On the general δ-shock model

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Abstract

The δ-shock model is one of the basic shock models which has a wide range of applications in reliability, finance and related fields. In existing literature, it is assumed that the recovery time of a system from the damage induced by a shock is constant as well as the shocks magnitude. However, as technical systems gradually deteriorate with time, it takes more time to recover from this damage, whereas the larger magnitude of a shock also results in the same effect. Therefore, in this paper, we introduce a general δ-shock model when the recovery time depends on both the arrival times and the magnitudes of shocks. Moreover, we also consider a more general and flexible shock process, namely, the Poisson generalized gamma process. It includes the homogeneous Poisson process, the non-homogeneous Poisson process, the Pólya process and the generalized Pólya process as the particular cases. For the defined survival model, we derive the relationships for the survival function and the mean lifetime and study some relevant stochastic properties. As an application, an example of the corresponding optimal replacement policy is discussed.

Keywords: Reliability, δ-shock model, Poisson generalized gamma process, homogeneous Poisson process.

1 Introduction

Shock models are widely used in reliability, finance and related fields for stochastic description of lifetimes of systems operating in random environments. In the literature, these models are usually classified into four major types, namely, the extreme shock model, the cumulative shock model, the run shock model and the δ-shock model. In the extreme shock model, a system fails due to a single shock of large magnitude (see Gut and Hüsler [15, 16], Shanthikumar and Sumita [33, 34], Cha and Finkelstein [6], and the references therein). In the cumulative shock model, a system fails if the additive damage due to shocks exceeds some predefined level (see A-Hameed and Proschan [2], Esary et al. [12], Gut [14],

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to name a few). Further, in the run shock model, a system fails due to multiple occurrences of large shocks (see, e.g., Mallor and Omey [27], Ozkut and Eryilmaz [30]). Lastly, in the classical δ-shock model, a system fails if the time lag between two successive shocks is less than a predefined threshold value δ (see Li, Chan and Yuan [22], Li, Huang and Wang [23], Li and Kong [24], and references therein). Besides, there are various mixed shock models that are combinations of two or more shock models (see Cha and Finkelstein [4], Finkelstein [13], Eryilmaz and Tekin [11], Wang and Zhang [39], Parvardeh and Balakrishnan [31], Mallor et al. [29], Eryilmaz [8], Goyal et al. [17], to name a few). Further, by considering different interrelations among model variables (e.g., shock's magnitude, shock's arrival time, intershock time, damage caused by a shock, deterioration process, etc.), various other shock models have been considered in the literature. Mallor and Santos [28] studied a general shock model, which generalizes the extreme, the cumulative and the run shock models, by considering correlation among the intershock time, the magnitude and the damage caused by the shock. Ranjkesh et al. [32] have considered a new cumulative shock model by assuming dependency between the damage caused by a shock and the inter-arrival time. Recently, Wang et al. [40] proposed an interdependency structure between random shocks and natural degradation process.

The δ-shock model based on the homogeneous Poisson process (HPP) was first studied by Li et al. [22, 23]. These authors have derived the corresponding reliability function and the mean lifetime of a system for this model. Later, Wang and Zhang [39] have considered two types of failures when a system fails due to either one single shock of large magnitude or the inter-arrival time between two successive shocks exceeds the predefined threshold value δ. For this model, they have studied some reliability indices (namely, the system's reliability function, the mean lifetime, etc.) and also have discussed the corresponding optimal replacement policy. Further, Parvardeh and Balakrishnan [31] have discussed the δ-shock model based on the renewal process of shocks. A δ-shock model based on the non-homogeneous Poisson process (NHPP) with periodic intensity of period δ was studied by Li and Kong [24]. Eryilmaz [8] have introduced the mixed shock model by combining the δ-shock model with the run shock model. Further, Eryilmaz and Bayramoglu [9] have studied a δ-shock model based on the renewal process of shocks with the uniformly distributed inter-arrival times. Wang and Peng [38] have discussed the generalized δ-shock model with two types of shocks with different threshold values δ_1 and δ_2. Furthermore, Eryilmaz [10] have studied the δ-shock model based on the Pólya process of shocks. Tuncel and Eryilmaz [37] considered the δ-shock model with non-identical inter-arrival times following the proportional hazard rates model. Jiang [19] have studied the generalized δ-shock model with multiple failure types. Lorvand [25, 26] have considered the mixed δ-shock model for multi-state systems. Recently, Kus et al. [20] have studied the δ-shock model, the extreme shock model and the mixed shock model under the assumption that the inter-arrival times between successive shocks follow the matrix-exponential distribution.

From the foregoing brief discussion and other numerous works on the subject, we can conclude that all published research on the δ-shock modeling has been carried out under the assumption of a constant recovery time (i.e, δ is fixed). However, this assumption is too restrictive and unrealistic to describe many real-life scenarios. Indeed, δ can obviously depend on other parameters, namely, magnitude of shocks, arrival times of shocks, etc. Note that, different systems, depending on their degradation states, have different recovery times even if they experience the same magnitude of shocks. Moreover, a system needs more (less) recovery time if a shock with large (small) magnitude occurs. For example, the recovery time of a bridge from the damage of an earthquake depends on the magnitude of a shock along with the age of the bridge (apart from other factors). If the magnitude of an earthquake is large, then the damage of the bridge is also substantial and hence, it takes more time to recover. Further, due to deterioration, the recovery time of the bridge from the damage caused by an earthquake increases as the age of the bridge increases. Moreover, if the magnitude of an earthquake exceeds a fixed threshold level, then the bridge entirely collapses. In our recent paper (Goyal et al. [18]), the first step in considering a more general model of the current paper was made when δ was assumed to depend only on the arrival time of a shock.

