Journal of Computational Mathematics
Vol.xx, No.x, 200x, 1–40.

http://www.global-sci.org/jcm
doi:??

# THE TRUNCATED EM METHOD FOR JUMP-DIFFUSION SDDES WITH SUPER-LINEARLY GROWING DIFFUSION AND JUMP COEFFICIENTS*

Shounian Deng
*Key Laboratory of Advanced Perception and Intelligent Control of High-end Equipment, Ministry of Education, and School of Mathematics-Physics and Finance, Anhui Polytechnic University, Wuhu* 241000*, China*
*Email: sndeng@mail.ahpu.edu.cn*
Chen Fei [1)]
*Business School, University of Shanghai for Science and Technology, Shanghai* 200093*, China*
*Email: chenfei@usst.edu.cn*
Weiyin Fei
*Key Laboratory of Advanced Perception and Intelligent Control of High-end Equipment, Ministry of Education, and School of Mathematics-Physics and Finance, Anhui Polytechnic University, Wuhu* 241000*, China*
*Email: wyfei@ahpu.edu.cn*
Xuerong Mao
*Department of Mathematics and Statistics, University of Strathclyde, Glasgow* G1 1XH*, UK*
*Email: x.mao@strath.ac.uk*

**Abstract**

This work is concerned with the convergence and stability of the truncated Euler-Maruyama (EM) method for super-linear stochastic differential delay equations (SDDEs) with time-variable delay and Poisson jumps. By constructing appropriate truncated functions to control the super-linear growth of the original coefficients, we present two types of the truncated EM method for such jump-diffusion SDDEs with time-variable delay, which is proposed to be approximated by the value taken at the nearest grid points on the left of the delayed argument. The first type is proved to have a strong convergence order which is arbitrarily close to 1/2 in mean-square sense, under the Khasminskii-type, global monotonicity with $U$ function and polynomial growth conditions. The second type is convergent in $q$th ($q < 2$) moment under the local Lipschitz plus generalized Khasminskii-type conditions. In addition, we show that the partially truncated EM method preserves the mean-square and $H_\infty$ stabilities of the true solutions. Lastly, we carry out some numerical experiments to support the theoretical results.

*Mathematics subject classification:* 60H10, 60H35, 34K34, 65L20.
*Key words:* SDDEs, Truncated EM method, Time-variable delay, Poisson jumps

## 1. Introduction

The stochastic differential delay equation (SDDE) models play a significant part in many application fields, such as economy, finance, automatic control and population dynamics (see, e.g., [4, 5, 10, 11, 26, 27, 29, 35, 41]). In general, such models rarely have explicit solutions available. Thus, there appears to be a practical need to estimate the true solution of the model

* Received xxx / Revised version received xxx / Accepted xxx /
1) Corresponding author.

via numerical approach. Moreover, many important SDDE models often possess super-linear growth coefficients in the real world, for example, stochastic delay Lotka-Volterra model arising in population dynamics of the form (see, e.g., [4])

$$dX(t) = \mathrm{diag}\Big(X_1(t), \cdots, X_d(t)\Big)\Big[(b + AX(t-\tau))dt + \sigma X(t)dB(t)\Big], \tag{1.1}$$

where $B(t)$ is a Brownian motion and $\sigma = (\sigma_{ij})_{d\times d}$ is a matrix representing the intensity of noise. If we apply the explicit Euler-Maruyama (EM) scheme to the model (1.1), it is well-documented that such EM approximation fails to converge in the strong sense to the true solution of (1.1) (see, e.g., [23]).

When the delay component vanishes, the underlying SDDEs reduce to the classical stochastic differential equations (SDEs), numerical methods for which have been extensively investigated for the past decades under the global Lipschitz condition (see, e.g., [25], [21], [39]). In the setting of SDEs whose coefficients can be allowed to grow super-linearly, several explicit schemes have been introduced, including tamed EM and Runge-Kutta schemes [18,24,40], balanced EM schemes [46,49] and truncated EM schemes [30,32,33]. Recently, the attention of some researches was attracted to the strong convergence of explicit numerical methods for super-linear SDEs with delay, i.e., SDDEs. Guo et al. [20] were the first to discuss the strong convergence of the truncated EM method for SDDEs under the local Lipschitz plus the generalized Khasminskii-type conditions. In a subsequent paper, Gao et al. [19] took the jumps into consideration and they extended convergence results from [15,20] to the case of SDDEs with Poisson jumps. By using a different estimate for the difference between the original and the truncated coefficients, Fei et al. [17] relaxed the restrictive condition on the step size which is required to extremely small and thus improved the convergence results of [20]. Moreover, Song et al. [43] achieved a better convergence order than [17,20] by adopting the truncation techniques from [30] for such SDDEs. Other explicit numerical methods for super-linear SDDEs, say tamed EM, balanced EM, truncated Milstein, projected EM, are discussed in [7,12,13,28,45,48], respectively.

Numerous studies suggest strong empirical evidences that there exist jumps within financial markets (see, e.g., [1–3]). Jumps risks can not be ignored in the pricing of financial assets (see, e.g., [38]). When it comes to the convergence of numerical schemes for SDEs or SDDEs with jumps, most of the existing works impose the linear growth assumption on the jump coefficient, such as [22], [39], and [12]. In the context of the super-linear growth assumption on the jump , Deng et al. [15] and Gao et al. [19] established the convergence results of the truncated EM method for jump-diffusion SDEs and SDDEs in small moment (i.e., $q$th moment with $q$ small than 2), respectively. However, when the mean-square convergence order is considered, some difficulties arise. Due to the fact that the higher moment bounds of the Poisson increments contribute to magnitude not more than $\mathcal{O}(\Delta)$, i.e.,

$$\mathbb{E}|N(t+\Delta) - N(t)|^p \le C\Delta,\ p \ge 2, \tag{1.2}$$

where $C$ is a positive constant independent of $\Delta$, the order of the one-step error in the $\mathcal{L}^p$-norm drops to $1/p$ and further decreases when we apply the truncated technology (3.8) or (4.7) to super-linearly growing jump coefficients (see, e.g., [6,8]). If the super-linear growth of the jump coefficient can not be addressed well, then the convergence rate in mean-square sense will not achieve the desired order. To overcome these difficulties, we design a truncated function depending on the Khasminskii parameter $p_0$ to control the super-linear growth of the jump coefficient, see (3.4) and (3.8). The selected function $\varphi(\Delta)$ or $\mu(R)$ in (3.4) must be controlled

by a function with power 1/3, which is just the order observed via applying Lemma 3.3 to the proof of the moment boundedness of truncated EM solution, see Lemma 3.5. Estimate (3.28) explains the origin of this controlled condition (3.5). According to the property (3.8) of the truncated functions and Lemma 3.2 on preserving the Khasminskii-type condition, a moment boundedness result of the truncated EM solution can be established, see Lemma 3.5. By adopting more refined moment estimation, the one-step error in our truncated EM method can arrive at the optimal order in the sense of $\mathcal{L}^q$, see Lemma 3.6. Further, we prove that the strong order of a type of truncated EM method is arbitrarily close to 1/2 in mean-square sense, without assuming linear growth in the drift, diffusion or jump coefficients, see Theorem 3.1.

In reality, delay can behave as a function of time, namely, time-variable delay. Mao and Sabanis [34] were the first authors to consider the strong convergence of numerical method for such SDDEs. In contrast to the constant delay, the main difficulty in the construction of the computational approach in case of SDDEs with time-variable delay is how to numerically approximate the values of the solution at the delayed instants. Mao and Sabanis [34] proposed to use the approximate values at the nearest grid points on the left of the delayed arguments to estimate the time-variable delay. Influenced by [34], several explicit and implicit variants of the EM method have been developed to study the convergence of the numerical solutions to stochastic equations with time-variable delay. For instance, we refer to [36] for the convergence in probability of EM method for highly nonlinear neutral stochastic differential equations with time-variable delay, and to [47] for the strong convergence and almost sure exponential stability of the backward EM method for nonlinear hybrid SDEs with time-variable delay. Further, Deng et al. [14] discussed the strong convergence rate for the split-step theta method applied to stochastic age-dependent population equations with Markovian switching and time-variable delay.

Based on the above discussions, the objective of this work is to study the strong convergence (including the stabilities) of explicit numerical method of super-linear SDDEs with time-variable delay and Poisson jumps. Following some ideas from Chen, Gan and Wang [9] who proposed a family of the tamed EM methods for SDEs with super-linearly growing diffusion and jump coefficients, we introduce two types of the truncated EM method for such jump-diffusion SDDEs with time-variable delay. Then we show that the methods are convergent in the sense of $\mathcal{L}^q$ $(q < 2)$ and $\mathcal{L}^2$, respectively. In addition, we discuss the mean-square and $H_\infty$ stabilities of the method. The main contributions of our paper are highlighted below.

- *Assumptions.* Our truncated EM method allows all the coefficients of SDDEs with jumps to grow super-linearly and thus requires a more relaxed assumptions than that in [19], where both drift and diffusion coefficients are allowed to grow super-linearly, but the jump coefficient should be required to satisfy a linear growth assumption.

- *Convergence order.* Our truncated EM method obtains a superior mean-square convergence order to those in [19, Theorem 3.9] and [9, Theorem 4.5], see Remarks 3.1 and 3.2 for details.

- *Techniques.* Our technical estimates are more refined than those of [20], [17] and [43], because we develop new techniques to overcome the challenges arising due to time-variable delay and super-linear growth of the jump coefficient, see Remark 4.1 for details.

The remainder of the paper is organised as follows. The second section introduces some basic notions and assumptions. Strong convergence order in mean-square sense and convergence

(without order) in $q$th ($q < 2$) moment are discussed in the third and the forth sections, respectively. In the fifth section, we prove some stability theorems. Numerical examples are provided in the final section.

## 2. Preliminaries and truncated EM scheme

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $\tau > 0$ be a constant and denote by $C([-\tau, 0]; \mathbb{R}^d)$ the space of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}^d$ with the norm $\|\phi\| = \sup_{-\tau\leq\theta\leq 0} |\phi(\theta)|$. Let $B(t)$ be an $m$-dimensional Brownian motion. Let $N(t)$ be a scalar Poisson process with the compensated Poisson process $\widetilde{N}(t) = N(t) - \lambda t$, where $\lambda \geq 0$ denotes the jump intensity. Further, we assume that $B(t)$ and $N(t)$ are independent. If $A$ is a vector or matrix, its transpose is denoted by $A^T$. If $X \in \mathbb{R}^d$, then $|X|$ is the Euclidean norm. If $A$ is a matrix, its trace norm is denoted by $|A| = \sqrt{A^T A}$. For two real numbers $a$ and $b$, $a \vee b := \max(a, b)$ and $a \wedge b := \min(a, b)$. Let $\mathbb{R}^+ = [0, +\infty)$. For a set $G$, its indicator function is denoted by $\mathbb{I}_G$. The scalar product of two vectors $X, Y \in \mathbb{R}^d$ is denoted by $\langle X, Y\rangle$ or $X^T Y$. The largest integer which is less or equal to a real number $a$ is denoted by $\lfloor a \rfloor$. In addition, we use $C$ to denote the generic constant that may change from place to place.

Let $\delta : [0, +\infty) \to [0, \tau]$ be the delay function which is Borel measurable. Consider the following SDDE with Poisson jumps of the form

$$\begin{aligned} dX(t) =& f(X(t)), X(t-\delta(t)))dt + g(X(t), X(t-\delta(t)))dB(t) \\ &+ h(X(t^-), X((t-\delta(t))^-))dN(t),\ t \geq 0, \end{aligned} \tag{2.1}$$

with the initial data

$$\{X(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C([-\tau, 0]; \mathbb{R}^d), \tag{2.2}$$

where $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$, and $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are Borel-measurable functions. Here, $X(t^-)$ denotes $\lim_{s\to t^-} X(s)$. We impose the following assumptions:

**Assumption 2.1 (Hölder Continuity of Initial Data)** *There is a pair of constants $K_0 > 0$ and $\varrho \in (0, 1]$ such that*

$$|\xi(t) - \xi(s)| \leq K_0 |t - s|^{\varrho},\ \forall s, t \in [-\tau, 0]. \tag{2.3}$$

**Assumption 2.2.** *Let $\delta : [0, +\infty) \to [0, \tau]$ is continuously differentiable and there is $\hat{\delta} \in [0, 1)$ such that $\left|\frac{d\delta(t)}{dt}\right| \leq \hat{\delta}$, for any $t \geq 0$.*

**Assumption 2.3 (Local Lipschitz Condition)** *For any $R > 0$, there exists a constant $L_R$ depending on $R$ such that*

$$|f(x_1, y_1) - f(x_2, y_2)|^2 \vee |g(x_1, y_1) - g(x_2, y_2)|^2 \vee |h(x_1, y_1) - h(x_2, y_2)|^2 \leq L_R(|x_1 - x_2|^2 + |y_1 - y_2|^2),$$

*for any $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$ with $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq R$.*

Assume that step size $\Delta \in (0,1]$ is a fraction of $\tau$. Take $\Delta = \tau/M$ for some sufficiently large integer $M$. Define $t_k = k\Delta$ and $\delta_k = \lfloor \delta(k\Delta)/\Delta \rfloor$, for any integer $k \geq 0$. Then the boundedness of $\delta$ gives

$$0 \leq \delta_k \leq \tau/\Delta = M. \tag{2.4}$$

Define $\kappa(t)=\lfloor t/\Delta \rfloor \Delta$, for any $t \geq -\tau$. Let $f_\Delta$, $g_\Delta$ and $h_\Delta$ be the truncated functions which are defined in the following section. Then the discrete-time truncated EM scheme is defined as follows:

$$\begin{aligned} &y_{k+1} = y_k + f_\Delta(y_k, y_{k-\delta_k})\Delta + g_\Delta(y_k, y_{k-\delta_k})\Delta B_k + h_\Delta(y_k, y_{k-\delta_k})\Delta N_k,\ k \geq 0, \\ &y_k = \xi(t_k),\ k = -M, -M+1, \cdots, 0, \end{aligned} \tag{2.5}$$

where $\Delta B_k = B(t_{k+1}) - B(t_k)$ and $\Delta N_k = N(t_{k+1}) - N(t_k)$. Define the continuous-time step approximations $Z_1(t)$ and $Z_2(t)$ by

$$Z_1(t) = \sum_{k=-M}^{\infty} y_k \mathbb{I}_{[t_k, t_{k+1})}(t),\ \forall\, t \geq -\tau, \tag{2.6}$$

$$Z_2(t) = \sum_{k=0}^{\infty} y_{k-\delta_k} \mathbb{I}_{[t_k, t_{k+1})}(t),\ \forall t \geq 0. \tag{2.7}$$

where $\mathbb{I}$ denotes the indicator function. Define the continuous-time approximation $Y_\Delta(t)$ on $t \in [-\tau, \infty)$ by

$$\begin{aligned} Y_\Delta(t) =& \xi(0) + \int_0^t f_\Delta(Z_1(s), Z_2(s))ds + \int_0^t g_\Delta(Z_1(s), Z_2(s))dB(s) \\ &+ \int_0^t h_\Delta(Z_1(s^-), Z_2(s^-))dN(s),\ t \geq 0, \\ Y_\Delta(t) =& \xi(t),\ -\tau \leq t \leq 0. \end{aligned} \tag{2.8}$$

Thus $Y_\Delta(t)$ is an Itô process on $t \geq 0$ with Itô differential

$$dY_\Delta(t) = f_\Delta(Z_1(t), Z_2(t))dt + g_\Delta(Z_1(t), Z_2(t))dB(t) + h_\Delta(Z_1(t^-), Z_2(t^-))dN(t). \tag{2.9}$$

It is useful to know that for any $t \in [t_k, t_{k+1})$ with $k \geq 0$,

$$Z_1(t) = Y_\Delta(t_k) = y_k \quad \text{and} \quad Z_2(t) = Y_\Delta(t_k - \lfloor \delta(t_k)/\Delta \rfloor \Delta) = y_{k-\delta_k}, \tag{2.10}$$

as well as

$$\begin{aligned} Y_\Delta(t) - Z_1(t) =& \int_{t_k}^t f_\Delta(Z_1(s), Z_2(s))ds + \int_{t_k}^t g_\Delta(Z_1(s), Z_2(s))dB(s) \\ &+ \int_{t_k}^t h_\Delta(Z_1(s^-), Z_2(s^-))dN(s), \end{aligned} \tag{2.11}$$

which means that the $Y_\Delta(t)$ and $Z_1(t)$ coincide with the discrete solution at the grid points.

**Lemma 2.1.** *Let Assumption 2.2 hold. For any $k \in \{0, 1, 2, \cdots\}$, let $k - \lfloor \delta(k\Delta)/\Delta \rfloor = u$, where $u \in \{-M, -M+1, \cdots, 0, 1, \cdots, k\}$. Then*

$$\#\Big\{ j \in \{0, 1, 2, \cdots\} : j - \lfloor \delta(j\Delta)/\Delta \rfloor = u \Big\} \leq \lfloor (1 - \hat{\delta})^{-1} \rfloor + 1, \tag{2.12}$$

*where $\#S$ denotes the number of elements of the set $S$.*

This lemma provides an upper bound for the maximal number of indices $k \in \{1, 2, \cdots\}$ for which the expressions $k - \delta_k$ are equal, the proof can be found in [37, Lemma 3].

## 3. Convergence order in $\mathcal{L}^2$

In this section, we mainly discuss the convergence order of the truncated EM method for (2.1) in mean-square sense.

