z)).$$

If $y < z$, then $I^T(x,y) = 0$ by $I^T(x,y) \leq I^T(x,z) = 0$. Thus

$$\begin{aligned} I^T(x, S_1(y,z)) &= I^T(x, \max(y,z)) \\ &= I^T(x,z) \\ &= 0 \\ &= S_2(0,0) \\ &= S_2(I^T(x,y), I^T(x,z)). \end{aligned}$$

Case 3: there exists an $i \in A$, such that $x \notin [a_i, b_i]$, $y \in [a_i, b_i]$ and $z \notin [a_i, b_i]$. The rest of the proof is similarly to Case 2.

Case 4: there exists an $i \in A$, such that $x \in [a_i, b_i]$, $y \notin [a_i, b_i]$ and $z \notin [a_i, b_i]$. Since $x > y$ and $x > z$, then $y < a_i$ and $z < a_i$. Thus $S_1(y,z) < a_i$ by (16). Therefore,

$$I^T(x, S_1(y,z)) = 0 = S_2(I^T(x,y), I^T(x,z)).$$

Case 5: there exists an $i \in A$, such that $x \notin [a_i, b_i]$, $y \in [a_i, b_i]$ and $z \in [a_i, b_i]$. It is easy to see that

$$I^T(x, S_1(y,z)) = 0 = S_2(I^T(x,y), I^T(x,z)).$$

Case 6: there exists an $i \in A$, such that $x \in [a_i, b_i]$, $y \in [a_i, b_i]$ and $z \notin [a_i, b_i]$. Since, $x > z$, then $z < a_i$. Thus

$$I^T(x, S_1(y,z)) = I^T(x,y) = S_2(I^T(x,y), 0) = S_2(I^T(x,y), I^T(x,z)).$$

Case 7: there exists an $i \in A$, such that $x \in [a_i, b_i]$, $y \notin [a_i, b_i]$ and $z \in [a_i, b_i]$. Similar to Case 6.
Case 8: there exists an $i \in A$, such that x, y, $z \in [a_i, b_i]$. The rest of the proof is analogue to the proof of Proposition 3.22. $\square$

Table 2 summarizes the distributivity solutions of the power based implication I^T. Here, T is a continuous Archimedean t-norm, or a non-trivial ordinal sum of continuous Archimedean t-norms.

Table 2: Distributivity solutions of the power based implication I^T $(T \neq T_M)$

Equation	Universal solution	Other solution
$I^T(S(x,y),z) = T_1(I^T(x,z), I^T(y,z))$	$S = S_M, T_1 = T_M$	None
$I^T(T_1(x,y),z) = S(I^T(x,z), I^T(y,z))$	$T_1 = T_M$, $S = S_M$	$T_1 = T$, $S = S_{LK}$ and $T_1 = T_1^*$, $S = S^*$, etc.
$I^T(x, T_1(y,z)) = T_2(I^T(x,y), I^T(x,z))$	$T_1 = T_M, T_2 = T_M$	None
$I^T(x, S_1(y,z)) = S_2(I^T(x,y), I^T(x,z))$	$S_1 = S_M, S_2 = S_M$	T is nilpotent: None T is $T^\star$: None T is strict: $S_1 = S_1^\star$, $S_2 = S^*$, etc. T is $T^{\star\star}$: $S_1 = S_1^{\star\star}$, $S_2 = S^*$, etc.

Note (i) T_1^* has an additive generator $t_1(x) = (k \cdot t(x))^{\frac{1}{\alpha}}$, $x \in [0,1]$ when T has a continuous additive generator t, or $T_1^* = (< a_i, b_i, T_{1i} >)_{i \in A}$ when $T = (< a_i, b_i, T_i >)_{i \in A}$, where T_{1i} has an additive generator $t_{1i}(x) = (k \cdot t_i(x))^{\frac{1}{\alpha}}$, $x \in [0,1]$, t_i is a continuous additive generator of T_i, $i \in A$, $k > 0$, $\alpha > 0$.
(ii) $S^*(x,y) = \min\left((x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}})^\alpha, 1\right)$, x, $y \in [0,1]$, where $\alpha > 0$.

(iii) $T^{\star}=(< a_i,b_i,T_i >)_{i\in A}$. There exists an $i\in A$ such that $a_i=0$, and T_i is nilpotent, or a_i is an idempotent point of S_1 and T_i is nilpotent.