Another key observation from the brief literature review is that most of the studies on the δ-shock model have been done by assuming that shocks occur according to the homogeneous Poisson process (HPP) or the non-homogeneous Poisson process (NHPP). Both of these processes have independent increments, which is indeed a very restrictive assumption in many applications. For example, if there is a larger number of shocks in the past, we may expect the same in the future as well (positive dependence).

Keeping the above research gaps in mind, we focus in this paper on proposing and investigating a *new general δ-shock model* that takes into account the discussed deficiency of existing approaches. To be more specific, what we consider here and what mostly constitutes the novelty and contribution of our paper, is as follows:

(a) We assume that the recovery time (δ) of a system from the damage of a shock depends on *both* the arrival times and the magnitudes of shocks;

(b) We consider a *more general shock process* (namely, the Poisson generalized gamma process (PGGP)) which has the dependent increments property. This process contains the HPP, the NHPP, the Pólya process and the generalized Polýa process (GPP) as the particular cases (Cha and Mercier [7]).

(c) We derive the *distribution of a fatal shock* (for a specific case) that causes the system's failure.

The rest of the paper is organized as follows. In Section 2, we first provide some preliminaries and then describe the model. In Section 3, we derive the survival function and the mean lifetime of a system for the defined model. Further, we study some stochastic comparisons results. In Section 4, we study a special case where the recovery time is assumed as a linear function of arrival times and magnitudes of shocks. For this special case, we derive the reliability function, the mean lifetime and the distribution for a fatal shock. In Section 5, as an example of a possible application, the optimal replacement policy for the proposed model is considered. Finally, the concluding remarks are given in Section 6.

All proofs of theorems and corollaries, where given, for convenience, are deferred to the Appendix.

2 Preliminaries and model formulation

In this section, we first discuss some relevant point processes and then provide the model formulation.

For any random variable U, we denote the cumulative distribution function by $F_U(\cdot)$, the survival/reliability function by $\bar{F}_U(\cdot)$, the probability density function (if exists) by $f_U(\cdot)$; here $\bar{F}_U(\cdot) \equiv 1 - F_U(\cdot)$. Further, we denote the set of natural numbers by $\mathbb{N}$. We use the notation $\lfloor \cdot \rfloor$ to mean the floor function.

2.1 Shock processes

Let $\{N(t),\ t \geq 0\}$ be an orderly point process where $N(t)$ represents the number of shocks arrived by the time t. In the literature, different point processes of shocks have been developed to model different real-life scenarios (see Cha and Finkelstein [5, 6], Teugels and Vynckier [36], and the references therein). Recently, a new point process, called the Poisson generalized gamma process (PGGP), has been introduced by Cha and Mercier [7]. This process possesses the dependent increments property with a rather general dependence structure and hence, it may be applicable to a wider class of problems. Moreover, it contains many well-known processes (namely, the HPP, the NHPP, the Pólya process, the generalized Pólya process (GPP)) as the particular cases. In what follows, we give the formal definitions of some point processes that will be used in this paper. First, we recall the definition of the generalized gamma distribution (see Agarwal and Kalla [1]).

Definition 2.1 *A random variable Q is said to have the generalized gamma distribution (GGD) with the set of parameters $\{\nu, k, \alpha, l\}$, $\nu \geq 0,\ k, \alpha, l > 0$, denoted by $Q \sim GG(\nu, k, \alpha, l)$, if its probability density function is given by*

$$f(q) = \frac{\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)} \frac{q^{k-1}\exp\{-\alpha q\}}{(q+l)^\nu}, \quad q > 0,$$

where

$$\Gamma_\nu(k, \alpha l) = \int_0^\infty \frac{y^{k-1}\exp\{-y\}}{(y+\alpha l)^\nu} dy = \int_0^\infty \frac{\alpha^{k-\nu} y^{k-1}\exp\{-\alpha y\}}{(y+l)^\nu} dy. \tag{2.1}$$

Definition 2.2 *A mixed Poisson process $\{N(t), t \geq 0\}$ is said to be the Pólya process with the set of parameters $\{\beta, b\}$, $\beta > 0,\ b > 0$, if the counting distribution is given by*

$$\begin{aligned} P(N(t) = n) &= \frac{\Gamma(\beta+n)}{\Gamma(\beta) n!}\left(\frac{t}{t+b}\right)^n \left(\frac{b}{t+b}\right)^\beta \\ &= \int_0^\infty \exp\{-\chi t\}\frac{(\chi t)^n}{n!} dH(\chi), \quad n = 0, 1, 2, \ldots, \end{aligned}$$

where H, to be called the structure distribution, is the gamma distribution with the density

$$dH(\chi) = \frac{b^\beta}{\Gamma(\beta)}\chi^{\beta-1}\exp\{-b\chi\} d\chi. \tag{2.2}$$

Definition 2.3 *A counting process $\{N(t), t \geq 0\}$ is said to be the GPP with the set of parameters $\{\lambda(t), \alpha, \beta\}$, $\alpha \geq 0$, $\beta > 0$, if*

(a) *N(0) = 0;*

(b) $\lambda_t = (\alpha N(t-) + \beta)\lambda(t)$,

where λ_t is the stochastic intensity of the counting process $\{N(t), t \geq 0\}$.

Definition 2.4 *A counting process $\{N(t), t \geq 0\}$ is said to be the PGGP with the set of parameters $\{\lambda(t), \nu, k, \alpha, l\}$, $\lambda(t) > 0$ for all $t \geq 0$, $\nu \geq\ 0$, $k, \alpha, l >\ 0$, if*

(a) $\{N(t), t \geq 0 | Q = q\} \sim NHPP(q\lambda(t))$;

(b) $Q \sim GG(\nu, k, \alpha, l)$.