**Assumption 3.1 (Khasminskii-type Condition)** *There exist positive constants $p_0 \geq 4$, $\varepsilon > 0$ and $\bar{K}_1 > 0$ such that*

$$\begin{aligned} &p_0|x|^{p_0-2}\Big(\langle x, f(x,y)\rangle + \frac{p_0-1}{2}|g(x,y)|^2\Big) + \lambda(1+\varepsilon)|h(x,y)|^{p_0} \\ &\leq \bar{K}_1(1+|x|^{p_0}+|y|^{p_0}),\ \forall x, y \in \mathbb{R}^d. \end{aligned}$$

**Lemma 3.1.** *Let Assumptions 2.3 and 3.1 hold. Then for any given initial data (2.2), there is a unique global solution $X(t)$ to (2.1) on $t \in [-\tau, +\infty)$. Moreover, the solution $X(t)$ has the property that*

$$\sup_{-\tau \leq t \leq T} \mathbb{E}|X(t)|^{p_0} < \infty,\ \forall T > 0.$$

The proof of the lemma is similar to that of [9, Theorem 2.4] and so is omitted. Let $\mathcal{U}$ be the family of continuous function $U : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+$ such that for any $b > 0$, there exists a positive constant $\kappa_b$ for which

$$U(x_1, x_2) \leq \kappa_b|x_1 - x_2|^2,\ \forall x_1, x_2 \in \mathbb{R}^d \text{ with } |x_1| \vee |x_2| \leq b.$$

**Assumption 3.2 (Global Monotonicity with $U$ Function and Polynomial Growth conditions)** *There exist constants $p_1 > 1$, $l \geq 0$ and $\bar{K}_2 > 0$ as well as a function $U \in \mathcal{U}$ such that*

$$\begin{aligned} &2\langle x_1 - x_2, f(x_1,y_1) - f(x_2,y_2)\rangle + p_1|g(x_1,y_1) - g(x_2,y_2)|^2 + p_1\lambda|h(x_1,y_1) - h(x_2,y_2)|^2 \\ &\leq \bar{K}_2(|x_1-x_2|^2 + |y_1-y_2|^2) - \frac{1}{1-\hat{\delta}}U(x_1,x_2) + U(y_1,y_2),\ \forall x_1,y_1,x_2,y_2 \in \mathbb{R}^d \end{aligned} \tag{3.1}$$

*and*

$$\begin{aligned} &|f(x_1,y_1) - f(x_2,y_2)|^2 \\ &\leq \bar{K}_2(1+|x_1|^l+|x_2|^l+|y_1|^l+|y_2|^l)(|x_1-x_2|^2+|y_1-y_2|^2),\ \forall x_1,y_1,x_2,y_2 \in \mathbb{R}^d, \end{aligned} \tag{3.2}$$

*as well as*

$$\begin{aligned} &|g(x_1,y_1) - g(x_2,y_2)|^2 \vee |h(x_1,y_1) - h(x_2,y_2)|^2 \\ &\leq \bar{K}_2(1+|x_1|^{l/2}+|x_2|^{l/2}+|y_1|^{l/2}+|y_2|^{l/2})(|x_1-x_2|^2+|y_1-y_2|^2),\ \forall x_1,y_1,x_2,y_2 \in \mathbb{R}^d. \end{aligned} \tag{3.3}$$

To define the first type of truncated EM scheme, we choose a strictly increasing continuous functions $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(R) \to \infty$ as $R \to \infty$ and

$$\sup_{|x|\vee|y| \leq R} \frac{|f(x,y)|}{(1+|x|+|y|)} \vee \left(\frac{|g(x,y)|}{1+|x|+|y|}\right)^2 \vee \left(\frac{|h(x,y)|}{1+|x|+|y|}\right)^{p_0} \leq \mu(R),\ \forall R \geq 1, \tag{3.4}$$

where constant $p_0$ comes from Assumption 3.1. Denote by $\mu^{-1}$ the inverse function of $\mu$ and we observe that $\mu^{-1} : [\mu(1), \infty) \to \mathbb{R}^+$ is a strictly increasing continuous function. Define a strictly decreasing function $\varphi : (0, 1] \to [\mu(1), +\infty)$ such that

$$\lim_{\Delta \to 0} \varphi(\Delta) = \infty \quad \text{and} \quad \varphi(\Delta) \leq \hat{L}\Delta^{-1/3}, \ \forall \Delta \in (0, 1], \tag{3.5}$$

where $\hat{L} = 1 \vee \mu(1)$. For a given step size $\Delta \in (0, 1]$, let us define a truncation mapping $\pi_\Delta : \mathbb{R}^d \to \{x \in \mathbb{R}^d : |x| \leq \mu^{-1}(\varphi(\Delta))\}$ by

$$\pi_\Delta(x) = \left(|x| \wedge \mu^{-1}(\varphi(\Delta))\right) \frac{x}{|x|}, \ \forall x \in \mathbb{R}^d, \tag{3.6}$$

when $x = 0$, we set $x/|x| = 0$. In other words, $\pi_\Delta$ maps to itself if $|x| \leq \mu^{-1}(h(\Delta))$ and to $\mu^{-1}(h(\Delta))x/|x|$ if $|x| > \mu^{-1}(h(\Delta))$. Define the following truncated functions

$$f_\Delta(x, y) = f(\pi_\Delta(x), \pi_\Delta(y)), \quad g_\Delta(x, y) = g(\pi_\Delta(x), \pi_\Delta(y)), \quad h_\Delta(x, y) = h(\pi_\Delta(x), \pi_\Delta(y)), \tag{3.7}$$

for any $x, y \in \mathbb{R}^d$. It is useful to note from (3.4) and (3.7) that

$$\begin{aligned} &|f_\Delta(x, y)| \leq \varphi(\Delta)(1 + |x| + |y|), \quad |g_\Delta(x, y)|^2 \leq \varphi(\Delta)(1 + |x| + |y|)^2, \\ &|h_\Delta(x, y)|^{p_0} \leq \varphi(\Delta)(1 + |x| + |y|)^{p_0}, \ \forall x, y \in \mathbb{R}^d, \end{aligned} \tag{3.8}$$

which means that the truncated coefficients $f_\Delta$, $g_\Delta$ and $h_\Delta$ grow at most linearly for a fixed step size $\Delta$, but original coefficients $f$, $g$ and $h$ may not. Note that

$$|\pi_\Delta(x)| \leq |x|, \quad |\pi_\Delta(y)| \leq |y|, \quad |\pi_\Delta(x) - \pi_\Delta(y)|^2 \leq |x - y|^2, \ \forall x, y \in \mathbb{R}^d. \tag{3.9}$$

Thus, from (3.2) and (3.3), we have the following growth condition

$$\begin{aligned} |f(x, y)| &\leq \bar{K}_3(1 + |x|^{1+l/2} + |y|^{1+l/2}) \\ |g(x, y)|^2 \vee |h(x, y)|^2 &\leq \bar{K}_3(1 + |x|^{2+l/2} + |y|^{2+l/2}), \ \forall x, y \in \mathbb{R}^d, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &|f_\Delta(x, y)| = |f(\pi_\Delta(x), \pi_\Delta(y))| \\ &\leq \bar{K}_3(1 + |\pi_\Delta(x)|^{1+l/2} + |\pi_\Delta(y)|^{1+l/2}) \leq \bar{K}_3(1 + |x|^{1+l/2} + |y|^{1+l/2}), \ \forall x, y \in \mathbb{R}^d, \end{aligned} \tag{3.11}$$

as well as

$$|g_\Delta(x, y)|^2 \vee |h_\Delta(x, y)|^2 \leq \bar{K}_3(1 + |x|^{2+l/2} + |y|^{2+l/2}), \ \forall x, y \in \mathbb{R}^d, \tag{3.12}$$

where $\bar{K}_3$ is a positive constant depending on $\bar{K}_2$.

The following theorem shows the strong convergence order of the truncated EM method for SDDEs with time-variable delay and Poisson jumps.

**Theorem 3.1 (Convergence Rate for SDDEs with Time-variable Delay and Poisson Jumps)** *Let Assumptions 2.1, 2.2, 3.1 and 3.2 hold with $p_0 \geq 4 \vee (2 + 2.5l)$. Then for any $\Delta \in (0, 1]$,*

$$\mathbb{E}|X(T) - Y_\Delta(T)|^2 \leq C \left( \Delta^{\left(1 - \frac{l}{p_0}\right) \wedge 2\varrho} \vee [\mu^{-1}(\varphi(\Delta))]^{-(p_0 - l - 2)} \right), \ \forall T > 0, \tag{3.13}$$

*and*

$$\mathbb{E}|X(T)-Z_1(T)|^2 \le C\left(\Delta^{\left(1-\frac{l}{p_0}\right)\wedge 2\varrho} \vee [\mu^{-1}(\varphi(\Delta))]^{-(p_0-l-2)}\right),\ \forall T>0, \tag{3.14}$$

*where $C$ is a positive constant independent of $\Delta$. In particular, let*

$$\mu(r)=\bar{K}_3 r^{l/2},\ \forall r\ge 1 \quad \text{and} \quad \varphi(\Delta)=(\bar{K}_3\vee 1)\Delta^{-1/3},\ \forall \Delta\in(0,1]. \tag{3.15}$$

*Then for any $\Delta\in(0,1]$,*

$$\mathbb{E}|X(T)-Y_\Delta(T)|^2 \le C\Delta^{\left(1-\frac{l}{p_0}\right)\wedge 2\varrho} \quad \text{and} \quad \mathbb{E}|X(T)-Z_1(T)|^2 \le C\Delta^{\left(1-\frac{l}{p_0}\right)\wedge 2\varrho},\ \forall T>0. \tag{3.16}$$

**Remark 3.1.** *Comparing our results Theorem 3.1 with that of [19], where the authors studied the strong convergence rate of the truncated EM method for SDDEs with Poisson jumps, we observe the following differences:*

- *Theorem 3.1 requires a slightly stronger condition on Khasminskii parameter $p_0$, namely, $p_0\ge 4\vee(2+2.5l)$, while [19, Theorem 3.9] requires $p_0>2+l$, approximately;*
- *Theorem 3.1 allows delay to be a function of time, while [19, Theorem 3.9] requires delay to be a constant;*
- *Theorem 3.1 allows the jump coefficient to grow super-linearly, but [19, Theorem 3.9] should require the jump coefficient to satisfy a linear growth bound;*
- *In mean-square sense, our truncated EM method has a better convergence order $\left(1-\frac{l}{p_0}\right)\wedge 2\varrho$ than that of [19, Theorem 3.9], where the corresponding convergence order is*

  $$\left(1-\frac{l}{p_0}\right)\wedge 2\varrho\wedge 2\varepsilon(p_0-l-2)/(2+l),\ \text{for some } \varepsilon\in(0,1/4],$$

  *see Example 6.1 for illustration.*

**Remark 3.2.** *When the delay term vanishes, i.e., $\delta(t)\equiv 0$, our convergence results reduce to that for SDEs with Poisson jumps. Under this situation, it is worth mentioning how our work compares with that of [9], who proposed a new version of the tamed EM for such SDEs. The differences between Theorem 3.1 and [9, Theorem 4.5] are listed below.*

- *Theorem 3.1 requires a weaker condition on Khasminskii parameter $p_0$, namely, $p_0\ge 4\vee(2+2.5l)$, while [9, Theorem 4.5] requires $p_0$ to be an even number and $p_0\ge(6+4l)\vee(2+6l)$;*
- *In mean-square sense, our truncated EM method has a better convergence order $1-\frac{l}{p_0}$ than that of [9, Theorem 4.5], where the corresponding convergence order is $1-2\frac{l}{p_0}$, see Example 6.2 for illustration.*

When $h\equiv 0$, Theorem 3.1 reduces to the following convergence theorem of the truncated EM method for non-jump SDEs with time-variable delay. In this case, the optimal rate of strong convergence can be recovered.

**Theorem 3.2 (Convergence Rate for Non-jump SDDEs)** *Assume that $h(x,y)\equiv 0,\ \forall x,y\in\mathbb{R}^d$. Let Assumptions 2.1, 2.2, 3.1 and 3.2 hold with $p_0\geq 4\vee(2+2.5l)$ and $\lambda=0$. Then for any $\Delta\in(0,1]$,*

$$\mathbb{E}|X(T)-Y_\Delta(T)|^2\leq C\left(\Delta^{1\wedge 2\varrho}\vee[\mu^{-1}(\varphi(\Delta))]^{-(p_0-l-2)}\right),\ \forall T>0, \tag{3.17}$$

*and*

$$\mathbb{E}|X(T)-Z_1(T)|^2\leq C\left(\Delta^{1\wedge 2\varrho}\vee[\mu^{-1}(\varphi(\Delta))]^{-(p_0-l-2)}\right),\ \forall T>0, \tag{3.18}$$

*where $C$ is a positive constant independent of $\Delta$. In particular, let*

$$\mu(r)=\bar{K}_3r^{l/2},\ \forall r\geq 1\quad\text{and}\quad\varphi(\Delta)=(\bar{K}_3\vee 1)\Delta^{-1/3},\ \forall\Delta\in(0,1]. \tag{3.19}$$

*Then for any $\Delta\in(0,1]$,*

$$\mathbb{E}|X(T)-Y_\Delta(T)|^2\leq C\Delta^{1\wedge 2\varrho}\quad\text{and}\quad\mathbb{E}|X(T)-Z_1(T)|^2\leq C\Delta^{1\wedge 2\varrho},\ \forall T>0. \tag{3.20}$$

In order to show the convergence order, we need some lemmas.

**Lemma 3.2.** *Let Assumption 3.1 hold. Then for any $\Delta\in(0,1]$,*

$$\begin{aligned}&p_0|x|^{p_0-2}\Big(\langle x,f_\Delta(x,y)\rangle+\frac{p_0-1}{2}|g_\Delta(x,y)|^2\Big)+\lambda(1+\varepsilon)|h_\Delta(x,y)|^{p_0}\\&\leq\widetilde{K}_1(1+|x|^{p_0}+|y|^{p_0}),\ \forall x,y\in\mathbb{R}^d,\end{aligned} \tag{3.21}$$

*where $\widetilde{K}_1=2\bar{K}_1\left(1\vee[1/\mu^{-1}(\varphi(1))]^{p_0-1}\right)$.*

We left the proof to the readers.

**Remark 3.3.** *By (3.8), we see from (2.5) that for a given step size $\Delta\in(0,1]$, any $p\geq 2$ and any $k\geq 1$,*

$$\mathbb{E}|y_k|^p\leq C_{p,\|\xi\|,\hat{L},\Delta}.$$

*Moreover, this and (2.11) guarantee that for a given step size $\Delta\in(0,1]$ and any $p\geq 2$,*

$$\mathbb{E}|Y_\Delta(t)|^p<\infty,\ \forall t\geq 0.$$

*However, we can not conclude that this bound is independent of $\Delta$. As a result of this observation, we need not apply stopping time arguments in the proof Lemma 3.5.*

For notational convenience, we write $Z_1(t)$ and $Z_2(t)$ instead of $Z_1(t^-)$ and $Z_2(t^-)$, respectively. This does not cause any problem since the compensators of the martingales are continuous (see, e.g., [12, Remark 2.1]).

**Lemma 3.3.** *Let (3.8) hold. Then*

$$\mathbb{E}\left[|Y_\Delta(t)-Z_1(t)|^p|\mathcal{F}_{\kappa(t)}\right]\leq C\varphi(\Delta)\Delta(1+|Z_1(t))|^p+|Z_2(t))|^p,\ 2\leq p\leq p_0, \tag{3.22}$$

$$\mathbb{E}\left[|Y_\Delta(t)-Z_1(t)|^p|\mathcal{F}_{\kappa(t)}\right]\leq C(\varphi(\Delta)\Delta)^{p/2}(1+|Z_1(t))|^p+|Z_2(t))|^p,\ 1\leq p<2, \tag{3.23}$$

*where $C$ is a positive constant independent of $\Delta$.*

*Proof.* For any $p \in [2, p_0]$, by (3.8), we observe from (2.11) that for any $t \geq 0$

$$
\begin{aligned}
&\mathbb{E}\Big[|Y_\Delta(t) - Z_1(t)|^p|\mathcal{F}_{\kappa(t)}\Big] \\
&= \mathbb{E}\left[\Big|\int_{\kappa(t)}^t f_\Delta(Z_1(s), Z_2(s))ds + \int_{\kappa(t)}^t g_\Delta(Z_1(s), Z_2(s))dB(s) + \int_{\kappa(t)}^t h_\Delta(Z_1(s^-), Z_2(s^-))dN(s)\Big|^p\Big|\mathcal{F}_{\kappa(t)}\right] \\
&\leq C\mathbb{E}\left[|f_\Delta(Z_1(t), Z_2(t))(t-\kappa(t))|^p|\mathcal{F}_{\kappa(t)}\right] + C\mathbb{E}\left[|g_\Delta(Z_1(t), Z_2(t))(B(t) - B(\kappa(t)))|^p|\mathcal{F}_{\kappa(t)}\right] \\
&\quad + C\mathbb{E}\left[|h_\Delta(Z_1(t), Z_2(t))(N(t) - N(\kappa(t)))|^p|\mathcal{F}_{\kappa(t)}\right] \\
&\leq C\Big(\Delta^p(\varphi(\Delta))^p + \Delta^{p/2}(\varphi(\Delta))^{p/2} + \Delta(\varphi(\Delta))^{p/p_0}\Big)(1 + |Z_1(t)|^p + |Z_2(t)|^p) \\
&\leq C\varphi(\Delta)\Delta(1 + |Z_1(t)|^p + |Z_2(t)|^p),
\end{aligned}
$$

which gives (3.22). Thus, (3.23) follows from (3.22) and the Hölder inequality. □

**Lemma 3.4.** *Let $p > 1$, $\hat{\eta} > 0$ and $x, y \in \mathbb{R}$. Then*

$$
|x + y|^p - |y|^p \leq (1 + \hat{\eta})|x|^p + K(\hat{\eta})|y|^p, \tag{3.24}
$$

*where $K(\hat{\eta})$ is a positive constant depending on $\hat{\eta}$.*

*Proof.* Let $p > 1$, $\varepsilon_1 > 0$, $\hat{\eta} > 0$ and $x, y \in \mathbb{R}$. By [31, Lemma 4.1, p.211], we have

$$
|x + y|^p \leq \left[1 + \varepsilon_1^{\frac{1}{p-1}}\right]^{p-1}\left(|x|^p + \frac{|y|^p}{\varepsilon_1}\right).
$$

Letting $\left[1 + \varepsilon_1^{\frac{1}{p-1}}\right]^{p-1} = 1 + \hat{\eta}$ gives (3.24). □

**Lemma 3.5.** *Let Assumptions 2.2, 2.3, 3.1 hold. Then*

$$
\sup_{0<\Delta\leq 1}\sup_{0\leq t\leq T}\mathbb{E}|Y_\Delta(t)|^{p_0} \leq C, \ \forall T > 0. \tag{3.25}
$$

*where $C$ is a positive constant independent of $\Delta$.*

*Proof.* By the Itô formula and Lemma 3.2 as well as Lemma 3.4, we have that for any $t \in [0, T]$,

$$
\begin{aligned}
\mathbb{E}|Y_\Delta(t)|^{p_0} - |y_0|^{p_0} &\leq p_0\mathbb{E}\int_0^t |Y_\Delta(s)|^{p_0-2}\Big(\langle Y_\Delta(s), f_\Delta(Z_1(s), Z_2(s))\rangle + \frac{p_0-1}{2}|g_\Delta(Z_1(s), Z_2(s))|^2\Big)ds \\
&\quad + \lambda\mathbb{E}\int_0^t \Big(|Y_\Delta(s) - h_\Delta(Z_1(s), Z_2(s))|^{p_0} - |Y_\Delta(s)|^{p_0}\Big)ds \\
&\leq p_0\mathbb{E}\int_0^t |Y_\Delta(s)|^{p_0-2}\Big(\langle Z_1(s), f_\Delta(Z_1(s), Z_2(s))\rangle + \frac{p_0-1}{2}|g_\Delta(Z_1(s), Z_2(s))|^2\Big)ds \\
&\quad + p_0\mathbb{E}\int_0^t |Y_\Delta(s)|^{p_0-2}\langle Y_\Delta(s) - Z_1(s), f_\Delta(Z_1(s), Z_2(s))\rangle ds \\
&\quad + C\mathbb{E}\int_0^t |Y_\Delta(s)|^{p_0}ds + \lambda(1+\varepsilon)\mathbb{E}\int_0^t |h_\Delta(Z_1(s), Z_2(s))|^{p_0}ds \\
&\leq p_0\mathbb{E}\int_0^t |Z_1(s)|^{p_0-2}\Big(\langle Z_1(s), f_\Delta(Z_1(s), Z_2(s))\rangle + \frac{p_0-1}{2}|g_\Delta(Z_1(s), Z_2(s))|^2\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda(1+\varepsilon)\mathbb{E}\int_0^t |h_\Delta(Z_1(s),Z_2(s))|^{p_0}ds + C\mathbb{E}\int_0^t |Y_\Delta(s)|^{p_0}ds + I_1 + I_2\\
&\le \widetilde{K}_1\mathbb{E}\int_0^t \Big(1+|Z_1(s)|^{p_0}+|Z_2(s)|^{p_0}\Big)ds + C\mathbb{E}\int_0^t |Y_\Delta(s)|^{p_0}ds + I_1 + I_2,
\end{aligned}
\tag{3.26}
$$

where

$$
I_1 := p_0\mathbb{E}\int_0^t |Y_\Delta(s)|^{p_0-2}\langle Y_\Delta(s)-Z_1(s), f_\Delta(Z_1(s),Z_2(s))\rangle ds
$$

and

$$
I_2 := p_0\mathbb{E}\int_0^t \Big(|Y_\Delta(s)|^{p_0-2}-|Z_1(s)|^{p_0-2}\Big)\Big(\langle Z_1(s), f_\Delta(Z_1(s),Z_2(s))\rangle + \frac{p_0-1}{2}|g_\Delta(Z_1(s),Z_2(s))|^2\Big)ds.
$$