(iv) $S_1^{\star}(x,y)=t^{-1}\left(\left(t(x)^{-\frac{1}{\alpha}}+t(y)^{-\frac{1}{\alpha}}\right)^{-\alpha}\right)$, $x,\ y\in[0,1]$, $\alpha>0$, where t is an additive generator of T.

(v) $T^{\star\star}=(< a_i,b_i,T_i >)_{i\in A}$, where $(T_i)_{i\in A}$ is a family of strict t-norms.

(vi) $S^{\star\star}=(< a_i,b_i,S_{1i} >)_{i\in A}$, where $S_{1i}(x,y)=t_i^{-1}\left(\left(t_i(x)^{-\frac{1}{\alpha}}+t_i(y)^{-\frac{1}{\alpha}}\right)^{-\alpha}\right)$, $x,y\in[0,1]$, t_i is an additive generator of T_i in $T^{\star\star}$, $i\in A$.

4 Conclusions

In this paper, four distributivity equations of T-power based implications are deeply studied respectively. This study shows that the equations (3) and (5) have a unique solution, while the the equations (4) and (6) have multiple solutions. This study has a certain significance for the application of T-power based implication in rule reduction. However, it is difficult to find all solutions for equations (4) and (6), this is a problem to be solved in the future.

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Appendix A: The distributivity laws of implication I^{T_M}.

(1) Let T be a t-norm, and S a t-conorm. Then the triple $(I^{T_M},S,\ T)$ satisfies (3) if and only if $S=S_M$.

Proof.(Necessity) Let the triple $(I^{T_M},S,\ T)$ satisfy (3). Then, for all $x,\ y,\ z\in[0,1]$, we get

$$I^{T_M}(S(x,y),z)=T(I^{T_M}(x,z),I^{T_M}(y,z)).$$

Putting $x=y=z$, then $I^{T_M}(S(x,x),x)=T(I^{T_M}(x,x),I^{T_M}(x,x))=1$. Since I^{T_M} satisfies (OP), then $S(x,x)\le x$. Since $S(x,x)\ge x$, thus $S(x,x)=x$ for all $x\in[0,1]$. Hence $S=S_M$.

(Sufficiency) Let $S=S_M$. It suffice to prove that

$$I^{T_M}(S(x,y),z)=T(I^{T_M}(x,z),I^{T_M}(y,z)) \tag{17}$$

for all $x,\ y,\ z\in[0,1]$.

If $x\le y\le z$, then $I^{T_M}(S(x,y),z)=I^{T_M}(y,z)=1$, $T(I^{T_M}(x,z),I^{T_M}(y,z))=T(1,1)=1$. Thus equation (17) holds.

If $x\le z<y$, then $I^{T_M}(S(x,y),z)=I^{T_M}(y,z)=0$, $T(I^{T_M}(x,z),I^{T_M}(y,z))=T(1,0)=0$. Thus equation (17) holds.

If $z<x\le y$, then $I^{T_M}(S(x,y),z)=I^{T_M}(y,z)=0$, $T(I^{T_M}(x,z),I^{T_M}(y,z))=T(0,0)=0$. Thus equation (17) holds.

If $x>y\ge z$, then $I^{T_M}(S(x,y),z)=I^{T_M}(x,z)=0$, $T(I^{T_M}(x,z),I^{T_M}(y,z))=T(0,I^{T_M}(y,z))=0$. Thus equation (17) holds.

If $x>z>y$, then $I^{T_M}(S(x,y),z)=I^{T_M}(x,z)=0$, $T(I^{T_M}(x,z),I^{T_M}(y,z))=T(0,I^{T_M}(y,z))=0$. Thus equation (17) holds.

If $z\ge x>y$, then $I^{T_M}(S(x,y),z)=I^{T_M}(x,z)=1$, $T(I^{T_M}(x,z),I^{T_M}(y,z))=T(1,1)=1$. Thus the equation (17) holds.

From the above discussion, equation (17) holds for all $x,\ y,\ z\in[0,1]$.

(2) Let T be a t-norm, and S a t-conorm. Then the triple $(I^{T_M},T,\ S)$ satisfies (4) if and only if $T=T_M$.

Proof. (Necessity) Let the triple $(I^{T_M},T,\ S)$ satisfy (4), i.e.,

$$I^{T_M}(T(x,y),z)=S(I^{T_M}(x,z),I^{T_M}(y,z)),\ \text{for all}\ x,\ y,\ z\in[0,1].$$

Assume that $T\ne T_M$, then there exists an $x_0\in(0,1)$ such that $T(x_0,x_0)<x_0$. Taking $z_0\in(0,1)$ such that $T(x_0,x_0)<z_0<x_0$. Thus

$$I^{T_M}(T(x_0,x_0),z_0)=1>0=S(I^{T_M}(x_0,z_0),I^{T_M}(x_0,z_0)).$$

A contradiction to the triple $(I^{T_M},T,\ S)$ satisfies (4).