Further, we define the homogeneous Poisson process (HPGGP) with the set of parameters $\{\lambda, \nu, k, \alpha, l\}$ as the PGGP with the set of parameters $\{\lambda(t), \nu, k, \alpha, l\}$, where $\lambda(t) = \lambda\ (> 0)$.

Note that the PGGP is a mixed Poisson process with a generalized gamma mixing distribution. That is, it is a Poisson process with a random intensity function defined as a product of a deterministic intensity function and a random variable.

Remark 2.1 *The following statements are true.*

(a) *The PGGP with the set of parameters* $\{\lambda(t), \nu, k, \alpha, l\}$*, where* $\lambda(t) = \lambda\ (> 0)$*,* $\nu = 0, \alpha = k$ *and* $k \to \infty$*, is the HPP with the intensity* λ*, regardless of* l*;*

(b) *The PGGP with the set of parameters* $\{\lambda(t), \nu, k, \alpha, l\}$*, where* $\nu = 0, \alpha = k$ *and* $k \to \infty$*, is the NHPP with the intensity function* $\lambda(\cdot)$*, regardless of* l*;*

(c) *The PGGP with the set of parameters* $\{\lambda(t), \nu, k, \alpha, l\}$*, where* $\lambda(t) = 1/b\ (> 0)$*,* $\nu = 0, k = \beta, \alpha = 1$*, is the Pólya process with the set of parameters* $\{\beta, b\}$*, regardless of* l*;*

(d) *The PGGP with the set of parameters* $\{\lambda(t), \nu, k, \alpha, l\}$*, where* $\nu = 0, k = \tau/\zeta, \alpha = 1/\zeta$ *and* $\lambda(t) = \eta(t)\exp\{\zeta \int_0^t \eta(x)dx\}$*, is the GPP with the set of parameters* $(\eta(t), \zeta, \tau)$*, regardless of* l*;*

2.2 Model formulation

Let L be a random variable representing the lifetime of a system which has started its operation at time $t = 0$. Assume that the system is subject to external shocks that arrive at random times. Let $0 = T_0 < T_1 < T_2 < \cdots < T_n$ be the sequence of random variables representing the arrival times of n shocks, and let $X_i = T_i - T_{i-1}$ be the random variable representing the time-lag between the i-th and the $(i-1)$-th shocks $(i \geq 1)$. Further, let Y_i be the random variable representing the magnitude of the i-th shock, $i = 1, 2, \ldots$. Assume that the system has the maximum threshold $\gamma\ (> 0)$ on shocks' magnitudes, i.e., the system fails if the magnitude of a single shock is larger than γ. In what follows, we give a list of model assumptions.

Assumptions:

1. Shocks occur according to the PGGP with the set of parameters $\{\lambda(t), \nu, k, \alpha, l\}$.

2. The shock process and $\{Y_i : i \in \mathbb{N}\}$ are independent.

3. The shock process and $\{X_i : i \in \mathbb{N}\}$ are independent with $\{Y_i : i \in \mathbb{N}\}$.

4. $\{Y_i : i \in \mathbb{N}\}$ is a sequence of independent and identically distributed (i.i.d.) random variables. However, the general case may be considered in the same line by assuming the corresponding joint density function.

5. $\delta : [0, \infty] \times [0, \infty] \to [\delta_0, \infty)$ is a recovery function such that

   (a) $\delta(0, 0) = \delta_0$, where δ_0 is a positive constant;

   (b) $\delta(t, y)$ is an increasing function in t, for each fixed y;

   (c) $\delta(t, y)$ is an increasing function in y, for each fixed t;

   (d) $\delta(t, y)$ is a continuous function of t and y.

Now, we assume that a system fails at i-th shock, $i \in \mathbb{N}$, in one of the following ways:

(i) $X_1 > \delta_0, Y_1 \le \gamma, X_2 > \delta(T_1, Y_1), Y_2 \le \gamma, \ldots, X_i > \delta(T_{i-1}, Y_{i-1}), Y_i > \gamma,$

(ii) $X_1 > \delta_0, Y_1 \le \gamma, X_2 > \delta(T_1, Y_1), Y_2 \le \gamma, \ldots, X_{i-1} > \delta(T_{i-1}, Y_{i-1}), Y_{i-1} \le \gamma, X_i < \delta(T_{i-1}, Y_{i-1}).$

Then, the probability of the event "the system survives n shocks in $[0, t)$" is given by

$$\begin{aligned} P(L > t | N(t) = n) &= P(X_1 > \delta_0, Y_1 \le \gamma, X_2 > \delta(T_1, Y_1), Y_2 \le \gamma, \ldots, X_n > \delta(T_{n-1}, Y_{n-1}), \\ &\quad Y_n < \gamma | N(t) = n), \end{aligned}$$

or equivalently,

$$\begin{aligned} P(L > t | N(t) = n) &= P(T_1 > \delta_0, Y_1 \le \gamma, T_2 > T_1 + \delta(T_1, Y_1), Y_2 \le \gamma, \ldots, T_n > T_{n-1} + \\ &\quad \delta(T_{n-1}, Y_{n-1}), Y_n < \gamma | N(t) = n). \end{aligned}$$

Let $g : [0, \infty) \times [0, \infty) \to [\delta_0, \infty)$ be a function given by $g(t, y) = t + \delta(t, y)$. Note that, for each fixed y, $g(\cdot, y)$ is a strictly increasing, surjective and continuous function. Consequently, $g(\cdot, y)$ is invertible, for each fixed y. Further, for each fixed y, let the inverse function of $g(\cdot, y)$ be given by $g^{-1}(\cdot, y) = h(\cdot, y)$. Then $g(h(t, y), y) = h(g(t, y), y) = t$, for all t, for each fixed y. For $n = 1, 2, \ldots,$ we write $g^n(t, y_1, y_2, \ldots, y_n) = g(g^{n-1}(t, y_1, y_2, \ldots, y_{n-1}), y_n)$ to mean that g is composed with itself $(n-1)$ times, where $g^0(t) = t$ for all $t \ge 0$.