We observe that

$$
\begin{aligned}
I_1 &\le p_0\mathbb{E}\int_0^t |Y_\Delta(s)|^{p_0-2}|Y_\Delta(s)-Z_1(s)||f_\Delta(Z_1(s),Z_2(s))|ds\\
&\le C\mathbb{E}\int_0^t |Z_1(s)|^{p_0-2}|Y_\Delta(s)-Z_1(s)||f_\Delta(Z_1(s),Z_2(s))|ds\\
&\quad + C\mathbb{E}\int_0^t |Y_\Delta(s)-Z_1(s)|^{p_0-2}|Y_\Delta(s)-Z_1(s)||f_\Delta(Z_1(s),Z_2(s))|ds\\
&=: I_{11}+I_{12}.
\end{aligned}
$$

By (3.8), Lemma 3.3 and the condition

$$
(\varphi(\Delta))^3\Delta \le \hat{L},
\tag{3.27}
$$

we have

$$
\begin{aligned}
I_{11} &= C\mathbb{E}\int_0^t |Z_1(s)|^{p_0-2}|Y_\Delta(s)-Z_1(s)||f_\Delta(Z_1(s),Z_2(s))|ds\\
&= C\int_0^t \mathbb{E}\Big[|Z_1(s)|^{p_0-2}|f_\Delta(Z_1(s),Z_2(s))|\mathbb{E}\Big(|Y_\Delta(s)-Z_1(s)||\mathcal{F}_{\kappa(s)}\Big)\Big]ds\\
&\le C\varphi(\Delta)[\varphi(\Delta)\Delta]^{1/2}\int_0^t \mathbb{E}\Big[|Z_1(s)|^{p_0-2}(1+|Z_1(s)|^2+|Z_2(s)|^2)\Big]ds\\
&\le C[\varphi(\Delta)]^{3/2}\Delta^{1/2}\int_0^t (1+\mathbb{E}|Z_1(s)|^{p_0}+\mathbb{E}|Z_2(s)|^{p_0})ds\\
&\le C\int_0^t (1+\mathbb{E}|Z_1(s)|^{p_0}+\mathbb{E}|Z_2(s)|^{p_0})ds.
\end{aligned}
\tag{3.28}
$$

For some $p_0 \ge 4$, Lemma 3.3 gives that

$$
\mathbb{E}\Big[|Y_\Delta(s)-Z_1(s)|^{p_0-1}|\mathcal{F}_{\kappa(s)}\Big] \le C\varphi(\Delta)\Delta(1+|Z_1(s)|^{p_0-1}+|Z_2(s)|^{p_0-1}).
$$

Thus, we have the following estimate

$$\begin{aligned} I_{12} &= C\mathbb{E}\int_0^t |Y_\Delta(s)-Z_1(s)|^{p_0-1}|f_\Delta(Z_1(s),Z_2(s))|ds \\ &\le C\varphi(\Delta)\int_0^t \mathbb{E}\Big[|Y_\Delta(s)-Z_1(s)|^{p_0-1}(1+|Z_1(s)|+|Z_2(s)|)\Big]ds \\ &\le C[\varphi(\Delta)]^2\Delta\int_0^t \mathbb{E}(1+|Z_1(s)|^{p_0}+|Z_2(s)|^{p_0})ds \\ &\le C\int_0^t (1+\mathbb{E}|Z_1(s)|^{p_0}+\mathbb{E}|Z_2(s)|^{p_0})ds. \end{aligned} \tag{3.29}$$

Thus, by the Taylor formula with integral remainder term and the elementary inequality, we have

$$\begin{aligned} &|y|^{p_0-2}-|\bar{y}|^{p_0-2} \le (p_0-2)\int_0^1 |\bar{y}+s(y-\bar{y})|^{p_0-3}|y-\bar{y}|ds \\ &\le C\Big(|\bar{y}|^{p_0-3}+|y-\bar{y}|^{p_0-3}\Big)|y-\bar{y}| = C\Big(|\bar{y}|^{p_0-3}|y-\bar{y}|+|y-\bar{y}|^{p_0-2}\Big),\ \forall y,\bar{y}\in\mathbb{R}^d. \end{aligned} \tag{3.30}$$

Moreover, by (3.8), we have

$$\langle x, f_\Delta(x,y)\rangle + \frac{p_0-1}{2}|g_\Delta(x,y)|^2 \le C\varphi(\Delta)(1+|x|^2+|y|^2),\ \forall x\in\mathbb{R}^d. \tag{3.31}$$

Therefore, by (3.30) and (3.31) as well as (3.8), we get that

$$\begin{aligned} I_2 &= p_0\mathbb{E}\int_0^t \Big(|Y_\Delta(s)|^{p_0-2}-|Z_1(s)|^{p_0-2}\Big)\Big(\langle Z_1(s), f_\Delta(Z_1(s),Z_2(s))\rangle + \frac{p_0-1}{2}|g_\Delta(Z_1(s),Z_2(s))|^2\Big)ds \\ &\le C\int_0^t \mathbb{E}\Big[\Big(|Z_1(s)|^{p_0-3}|Y_\Delta(s)-Z_1(s)|+|Y_\Delta(s)-Z_1(s)|^{p_0-2}\Big)\varphi(\Delta)(1+|Z_1(s)|^2+|Z_2(s)|^2)\Big]ds \\ &\le C\varphi(\Delta)\int_0^t \mathbb{E}\Big[|Z_1(s)|^{p_0-3}(1+|Z_1(s)|^2+|Z_2(s)|^2)\mathbb{E}[|Y_\Delta(s)-Z_1(s)||\mathcal{F}_{\kappa(s)}]\Big]ds \\ &\quad + C\varphi(\Delta)\int_0^t \mathbb{E}\Big[(1+|Z_1(s)|^2+|Z_2(s)|^2)\mathbb{E}[|Y_\Delta(s)-Z_1(s)|^{p_0-2}|\mathcal{F}_{\kappa(s)}]\Big]ds \\ &\le C[\varphi(\Delta)]^{3/2}\Delta^{1/2}\int_0^t (1+\mathbb{E}|Z_1(s)|^{p_0}+\mathbb{E}|Z_2(s)|^{p_0})ds \\ &\quad + C[\varphi(\Delta)]^2\Delta\int_0^t (1+\mathbb{E}|Z_1(s)|^{p_0}+\mathbb{E}|Z_2(s)|^{p_0})ds \\ &\le C\int_0^t (1+\mathbb{E}|Z_1(s)|^{p_0}+\mathbb{E}|Z_2(s)|^{p_0})ds. \end{aligned} \tag{3.32}$$

Note that

$$\begin{aligned} &\mathbb{E}|Z_1(s)|^{p_0} \le \sup_{0\le u\le s}\mathbb{E}|Z_1(u)|^{p_0} \le \sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)|^{p_0}, \\ &\mathbb{E}|Z_2(s)|^{p_0} \le \sup_{0\le u\le s}\mathbb{E}|Z_2(u)|^{p_0} \le \|\xi\|^{p_0} + \sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)|^{p_0}. \end{aligned} \tag{3.33}$$

Thus, from (3.26), (3.28), (3.29) to (3.32), we observe that

$$\mathbb{E}|Y_\Delta(t)|^{p_0} \le C\int_0^t (1+\mathbb{E}|Y_\Delta(s)|^{p_0}+\mathbb{E}|Z_1(s)|^{p_0}+\mathbb{E}|Z_2(s)|^{p_0})ds \le C + C\int_0^t \sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)|^{p_0}ds.$$

Thus,

$$\sup_{0\le u\le t}\mathbb{E}|Y_\Delta(u)|^{p_0}\le C+C\int_0^t\sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)|^{p_0}ds.$$

By the Gronwall inequality, we obtain the desired assertion. □

**Lemma 3.6.** *Let Assumptions 2.1, 2.2, 3.1 and 3.2 hold with $p_0\ge 4\vee(2+2.5l)$. Then for any $\Delta\in(0,1]$ and any $p\in\left[2,\dfrac{p_0}{1+l/2}\right]$,*

$$\mathbb{E}|Y_\Delta(t)-Z_1(t)|^p\le C\Delta,\ \forall t\ge 0, \tag{3.34}$$

*and*

$$\mathbb{E}|Y_\Delta(t-\delta(t))-Z_2(t)|^p\le C\Delta^{1\wedge\varrho p},\ \forall t\ge 0. \tag{3.35}$$

*Moreover,*

$$\begin{aligned}&\mathbb{E}\int_0^T|f(Y_\Delta(s)-f_\Delta(Y_\Delta(s))|^2ds\le C[\mu^{-1}(\varphi(\Delta))]^{-(p_0-l-2)},\\&\mathbb{E}\int_0^T\Big(|g(Y_\Delta(s)-g_\Delta(Y_\Delta(s))|^2+|h(Y_\Delta(s)-h_\Delta(Y_\Delta(s))|^2\Big)ds\le C[\mu^{-1}(\varphi(\Delta))]^{-(p_0-l/2-2)},\end{aligned} \tag{3.36}$$

*where $C$ is a positive constant independent of $\Delta$.*

*Proof.* Let $p\in\left[2,\dfrac{p_0}{1+l/2}\right]$. For any $t\in[t_k,t_{k+1})$ with $k\ge 0$, we see from (2.11) that

$$\begin{aligned}&\mathbb{E}|Y_\Delta(t)-Z_1(t)|^p=\mathbb{E}|Y_\Delta(t)-Y_\Delta(t_k)|^p\\&\le C\left(\Delta^p\mathbb{E}|f_\Delta(y_k,y_{k-\delta_k})|^p+\Delta^{0.5p}\mathbb{E}|g_\Delta(y_k,y_{k-\delta_k})|^p+\Delta\mathbb{E}|h_\Delta(y_k,y_{k-\delta_k})|^p\right).\end{aligned}$$

By (3.11), we have

$$\mathbb{E}|f_\Delta(y_k,y_{k-\delta_k})|^p\le C\left(1+\mathbb{E}|y_k|^{(1+l/2)p}+\mathbb{E}|y_{k-\delta_k}|^{(1+l/2)p}\right)\le C, \tag{3.37}$$

where Lemma 3.1 has been used. Similarly, we can show that

$$\mathbb{E}|g_\Delta(y_k,y_{k-\delta_k})|^p\vee\mathbb{E}|h_\Delta(y_k,y_{k-\delta_k})|^p\le C. \tag{3.38}$$

Thus,

$$\mathbb{E}|Y_\Delta(t)-Z_1(t)|^p\le C\Delta^p+C\Delta^{0.5p}+C\Delta\le C\Delta.$$

We now begin to establish assertion (3.35). Recall that $\delta_k=\lfloor\delta(t_k)/\Delta\rfloor$ and

$$Y_\Delta(t-\delta(t))-Z_2(t)=Y_\Delta(t-\delta(t))-Y_\Delta((k-\delta_k)\Delta). \tag{3.39}$$

By Assumption 2.2, we have the following useful estimate

$$|(t-\delta(t))-(k-\delta_k)\Delta|\le(\lfloor\hat{\delta}\rfloor+4)\Delta, \tag{3.40}$$

see [16, Lemma 4.6].
Now consider the following four possible cases.

**Case 1:** If $t-\delta(t)\geq(k-\delta_k)\Delta\geq 0$ or $(k-\delta_k)\Delta\geq t-\delta(t)\geq 0$, then it follows from (3.39) that

$$\begin{aligned}
&\mathbb{E}|Y_\Delta(t-\delta(t))-Z_2(t)|^p\\
&=\mathbb{E}\Big|\int_{(k-\delta_k)\Delta}^{t-\delta(t)} f_\Delta(Z_1(s),Z_2(s))ds+\int_{(k-\delta_k)\Delta}^{t-\delta(t)} g_\Delta(Z_1(s),Z_2(s))dB(s)\\
&\quad+\int_{(k-\delta_k)\Delta}^{t-\delta(t)} h_\Delta(Z_1(s^-),Z_2(s^-))dN(s)\Big|^p\\
&\leq C|t-\delta(t)-(k-\delta_k)\Delta|^{p-1}\Big|\int_{(k-\delta_k)\Delta}^{t-\delta(t)}\mathbb{E}|f_\Delta(Z_1(s),Z_2(s))|^p ds\Big|\\
&\quad+C|t-\delta(t)-(k-\delta_k)\Delta|^{0.5p-1}\Big|\int_{(k-\delta_k)\Delta}^{t-\delta(t)}\mathbb{E}|g_\Delta(Z_1(s),Z_2(s))|^p ds\Big|\\
&\quad+C\mathbb{E}\Big|\int_{(k-\delta_k)\Delta}^{t-\delta(t)} h_\Delta(Z_1(s^-),Z_2(s^-))dN(s)\Big|^p\\
&\leq C|t-\delta(t)-(k-\delta_k)\Delta|^{0.5p}+C\mathbb{E}\Big|\int_{(k-\delta_k)\Delta}^{t-\delta(t)} h_\Delta(Z_1(s^-),Z_2(s^-))dN(s)\Big|^p\\
&\leq C\Delta,
\end{aligned}\tag{3.41}$$

where the Burkholder-Davis-Gundy inequality, (3.37) (3.38) and (3.40) have been used.

**Case 2:** If $t-\delta(t)\leq(k-\delta_k)\Delta\leq 0$ or $(k-\delta_k)\Delta\leq t-\delta(t)\leq 0$, by Assumption 2.1 and (3.40) we have

$$\begin{aligned}
&\mathbb{E}|Y_\Delta(t-\delta(t))-Z_2(t)|^p=|\xi(t-\delta(t))-\xi((k-\delta_k)\Delta)|^p\\
&\leq\Big(K_0^p(\lfloor\hat{\delta}\rfloor+4)^{\varrho p}\Big)\Delta^{\varrho p}.
\end{aligned}\tag{3.42}$$

**Case 3:** If $t-\delta(t)\geq 0\geq(k-\delta_k)\Delta$, then

$$t-\delta(t)\leq(\lfloor\hat{\delta}\rfloor+4)\Delta\quad\text{and}\quad-(k-\delta_k)\Delta\leq(\lfloor\hat{\delta}\rfloor+4)\Delta.$$

Thus, we have

$$\begin{aligned}
\mathbb{E}|Y_\Delta(t-\delta(t))-Z_2(t)|^p&=\mathbb{E}|Y_\Delta(t-\delta(t))-Y_\Delta((k-\delta_k)\Delta)|^p\\
&\leq 2^{p-1}\mathbb{E}|Y_\Delta(t-\delta(t))-\xi(0)|^p+2^{p-1}|\xi(0)-\xi((k-\delta_k)\Delta)|^p\\
&\leq C\Delta^{1\wedge\varrho p}.
\end{aligned}\tag{3.43}$$

**Case 4:** If $(k-\delta_k)\Delta\geq 0\geq t-\delta(t)$, in a similar way as (3.43) is obtained, we also have

$$\mathbb{E}|Y_\Delta(t-\delta(t))-Z_2(t)|^p\leq C\Delta^{1\wedge\varrho p}.\tag{3.44}$$

Combining these different cases together, we obtain the desired assertion (3.35). Noting that $p_0\geq 2+2.5l>2+l$, in a similar way as [17, (3.15)] was obtained, we also can show that (3.36) holds. □

**Proof of Theorem 3.1.** Let $p_0\geq 2+2.5l$. Set $R\geq\|\xi\|$ and define the stopping time

$$\rho_R=\inf\{t\geq 0:|X(t)|\wedge|Y_\Delta(t)|\geq R\}.$$

Set $e_\Delta(t) = X(t) - Y_\Delta(t)$, for any $t \in [-\tau, T]$, which means that $e_\Delta(t) = 0$, for any $t \in [-\tau, 0]$. By the Itô formula and the elementary inequality, we have that for any $t \in [0, T]$,

$$
\begin{aligned}
&\mathbb{E}|e_\Delta(t\wedge\rho_R)|^2\\
&= \mathbb{E}\int_0^{t\wedge\rho_R}\Big(2\langle X(s)-Y_\Delta(s), f(X(s),X(s-\delta(s)))-f_\Delta(Z_1(s),Z_2(s))\rangle\\
&\quad + |g(X(s),X(s-\delta(s)))-g_\Delta(Z_1(s),Z_2(s))|^2+\lambda|h(X(s),X(s-\delta(s)))-h_\Delta(Z_1(s),Z_2(s))|^2\Big)ds\\
&\le \mathbb{E}\int_0^{t\wedge\rho_R}\Big(2\langle X(s)-Y_\Delta(s), f(X(s),X(s-\delta(s)))-f(Y_\Delta(s),Y_\Delta(s-\delta(s)))\rangle\\
&\quad + p_1|g(X(s),X(s-\delta(s)))-g(Y_\Delta(s),Y_\Delta(s-\delta(s)))|^2\\
&\quad + p_1\lambda|h(X(s),X(s-\delta(s)))-h(Y_\Delta(s),Y_\Delta(s-\delta(s)))|^2\Big)ds\\
&\quad + \mathbb{E}\int_0^{t\wedge\rho_R}\Big(2\langle X(s)-Y_\Delta(s), f(Y_\Delta(s),Y_\Delta(s-\delta(s)))-f_\Delta(Z_1(s),Z_2(s))\rangle+|X(s)-Y_\Delta(s)|^2\\
&\quad + C|g(Y_\Delta(s),Y_\Delta(s-\delta(s)))-g_\Delta(Z_1(s),Z_2(s))|^2+C|h(Y_\Delta(s),Y_\Delta(s-\delta(s)))-h_\Delta(Z_1(s),Z_2(s))|^2\Big)ds.
\end{aligned}
$$