(Sufficiency) Let $T = T_M$, and $x,\ y,\ z \in [0,1]$. If $x \le z$ or $y \le z$, then $T(x,y) = T_M(x,y) \le z$. Thus

$$I^{T_M}(T(x,y),z) = 1 = S(I^{T_M}(x,z), I^{T_M}(y,z)).$$

If $x > z$ and $y > z$, then $T(x,y) = T_M(x,y) > z$. Thus

$$I^{T_M}(T(x,y),z) = 0 = S(0,0) = S(I^{T_M}(x,z), I^{T_M}(y,z)).$$

From the above discussion, we get that the triple $(I^{T_M}, T,\ S)$ satisfies (4).

(3) Let $T_1,\ T_2$ be t-norms. Then the triple $(I^{T_M}, T_1,\ T_2)$ satisfies (5) if and only if $T_1 = T_M$.

Proof. (Necessity) Let the triple $(I^{T_M}, T_1,\ T_2)$ satisfy (5). Then

$$I^{T_M}(x, T_1(y,z)) = T_2(I^{T_M}(x,y), I^{T_M}(x,z)) \text{ for all } x,\ y,\ z \in [0,1].$$

Taking $x = y = z$. Then $I^{T_M}(x, T_1(x,x)) = T_2(I^{T_M}(x,x), I^{T_M}(x,x)) = 1$. Since I^{T_M} satisfies (OP), then $x \le T_1(x,x)$ for all $x \in [0,1]$. Thus $T_1(x,x) = x$, i.e., $T_1 = T_M$.

(Sufficiency) Let $T_1 = T_M$. If $x > y$ or $x > z$, then $x > T_1(y,z)$. Thus

$$I^{T_M}(x, T_1(y,z)) = 0 = T_2(I^{T_M}(x,y), I^{T_M}(x,z)).$$

If $x \le y$ and $x \le z$, then $x \le T_M(y,z) = T_1(y,z)$. Thus

$$I^{T_M}(x, T_1(y,z)) = 1 = T_2(I^{T_M}(x,y), I^{T_M}(x,z)).$$

From the above discussion, we get $I^{T_M}(x, T_1(y,z)) = T_2(I^{T_M}(x,y), I^{T_M}(x,z))$ for all $x,\ y,\ z \in [0,1]$, i.e., the triple $(I^{T_M}, T_1,\ T_2)$ satisfies (5).

(4) Let $S_1,\ S_2$ be t-conorms. Then the triple $(I^{T_M}, S_1,\ S_2)$ satisfies (6) if and only if $S_1 = S_M$.

Proof. (Necessity) Let the triple $(I^{T_M}, S_1,\ S_2)$ satisfy (6), then

$$I^{T_M}(x, S_1(y,z)) = S_2(I^{T_M}(x,y), I^{T_M}(x,z)) \text{ for all } x,\ y,\ z \in [0,1].$$

Assume that $S_1 \ne S_M$, then there exists a $y_0 \in (0,1)$ such that $y_0 < S_1(y_0,y_0)$. Taking $x_0 \in (0,1)$ such that $y_0 < x_0 < S_1(y_0,y_0)$. Thus

$$I^{T_M}(x_0, S_1(y_0,y_0)) = 1 > 0 = S_2(0,0) = S_2(I^{T_M}(x_0,y_0), I^{T_M}(x_0,y_0)).$$

A contradiction to the triple $(I^{T_M}, S_1,\ S_2)$ satisfies (6).

(Sufficiency) Let $S_1 = S_M$, and $x,\ y,\ z \in [0,1]$. If $x \le y$ or $x \le z$, then $x \le S_1(y,z)$. Thus

$$I^{T_M}(x, S_1(y,z)) = 1 = S_2(I^{T_M}(x,y), I^{T_M}(x,z)).$$

If $x > y$ and $x > z$, then $x > S_M(y,z) = S_1(y,z)$. Thus

$$I^{T_M}(x, S_1(y,z)) = 0 = S_2(0,0) = S_2(I^{T_M}(x,y), I^{T_M}(x,z)).$$

From the above discussion, it is easy to see that the triple $(I^{T_M}, S_1,\ S_2)$ satisfies (6)

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