3 Reliability characteristics of the model

In this section, we first discuss some reliability indices (namely, the reliability function and the mean lifetime) for the defined model and then provide some results on relevant stochastic comparisons. We assume the general form of the recovery function and hence, the results discussed in this section are quite general in nature and contain relatively cumbersome expressions. More practical results for a specific case (i.e., the linear recovery function) of this model are given in the next section. We begin this section with the following lemma obtained in Cha and Mercier [7].

Lemma 3.1 *For the PGGP with the set of parameters* $\{\lambda(t), \nu, k, \alpha, l\}$, $\lambda(t) > 0$ *for all* $t \ge 0$, $\nu \ge 0, k, \alpha, l > 0$, *the following results hold.*

(a) *The distribution of* $N(t)$ *is given by*

$$P(N(t) = n) = \frac{\alpha^{k-\nu}}{(\alpha + \Lambda(t))^{k+n-\nu}} \frac{\Gamma_\nu(k + n, (\alpha + \Lambda(t))l)}{\Gamma_\nu(k, \alpha l)} \frac{(\Lambda(t))^n}{n!}, \quad n = 0, 1, 2, \ldots;$$

(b) *The conditional joint probability density function of* $(T_1, T_2, \ldots, T_{N(t)})$, *given that* $N(t) = n$, *is given by*

$$f_{(T_1, T_2, \cdots, T_{N(t)} | N(t))}(t_1, t_2, \cdots, t_n | n) = n! \prod_{i=1}^{n} \left(\frac{\lambda(t_i)}{\Lambda(t)} \right), \; 0 < t_1 \le t_2 \le \cdots \le t_n \le t,$$

where $\Lambda(t) = \int_0^t \lambda(u) du$.

3.1 Reliability indices

In the following theorem, we derive the reliability function for the defined model.

Theorem 3.1 *Let shocks occur according to the PGGP with the set of parameters* $\{\lambda(t), \nu, k, \alpha, l\}$, $\lambda(t) > 0$ *for all* $t \geq 0$, $\nu \geq 0, k, \alpha, l > 0$. *Then the survival function of a system for the general* δ*-shock model with the recovery function* $\delta(t, y)$ *is given by*

$$\bar{F}_L(t) = \frac{\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)(\alpha + \Lambda(t))^{k-\nu}} \left[\Gamma_\nu(k, (\alpha + \Lambda(t))l) + \sum_{n=1}^{K_0(t)} \frac{\Gamma_\nu(k+n, (\alpha + \Lambda(t))l)}{(\alpha + \Lambda(t))^n} u(t, n) \right],$$

where

$$\begin{aligned} u(t,n) &= \underbrace{\overbrace{\int\int\cdots\int}^{n\ times}}_{A_0(t,n)} \Bigg\{ \int_{g^{n-1}(\delta_0, y_1, y_2, \ldots, y_{n-1})}^{t} \int_{g^{n-2}(\delta_0, y_1, y_2, \ldots, y_{n-2})}^{h(t_n, y_{n-1})} \cdots \int_{g(\delta_0, y_1)}^{h(t_3, y_2)} \int_{\delta_0}^{h(t_2, y_1)} \prod_{i=1}^{n} \lambda(t_i) \\ &\qquad dt_1 dt_2 \ldots dt_n \Bigg\} \prod_{i=1}^{n} f_{Y_1}(y_i) dy_1 dy_2 \ldots dy_n \end{aligned}$$

and

$$\begin{aligned} &A_0(t, n) = \{(y_1, y_2, \ldots, y_n) : g^{n-1}(\delta_0, y_1, y_2, \ldots, y_{n-1}) < t, 0 \leq y_i \leq \gamma, 1 \leq i \leq n\}, \\ &K_0(t) = max\{n \geq 1 | g^{n-1}(\delta_0, \underbrace{0, 0, \ldots, 0}_{n-1 \text{ times}}) < t\}. \end{aligned} \qquad \square$$

In the next theorem, we derive the mean lifetime of the system for the defined model.

Theorem 3.2 *Let shocks occur according to the PGGP with the set of parameters* $\{\lambda(t), \nu, k, \alpha, l\}$, $\lambda(t) > 0$ *for all* $t \geq 0$, $\nu \geq 0, k, \alpha, l > 0$. *Then the mean lifetime of a system for the general* δ*-shock model with the recovery function* $\delta(t, y)$ *is given by*

$$\begin{aligned} E(L) &= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)} \int_0^\infty \frac{\Gamma_\nu(k, (\alpha + \Lambda(t))l)}{(\alpha + \Lambda(t))^{k-\nu}} dt \\ &\quad + \frac{\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)} \sum_{n=1}^{\infty} \int_{g^{n-1}(\delta_0, \underbrace{0, 0, \ldots, 0}_{n-1\ times})}^{\infty} \left(\frac{\Gamma_\nu(k+n, (\alpha + \Lambda(t))l)}{(\alpha + \Lambda(t))^{k+n-\nu}} \right) u(t, n) dt, \end{aligned}$$

provided the expectation exists; here $u(t, n)$ *is the same as in Theorem 3.1.*

3.2 Stochastic comparisons

In this subsection, we study some stochastic comparisons results for systems subject to external shocks. Before discussing the results, we give the following useful definition.