By Assumption 3.2 and the elmemwntary inequality, we have

$$
\begin{aligned}
&\mathbb{E}|e_\Delta(t\wedge\rho_R)|^2\\
&\le C\mathbb{E}\int_0^{t\wedge\rho_R}\Big(|X(s)-Y_\Delta(s)|^2+|X(s-\delta(s))-Y_\Delta(s-\delta(s))|^2\Big)ds\\
&\quad + \mathbb{E}\int_0^{t\wedge\rho_R}\Big(-\frac{1}{1-\hat{\delta}}U(X(s),Y_\Delta(s))+U(X(s-\delta(s)),Y_\Delta(s-\delta(s)))\Big)ds\\
&\quad + C\mathbb{E}\int_0^{T}\Big(|f(Y_\Delta(s),Y_\Delta(s-\delta(s)))-f_\Delta(Z_1(s),Z_2(s))|^2\\
&\quad + |g(Y_\Delta(s),Y_\Delta(s-\delta(s)))-g_\Delta(Z_1(s),Z_2(s))|^2\\
&\quad + |h(Y_\Delta(s),Y_\Delta(s-\delta(s)))-h_\Delta(Z_1(s),Z_2(s))|^2\Big)ds\\
&\le C\mathbb{E}\int_0^{t\wedge\rho_R}\Big(|X(s)-Y_\Delta(s)|^2+|X(s-\delta(s))-Y_\Delta(s-\delta(s))|^2\Big)ds+\Pi_1+\Pi_2+\Pi_3, \qquad (3.45)
\end{aligned}
$$

where

$$
\begin{aligned}
\Pi_1 = C\mathbb{E}\int_0^T &|f(Y_\Delta(s),Y_\Delta(s-\delta(s)))-f_\Delta(Y_\Delta(s),Y_\Delta(s-\delta(s)))|^2ds\\
&+ C\mathbb{E}\int_0^T |g(Y_\Delta(s),Y_\Delta(s-\delta(s)))-g_\Delta(Y_\Delta(s),Y_\Delta(s-\delta(s)))|^2ds\\
&+ C\mathbb{E}\int_0^T |h(Y_\Delta(s),Y_\Delta(s-\delta(s)))-h_\Delta(Y_\Delta(s),Y_\Delta(s-\delta(s)))|^2ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi_2 = C\mathbb{E}\int_0^T &|f_\Delta(Y_\Delta(s),Y_\Delta(s-\delta(s)))-f_\Delta(Z_1(s),Z_2(s))|^2ds\\
&+ C\mathbb{E}\int_0^T |g_\Delta(Y_\Delta(s),Y_\Delta(s-\delta(s)))-g_\Delta(Z_1(s),Z_2(s))|^2ds\\
&+ C\mathbb{E}\int_0^T |h_\Delta(Y_\Delta(s),Y_\Delta(s-\delta(s)))-h_\Delta(Z_1(s),Z_2(s))|^2ds,
\end{aligned}
$$

as well as

$$\Pi_3 := \mathbb{E}\int_0^{t\wedge\rho_R}\Big(-\frac{1}{1-\hat{\delta}}U(X(s),Y_\Delta(s))+U(X(s-\delta(s)),Y_\Delta(s-\delta(s)))\Big)ds.$$

Noticing that $U(X(s),Y_\Delta(s))=0$ for any $s\in[-\tau,0]$, we then have

$$\int_0^{t\wedge\rho_R}U(X(s-\delta(s)),Y_\Delta(s-\delta(s)))ds\le\frac{1}{1-\hat{\delta}}\int_0^{t\wedge\rho_R}U(X(s),Y_\Delta(s))ds$$

and

$$\int_0^{t\wedge\rho_R}|X(s-\delta(s))-Y_\Delta(s-\delta(s))|^2ds\le\frac{1}{1-\hat{\delta}}\int_0^{t\wedge\rho_R}|X(s)-Y_\Delta(s)|^2ds.$$

Consequently, $\Pi_3\le 0$ and thus

$$\mathbb{E}|e_\Delta(t\wedge\rho_R)|^2\le C\mathbb{E}\int_0^{t\wedge\rho_R}|e_\Delta(s)|^2ds+\Pi_1+\Pi_2\le C\int_0^t\mathbb{E}|e_\Delta(s\wedge\rho_R)|^2ds+\Pi_1+\Pi_2. \tag{3.46}$$

Noting that $p_0\ge 2+2.5l>2+l$ and using Lemma 3.6, we have

$$\Pi_1\le C[\mu^{-1}(\varphi(\Delta))]^{-(p_0-l-2)}. \tag{3.47}$$

Recall [17, Lemma 3.3] that

$$|f_\Delta(x_1,y_1)-f_\Delta(x_2,y_2)|^2\le\bar{K}_2(1+|x_1|^l+|x_2|^l+|y_1|^l+|y_2|^l)(|x_1-x_2|^2+|y_1-y_2|^2),$$

for any $x_1,y_1,x_2,y_2\in\mathbb{R}^d$. By the condition that $p_0\ge 2+2.5l$ which implies $\dfrac{2p_0}{p_0-l}\le\dfrac{p_0}{1+l/2}$, we have that for $s\in[0,T]$,

$$\begin{aligned}
&\mathbb{E}|f_\Delta(Y_\Delta(s),Y_\Delta(s-\delta(s)))-f_\Delta(Z_1(s),Z_2(s))|^2\\
&\le\bar{K}_2\mathbb{E}\Big(\big[|Y_\Delta(s)-Z_1(s)|^2+|Y_\Delta(s-\delta(s))-Z_2(s)|^2\big]\\
&\quad\times\big[1+|Y_\Delta(s)|^l+|Y_\Delta(s-\delta(s))|^l+|Z_1(s)|^l+|Z_2(s)|^l\big]\Big)\\
&\le C\Big(\mathbb{E}|Y_\Delta(s)-Z_1(s)|^{2p_0/(p_0-l)}+\mathbb{E}|Y_\Delta(s-\delta(s))-Z_2(s)|^{2p_0/(p_0-l)}\Big)^{(p_0-l)/p_0}\\
&\quad\times(1+\mathbb{E}|Y_\Delta(s)|^{p_0}+\mathbb{E}|Y_\Delta(s-\delta(s))|^{p_0}+\mathbb{E}|Z_1(s)|^{p_0}+\mathbb{E}|Z_2(s)|^{p_0})^{l/p_0}\\
&\le C\Big(\mathbb{E}|Y_\Delta(s)-Z_1(s)|^{2p_0/(p_0-l)}\Big)^{(p_0-l)/p_0}+C\Big(\mathbb{E}|Y_\Delta(s-\delta(s))-Z_2(s)|^{2p_0/(p_0-l)}\Big)^{(p_0-l)/p_0}\\
&\le C\Delta^{\frac{p_0-l}{p_0}\wedge 2\varrho},
\end{aligned} \tag{3.48}$$

where (3.2) and the Hölder inequality as well as Lemma 3.6 have been used. Similarly, we can show that

$$\begin{aligned}
\mathbb{E}|g_\Delta(Y_\Delta(s),Y_\Delta(s-\delta(s)))-g_\Delta(Z_1(s),Z_2(s))|^2&\le C\Delta^{\frac{p_0-l}{p_0}\wedge 2\varrho},\\
\mathbb{E}|h_\Delta(Y_\Delta(s),Y_\Delta(s-\delta(s)))-h_\Delta(Z_1(s),Z_2(s))|^2&\le C\Delta^{\frac{p_0-l}{p_0}\wedge 2\varrho}.
\end{aligned} \tag{3.49}$$

Combining (3.45)-(3.49), we get

$$\mathbb{E}|e_\Delta(t\wedge\rho_R)|^2 \le C\int_0^t \mathbb{E}|e_\Delta(s\wedge\rho_R)|^2 ds + C\left(\Delta^{2\varrho\wedge 1}\vee[\mu^{-1}(\varphi(\Delta))]^{-(p_0-l-2)}\right). \tag{3.50}$$

The Gronwall inequality gives

$$\mathbb{E}|e_\Delta(t\wedge\rho_R)|^2 \le C\left(\Delta^{\frac{p_0-l}{p_0}\wedge 2\varrho}\vee[\mu^{-1}(\varphi(\Delta))]^{-(p_0-l-2)}\right). \tag{3.51}$$

Letting $R\to\infty$ gives assertion (3.17). While (3.18) follows from (3.17) and Lemma 3.6. Finally, from (3.10) and (3.4), we may define $\mu(R)$ and $\varphi(\Delta)$ by (3.19), e.g.,

$$\mu(r)=\bar{K}_3 r^{l/2},\ \forall r\ge 1 \quad\text{and}\quad \varphi(\Delta)=(\bar{K}_3\vee 1)\Delta^{-1/3},\ \forall\Delta\in(0,1].$$

Then

$$[\mu^{-1}(\varphi(\Delta))]^{-(p_0-l-2)} = C\Delta^{2(p_0-l-2)/(3l)} \le C\Delta, \tag{3.52}$$

due to $p_0\ge 2+2.5l$, which implies that $\frac{2(p_0-l-2)}{3l}\ge 1$. From (3.52) and (3.17) as well as (3.18), we obtain the assertion (3.20). Thus, the proof is complete. □

**Proof of Theorem 3.2.** When jump term is absent in (2.1), we set $h(x,y)\equiv 0,\ \forall x,y\in\mathbb{R}^d$. Let Assumptions 2.1, 2.2, 3.1 and 3.2 hold with $p_0\ge 4\vee(2+2.5l)$ and $\lambda=0$. Then (3.34) and (3.35) will become

$$\mathbb{E}|Y_\Delta(t)-Z_1(t)|^p \le C\Delta^{0.5p},\ \forall t\ge 0, \tag{3.53}$$

and

$$\mathbb{E}|Y_\Delta(t-\delta(t))-Z_2(t)|^p \le C\Delta^{(0.5\wedge\varrho)p},\ \forall t\ge 0, \tag{3.54}$$

respectively. In a similar way as Theoem 3.1 is proved, we conclude that Theorem 3.2 holds. □

## 4. Convergence in $\mathcal{L}^q$ for $0<q<2$

Under certain circumstance, such as discussing stability in distribution of numerical approximation, we only need the convergence in small moment of the numerical method rather than convergence in mean-square sense. In this situation, Assumptions 3.1 and 3.2 required in Theorem 3.1 can be replaced by weaker Assumptions 2.3 and 4.1. Noting that under generalized Khasminskii-type condition, i.e., Assumption 4.1, we only obtain the boundedness of the numerical solution in second moment, see Lemma 4.3. Thus we can establish the convergence (without order) in $q$th ($q<2$) moment, see Theorem 4.1.

**Assumption 4.1 (Generalized Khasminskii-type Condition)** *There exist constants $K_1>0$, $K_2\ge 0, K_3\ge 0$ and $\beta>2$ such that*

$$\begin{aligned}&2\langle x,f(x,y)\rangle+|g(x,y)|^2+\lambda(2\langle x,h(x,y)\rangle+|h(x,y)|^2)\\&\le K_1(1+|x|^2+|y|^2)-K_2|x|^\beta+K_3|y|^\beta,\quad \forall x,y\in\mathbb{R}^d.\end{aligned} \tag{4.1}$$

Applying the Itô formula and using the techniques in [44] to deal with the time-variable delay, we have the following lemma.

**Lemma 4.1.** *Suppose that Assumptions 2.2, 2.3 and 4.1 hold with* $K_2 > \dfrac{K_3}{(1-\hat{\delta})} \geq 0$. *Then for any given initial data* (2.2), *there is a unique global solution* $X(t)$ *to* (2.1) *on* $t \in [-\tau, +\infty)$. *Moreover, the solution* $X(t)$ *has the property that*

$$\sup_{-\tau \leq t \leq T} \mathbb{E}|X(t)|^2 < \infty, \ \forall T > 0. \tag{4.2}$$

In this section, to define the second numerical scheme, we also choose a strictly increasing continuous functions $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(R) \to \infty$ as $R \to \infty$ and

$$\sup_{|x| \vee |y| \leq R} \frac{|f(x,y)|}{(1+|x|+|y|)} \vee \frac{|g(x,y)|}{(1+|x|+|y|)} \vee \frac{|h(x,y)|}{(1+|x|+|y|)} \leq \mu(R), \ \forall R \geq 1. \tag{4.3}$$

Denote by $\mu^{-1}$ the inverse function of $\mu$ and we see that $\mu^{-1} : [\mu(1), \infty) \to \mathbb{R}^+$ is a strictly increasing continuous function. We then choose a constant $\hat{L} \geq 1 \vee \mu(1)$ and a strictly decreasing function $\varphi : (0,1] \to [\mu(1), +\infty)$ such that

$$\lim_{\Delta \to 0} \varphi(\Delta) = \infty \quad \text{and} \quad \varphi(\Delta) \leq \hat{L}\Delta^{-1/4}, \ \forall \Delta \in (0,1]. \tag{4.4}$$

For a given step size $\Delta \in (0,1]$, let us define a truncation mapping $\pi_\Delta : \mathbb{R}^d \to \{x \in \mathbb{R}^d : |x| \leq \mu^{-1}(\varphi(\Delta))\}$ by

$$\pi_\Delta(x) = \left(|x| \wedge \mu^{-1}(\varphi(\Delta))\right) \frac{x}{|x|}, \ \forall x \in \mathbb{R}^d, \tag{4.5}$$

where $\mu$ and $\varphi$ are from (4.3) and (4.4), respectively. Define the following truncated functions for any $x, y \in \mathbb{R}^d$,

$$f_\Delta(x,y) = f(\pi_\Delta(x), \pi_\Delta(y)), \quad g_\Delta(x,y) = g(\pi_\Delta(x), \pi_\Delta(y)), \quad h_\Delta(x,y) = h(\pi_\Delta(x), \pi_\Delta(y)). \tag{4.6}$$

From (3.4) and (4.6), we have

$$\begin{aligned} &|f_\Delta(x,y)| \vee |g_\Delta(x,y)| \vee |h_\Delta(x,y)| \\ &\leq \varphi(\Delta)(1 + |\pi_\Delta(x)| + |\pi_\Delta(y)|) \leq \varphi(\Delta)(1+|x|+|y|), \ \forall x, y \in \mathbb{R}^d. \end{aligned} \tag{4.7}$$

The following lemma shows that these truncated coefficients conserve the generalized Khasminskii-type condition for any $\Delta \in (0,1]$, the proof of the lemma is similar to that of [15, Lemma 3.18] and so is omitted.

**Lemma 4.2.** *Let Assumptions 4.1 hold with* $K_2 \geq K_3 \geq 0$. *Then for any* $\Delta \in (0,1]$,

$$\begin{aligned} &2\langle x, f_\Delta(x,y)\rangle + |g_\Delta(x,y)|^2 + \lambda(2\langle x, h_\Delta(x,y)\rangle + |h_\Delta(x,y)|^2) \\ &\leq \hat{K}_1(1+|x|^2+|y|^2) - K_2|\pi_\Delta(x)|^\beta + K_3|\pi_\Delta(y)|^\beta, \ \forall x, y \in \mathbb{R}^d, \end{aligned} \tag{4.8}$$

*where* $\hat{K}_1 = 2K_1\left(1 \vee [1/\mu^{-1}(\varphi(1))]\right)$.

**Lemma 4.3.** *Let Assumptions 2.2, 2.3 and 4.1 hold with* $K_2 \geq (\lfloor (1-\hat{\delta})^{-1} \rfloor + 1)K_3 \geq 0$. *Then for any* $\Delta \in (0,1]$,

$$\sup_{0 \leq k\Delta \leq T} \mathbb{E}|y_k|^2 \leq C, \quad \forall T > 0, \tag{4.9}$$

*where* $C$ *is a positive constant independent of* $\Delta$.

*Proof.* For any integer $k \geq 0$, we conclude from (2.5) that

$$\begin{aligned}|y_{k+1}|^2 &= |y_k|^2 + 2\langle y_k, f_\Delta(y_k, y_{k-\delta_k})\rangle\Delta + |g_\Delta(y_k, y_{k-\delta_k})|^2\Delta + 2\langle y_k, h_\Delta(y_k, y_{k-\delta_k})\rangle\Delta N_k \\ &\quad + |h_\Delta(y_k, y_{k-\delta_k})\Delta N_k|^2 + |f_\Delta(y_k, y_{k-\delta_k})|^2\Delta^2 \\ &\quad + 2\langle f_\Delta(y_k, y_{k-\delta_k}), h_\Delta(y_k, y_{k-\delta_k})\rangle\Delta N_k\Delta + J_k, \end{aligned} \tag{4.10}$$

where

$$\begin{aligned}J_k &= 2\langle y_k, g_\Delta(y_k, y_{k-\delta_k})\Delta B_k\rangle + 2\langle f_\Delta(y_k, y_{k-\delta_k}), g_\Delta(y_k, y_{k-\delta_k})\Delta B_k\rangle\Delta \\ &\quad + |g_\Delta(y_k, y_{k-\delta_k})|^2(|\Delta B_k|^2 - \Delta) + 2\langle g_\Delta(y_k, y_{k-\delta_k})\Delta B_k, h_\Delta(y_k, y_{k-\delta_k})\Delta N_k\rangle.\end{aligned}$$

Obviously, $\mathbb{E}J_k = 0$. Note that

$$\mathbb{E}\Delta N_k = \lambda\Delta \quad \text{and} \quad \mathbb{E}|\Delta N_k|^2 = \lambda\Delta + \lambda^2\Delta^2. \tag{4.11}$$

By Lemma 4.2, we have

$$\begin{aligned}\mathbb{E}|y_{k+1}|^2 &\leq \mathbb{E}|y_k|^2 + \hat{K}_1\mathbb{E}(1 + |y_k|^2 + |y_{k-\delta_k}|^2)\Delta + \mathbb{E}|f_\Delta(y_k, y_{k-\delta_k})|^2\Delta^2 \\ &\quad + 2\lambda\mathbb{E}\langle f_\Delta(y_k, y_{k-\delta_k}), h_\Delta(y_k, y_{k-\delta_k})\rangle\Delta^2 + \lambda^2\mathbb{E}|h_\Delta(y_k, y_{k-\delta_k})|^2\Delta^2 \\ &\quad + \mathbb{E}\Big[-K_2|\pi_\Delta(y_k)|^\beta + K_3|\pi_\Delta(y_{k-\delta_k})|^\beta\Big]\Delta, \ \forall k \geq 0.\end{aligned} \tag{4.12}$$

Moreover, by (4.7), we have

$$\begin{aligned}&\mathbb{E}|f_\Delta(y_k, y_{(k-\delta_k)})|^2\Delta^2 + 2\lambda\mathbb{E}\langle f_\Delta(y_k, y_{k-\delta_k}), h_\Delta(y_k, y_{k-\delta_k})\rangle\Delta^2 + \lambda^2\mathbb{E}|h_\Delta(y_k, y_{k-\delta_k})|^2\Delta^2 \\ &\leq (1+\lambda)^2(\varphi(\Delta))^2\mathbb{E}(1 + |y_k|^2 + |y_{k-\delta_k}|^2)\Delta^2 \\ &\leq (1+\lambda)^2\hat{L}^2\mathbb{E}(1 + |y_k|^2 + |y_{k-\delta_k}|^2)\Delta^{3/2} \\ &\leq (1+\lambda)^2\hat{L}^2\mathbb{E}(1 + |y_k|^2 + |y_{k-\delta_k}|^2)\Delta.\end{aligned} \tag{4.13}$$

Inserting (4.13) into (4.12) gives

$$\begin{aligned}\mathbb{E}|y_{k+1}|^2 &\leq \mathbb{E}|y_k|^2 + (\hat{K}_1 + (1+\lambda)^2\hat{L}^2)\mathbb{E}(1 + |y_k|^2 + |y_{k-\delta_k}|^2)\Delta \\ &\quad + \mathbb{E}\Big[-K_2|\pi_\Delta(y_k)|^\beta + K_3|\pi_\Delta(y_{k-\delta_k})|^\beta\Big]\Delta, \ \forall k \geq 0.\end{aligned} \tag{4.14}$$

Thus, we have

$$\begin{aligned}\mathbb{E}|y_k|^2 &\leq \|\xi\|^2 + (\hat{K}_1 + (1+\lambda)^2\hat{L}^2)\sum_{j=0}^{k-1}\mathbb{E}(1 + |y_j|^2 + |y_{j-\delta_j}|^2)\Delta \\ &\quad + \mathbb{E}\left[\sum_{j=0}^{k-1}\Big(-K_2|\pi_\Delta(y_j)|^\beta + K_3|\pi_\Delta(y_{j-\delta_j})|^\beta\Big)\right]\Delta, \ \forall k \geq 1.\end{aligned} \tag{4.15}$$

By Lemma 2.1, we yields that

$$\begin{aligned}&\sum_{i=0}^{k-1}|\pi_\Delta(y_{i-\delta_i})|^\beta\Delta \leq (\lfloor(1-\hat{\delta})^{-1}\rfloor + 1)\sum_{j=-M}^{k-1}|\pi_\Delta(y_j)|^\beta\Delta \\ &= (\lfloor(1-\hat{\delta})^{-1}\rfloor + 1)\sum_{j=-M}^{-1}|\pi_\Delta(y_j)|^\beta\Delta + (\lfloor(1-\hat{\delta})^{-1}\rfloor + 1)\sum_{j=0}^{k-1}|\pi_\Delta(y_j)|^\beta\Delta \\ &\leq (\lfloor(1-\hat{\delta})^{-1}\rfloor + 1)\tau\|\xi\|^\beta + (\lfloor(1-\hat{\delta})^{-1}\rfloor + 1)\sum_{j=0}^{k-1}|\pi_\Delta(y_j)|^\beta\Delta, \ \forall k \geq 1,\end{aligned} \tag{4.16}$$

where $\delta_k = \lfloor \delta(k\Delta)/\Delta \rfloor$. Thus,

$$\begin{aligned}
&\sum_{j=0}^{k-1}\Big[-K_2|\pi_\Delta(y_j)|^\beta + K_3|\pi_\Delta(y_{j-\delta_j})|^\beta\Big]\Delta \\
&\le K_3(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)\tau\|\xi\|^\beta - (K_2 - K_3(\lfloor(1-\hat{\delta})^{-1}\rfloor+1))\sum_{j=0}^{k-1}|\pi_\Delta(y_j)|^\beta\Delta \\
&\le K_3(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)\tau\|\xi\|^\beta. \qquad (4.17)
\end{aligned}$$

Inserting this into (4.15), we have

$$\begin{aligned}
\mathbb{E}|y_k|^2 &\le (\|\xi\|^2 + K_3(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)\tau\|\xi\|^\beta) \\
&\quad + (\hat{K}_1 + (1+\lambda)^2\hat{L}^2)\sum_{j=0}^{k-1}\Big[1+\mathbb{E}|y_j|^2+\mathbb{E}|y_{j-\delta_j}|^2\Big]\Delta \\
&\le (\|\xi\|^2 + K_3(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)\tau\|\xi\|^\beta) \\
&\quad + (\hat{K}_1 + (1+\lambda)^2\hat{L}^2)\sum_{j=0}^{k-1}\Big[1+2\sup_{-M\le i\le j}\mathbb{E}|y_j|^2\Big]\Delta,\ \forall k\ge 1. \qquad (4.18)
\end{aligned}$$

As this holds for any integer $k$ satisfying $1 \le k \le \lfloor T/\Delta \rfloor$, while the sum of the right-hand-side (RHS) terms is non-decreasing in $k$, we then have

$$\begin{aligned}
\sup_{1\le i\le k}\mathbb{E}|y_i|^2 &\le (\|\xi\|^2 + K_3(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)\tau\|\xi\|^\beta) \\
&\quad + (\hat{K}_1 + (1+\lambda)^2\hat{L}^2)\sum_{j=0}^{k-1}\Big[1+2\sup_{-M\le i\le j}\mathbb{E}|y_j|^2\Big]\Delta, \qquad (4.19)
\end{aligned}$$

which implies that

$$\begin{aligned}
\sup_{-M\le i\le k}\mathbb{E}|y_i|^2 &\le (\|\xi\|^2 + K_3(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)\tau\|\xi\|^\beta) \\
&\quad + (\hat{K}_1 + (1+\lambda)^2\hat{L}^2)\sum_{j=0}^{k-1}\Big[1+2\sup_{-M\le i\le j}\mathbb{E}|y_j|^2\Big]\Delta,\ \forall k = 1,2,\cdots,\lfloor T/\Delta\rfloor. \qquad (4.20)
\end{aligned}$$

By the discrete Gronwall inequality, we get the desired assertion (4.9). □

**Lemma 4.4.** *Let Assumptions 2.2, 2.3 and 4.1 hold with $K_2 \ge (\lfloor(1-\hat{\delta})^{-1}\rfloor+1)K_3 \ge 0$. For any $R \ge \|\xi\|$, define $\tau_R = \inf\{t\ge 0 : |X(t)| \ge R\}$ and $\hat{\rho}_{\Delta,R} = \inf\{t \ge 0 : |Z_1(t)| \ge R\}$. Then*

$$\mathbb{P}(\tau_R \le T) \le \frac{C}{R^2} \quad \text{and} \quad \mathbb{P}(\hat{\rho}_{\Delta,R} \le T) \le \frac{C}{R^2},\ \forall T > 0, \qquad (4.21)$$

*where $C$ is a positive constant independent of $\Delta$.*

*Proof.* By the Itô formula and Assumption 4.1, we have that for any $t \in [0,T]$

$$\begin{aligned}\mathbb{E}|X(t\wedge\tau_R)|^2 &\le |\xi(0)|^2 + K_1\mathbb{E}\int_0^{t\wedge\tau_R}\Big(1+|X(s)|^2+|X(s-\delta(s))|^2\Big)ds\\ &\quad+\mathbb{E}\int_0^{t\wedge\tau_R}\Big(-K_2|X(s)|^\beta+K_3|X(s-\delta(s))|^\beta\Big)ds\\ &\le |\xi(0)|^2+K_1T+K_1\int_0^t\Big(\mathbb{E}|X(s\wedge\tau_R)|^2+\mathbb{E}|X((s-\delta(s))\wedge\tau_R)|^2\Big)ds\\ &\quad+\Big(\frac{K_3}{1-\hat{\delta}}-K_2\Big)\mathbb{E}\int_0^{t\wedge\tau_R}|X(s)|^\beta ds+\frac{\tau\|\xi\|^\beta}{1-\hat{\delta}}\\ &\le C+2K_1\int_0^t\Big(\sup_{0\le u\le s}\mathbb{E}|X(u\wedge\tau_R)|^2\Big)ds,\end{aligned}$$

where we have used the following estimates

$$\begin{aligned}&\int_0^{t\wedge\tau_R}|X(s-\delta(s))|^\beta ds \le \frac{1}{1-\hat{\delta}}\int_{-\delta(0)}^{(t\wedge\tau_R)-\delta(t\wedge\tau_R)}|X(u)|^\beta du\\ &\le \frac{1}{1-\hat{\delta}}\int_{-\tau}^{t\wedge\tau_R}|X(u)|^\beta du \le \frac{\tau\|\xi\|^\beta}{1-\hat{\delta}}+\frac{1}{1-\hat{\delta}}\int_0^{t\wedge\tau_R}|X(s)|^\beta ds,\end{aligned}$$

and

$$\begin{aligned}\int_0^t\mathbb{E}|X((s-\delta(s))\wedge\tau_R)|^2ds &\le \int_0^t\Big(\sup_{-\tau\le u\le s}\mathbb{E}|X(u\wedge\tau_R)|^2\Big)ds\\ &\le T\|\xi\|^2+\int_0^t\Big(\sup_{0\le u\le s}\mathbb{E}|X(u\wedge\tau_R)|^2\Big)ds. \qquad (4.22)\end{aligned}$$

Consequently,

$$\sup_{0\le u\le t}\mathbb{E}|X(u\wedge\tau_R)|^2 \le C+4K_1\int_0^t\Big(\sup_{0\le u\le s}\mathbb{E}|X(u\wedge\tau_R)|^2\Big)ds.$$

The Gronwall inequality gives

$$\sup_{0\le u\le t}\mathbb{E}|X(u\wedge\tau_R)|^2 \le C.$$

Thus,

$$\mathbb{E}|X(T\wedge\tau_R)|^2 \le C.$$

Finally, using the Chebyshev inequality gives

$$\mathbb{P}(\tau_R\le T)\le\frac{C}{R^2}.$$

Now, we begin to establish the second assertion in (4.21). The remaining proof of this lemma is similar to that of [30, Lemma 3.2], but more refined techniques are needed to overcome the difficilty due to time-variable delay. We observe that $\hat{\rho}_{\Delta,R}=\vartheta_{\Delta,R}\Delta$, where

$$\vartheta_{\Delta,R}:=\inf\{k>0:|y_k|\ge R\}.$$

Clearly, $\hat{\rho}_{\Delta,R}$ and $\vartheta_{\Delta,R}$ are $\mathcal{F}_t$ and $\mathcal{F}_{t_k}$ stopping times, respectively. It is useful to know that

$$y_{(k+1)\wedge\vartheta_{\Delta,R}} - y_{k\wedge\vartheta_{\Delta,R}} = \mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}(y_{k+1} - y_k),\ \forall k \geq 0, \tag{4.23}$$

see [42, p. 477]. Thus, from (2.5) and (4.23), we have

$$y_{(k+1)\wedge\vartheta_{\Delta,R}} = y_{k\wedge\vartheta_{\Delta,R}} + \Big[f_\Delta(y_k, y_{k-\delta_k})\Delta + g_\Delta(y_k, y_{k-\delta_k})\Delta B_k + h_\Delta(y_k, y_{k-\delta_k})\Delta N_k\Big]\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}.$$

Note that

$$\begin{aligned}\Delta B_k\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}} &= B(t_{(k+1)\wedge\vartheta_{\Delta,R}}) - B(t_{k\wedge\vartheta_{\Delta,R}}) =: \Delta B_{k\wedge\vartheta_{\Delta,R}},\\ \Delta N_k\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}} &= N(t_{(k+1)\wedge\vartheta_{\Delta,R}}) - N(t_{k\wedge\vartheta_{\Delta,R}}) =: \Delta N_{k\wedge\vartheta_{\Delta,R}}.\end{aligned} \tag{4.24}$$

Consequently,

$$\begin{aligned}&\mathbb{E}|y_{(k+1)\wedge\vartheta_{\Delta,R}}|^2\\ &= \mathbb{E}\Big[|y_{k\wedge\vartheta_{\Delta,R}}|^2 + 2\langle y_{k\wedge\vartheta_{\Delta,R}}, f_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\rangle\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Delta\Big]\\ &\quad+\mathbb{E}\Big[|g_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\Delta B_k|^2\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Big]\\ &\quad+\mathbb{E}\Big[2\langle y_{k\wedge\vartheta_{\Delta,R}}, h_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\rangle\Delta N_k\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Big]\\ &\quad+\mathbb{E}\Big[|h_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\Delta N_k|^2\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Big]\\ &\quad+\mathbb{E}\Big[|f_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})|^2\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Delta^2\Big]\\ &\quad+\mathbb{E}\Big[2\langle f_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}}), h_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\rangle\Delta N_k\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Delta + \hat{J}_k\Big],\ \forall k \geq 0,\end{aligned} \tag{4.25}$$

where

$$\begin{aligned}\hat{J}_k :&= 2\langle y_{k\wedge\vartheta_{\Delta,R}}, g_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\Delta B_{k\wedge\vartheta_{\Delta,R}}\rangle\\ &\quad+2\langle f_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}}), g_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\Delta B_{k\wedge\vartheta_{\Delta,R}}\rangle\Delta\\ &\quad+2\langle g_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\Delta B_{k\wedge\vartheta_{\Delta,R}}, h_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\Delta N_{k\wedge\vartheta_{\Delta,R}}\rangle.\end{aligned}$$

Since $B(t)$ is a continuous martingale, by the Doob martingale stopping time theorem, we have that $\mathbb{E}\Big[\Delta B_k\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}|\mathcal{F}_{t_{k\wedge\vartheta_{\Delta,R}}}\Big] = 0$ and for any $A \in \mathbb{R}^{d\times m}$

$$\begin{aligned}\mathbb{E}\Big[|A\Delta B_k|^2\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}|\mathcal{F}_{t_{k\wedge\vartheta_{\Delta,R}}}\Big] &= |A|^2\mathbb{E}\Big[(t_{(k+1)\wedge\vartheta_{\Delta,R}} - t_{k\wedge\vartheta_{\Delta,R}})|\mathcal{F}_{t_{k\wedge\vartheta_{\Delta,R}}}\Big]\\ &= |A|^2\mathbb{E}\Big[\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}|\mathcal{F}_{t_{k\wedge\vartheta_{\Delta,R}}}\Big]\Delta,\end{aligned} \tag{4.26}$$

see [30, p.12]. Then

$$\begin{aligned}\mathbb{E}\hat{J}_k &= 2\mathbb{E}\Big[y^T_{k\wedge\vartheta_{\Delta,R}}g_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\mathbb{E}\big[\Delta B_k\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}|\mathcal{F}_{t_{k\wedge\vartheta_{\Delta,R}}}\big]\Big]\\ &\quad+2\mathbb{E}\Big[f^T_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})g_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\mathbb{E}\big[\Delta B_k\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}|\mathcal{F}_{t_{k\wedge\vartheta_{\Delta,R}}}\big]\Big]\Delta\\ &\quad+\mathbb{E}\big[2\langle g_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\Delta B_{k\wedge\vartheta_{\Delta,R}}, h_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\Delta N_{k\wedge\vartheta_{\Delta,R}}\rangle\big]\\ &= 0,\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}
&\mathbb{E}\Big[|h_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})|^2|\Delta N_k|^2\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Big]\\
&= \mathbb{E}\Big[|h_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})|^2\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}(\lambda\Delta+\lambda^2\Delta^2)\Big],\\
&\mathbb{E}\Big[2\langle y_{k\wedge\vartheta_{\Delta,R}}, h_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\rangle\Delta N_k\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Big]\\
&= \mathbb{E}\Big[2\langle y_{k\wedge\vartheta_{\Delta,R}}, h_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\rangle\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\lambda\Delta\Big]. \qquad (4.28)
\end{aligned}$$

Moreover, by (4.7) and Lemma 4.3, we have

$$\begin{aligned}
&\mathbb{E}\Big[|f_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})|^2\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Big]\Delta^2 = \mathbb{E}\Big[|f_\Delta(y_k, y_{k-\delta_k})|^2\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Big]\Delta^2\\
&\le \mathbb{E}\Big[|f_\Delta(y_k, y_{k-\delta_k})|^2\Big]\Delta^2 \le C(\varphi(\Delta))^2\mathbb{E}\Big[1+|y_k|^2+|y_{k-\delta_k}|^2\Big]\Delta^2\\
&\le C\Delta^{3/2} \le C\Delta. \qquad (4.29)
\end{aligned}$$

Plugging (4.27)-(4.29) into (4.25) and using Lemma 4.2, Lemma 4.3, we have

$$\begin{aligned}
&\mathbb{E}|y_{(k+1)\wedge\vartheta_{\Delta,R}}|^2\\
&\le \mathbb{E}|y_{k\wedge\vartheta_{\Delta,R}}|^2 + \mathbb{E}\Big[\big(2\langle y_{k\wedge\vartheta_{\Delta,R}}, f_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\rangle + |g_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}}|^2\big)\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Big]\Delta\\
&\quad + \mathbb{E}\Big[\lambda\big(2\langle y_{k\wedge\vartheta_{\Delta,R}}, h_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\rangle + |h_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})|^2\big)\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Big]\Delta\\
&\quad + \mathbb{E}\Big[|f_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})|^2 + \lambda^2|h_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})|^2\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Big]\Delta^2\\
&\quad + \mathbb{E}\Big[2\lambda\langle f_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}}), h_\Delta(y_{k\wedge\vartheta_{\Delta,R}}, y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}})\rangle\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Big]\Delta^2\\
&\le \mathbb{E}|y_{k\wedge\vartheta_{\Delta,R}}|^2 + C\Delta + \hat{K}_1\mathbb{E}\Big[\big(1+|y_{k\wedge\vartheta_{\Delta,R}}|^2+|y_{(k-\delta_k)\wedge\vartheta_{\Delta,R}}|^2\big)\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Big]\Delta + \hat{H}_k\\
&= \mathbb{E}|y_{k\wedge\vartheta_{\Delta,R}}|^2 + C\Delta + \hat{K}_1\mathbb{E}\Big[\big(1+|y_k|^2+|y_{(k-\delta_k)}|^2\big)\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Big]\Delta + \hat{H}_k\\
&\le \mathbb{E}|y_{k\wedge\vartheta_{\Delta,R}}|^2 + C\Delta + \hat{K}_1\mathbb{E}\Big[\big(1+|y_k|^2+|y_{k-\delta_k}|^2\big)\Big]\Delta + \hat{H}_k\\
&\le \mathbb{E}|y_{k\wedge\vartheta_{\Delta,R}}|^2 + C\Delta + \hat{H}_k,\ \forall k\ge 0, \qquad (4.30)
\end{aligned}$$

where

$$\hat{H}_k = \mathbb{E}\Big[\big(-K_2|\pi_\Delta(y_k)|^\beta + K_3|\pi_\Delta(y_{k-\delta_k})|^\beta\big)\mathbb{I}_{\{k<\vartheta_{\Delta,R}\}}\Big]\Delta.$$

Thus, for any integer $k$ satisfying $1\le k\le\lfloor T/\Delta\rfloor$, we conclude from (4.30) that

$$\begin{aligned}
\mathbb{E}|y_{k\wedge\vartheta_{\Delta,R}}|^2 &\le \|\xi\|^2 + Ck\Delta + \sum_{j=0}^{k-1}\mathbb{E}\Big[\big(-K_2|\pi_\Delta(y_j)|^\beta + K_3|\pi_\Delta(y_{j-\delta_j})|^\beta\big)\mathbb{I}_{\{j<\vartheta_{\Delta,R}\}}\Big]\Delta\\
&\le \|\xi\|^2 + CT + \mathbb{E}\left[\sum_{j=0}^{k-1}\big(-K_2|\pi_\Delta(y_j)|^\beta + K_3|\pi_\Delta(y_{j-\delta_j})|^\beta\big)\mathbb{I}_{\{j<\vartheta_{\Delta,R}\}}\right]\Delta\\
&= \|\xi\|^2 + CT + \mathbb{E}\left[\sum_{j=0}^{(k-1)\wedge(\vartheta_{\Delta,R}-1)}\big(-K_2|\pi_\Delta(y_j)|^\beta + K_3|\pi_\Delta(y_{j-\delta_j})|^\beta\big)\right]\Delta. \qquad (4.31)
\end{aligned}$$

By Lemma 2.1, we get

$$\begin{aligned}
&\sum_{j=0}^{(k-1)\wedge(\vartheta_{\Delta,R}-1)} |\pi_\Delta(y_{j-\delta_j})|^\beta \\
&\le (\lfloor(1-\hat{\delta})^{-1}\rfloor+1) \sum_{i=-M}^{(k-1)\wedge(\vartheta_{\Delta,R}-1)} |\pi_\Delta(y_i)|^\beta \\
&= (\lfloor(1-\hat{\delta})^{-1}\rfloor+1) \sum_{i=-M}^{-1} |\pi_\Delta(y_i)|^\beta + (\lfloor(1-\hat{\delta})^{-1}\rfloor+1) \sum_{i=0}^{(k-1)\wedge(\vartheta_{\Delta,R}-1)} |\pi_\Delta(y_i)|^\beta \\
&\le (\lfloor(1-\hat{\delta})^{-1}\rfloor+1)M\|\xi\|^\beta + (\lfloor(1-\hat{\delta})^{-1}\rfloor+1) \sum_{i=0}^{(k-1)\wedge(\vartheta_{\Delta,R}-1)} |\pi_\Delta(y_i)|^\beta,\ \forall k\ge 1. \qquad (4.32)
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sum_{j=0}^{(k-1)\wedge(\vartheta_{\Delta,R}-1)} \Big[\big(-K_2|\pi_\Delta(y_j)|^\beta + K_3|\pi_\Delta(y_{j-\delta_j})|^\beta\big)\Big]\Delta \\
&\le K_3(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)(M\Delta)\|\xi\|^\beta + (K_3(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)-K_2) \sum_{j=0}^{(k-1)\wedge(\vartheta_{\Delta,R}-1)} |\pi_\Delta(y_j)|^\beta\Delta \\
&\le K_3(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)\tau\|\xi\|^\beta,\ \forall k\ge 1. \qquad (4.33)
\end{aligned}$$

Inserting this into (4.31) gives

$$\mathbb{E}|y_{k\wedge\vartheta_{\Delta,R}}|^2 \le \|\xi\|^2 + CT + K_3(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)\tau\|\xi\|^\beta,\ \forall k=1,2,\cdots,\lfloor T/\Delta\rfloor.$$

In particular, we have

$$\mathbb{E}|y_{\lfloor T/\Delta\rfloor\wedge\vartheta_{\Delta,R}}|^2 \le C,$$

or equivalently,

$$\mathbb{E}|Z_1(T\wedge\hat{\rho}_{\Delta,R})|^2 \le C$$

which implies that

$$R^2\mathbb{P}(\hat{\rho}_{\Delta,R}\le T) \le \mathbb{E}\Big[\mathbb{I}_{\{\hat{\rho}_{\Delta,R}\le T\}}|Z_1(\hat{\rho}_{\Delta,R})|^2\Big] \le \mathbb{E}|Z_1(T\wedge\hat{\rho}_{\Delta,R})|^2 \le C.$$

Thus, the proof is finished. □

**Remark 4.1.** *It should be pointed out that if we use the usual continuous proof of [20, Lemma 3.3] to estimate the second assertion in* (4.21)*, then there will be a term $J^\star$ with the following form we have to address,*

$$J^\star := \int_0^t |\pi_\Delta(Z_2(s))|^\beta ds - (\lfloor(1-\hat{\delta})^{-1}\rfloor+1)\int_0^t |\pi_\Delta(Z_1(s))|^\beta ds, \qquad (4.34)$$

*Of course, by the known conditions, we can show that*

$$\begin{aligned}
J^\star &\le (\lfloor(1-\hat{\delta})^{-1}\rfloor+1)\tau\|\xi\|^\beta \\
&\quad + \Big(|\pi_\Delta(y_{\lfloor t/\Delta\rfloor-\delta_{\lfloor t/\Delta\rfloor}})|^\beta - (\lfloor(1-\hat{\delta})^{-1}\rfloor+1)|\pi_\Delta(y_{\lfloor t/\Delta\rfloor})|^\beta\Big)(t-\lfloor t/\Delta\rfloor). \qquad (4.35)
\end{aligned}$$

*But we see from* (4.35) *that this estimate remains a tail, namely, the second term on the right hand side of* (4.35) *and we have no other method to address $J^\star$ properly. However, if we take $t$ to be the grid point, then the tail will vanish. This motivates the above discrete proof in Lemma 4.4.*

The following theorem establishes the strong convergence (without order) results of the truncated EM method.