Definition 3.1 *A random variable* V *is said to be smaller than* W *in the usual stochastic order, denoted by* $V \leq_{st} W$, *if* $\bar{F}_V(t) \leq \bar{F}_W(t)$ *for all* $t > 0$. $\square$

In the following theorems, we compare the lifetimes of two systems with respect to the usual stochastic order. Here, we assume that both systems are subject to the same external shocks. The proofs of the first two theorems immediately follow from Theorem 3.1 and hence, omitted.

Theorem 3.3 *Let L_1 and L_2 be the lifetimes of two systems with the same recovery function $\delta(\cdot,\cdot)$. Further, let γ_1 and γ_2 be the maximum thresholds on shock's magnitudes for the first and the second systems, respectively. Assume that both systems are subject to the same external shocks that occur according to the PGGP with the set of parameters $\{\lambda(t), \nu, k, \alpha, l\}$, $\lambda(t) > 0$ for all $t \geq 0$, $\nu \geq 0, k, \alpha, l > 0$. If $\gamma_1 \leq \gamma_2$ then $L_1 \leq_{st} L_2$.*

Theorem 3.4 *Let L_1 and L_2 be the lifetimes of two systems with the same recovery function $\delta(\cdot,\cdot)$. Further, let $\{Y_{1n}|\ n \in \mathbb{N}\}$ and $\{Y_{2n}|\ n \in \mathbb{N}\}$ be two sequences of i.i.d. random variables representing the magnitudes of shocks that are experienced by the first and the second systems, respectively. Assume that both systems are subject to the same external shocks that occur according to the PGGP with the set of parameters $\{\lambda(t), \nu, k, \alpha, l\}$, $\lambda(t) > 0$ for all $t \geq 0$, $\nu \geq 0, k, \alpha, l > 0$. Further, assume that $Y_{1n} \sim U[0, \xi_1]$ and $Y_{2n} \sim U[0, \xi_2]$, for all $n \in \mathbb{N}$, for some $\xi_1, \xi_2 > 0$. If $Y_{1n} \geq_{st} Y_{2n}$, for all $n \in \mathbb{N}$, then $L_1 \leq_{st} L_2$.*

Theorem 3.5 *Let L_1 and L_2 be the lifetimes of two systems with the recovery functions $\delta_1(\cdot,\cdot)$ and $\delta_2(\cdot,\cdot)$, respectively. Assume that both systems are subject to the same external shocks that occur according to the PGGP with the set of parameters $\{\lambda(t), \nu, k, \alpha, l\}$, $\lambda(t) > 0$ for all $t \geq 0$, $\nu \geq 0, k, \alpha, l > 0$. Further, assume that $\delta_1(0,0) = \delta_2(0,0) = \delta_0$. If $\delta_1(t,y) \leq \delta_2(t,y)$, for all $(t,y) \in [0,\infty) \times [0,\infty)$, then $L_2 \leq_{st} L_1$.*

4 A special case: δ-shock model with the linear recovery function

In this section, we assume the linear (with respect to both arrival times and magnitudes of shocks) recovery function, i.e., $\delta(t,y) = \delta_0 + \sigma_1 t + \sigma_2 y$, for all $(t,y) \in [0,\infty) \times [0,\infty)$, for some $\delta_0 > 0$ and $\sigma_1, \sigma_2 \geq 0$. Here, we specifically study the reliability function, the distribution for a fatal shock and the mean lifetime for the defined model by assuming that shocks occur according to the HPGGP.

4.1 Reliability function

Theorem 4.1 *Let shocks occur according to the HPGGP with the set of parameters $\{\lambda, \nu, k, \alpha, l\}$, $\nu \geq 0$, $\lambda, k, \alpha, l > 0$. Assume that $Y_i \sim U[0,\xi]$, $i \in \mathbb{N}$, for some $\xi > 0$. Then the survival function of a system for the general δ-shock model with the recovery function $\delta(t,y) = \delta_0 + \sigma_1 t + \sigma_2 y$ is given by*

$$\begin{aligned}\bar{F}_L(t) &= \frac{\alpha^{k-\nu}}{(\alpha+\lambda t)^{k-\nu}} \frac{\Gamma_\nu(k, (\alpha+\lambda t)l)}{\Gamma_\nu(k, \alpha l)} \\ &\quad + \frac{\gamma \alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)(\alpha+\lambda t)^{k-\nu}} \sum_{n=1}^{S_0(t)} \left(\frac{\Gamma_\nu(k+n, (\alpha+\lambda t)l)}{n!(\alpha+\lambda t)^n} \left(\frac{\lambda}{\xi}\right)^n \frac{v(t,n)}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \right),\end{aligned}$$

where

$$v(t,n) = \underbrace{\overbrace{\int\int\cdots\int}^{n-1\ times}}_{B_0(t,n)} \left(t - g^{n-1}(\delta_0, y_1, y_2, \ldots, y_{n-1})\right)^n dy_1 dy_2 \ldots dy_{n-1},$$

$B_0(t,n) = \{(y_1, y_2, \ldots, y_{n-1}) : g^{n-1}(\delta_0, y_1, y_2, \ldots, y_{n-1}) < t,\ \ 0 \leq y_i \leq \gamma,\ 1 \leq i \leq (n-1)\}$,

$$g^{n-1}(\delta_0, y_1, y_2, \ldots, y_{n-1}) = \delta_0 \left(\frac{(1+\sigma_1)^n - 1}{\sigma_1}\right) + \sigma_2 \left[(1+\sigma_1)^{n-2} y_1 + (1+\sigma_1)^{n-3} y_2 + \cdots + y_{n-1}\right],$$

and

$$S_0(t) = \left\lfloor \frac{\ln\left(\frac{\delta_0+\sigma_1 t}{\delta_0}\right)}{\ln(1+\sigma_1)} \right\rfloor.$$

The following corollary immediately follows from Theorem 4.1.

Corollary 4.1 *Assume that $\sigma_2 \to 0$ (i.e., the recovery function depends only on the arrival times of shocks) in Theorem 4.1. Then*

$$\bar{F}_L(t) = \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)} \sum_{n=0}^{S_0(t)} \frac{(\lambda\gamma)^n}{\xi^n n!} \frac{\Gamma_\nu(k+n,(\alpha+\lambda t)l)}{(\alpha+\lambda t)^{k+n-\nu}} \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \left[t - \left(\frac{(1+\sigma_1)^n - 1}{\sigma_1}\right)\delta_0\right]^n,$$

where $S_0(t)$ is the same as in Theorem 4.1. □

Some special cases of the above corollary are as follows. When $\lambda = 1/b$, $\nu = 0, k = \beta, \alpha = 1$ (i.e., the Pólya process with the set of parameters $\{\beta, b\}$), we have

$$\bar{F}_L(t) = \left(\frac{b}{t+b}\right)^\beta \sum_{n=0}^{S_0(t)} \left(\frac{\gamma}{\xi}\right)^n \frac{\Gamma(\beta+n)}{\Gamma(\beta)n!} \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \left[\frac{t - \left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0}{t+b}\right]^n.$$

Further, when $\nu = 0, \alpha = k$ and $k \to \infty$ (i.e., the HPP with the intensity λ), we have

$$\bar{F}_L(t) = \exp\{-\lambda t\} \sum_{n=0}^{S_0(t)} \left(\frac{\gamma}{\xi}\right)^n \left(\frac{\lambda^n}{n!}\right) \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \left[t - \left(\frac{(1+\sigma_1)^n - 1}{\sigma_1}\right)\delta_0\right]^n.$$

The next corollary also follows from Theorem 4.1.

Corollary 4.2 *Assume that $\sigma_1 \to 0$ (i.e., the recovery function depends only on the magnitudes of shocks) in Theorem 4.1. Then*

$$\bar{F}_L(t) = \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)(\alpha+\lambda t)^{k-\nu}} \left[\Gamma_\nu(k,(\alpha+\lambda t)l) + \gamma \sum_{n=1}^{\left\lfloor \frac{t}{\delta_0} \right\rfloor} \left(\frac{\lambda}{\xi}\right)^n \frac{\Gamma_\nu(k+n,(\alpha+\lambda t)l)}{n!(\alpha+\lambda t)^n} w(t,n)\right],$$

where

$$w(t,n) = \begin{cases} \left(\frac{n!}{\sigma_2^{n-1}}\right)\left\{\frac{(t-n\delta_0)^{(2n-1)}}{(2n-1)!}\right\}, & \text{if } n\delta_0 < t \le n\delta_0 + \gamma\sigma_2 \\ \left(\frac{n!}{\sigma_2^{n-1}}\right)\left\{\frac{(t-n\delta_0)^{(2n-1)} - \binom{n-1}{1}(t-n\delta_0-\gamma\sigma_2)^{(2n-1)}}{(2n-1)!}\right\}, & \text{if } n\delta_0+\gamma\sigma_2 < t \le n\delta_0 + 2\gamma\sigma_2 \\ \vdots & \vdots \\ \left(\frac{n!}{\sigma_2^{n-1}}\right)\left\{\sum_{i=0}^{k}(-1)^i\binom{n-1}{i}\frac{(t-n\delta_0-i\gamma\sigma_2)^{(2n-1)}}{(2n-1)!}\right\}, & \text{if } n\delta_0+k\gamma\sigma_2 < t \le n\delta_0 + (k+1)\gamma\sigma_2 \\ \vdots & \vdots \\ \left(\frac{n!}{\sigma_2^{n-1}}\right)\left\{\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\frac{(t-n\delta_0-i\gamma\sigma_2)^{(2n-1)}}{(2n-1)!}\right\}, & \text{if } n\delta_0+(n-1)\gamma\sigma_2 < t. \end{cases}$$

□

The following observations can be made from Corollary 4.2. When $\lambda = 1/b$, $\nu = 0, k = \beta, \alpha = 1$ (i.e., the Pólya process with the set of parameters $\{\beta, b\}$), we have

$$\bar{F}_L(t) = \left(\frac{b}{t+b}\right)^\beta \left[1 + \gamma \sum_{n=1}^{\left\lfloor \frac{t}{\delta_0} \right\rfloor} \frac{\Gamma(\beta+n)}{\Gamma(\beta)n!} \left(\frac{1}{\xi}\right)^n \left(\frac{1}{t+b}\right)^n w(t,n)\right].$$

Further, when $\nu = 0, \alpha = k$ and $k \to \infty$ (i.e., the HPP with the intensity λ), we have

$$\bar{F}_L(t) = \exp\{-\lambda t\} \left[1 + \gamma \sum_{n=1}^{\lfloor \frac{t}{\delta_0} \rfloor} \frac{1}{n!} \left(\frac{\lambda}{\xi} \right)^n w(t,n) \right].$$

Below we give another corollary of Theorem 4.1.