**Theorem 4.1.** *Let Assumptions 2.1, 2.2, 2.3 and 4.1 hold with $K_2 \geq (\lfloor (1-\hat{\delta})^{-1} \rfloor + 1)K_3 \geq 0$. Then for any $q \in [1,2)$,*

$$\lim_{\Delta \to 0} \mathbb{E}|X(T) - Z_1(T)|^q = 0, \ \forall T > 0. \tag{4.36}$$

*Proof.* Let $\tau_R$ and $\hat{\rho}_{\Delta,R}$ be the same as before. Let $q \in [1,2)$. Define $\hat{\theta}_{\Delta,R} = \tau_R \wedge \hat{\rho}_{\Delta,R}$ and $\hat{e}_\Delta(T) = X(T) - Z_1(T)$, for any $T > 0$. By the Young inequality, for any $\eta > 0$, we have

$$\begin{aligned}\mathbb{E}|\hat{e}_\Delta(T)|^q &= \mathbb{E}\big[|\hat{e}_\Delta(T)|^q \mathbb{I}_{\{\hat{\theta}_{\Delta,R} > T+1\}}\big] + \mathbb{E}\big[|\hat{e}_\Delta(T)|^q \mathbb{I}_{\{\hat{\theta}_{\Delta,R} \leq T+1\}}\big] \\ &\leq \mathbb{E}\big[|\hat{e}_\Delta(T)|^q \mathbb{I}_{\{\hat{\theta}_{\Delta,R} > T+1\}}\big] + \frac{q\eta}{2}\mathbb{E}|\hat{e}_\Delta(T)|^2 + \frac{2-q}{2\eta^{q/(2-q)}}\mathbb{P}(\hat{\theta}_{\Delta,R} \leq T+1).\end{aligned}$$

In this theorem, $C_R$ denotes a positive constant depending on $R$ but independent of $\Delta$ and its value may be different for different appearance. By Lemmas 4.1 and 4.3, we get that

$$\mathbb{E}|\hat{e}_\Delta(T)|^2 \leq 2\mathbb{E}|X(T)|^2 + 2\mathbb{E}|Z_1(T)|^2 \leq C.$$

From Lemma 4.4, we have

$$\mathbb{P}(\theta_{\Delta,R} \leq T+1) \leq \mathbb{P}(\tau_R \leq T+1) + \mathbb{P}(\hat{\rho}_{\Delta,R} \leq T+1) \leq \frac{C}{R^2}.$$

Consequently, we have

$$\mathbb{E}|\hat{e}_\Delta(T)|^q \leq \frac{q\eta C}{2} + \frac{(2-q)C}{2R^2\eta^{q/(2-q)}} + \mathbb{E}\big[|\hat{e}_\Delta(T)|^q \mathbb{I}_{\{\hat{\theta}_{\Delta,R} > T+1\}}\big]. \tag{4.37}$$

Let $\hat{\varepsilon} > 0$ be arbitrary. Choose $\eta > 0$ sufficiently small for $\frac{q\eta C}{2} \leq \hat{\varepsilon}$ and then choose $R$ sufficiently large for $\frac{(2-q)C}{2R^2\eta^{q/(2-q)}} \leq \hat{\varepsilon}$. Then for such chosen $R$, we see from (4.37) that

$$\mathbb{E}|\hat{e}_\Delta(T)|^q \leq \mathbb{E}\big[|\hat{e}_\Delta(T)|^q \mathbb{I}_{\{\hat{\theta}_{\Delta,R} > T+1\}}\big] + 2\hat{\varepsilon}.$$

If we can show that

$$\lim_{\Delta \to 0} \mathbb{E}\big[|\hat{e}_\Delta(T)|^q \mathbb{I}_{\{\hat{\theta}_{\Delta,R} > T+1\}}\big] = 0, \tag{4.38}$$

then the desired assertion (4.36) follows. Define the truncated functions

$$F_R(x,y) = f\Big((|x| \wedge R)\frac{x}{|x|}, (|y| \wedge R)\frac{y}{|y|}\Big), \quad G_R(x,y) = g\Big((|x| \wedge R)\frac{x}{|x|}, (|y| \wedge R)\frac{y}{|y|}\Big)$$

and

$$H_R(x,y) = h\Big((|x| \wedge R)\frac{x}{|x|}, (|y| \wedge R)\frac{y}{|y|}\Big),$$

for any $x, y \in \mathbb{R}^d$. Without loss of any generality, we assume that $\Delta^*$ is sufficiently small for $\mu^{-1}(\varphi(\Delta^*)) \geq R$ . Then, for any $\Delta \in (0, \Delta^*]$, we get that

$$f_\Delta(x,y) = F_R(x,y), \quad g_\Delta(x,y) = G_R(x,y) \quad \text{and} \quad h_\Delta(x,y) = H_R(x,y),$$

for any $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq R$. Consider the following SDDE

$$\begin{aligned} dz(t) =& F_R(z(t), z(t-\delta(t)))dt + G_R(z(t), z(t-\delta(t)))dB(t) \\ &+ H_R(z(t^-), z((t-\delta(t))^-))dN(t), \ t \geq 0, \end{aligned} \tag{4.39}$$

with the initial data $z(t) = \xi(t)$ on $t \in [-\tau, 0]$. By Assumption 2.3, we observe that $F_R(x,y)$ and $G_R(x,y)$ as well as $H_R(x,y)$ are globally Lipschitz continuous with the Lipschitz constant $L_R$. Hence, SDDE (4.39) has a unique global solution $z(t)$ on $t \geq -\tau$ satisfying

$$\mathbb{P}(z(t \wedge \tau_R) = X(t \wedge \tau_R) \text{ for any } t \in [0,T]) = 1.$$

On the other hand, for any $\Delta \in (0, \Delta^*]$, we apply the (classical) EM method to the SDDE (4.39) and we denote $z_\Delta(t)$ and $\bar{z}_\Delta(t)$ by the continuous-time continuous-sample and the piecewise constant EM solutions, respectively. Then we see from [34, Theorem 2.1] that continuous-time continuous-sample EM solution $z_\Delta(t)$ has the property

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |z(t) - z_\Delta(t)|^q\Big] \leq c_2 \Delta^{q(0.5\wedge\varrho)}, \tag{4.40}$$

where $c_2$ is a positive constant dependent of $L_R$, $T$, $\xi$, $q$ but independent of $\Delta$. From this and the fact that

$$\mathbb{E}|z_\Delta(T \wedge (\hat{\rho}_{\Delta,R} - 1)) - \bar{z}_\Delta(T \wedge (\hat{\rho}_{\Delta,R} - 1))|^q \leq C_R \Delta^{0.5(q\wedge\varrho)},$$

see [34, Corollary 3.4], we conclude that

$$\mathbb{E}|z(T \wedge (\hat{\rho}_{\Delta,R} - 1)) - \bar{z}_\Delta(T \wedge (\hat{\rho}_{\Delta,R} - 1))|^q \leq C_R \Delta^{0.5(q\wedge\varrho)}.$$

Moreover,

$$\mathbb{P}\Big(Z_1(t \wedge (\hat{\rho}_{\Delta,R} - 1)) = \bar{z}_\Delta(t \wedge (\hat{\rho}_{\Delta,R} - 1)) \text{ for any } t \in [0,T]\Big) = 1.$$

Consequently,

$$\begin{aligned} &\mathbb{E}\big[|\hat{e}_\Delta(T)|^q \mathbb{I}_{\{\hat{\theta}_{\Delta,R} > T+1\}}\big] = \mathbb{E}\big[|\hat{e}_\Delta(T \wedge (\hat{\theta}_{\Delta,R} - 1))|^q \mathbb{I}_{\{\hat{\theta}_{\Delta,R} - 1 > T\}}\big] \\ &\leq \mathbb{E}\big[|X(T \wedge (\hat{\theta}_{\Delta,R} - 1)) - Z_1(T \wedge (\hat{\theta}_{\Delta,R} - 1))|^q\big] \\ &= \mathbb{E}\big[|z(T \wedge (\hat{\theta}_{\Delta,R} - 1)) - \bar{z}_\Delta(T \wedge (\hat{\theta}_{\Delta,R} - 1))|^q\big] \\ &\leq C_R \Delta^{q(0.5\wedge\varrho)}, \end{aligned}$$

which establishes (4.38). Thus, the proof is finished. □

## 5. Mean-square and $H_\infty$ stabilities

In this section, we mainly discuss the mean-square and $H_\infty$ stabilities of the truncated EM method for SDDE (2.1). Noting that the truncated functions can preserve the Khasminskii-type

condition (4.8), unfortunately they may not preserve the stability condition. Consequently, we hope that the terms that work for the stability in the coefficients grow at most linearly, while those terms which grow super-linearly have no stabilizing effect. In our truncated method for stability, we only use the truncation technique to those super-linear terms in the coefficients. Lemma 5.2 shows that partially truncated functions defined by (5.10) have the property of preserving the stability condition. Thus, we assume that $f$, $g$ and $h$ can be decomposed as

$$f(x,y) = F_1(x,y) + F(x,y),\ g(x,y) = G_1(x,y) + G(x,y),\ h(x,y) = H_1(x,y) + H(x,y),$$

where $F_1, F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $G_1, G : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$, and $H_1, H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. Moreover,

$$F_1(0,0) = F(0,0) = G_1(0,0) = G(0,0) = H_1(0,0) = H(0,0) = 0,$$

the coefficients $F_1$, $F$, $G_1$, $G$, and $H_1$, $H$ satisfy the following conditions.

**Assumption 5.1.** *For any $R > 0$, there exists constants $\bar{L}$ and $\bar{L}_R$ depending on $R$ such that*

$$\begin{aligned}&|F_1(x_1,y_1) - F_1(x_2,y_2)|^2 \vee |G_1(x_1,y_1) - G_1(x_2,y_2)|^2 \vee |H_1(x_1,y_1) - H_1(x_2,y_2)|^2\\ &\le \bar{L}(|x_1-x_2|^2 + |y_1-y_2|^2),\ \forall x_1,x_2,y_1,y_2 \in \mathbb{R}^d,\end{aligned} \tag{5.1}$$

*and*

$$\begin{aligned}&|F(x_1,y_1) - F(x_2,y_2)|^2 \vee |G(x_1,y_1) - G(x_2,y_2)|^2 \vee |H(x_1,y_1) - H(x_2,y_2)|^2\\ &\le \bar{L}_R(|x_1-x_2|^2 + |y_1-y_2|^2),\ \forall x_1,x_2,y_1,y_2 \in \mathbb{R}^d,\ \textit{with } |x_1| \vee |x_2| \vee |y_1| \vee |y_2| \le R\end{aligned} \tag{5.2}$$

*for any $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$ with $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \le R$.*

**Assumption 5.2.** *There exist nonnegative constants $\theta$, $\nu_1$, $\nu_2$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ and $\beta > 2$ such that*

$$\begin{aligned}&2\langle x, F_1(x,y)\rangle + (1+\theta)|G_1(x,y)|^2 + \lambda\Big(2\langle x, H_1(x,y)\rangle + (1+\theta)|H_1(x,y)|^2\Big)\\ &\le -\nu_1|x|^2 + \nu_2|y|^2,\ \forall x,y \in \mathbb{R}^d,\\ &2\langle x, F(x,y)\rangle + (1+\theta^{-1})|G(x,y)|^2 + \lambda\Big(2\langle x, H(x,y)\rangle + (1+\theta^{-1})|H(x,y)|^2\Big)\\ &\le \alpha_1|x|^2 + \alpha_2|y|^2 - \alpha_3|x|^\beta + \alpha_4|y|^\beta,\ \forall x,y \in \mathbb{R}^d.\end{aligned} \tag{5.3}$$

When $\theta = 0$, we set $\theta^{-1}|G(x,y)|^2 = \theta^{-1}|H(x,y)|^2 = 0$, when $\theta = \infty$, we set $\theta|G_1(x,y)|^2 = \theta|H_1(x,y)|^2 = 0$. Clearly, Assumption 5.2 implies that

$$\begin{aligned}&2\langle x, f(x,y)\rangle + |g(x,y)|^2 + \lambda\Big(2\langle x, h(x,y)\rangle + |h(x,y)|^2\Big)\\ &\le -(\nu_1-\alpha_1)|x|^2 + (\nu_2+\alpha_2)|y|^2 - \alpha_3|x|^\beta + \alpha_4|y|^\beta,\ \forall x,y \in \mathbb{R}^d.\end{aligned} \tag{5.4}$$

It is not difficult to show that the SDDE (2.1) is stable in mean square sense, which can be stated by the following lemma, see, e.g., [44, Theorem 3.6].

**Lemma 5.1.** *Let Assumptions 2.2, 5.1 and 5.2 hold with*

$$\nu_1 > \alpha_1 + \frac{1}{1-\hat{\delta}}(\nu_2+\alpha_2) \quad \textit{and} \quad \alpha_3 > \frac{1}{1-\hat{\delta}}\alpha_4 \ge 0. \tag{5.5}$$

*Then for any given initial data* (2.2)*, the unique global solution* $X(t)$ *to* (2.1) *has the property that*

$$\limsup_{t\to\infty} \frac{\log \mathbb{E}|X(t)|^2}{t} \leq -\left(\gamma^\star \wedge \frac{1}{\tau}\log\frac{(1-\hat{\delta})\alpha_3}{\alpha_4}\right), \tag{5.6}$$

*and*

$$\int_0^\infty \mathbb{E}|X(t)|^2 dt < \infty, \tag{5.7}$$

*where* $\gamma^\star > 0$ *is the unique root to the following equation*

$$\nu_1 = \alpha_1 + \frac{1}{1-\hat{\delta}}(\nu_2+\alpha_2)e^{\gamma^\star\tau} + \gamma^\star. \tag{5.8}$$

**Lemma 5.2.** *Let Assumptions 2.2, 5.1 and 5.2 hold with*

$$\nu_1 > \alpha_1 + \frac{1}{4}\nu_2 + (\lfloor(1-\hat{\delta})^{-1}\rfloor+1)(\nu_2+\alpha_2) \quad \text{and} \quad \alpha_3 > (\lfloor(1-\hat{\delta})^{-1}\rfloor+1)\alpha_4 \geq 0. \tag{5.9}$$

*For any* $x, y \in \mathbb{R}^d$*, define*

$$\begin{aligned} &f_\Delta(x,y) = F_1(x,y) + F_\Delta(x,y), \quad g_\Delta(x,y) = G_1(x,y) + G_\Delta(x,y), \\ &h_\Delta(x,y) = G_1(x,y) + H_\Delta(x,y), \end{aligned} \tag{5.10}$$

*where* $F_\Delta(x,y) = F(\pi_\Delta(x),\pi_\Delta(y))$*,* $G_\Delta(x,y) = G(\pi_\Delta(x),\pi_\Delta(y))$ *and* $H_\Delta(x,y) = H(\pi_\Delta(x),\pi_\Delta(y))$*. Then*

$$\begin{aligned} &2\langle x, f_\Delta(x,y)\rangle + |g_\Delta(x,y)|^2 + \lambda\Big(2\langle x, h_\Delta(x,y)\rangle + |h_\Delta(x,y)|^2\Big) \\ &\leq -(\nu_1 - \alpha_1 - \frac{1}{4}\alpha_2)|x|^2 + (\nu_2+\alpha_2)|y|^2 - \alpha_3|\pi_\Delta(x)|^\beta + \alpha_4|\pi_\Delta(y)|^\beta, \ \forall x,y\in\mathbb{R}^d, \end{aligned} \tag{5.11}$$

*and*

$$|f_\Delta(x,y)|^2 + 2\lambda\langle f_\Delta(x,y), h_\Delta(x,y)\rangle + \lambda^2|h_\Delta(x,y)|^2 \leq \epsilon_\Delta(|x|^2+|y|^2), \ \forall x,y\in\mathbb{R}^d, \tag{5.12}$$

*where*

$$\epsilon_\Delta = (1+\lambda)^2(4\bar{L}+2\bar{L}_1)\Delta + 8(1+\lambda)^2(\varphi(\Delta))^2\Delta. \tag{5.13}$$

*Proof.* For any $x\in\mathbb{R}^d$ with $|x|\leq\mu^{-1}(\varphi(\Delta))$ and any $y\in\mathbb{R}^d$, (5.11) follows from Assumption 5.2. While for any $x\in\mathbb{R}^d$ with $|x|>\mu^{-1}(\varphi(\Delta))$ and any $y\in\mathbb{R}^d$, in a similar way as [20, Inequality (2.12)] and [15, Appendix C] were obtained, we have

$$\begin{aligned} &2\langle x, F_\Delta(x,y)\rangle + |G_\Delta(x,y)|^2 + \lambda\Big(2\langle x, H_\Delta(x,y)\rangle + |H_\Delta(x,y)|^2\Big) \\ &\leq \frac{|x|}{\mu^{-1}(\varphi(\Delta))}\left(\alpha_1|\pi_\Delta(x)|^2 + \alpha_2|\pi_\Delta(y)|^2\right) - \alpha_3|\pi_\Delta(x)|^\beta + \alpha_4|\pi_\Delta(y)|^\beta \\ &\leq \alpha_1|x|^2 + \alpha_2|x||y| - \alpha_3|\pi_\Delta(x)|^\beta + \alpha_4|\pi_\Delta(y)|^\beta. \end{aligned}$$

By the elementary inequality and Assumption 5.2, we have

$$\begin{aligned}
&2\langle x, f_\Delta(x,y)\rangle + |g_\Delta(x,y)|^2 + \lambda\Big(2\langle x, h_\Delta(x,y)\rangle + |h_\Delta(x,y)|^2\Big)\\
&= 2\langle x, F_1(x,y) + F_\Delta(x,y)\rangle + |G(x,y) + G_\Delta(x,y)|^2\\
&\quad + \lambda\Big(2\langle x, H_1(x,y) + H_\Delta(x,y)\rangle + |H_1(x,y) + H_\Delta(x,y)|^2\Big)\\
&\le 2\langle x, F_1(x,y)\rangle + (1+\theta)|G_1(x,y)|^2 + \lambda\Big(2\langle x, H_1(x,y)\rangle + (1+\theta)|H_1(x,y)|^2\Big)\\
&\quad + 2\langle x, F(x,y)\rangle + (1+\theta^{-1})|G(x,y)|^2 + \lambda\Big(2\langle x, H(x,y)\rangle + (1+\theta^{-1})|H(x,y)|^2\Big)\\
&\le (-\nu_1|x|^2 + \nu_2|y|^2) + (\alpha_1|x|^2 + \alpha_2|x||y| - \alpha_3|\pi_\Delta(x)|^\beta + \alpha_4|\pi_\Delta(y)|^\beta)\\
&\le -(\nu_1 - \alpha_1 - \frac{1}{4}\alpha_2)|x|^2 + (\nu_2 + \alpha_2)|y|^2 - \alpha_3|\pi_\Delta(x)|^\beta + \alpha_4|\pi_\Delta(y)|^\beta, \qquad (5.14)
\end{aligned}$$

where inequality $|x||y| \le \frac{1}{4}|x|^2 + |y|^2$ for any $x, y \in \mathbb{R}^d$ has been used. Now let us establish (5.12). From Assumption 5.1 and the property of truncated function, we have that

$$\begin{aligned}
&|f_\Delta(x,y)|^2 = |F_1(x,y) + F_\Delta(x,y)|^2\\
&\le 2|F_1(x,y)|^2 + 2|F_\Delta(x,y)|^2 \le 2(\bar{L} + \bar{L}_1)(|x|^2 + |y|^2), \text{ if } |x| \le 1,\ |y| \le 1, \qquad (5.15)
\end{aligned}$$

and

$$\begin{aligned}
|f_\Delta(x,y)|^2 &\le 2\bar{L}(|x|^2 + |y|^2) + 2(\varphi(\Delta))^2(1 + |x| + |y|)^2\\
&\le (2\bar{L} + 8(\varphi(\Delta))^2)(|x|^2 + |y|^2), \text{ if } |x| \le 1,\ |y| > 1. \qquad (5.16)
\end{aligned}$$

This also holds for $|x| > 1$, $|y| \le 1$ or $|x| \ge 1$, $|y| \ge 1$. Thus,

$$|f_\Delta(x,y)|^2\Delta \le (4\bar{L} + 2\bar{L}_1)\Delta + 8(\varphi(\Delta))^2\Delta(|x|^2 + |y|^2),\ \forall x, y \in \mathbb{R}^d.$$

Similarly,

$$|h_\Delta(x,y)|^2\Delta \le (4\bar{L} + 2\bar{L}_1)\Delta + 8(\varphi(\Delta))^2\Delta(|x|^2 + |y|^2),\ \forall x, y \in \mathbb{R}^d.$$

Consequently,

$$\begin{aligned}
&2\lambda\langle f_\Delta(x,y), h_\Delta(x,y)\rangle + |f_\Delta(x,y)|^2 + \lambda^2|h_\Delta(x,y)|^2\\
&\le \lambda|f_\Delta(x,y)|^2 + \lambda|h_\Delta(x,y)|^2 + |f_\Delta(x,y)|^2 + \lambda^2|h_\Delta(x,y)|^2\\
&\le (1+\lambda)^2(|f_\Delta(x,y)|^2 \vee |h_\Delta(x,y)|^2)\\
&\le (1+\lambda)^2(4\bar{L} + 2\bar{L}_1)\Delta + 8(1+\lambda)^2(\varphi(\Delta))^2\Delta(|x|^2 + |y|^2).
\end{aligned}$$

Thus, the proof is finished. □

The following theorem shows that the partially truncated EM solution can share the mean-square and $H_\infty$ stabilities of the true solution.

**Theorem 5.1.** *Let Assumptions 2.2, 5.1 and 5.2 hold with*

$$\nu_1 > \alpha_1 + \frac{1}{4}\nu_2 + (\lfloor(1-\hat{\delta})^{-1}\rfloor + 1)(\nu_2 + \alpha_2) \quad \textit{and} \quad \alpha_3 > (\lfloor(1-\hat{\delta})^{-1}\rfloor + 1)\alpha_4 \ge 0.$$

*Choose* $\Delta^\star \in (0,1]$ *such that*

$$\epsilon_{\Delta^\star} = \frac{\nu_1 - \alpha_1 - \frac{1}{4}\nu_2 - (\lfloor (1-\hat{\delta})^{-1}\rfloor + 1)(\alpha_2 + \nu_2)}{1 + (\lfloor (1-\hat{\delta})^{-1}\rfloor + 1)}, \tag{5.17}$$

*where* $\epsilon_{\Delta^\star}$ *is defined in* (5.13). *Then for any* $\Delta \in (0, \Delta^\star)$ *and any initial data* (2.2), *the truncated EM approximation* $\{y_k\}_{k\geq 0}$ *with the truncated coefficients* $f_\Delta$ *and* $g_\Delta$ *as well as* $h_\Delta$ *given by* (5.10) *has the property that*

$$\limsup_{k\to\infty} \frac{\log \mathbb{E}|y_k|^2}{t_k} \leq -\left(\gamma_\Delta^\star \wedge \frac{1}{\tau}\log \frac{\alpha_3}{(\lfloor (1-\hat{\delta})^{-1}\rfloor + 1)\alpha_4}\right), \tag{5.18}$$

*where* $\gamma_\Delta^\star$ *is the unique root to the following equation*

$$\nu_1 = \left(\alpha_1 + \frac{1}{4}\nu_2 + \epsilon_\Delta\right) + (\lfloor (1-\hat{\delta})^{-1}\rfloor + 1)(\nu_2 + \alpha_2 + \epsilon_\Delta)e^{\gamma_\Delta^\star \tau} + \frac{1 - e^{-\gamma_\Delta^\star \Delta}}{\Delta}. \tag{5.19}$$

*Moreover,*

$$\lim_{\Delta\to 0} \gamma_\Delta^\star < \gamma^\star, \tag{5.20}$$

*and*

$$\int_0^\infty \mathbb{E}|Z_1(s)|^2 ds < \infty. \tag{5.21}$$

*Proof.* Recall (4.10) that

$$\begin{aligned}|y_{k+1}|^2 = {} & |y_k|^2 + 2\langle y_k, f_\Delta(y_k, y_{k-\delta_k})\rangle\Delta + |g_\Delta(y_k, y_{k-\delta_k})|^2\Delta + 2\langle y_k, h_\Delta(y_k, y_{k-\delta_k})\rangle\Delta N_k \\ & + |h_\Delta(y_k, y_{k-\delta_k})\Delta N_k|^2 + |f_\Delta(y_k, y_{k-\delta_k})|^2\Delta^2 \\ & + 2\langle f_\Delta(y_k, y_{k-\delta_k}), h_\Delta(y_k, y_{k-\delta_k})\rangle\Delta N_k\Delta + J_k,\end{aligned}$$

where $J_k$ has been defined in Lemma 4.3. Obviously, $\mathbb{E}J_k = 0$. Using Lemma 5.2 yields

$$\begin{aligned}& 2\langle y_k, f_\Delta(y_k, y_{k-\delta_k})\rangle + |g_\Delta(y_k, y_{k-\delta_k})|^2 + \lambda\Big(2\langle y_k, h_\Delta(y_k, y_{k-\delta_k})\rangle + |h_\Delta(y_k, y_{k-\delta_k})|^2\Big) \\ & \leq -\Big(\nu_1 - \alpha_1 - \frac{1}{4}\alpha_2\Big)|y_k|^2 + (\nu_2 + \alpha_2)|y_{k-\delta_k}|^2 - \alpha_3|\pi_\Delta(y_k)|^\beta + \alpha_4|\pi_\Delta(y_{k-\delta_k})|^\beta.\end{aligned} \tag{5.22}$$

Thus,

$$\begin{aligned}\mathbb{E}|y_{k+1}|^2 \leq {} & \mathbb{E}|y_k|^2 - \Big(\nu_1 - \alpha_1 - \frac{1}{4}\alpha_2\Big)\mathbb{E}|y_k|^2\Delta + (\nu_2 + \alpha_2)\mathbb{E}|y_{k-\delta_k}|^2\Delta \\ & + \mathbb{E}|f_\Delta(y_k, y_{k-\delta_k})|^2\Delta^2 + 2\lambda\mathbb{E}\langle f_\Delta(y_k, y_{k-\delta_k}), h_\Delta(y_k, y_{k-\delta_k})\rangle\Delta^2 \\ & + \lambda^2\mathbb{E}|h_\Delta(y_k, y_{k-\delta_k})|^2\Delta^2 + \mathbb{E}\Big[-\alpha_3|\pi_\Delta(y_k)|^\beta + \alpha_4|\pi_\Delta(y_{k-\delta_k})|^\beta\Big]\Delta, \ \forall k \geq 0.\end{aligned} \tag{5.23}$$

Using (5.12), we have

$$\begin{aligned}\mathbb{E}|y_{k+1}|^2 \leq {} & \mathbb{E}|y_k|^2 - \Big(\nu_1 - \alpha_1 - \frac{1}{4}\alpha_2 - \epsilon_\Delta\Big)\mathbb{E}|y_k|^2\Delta + (\nu_2 + \alpha_2 + \epsilon_\Delta)\mathbb{E}|y_{k-\delta_k}|^2\Delta \\ & + \mathbb{E}\Big(-\alpha_3|\pi_\Delta(y_k)|^\beta\Delta + \alpha_4|\pi_\Delta(y_{k-\delta_k})|^\beta\Delta\Big), \ \forall k \geq 0.\end{aligned} \tag{5.24}$$

For an arbitrary constant $r > 1$, we see from (5.24) that

$$\begin{aligned}
&r^{(k+1)\Delta}\mathbb{E}|y_{k+1}|^2 - r^{k\Delta}\mathbb{E}|y_k|^2 \\
&\le (r^{(k+1)\Delta} - r^{k\Delta})\mathbb{E}|y_k|^2 - \Big(\nu_1 - \alpha_1 - \frac{1}{4}\alpha_2 - \epsilon_\Delta\Big) r^{(k+1)\Delta}\mathbb{E}|y_k|^2\Delta \\
&\quad + (\nu_2 + \alpha_2 + \epsilon_\Delta) r^{(k+1)\Delta}\mathbb{E}|y_{k-\delta_k}|^2\Delta \\
&\quad + \mathbb{E}\Big[ - \alpha_3 r^{(k+1)\Delta}|\pi_\Delta(y_k)|^\beta\Delta + \alpha_4 r^{(k+1)\Delta}|\pi_\Delta(y_{k-\delta_k})|^\beta\Big]\Delta,\ \forall k \ge 0. \qquad (5.25)
\end{aligned}$$

Consequently,

$$\begin{aligned}
&r^{(k+1)\Delta}\mathbb{E}|y_{k+1}|^2 \\
&\le |\xi(0)|^2 + \left(-\Big(\nu_1 - \alpha_1 - \frac{1}{4}\alpha_2 - \epsilon_\Delta\Big)\Delta + 1 - r^{-\Delta}\right)\sum_{j=0}^{k} r^{(j+1)\Delta}\mathbb{E}|y_j|^2 \\
&\quad + (\nu_2 + \alpha_2 + \epsilon_\Delta)\sum_{j=0}^{k} r^{(j+1)\Delta}\mathbb{E}|y_{j-\delta_j}|^2\Delta \\
&\quad + \mathbb{E}\left[-\alpha_3\sum_{j=0}^{k} r^{(j+1)\Delta}|\pi_\Delta(y_j)|^\beta + \alpha_4\sum_{j=0}^{k} r^{(j+1)\Delta}|\pi_\Delta(y_{j-\delta_j})|^\beta\right]\Delta,\ \forall k \ge 0. \qquad (5.26)
\end{aligned}$$

By Lemma 2.1, we get

$$\begin{aligned}
&\sum_{j=0}^{k} r^{(j+1)\Delta}|y_{j-\delta_j}|^2 \\
&\le (\lfloor(1-\hat{\delta})^{-1}\rfloor + 1)r^\tau \sum_{j=-M}^{k} r^{(j+1)\Delta}|y_j|^2 \\
&= (\lfloor(1-\hat{\delta})^{-1}\rfloor + 1)r^\tau \sum_{j=-M}^{-1} r^{(j+1)\Delta}|y_j|^2 + (\lfloor(1-\hat{\delta})^{-1}\rfloor + 1)r^\tau \sum_{j=0}^{k} r^{(j+1)\Delta}|y_j|^2 \\
&\le \frac{(\lfloor(1-\hat{\delta})^{-1}\rfloor + 1)r^\tau}{1 - r^{-\Delta}}\|\xi\|^2 + \bar{\kappa} r^\tau \sum_{j=0}^{k} r^{(j+1)\Delta}|y_j|^2,\ \forall k \ge 0, \qquad (5.27)
\end{aligned}$$

and

$$\sum_{j=0}^{k} r^{(j+1)\Delta}|\pi_\Delta(y_{j-\delta_j})|^\beta \le \frac{(\lfloor(1-\hat{\delta})^{-1}\rfloor + 1)r^\tau}{1 - r^{-\Delta}}\|\xi\|^\beta + \bar{\kappa} r^\tau \sum_{j=0}^{k} r^{(j+1)\Delta}|\pi_\Delta(y_j)|^\beta,\ \forall k \ge 0. \qquad (5.28)$$

Inserting (5.27) and (5.28) into (5.26) gives that

$$\begin{aligned}
r^{(k+1)\Delta}\mathbb{E}|y_{k+1}|^2 &\le H_0(r,\Delta) - H_1(r,\Delta)\sum_{j=0}^{k} r^{(j+1)\Delta}\mathbb{E}|y_j|^2\Delta \\
&\quad - H_2(r)\sum_{j=0}^{k} r^{(j+1)\Delta}\mathbb{E}|\pi_\Delta(y_j)|^2\Delta,\ \forall k \ge 0, \qquad (5.29)
\end{aligned}$$

where

$$\begin{aligned} H_0(r,\Delta) &= \|\xi\|^2 + (\lfloor(1-\hat{\delta})^{-1}\rfloor+1)r^\tau\Big[(\nu_2+\alpha_2+\epsilon_\Delta)\|\xi\|^2+\alpha_4\|\xi\|^\beta\Big]\frac{\Delta}{1-r^{-\Delta}},\\ H_1(r,\Delta) &= \Big[\nu_1-\alpha_1-\frac{1}{4}\alpha_2-\epsilon_\Delta-(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)r^\tau(\nu_2+\alpha_2+\epsilon_\Delta)\Big]-\frac{1-r^{-\Delta}}{\Delta},\\ H_2(r) &= \alpha_3-(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)r^\tau\alpha_4. \end{aligned} \tag{5.30}$$

Choose $\Delta^\star\in(0,1]$ such that (5.17) holds, i.e.,

$$\epsilon_{\Delta^\star}=\frac{\nu_1-\alpha_1-\frac{1}{4}\nu_2-(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)(\alpha_2+\nu_2)}{1+(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)}.$$

Then for any $\Delta\in(0,\Delta^\star)$, we have

$$H_1(1,\Delta)=\nu_1-\alpha_1-\frac{1}{4}\nu_2-\epsilon_\Delta-(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)(\nu_2+\alpha_2+\epsilon_\Delta)>0 \tag{5.31}$$

$$H_1(\bar{r},\Delta)=-\frac{1-\bar{r}^{-\Delta}}{\Delta}<0,\ \text{with}\ \left(\frac{\nu_1-\alpha_1-\frac{1}{4}\nu_2-\epsilon_\Delta}{(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)(\nu_2+\alpha_2+\epsilon_\Delta)}\right)^{1/\tau}>1 \tag{5.32}$$

and

$$\frac{dH_1(r,\Delta)}{dr}<0. \tag{5.33}$$

From (5.31), (5.32) and (5.33), there is a positive constant $r_1^\star=r_1^\star(\Delta)\in(1,\bar{r})$ such that $H_1(r_1^\star,\Delta)=0$. Let $r^\star=r^\star(\Delta)=r_1^\star(\Delta)\wedge r_2^\star$ with $r_2^\star=\left(\frac{\alpha_3}{(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)\alpha_4}\right)^{1/\tau}>1$, then for any $1<r<r^\star(\Delta)$, we have $H_1(r,\Delta)>0$ and $H_2(r)>0$, thus we conclude from (5.29) that

$$r^{(k+1)\Delta}\mathbb{E}|y_{k+1}|^2\le H_0(r,\Delta)<\infty,\ \forall k\ge 0.$$

Therefore,

$$\limsup_{k\to\infty}\frac{\log\mathbb{E}|y_k|^2}{t_k}\le-\log r. \tag{5.34}$$

Bearing in mind that $r_2^\star=\mathrm{e}^{\gamma_2^\star}$ and setting $r=\mathrm{e}^\gamma$, $r_1^\star=\mathrm{e}^{\gamma_1}$, then (5.34) becomes (5.18). Moreover, notice that $\epsilon_\Delta\to 0$ and $\frac{1-\mathrm{e}^{-r_1^\star\Delta}}{\Delta}\to r_1^\star$ as $\Delta\to 0$. Due to $(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)>(1-\hat{\delta})^{-1}$, we obtain the assertion (5.20) by comparing (5.8) with (5.19).
Finally, we begin to establish (5.21). By (5.24), we have

$$\begin{aligned}\mathbb{E}|y_{k+1}|^2 &\le |\xi(0)|^2-\Big(\nu_1-\alpha_1-\frac{1}{4}\alpha_2-\epsilon_\Delta\Big)\sum_{j=0}^k\mathbb{E}|y_j|^2\Delta+(\nu_2+\alpha_2+\epsilon_\Delta)\sum_{j=0}^k\mathbb{E}|y_{j-\delta_j}|^2\Delta\\ &\quad+\mathbb{E}\left[-\alpha_3\sum_{j=0}^k|\pi_\Delta(y_j)|^\beta+\alpha_4\sum_{j=0}^k|\pi_\Delta(y_{j-\delta_j})|^\beta\right]\Delta,\ \forall k\ge 0.\end{aligned} \tag{5.35}$$

Thus, Lemma 2.1 gives that

$$\sum_{j=0}^{k}|y_{j-\delta_j}|^2 \le (\lfloor(1-\hat{\delta})^{-1}\rfloor+1)M\|\xi\|^2+\sum_{j=0}^{k}|y_j|^2 \tag{5.36}$$

and

$$\sum_{j=0}^{k}|\pi_\Delta(y_{j-\delta_j})|^\beta \le (\lfloor(1-\hat{\delta})^{-1}\rfloor+1)M\|\xi\|^\beta+\sum_{j=0}^{k}|\pi_\Delta(y_j)|^\beta. \tag{5.37}$$

Substituting (5.36) and (5.37) into (5.35), we obtain that for any $\Delta\in(0,\Delta^\star)$,

$$\begin{aligned}0\le\mathbb{E}|y_{k+1}|^2 &\le \|\xi\|^2+(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)\tau\Big((\nu_2+\alpha_2+\epsilon_\Delta)\|\xi\|^2+\alpha_4\|\xi\|^\beta\Big)\\ &\quad-(\alpha_3-(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)\alpha_4)\mathbb{E}\left[\sum_{j=0}^{k}|\pi_\Delta(y_j)|^\beta\right]\Delta\\ &\quad-\Big[\nu_1-\alpha_1-\frac{1}{4}\alpha_2-\epsilon_\Delta-(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)(\nu_2+\alpha_2+\epsilon_\Delta)\Big]\sum_{j=0}^{k}\mathbb{E}|y_j|^2\Delta,\ \forall k\ge 0,\end{aligned}$$

which means

$$\sum_{j=0}^{k}\mathbb{E}|y_j|^2\Delta \le \frac{\|\xi\|^2+(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)\tau\Big((\nu_2+\alpha_2+\epsilon_\Delta)\|\xi\|^2+\alpha_4\|\xi\|^\beta\Big)}{\nu_1-\alpha_1-\frac{1}{4}\alpha_2-\epsilon_\Delta-(\lfloor(1-\hat{\delta})^{-1}\rfloor+1)(\nu_2+\alpha_2+\epsilon_\Delta)}<\infty,$$

holds for any $k\ge 0$. Letting $k\to\infty$ gives (5.21). Thus, the proof is finished. □

## 6. Numerical examples

In this section, we provide some applications of our method under different situations. Examples 6.1 and 6.2 show the superiority of our method for SDEs and SDDEs with jumps in convergence order. Examples 6.3 and 6.4 illustrate the convergence rate and stability of our method for non-jump SDDEs with time-variable delay, respectively.