Corollary 4.3 *Assume that $\sigma_1 \to 0$ and $\sigma_2 \to 0$ (i.e., the recovery function is constant) in Theorem 4.1. Then*

$$\bar{F}_L(t) = \frac{\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)(\alpha+\lambda t)^{k-\nu}} \left[\Gamma_\nu(k, (\alpha+\lambda t)l) + \sum_{n=1}^{\lfloor \frac{t}{\delta_0} \rfloor} \left(\frac{\gamma\lambda}{\xi} \right)^n \frac{\Gamma_\nu(k+n, (\alpha+\lambda t)l)}{n!(\alpha+\lambda t)^n} (t - n\delta_0)^n \right].$$

Some special cases of Corollary 4.3 are as follows. When $\lambda = 1/b$, $\nu = 0, k = \beta, \alpha = 1$ (i.e., the Pólya process with the set of parameters $\{\beta, b\}$), we have

$$\bar{F}_L(t) = \left(\frac{b}{t+b} \right)^\beta \sum_{n=0}^{\lfloor \frac{t}{\delta_0} \rfloor} \frac{\Gamma(\beta+n)}{\Gamma(\beta)n!} \left(\frac{\gamma}{\xi} \right)^n \left(\frac{t-n\delta_0}{t+b} \right)^n.$$

Further, when $\nu = 0, \alpha = k$ and $k \to \infty$ (i.e., the HPP with the intensity λ), we have

$$\bar{F}_L(t) = \exp\{-\lambda t\} \sum_{n=0}^{\lfloor \frac{t}{\delta_0} \rfloor} \frac{\lambda^n}{n!} \left(\frac{\gamma}{\xi} \right)^n (t - n\delta_0)^n.$$

4.2 Distribution for a fatal shock

In this subsection, we first derive the distribution for a fatal shock and then, find out the expected number of shocks that the system may experience by its failure. Let M be a random variable representing the fatal shock that causes the system's failure.

Due to mathematical complexity, it is not possible to obtain the distribution of M for the case when shocks occur according to the HPGGP. Thus, in this subsection, we derive all results under the assumption that shocks occurs according to the Pólya process. As mentioned in Remark 2.1, the HPGGP with the set of parameters $\{1/b, 0, \beta/\omega, 1/\omega, l\}$ is indeed the Pólya process with the set of parameters $\{\beta/\omega, b/\omega\}$, regardless of l. Here, we follow the terminology of the HPGGP for notational convenience. In what follows, we introduce some notion which will be be used in the subsequent study.

Let C be any finite sequence with the cardinality n ($\in \mathbb{N} \cup \{0\}$), where the elements of C may occur multiple times. Define $S_{C,i}$ as the collection of all subsequences of C containing exactly i ($\in \mathbb{N} \cup \{0\}$) elements, where $0 \le i \le n$. Clearly, the cardinality of $S_{C,i}$ is $\binom{n}{i}$. Further, define $S^l_{C,i}$ as the l-th element of $S_{C,i}$, where $1 \le l \le \binom{n}{i}$. We write $\prod_{i=1}^{0} a_i = 1$, for any a_i's. Further, we denote the null set by $\{\}$.

In the following theorem, we obtain the distribution and the expectation of M. The proof of the second part of this theorem immediately follows from the first part and hence, omitted.

Theorem 4.2 *Let shocks occur according to the HPGGP with the set of parameters $\{1/b, 0, \beta/\omega, 1/\omega, l\}$, where $\omega, \beta, l, b > 0$ and β/ω is not an integer. Assume that $Y_i \sim U[0, \xi]$, $i \in \mathbb{N}$, for some $\xi > 0$. Then* *the probability mass function (pmf) and the mean of M, for the general δ-shock model with the recovery function $\delta(t,y)=\delta_0+\sigma_1 t+\sigma_2 y$, are given by*

$$P(M=m)=\Upsilon_{m-1}-\Upsilon_m, \quad m=1,2,\ldots,$$

and

$$E(M)=\sum_{m=0}^{\infty}\Upsilon_m,$$

respectively, where $\Upsilon_0=1$ and, for $m=1,2,\ldots,$

$$\begin{aligned}\Upsilon_m &= \frac{\gamma b^{\frac{\beta}{\omega}}}{\xi^m\sigma_2^{m-1}(1+\sigma_1)^{(m-1)(m-1)}\left\{\prod_{i=1}^{m-1}(i\omega-\beta)\right\}} \\ &\quad\times\left[\sum_{i=0}^{m-1}(-1)^{m-1-i}\sum_{l=1}^{\binom{m-1}{i}}\left(\omega\delta_0\left(\frac{(1+\sigma_1)^m-1}{\sigma_1}\right)+\left(\sum_{s\in S^l_{A_{m-1},i}}s\right)+b\right)^{-\frac{\beta}{\omega}+m-1}\right]\end{aligned}$$

and

$$A_0=\{\},\ \ A_{m-1}=\{\omega\sigma_2\gamma,\omega\sigma_2\gamma(1+\sigma_1),\ldots,\omega\sigma_2\gamma(1+\sigma_1)^{m-2}\}, \quad m=2,3,\ldots.$$

Below we give three consecutive corollaries which immediately follow from Theorem 4.2.