**Example 6.1.** *Consider the super-linear scalar SDDE with Poisson jumps (see, [19, Example 1])*

$$\begin{aligned}dX(t) &= \left[2X(t)-5X^3(t)+\frac{1}{8}|X(t-\tau)|^{5/4}\right]dt+\left[\frac{1}{2}|X(t)|^{3/2}+X(t-\tau)\right]dB(t)\\ &\quad+\left[X(t^-)+X((t-\tau)^-)\right]dN(t),\ t\ge 0,\end{aligned} \tag{6.1}$$

*with initial data* $\{X(t):-\tau\le t\le 0\}=\xi\in C([-\tau,0];\mathbb{R})$*, where* $B(t)$ *is a one-dimensional Brownian motion and* $N(t)$ *is a Poisson process with intensity* $\lambda=0.2$*. We conclude from [19, Example 1] that Assumptions 3.1 and 3.2 are satisfied with* $l=4$*,* $p_0=26$*,* $p_1=2$*,* $\varepsilon=25$*,* $\hat{\delta}=0$ *and* $U(x_1,x_2)=0.5(x_1^2+x_2^2)|x_1-x_2|^2$*. By [19, (5.2), Example 1], we set*

$$\mu(R)=10R^2,\ \forall R\ge 1 \quad \textit{and} \quad \varphi(\Delta)=10\Delta^{-1/3},\ \forall\Delta\in(0,1].$$

*Set $h_\Delta = h$, but define $f_\Delta$ and $g_\Delta$ by (3.7). Then, according to Theorem 3.1, the truncated EM method has the $\mathcal{L}^2$-convergence order*

$$2\varrho \wedge \left(1 - \frac{l}{p_0}\right) = 2\varrho \wedge \frac{22}{26} \approx 2\varrho \wedge 0.85. \tag{6.2}$$

*However, if we apply the truncated EM method from [19] to (6.1), then the convergence order is*

$$2\varrho \wedge 0.75. \tag{6.3}$$

*From (6.2) and (6.3), we conclude that our method has a better convergence order than that of [19] even for the jump-diffusion SDDEs with linear jump coefficient and constant delay.*

**Example 6.2.** *Consider a jump extended version of the 3/2-volatility model (see, e.g., [9, 40])*

$$\begin{aligned} dX(t) &= 3X(t)(1-|X(t)|)dt + 0.5|X(t)|^{3/2}dB(t) + 0.1X(t^-)\log(1+X^2(t^-))dN(t), t \geq 0 \\ X(0) &= 10, \end{aligned} \tag{6.4}$$

*where $B(t)$ is a one-dimensional Brownian motion and $N(t)$ is a Poisson process with intensity $\lambda = 1$. By [9, Example 1], we deduce that Assumptions 3.1 and 3.2 hold with $p_1 = 2$, $U \equiv 0$, $p_0 = 20$, $\varepsilon = 19$, $l = 2$. Set $\delta(t) \equiv 0$ and*

$$\mu(R) = 4R,\ \forall R \geq 1 \quad \text{and} \quad \varphi(\Delta) = 4\Delta^{-1/3},\ \forall \Delta \in (0,1],$$

*where $f_\Delta$, $g_\Delta$ and $h_\Delta$ are defined by (3.7). Then, by Theorem 3.1, the truncated EM method has the $\mathcal{L}^2$-convergence order $1 - \frac{l}{p_0} = 0.9$. However, if we apply the tamed EM method from [9] to (6.4), then the convergence order will become $1 - 2\frac{l}{p_0} = 0.8$. Compared with the method of [9, Theorem 4.5], our method has a slightly better convergence order for the SDEs with Poisson jumps under almost the same conditions.*

**Example 6.3.** *Consider the super-linear non-jump SDDE with time-variable delay (see, [17])*

$$dX(t) = \left[-9X^3(t) + |X(t-\delta(t))|^{3/2}\right]dt + X^2(t)dB(t),\ t \geq 0, \tag{6.5}$$

*with initial data $\{X(t) : -\tau \leq t \leq 0\} = \xi \in C([-\tau, 0]; \mathbb{R})$, where $B(t)$ is a one-dimensional Brownian motion. Assume that $\delta$ satisfies Assumption 2.2. Clearly, the coefficients*

$$f(x,y) = -9x^3 + |y|^{3/2} \quad \text{and} \quad g(x,y) = x^2,\ \forall x, y \in \mathbb{R}, \tag{6.6}$$

*are locally Lipschitz continuous. Moreover, if $p_0 = 18.5$, then*

$$\begin{aligned} xf(x,y) + \frac{p_0 - 1}{2}|g(x,y)|^2 &= -9x^4 + x|y|^{3/2} + 8.75x^4 \\ &\leq -9x^4 + 8.75x^4 + 0.25x^4 + 0.75y^2 = 0.75y^2, \end{aligned}$$

*which means that Assumption 3.1 is satisfied. For any $x_1, x_2, y_1, y_2, \in \mathbb{R}$, we have*

$$\begin{aligned} &(x_1 - x_2)(f(x_1,y_1) - f(x_2,y_2)) \\ &\leq -4.5(x_1^2 + x_2^2)|x_1 - x_2|^2 + 0.5|x_1 - x_2|^2 + 2.25|y_1 - y_2|^2 + 2.25(y_1^2 + y_2^2)|y_1 - y_2|^2 \end{aligned}$$

*and*

$$|g(x_1,y_1)-g(x_2,y_2)|^2=|x_1^2-x_2^2|^2\le 2(x_1^2+x_2^2)|x_1-x_2|^2,$$

*see [17, p. 2086]. Consequently,*

$$\begin{aligned}&(x_1-x_2)(f(x_1,y_1)-f(x_2,y_2))+\frac{p_1-1}{2}|g(x_1,y_1)-g(x_2,y_2)|^2\\&\le 0.5|x_1-x_2|^2+2.25|y_1-y_2|^2-(5.5-p_1)(x_1^2+x_2^2)|x_1-x_2|^2+1.125(y_1^2+y_2^2)|y_1-y_2|^2.\end{aligned}\tag{6.7}$$

*If we set* $p_1=3.25$, $\hat{\delta}=0.5$ *and* $U(x_1,x_2)=1.125(x_1^2+x_2^2)|x_1-x_2|^2$, *then* (6.7) *becomes*

$$\begin{aligned}&(x_1-x_2)(f(x_1,y_1)-f(x_2,y_2))+\frac{p_1-1}{2}|g(x_1,y_1)-g(x_2,y_2)|^2\\&\le 0.5|x_1-x_2|^2+2.25|y_1-y_2|^2-\frac{1}{1-\hat{\delta}}U(x_1,x_2)+U(y_1,y_2).\end{aligned}$$

*Moreover, it is straightforward to show that* (3.2) *is satisfied with* $l=4$. *Thus, we have verified Assumption* (3.2) *with* $p_0\ge 2+2.5l$. *From* (6.6) *and* (3.4), *we may set*

$$\mu(R)=10R^2,\ \forall R\ge 1\quad \textit{and}\quad \varphi(\Delta)=10\Delta^{-1/3},\ \forall\Delta\in(0,1].$$

*Then, by Theorem 3.2, for any* $\Delta\in(0,1]$ *the truncated EM solution* $Y_\Delta$ *defined in* (2.11) *will converge to the true solution* $X$ *in the sense that*

$$\mathbb{E}|X(T)-Y_\Delta(T)|^2\le C\Delta,\ \forall T>0.\tag{6.8}$$

*However, if constant delay is considered in SDDE* (6.5), *we may also apply the truncated EM method from [17] to SDDE* (6.5) *by setting*

$$\mu(R)=10R^3,\ \forall R\ge 1\quad \textit{and}\quad \varphi(\Delta)=10\Delta^{-1/5},\ \forall\Delta\in(0,1],$$

*due to*

$$\sup_{|x|\vee|y|\le R}\Big(|f(x,y)|\vee|g(x,y)|\Big)\le 10R^3,\ \forall R\ge 1.$$

*Thus according to [17, Corollary 3.7], for any* $\Delta\in(0,1]$ *the truncated EM solution* $\widetilde{Y}_\Delta$ *defined in [17] will converge to the true solution* $X$ *in the sense that*

$$\mathbb{E}|X(T)-\widetilde{Y}_\Delta(T)|^2\le C\Delta^{3/5},\ \forall T>0,\tag{6.9}$$

*see [17, p. 2086]. We observe from* (6.8) *and* (6.9) *that the* $\mathcal{L}^2$*-convergence order of the truncated EM method in this paper can arrive at* 1.0, *however if we use the method from [17], the corresponding order will be only* 0.6.

*Now, set* $\tau=1$, $\delta(t)=0.5-0.5\sin(t)$ *and* $X(t)=2$ *for any* $t\in[-\tau,0]$. *Truncated EM solution with step size* $\Delta=2^{-14}$ *is taken as the replacement of the true solution. Fig. 6.1(a) illustrates the root of mean-square errors with different step sizes* $2^{-7}$, $2^{-8},\cdots$ $2^{-11}$ *at time* $T=10$ *for* 500 *simulations. A least square fit of errors yields the strong convergence order* 0.5134 *and thus is close to the theoretical value* 0.5.

**Example 6.4.** *Let us consider the stochastic delay power logistic model (see, e.g., [4, 5])*

$$dX(t) = X(t)[a + bX(t-\delta(t)) - X^2(t)]dt + cX(t)X(t-\delta(t))dB(t),\ t \geq 0 \tag{6.10}$$

*with initial data* $\{X(t) : -\tau \leq t \leq 0\} = \xi \in C([-\tau, 0]; \mathbb{R})$, *where* $B(t)$ *is a one-dimensional Brownian motion and* $a, b, c$ *are all constants. Assume that* $\xi$ *satisfies Assumption 2.1 with* $K_0 = 2$, $\varrho = 0.5$ *and* $\delta$ *satisfies Assumption 2.2. Set*

$$f(x,y) = F_1(x,y) + F(x,y) \quad \text{and} \quad g(x,y) = G_1(x,y) + G(x,y)$$

*where*

$$F_1(x,y) = ax, \quad G_1(x,y) = 0, \quad F(x,y) = bxy - x^2, \quad G(x,y) = cxy,$$

*for any* $x, y \in \mathbb{R}$. *Clearly, Assumption 5.1 holds. Put* $\theta = \infty$. *Then*

$$2xF_1(x,y) + (1+\theta)|G_1(x,y)|^2 = 2ax^2,$$

*and the elementary inequality yields that*

$$\begin{aligned} &2xF(x,y) + (1+\theta^{-1})|G(x,y)|^2 = 2x(bxy - x^3) + (cxy)^2 \\ &\leq 0.5x^4 + 0.5(2by)^2 - 2x^4 + 0.5x^4 + 0.5(c^2y^2)^2 = 2b^2y^2 - x^4 + 0.5c^4y^4. \end{aligned}$$

*Thus,*

$$\lambda_1 = -2a, \quad \nu_2 = 0, \quad \alpha_1 = 0, \quad \alpha_2 = 2b^2, \quad \alpha_3 = 1, \quad \alpha_4 = 0.5c^4, \quad \beta = 4.$$

*If we let*

$$0.5c^4(\lfloor(1-\hat{\delta})^{-1}\rfloor + 1) < 1 \quad \text{and} \quad -2a > 2b^2(\lfloor(1-\hat{\delta})^{-1}\rfloor + 1),$$

*which means that condition* (5.9) *is satisfied, then by Lemma 5.1, the true solution* $X(t)$ *to SDDE* (6.10) *has the property that*

$$\limsup_{t\to\infty} \frac{\log \mathbb{E}|X(t)|^2}{t} \leq -\left(\gamma^\star \wedge \frac{1}{\tau}\log\frac{(1-\hat{\delta})}{0.5c^4}\right), \tag{6.11}$$

*and* $\int_0^\infty \mathbb{E}|X(t)|^2 dt < \infty$, *where* $\gamma^\star > 0$ *is the unique root to the following equation*

$$-2a = \gamma^\star + 2b^2\frac{1}{(1-\hat{\delta})}e^{\gamma^\star\tau}.$$

*On the other hand, take*

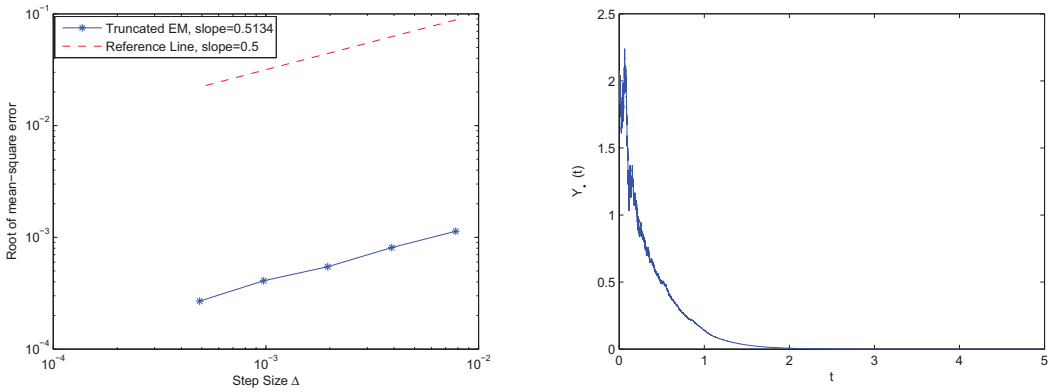
$$\mu(R) = \Big((|b|+1)\vee|c|\Big)R^2,\ \forall R \geq 1 \quad \text{and} \quad \varphi(\Delta) = \Big((|b|+1)\vee|c|\Big)\Delta^{-1/4},\ \forall \Delta \in (0,1].$$

*Then, we apply the EM scheme* (2.5) *with partially truncated coefficients* $f_\Delta$ *and* $g_\Delta$ *given by* (5.10) *to SDDE* (6.10). *Let* $\{y_k\}_{k\geq 0}$ *be the discrete truncated EM approximation. According to Theorem 5.1, for any* $\Delta \in (0, \Delta^\star)$ *and any initial data, the truncated EM approximation* $\{y_k\}_{k\geq 0}$ *has the property that*

$$\limsup_{k\to\infty} \frac{\log \mathbb{E}|y_k|^2}{t_k} \leq -\left(\gamma_\Delta^\star \wedge \frac{1}{\tau}\log\frac{1}{0.5c^4(\lfloor(1-\hat{\delta})^{-1}\rfloor + 1)}\right), \tag{6.12}$$

Table 6.1:: $\epsilon_\Delta$ and $\gamma^\star_\Delta$ with different step sizes for solving (6.13)

| $\Delta$ | $10^{-4}$ | $10^{-5}$ | $10^{-6}$ | $10^{-7}$ | $10^{-8}$ | $10^{-9}$ |
|---|---|---|---|---|---|---|
| $\epsilon_\Delta$ | 0.3220 | 0.1014 | 0.0320 | 0.0101 | 0.0032 | 0.0010 |
| $\gamma^\star_\Delta$ | 0.6982 | 1.1728 | 1.3272 | 1.3764 | 1.3920 | 1.3970 |




(a) Root of mean-square errors for (6.5) (b) A sample path of $Y_\Delta(t)$ for (6.10)

Fig. 6.1.: Numerical simulations for (6.5) and (6.10)

*and* $\sum_{k=0}^{\infty} \mathbb{E}|y_k|^2\Delta < \infty$, *where* $\gamma^\star_\Delta > 0$ *is the unique root to the following equation*

$$-2a = \frac{1-e^{-\gamma^\star_\Delta \Delta}}{\Delta} + \epsilon_\Delta + (\lfloor (1-\hat{\delta})^{-1} \rfloor + 1)(2b^2 + \epsilon_\Delta)e^{\gamma^\star \tau}. \tag{6.13}$$

*Numerically, if we set*

$$\hat{\delta} = 0.05, \quad \tau = 0.1, \quad a = -3, \quad b = 1, \quad c = 0.5, \quad \varphi(\Delta) = 2\Delta^{-1/4},$$

*then*

$$(\lfloor (1-\hat{\delta})^{-1} \rfloor + 1) = 2, \quad \gamma^\star = 3.1230, \quad \bar{L} = 5, \quad \bar{L}_R = (3R^2+1), \quad \epsilon_\Delta = 20\Delta + 32\Delta^{1/2}. \tag{6.14}$$

*Solving* (5.17) *gives* $\Delta^\star = 4.2308 \times 10^{-4}$. *Computational results for* $\epsilon_\Delta$ *and* $\gamma^\star_\Delta$ *with different step sizes* $\Delta$ *are shown in Table 6.1. Fig. 6.1 (b) illustrates a simple path of the truncated EM solution* $Y_\Delta(t)$ *with step size* $\Delta = 10^{-4}$ *and* $\delta(t) = 0.05 - 0.05\sin(t)$. *We observe from Table 6.1 and Fig. 6.1 (b) that numerical experiments support the findings from Theorem 5.1.*

**Acknowledgments.**

The authors would like to thank the Open Project of Anhui Province Center for International Research of Intelligent Control of High-end Equipment (IRICHE-01), the Natural Science Foundation of Universities in Anhui Province (KJ2020A0367, KJ2020A0368, KJ2021A0515), the Startup Foundation for Introduction Talent of AHPU (2021YQQ058, 2020YQQ066), the Pre-research Foundation of National Natural Science Foundation of China of AHPU (Xjky08201906),

the Royal Society (WM160014, Royal Society Wolfson Research Merit Award), the Royal Society and the Newton Fund (NA160317, Royal Society-Newton Advanced Fellowship), the Royal Society of Edinburgh (RSE1832) for their financial support.

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