Corollary 4.4 *Assume that $\beta=1$, $\omega\to 0$ and $b=1/\lambda$ (i.e., the shocks occur according to the HPP with intensity $\lambda>0$) in Theorem 4.2. Then the pmf and the mean of M are given by*

$$P(M=m)=\Omega_{m-1}-\Omega_m, \quad m=1,2,\ldots,$$

and

$$E(M)=\sum_{m=0}^{\infty}\Omega_m,$$

respectively, where $\Omega_0=1$ and, for $m=1,2,\ldots,$

$$\Omega_m=\left(\frac{\gamma\exp\left\{-\lambda\delta_0\left(\frac{(1+\sigma_1)^m-1}{\sigma_1}\right)\right\}}{\xi^m(1+\sigma_1)^{(m-1)^2}\lambda^{m-1}\sigma_2^{m-1}}\right)\left[\prod_{i=1}^{m-1}\left(1-\exp\{-\lambda\sigma_2(1+\sigma_1)^{i-1}\gamma\}\right)\right].$$

Corollary 4.5 *Assume that $\sigma_1\to 0$ in Theorem 4.2. Then the pmf and the mean of M are given by*

$$P(M=m)=\Psi_{m-1}-\Psi_m, \quad m=1,2,\ldots,$$

and

$$E(M)=\sum_{m=0}^{\infty}\Psi_m,$$

respectively, where $\Psi_0=1$ and, for $m=1,2,\ldots,$

$$\Psi_m=\frac{\gamma b^{\frac{\beta}{\omega}}}{\xi^m\sigma_2^{m-1}\left(\prod_{i=1}^{m-1}(i\omega-\beta)\right)}\sum_{i=0}^{m-1}(-1)^{m-1-i}\binom{m-1}{i}\left(m\omega\delta_0+i\omega\gamma\sigma_2+b\right)^{\left(-\frac{\beta}{\omega}+m-1\right)}.$$

Corollary 4.6 *Assume that $\sigma_2 \to 0$ in Theorem 4.2. Then the pmf and the mean of M are given by*

$$P(M=m) = \Phi_{m-1} - \Phi_m, \quad m = 1, 2, \ldots,$$

and

$$E(M) = \sum_{m=0}^{\infty} \Phi_m,$$

respectively, where

$$\Phi_m = \frac{\gamma^m}{\xi^m (1+\sigma_1)^{\frac{m(m-1)}{2}}} \left[\frac{b}{b + \omega\delta_0 \left(\frac{(1+\sigma_1)^m - 1}{\sigma_1} \right)} \right]^{\frac{\beta}{\omega}}, \quad m = 0, 1, 2, \ldots.$$

In the following theorem we derive the the same result, as in Theorem 4.2, under the assumption that shocks' magnitudes follow the exponential distribution. The proof of the second part immediately follows from the first part and hence, omitted.

Theorem 4.3 *Let shocks occur according to the HPP with intensity $\lambda > 0$. Assume that $Y_i \sim \exp(\theta)$, $i \in \mathbb{N}$, for some $\theta > 0$. Then the pmf and the mean of M, for the general δ-shock model with the recovery function $\delta(t, y) = \delta_0 + \sigma_1 t + \sigma_2 y$, are given by*

$$P(M=m) = \Delta_{m-1} - \Delta_m, \quad m = 1, 2, \ldots,$$

and

$$E(M) = \sum_{i=0}^{\infty} \Delta_m,$$

respectively, where $\Delta_0 = 1$ and

$$\begin{aligned} \Delta_m &= \theta^{m-1} \left(\frac{(1 - \exp\{-\theta\gamma\})}{(1+\sigma_1)^{\frac{m(m-1)}{2}}} \right) \exp\left\{ -\lambda\delta_0 \left(\frac{(1+\sigma_1)^m - 1}{\sigma_1} \right) \right\} \\ &\quad \times \prod_{i=1}^{m-1} \left(\frac{1 - \exp\left\{ -\gamma(\lambda\sigma_2(1+\sigma_1)^{m-1-i} + \theta) \right\}}{\lambda\sigma_2(1+\sigma_1)^{m-1-i} + \theta} \right), \quad m = 1, 2, \ldots. \end{aligned}$$

The following two corollaries follow from Theorem 4.3.

Corollary 4.7 *Assume that $\sigma_1 \to 0$ in Theorem 4.3. Then the pmf and the mean of M are given by*

$$P(M=m) = \Theta_{m-1} - \Theta_m, \quad m = 1, 2, \ldots,$$

and

$$E(M) = 1 + \left[\frac{(\lambda\sigma_2 + \theta)\exp\{-\lambda\delta_0\}(1 - \exp\{-\theta\gamma\})}{\lambda\sigma_2 + \theta(1 - \exp\{-\lambda\delta_0\}) + \theta\exp\{-(\lambda\delta_0 + \lambda\sigma_2\gamma + \theta\gamma)\}} \right],$$

respectively, where $\Theta_0 = 1$ and

$$\Theta_m = \theta^{m-1}(1 - \exp\{-\theta\gamma\})\exp\{-m\lambda\delta_0\} \left(\frac{1 - \exp\{-(\lambda\sigma_2 + \theta)\gamma\}}{\lambda\sigma_2 + \theta} \right)^{m-1}, \quad m = 1, 2, \ldots.$$

Corollary 4.8 *Assume that $\sigma_2 \to 0$ in Theorem 4.3. Then the pmf and the mean of M are given by*

$$P(M=m) = \Xi_{m-1} - \Xi_m, \quad m = 1, 2, \ldots,$$

and

$$E(M) = \sum_{m=0}^{\infty} \Xi_m,$$

respectively, where

$$\Xi_m = \left(\frac{(1-\exp\{-\theta\gamma\})^m}{(1+\sigma_1)^{\frac{m(m-1)}{2}}} \right) \exp\left\{ -\lambda\delta_0 \left(\frac{(1+\sigma_1)^m - 1}{\sigma_1} \right) \right\}, \quad m = 0, 1, 2, \ldots.$$

4.3 Mean lifetime

In the following theorem, we derive the mean lifetime of a system for the defined model. The proof is similar to Theorem 3.2 and hence, omitted.

Theorem 4.4 *Let shocks occur according to the HPGGP with the set of parameters $\{\lambda, \nu, k, \alpha, l\}$, $\nu \geq 0,\ k, \alpha, l > 0$. Assume that $Y_i \sim U[0, \xi]$, $i \in \mathbb{N}$, for some $\xi > 0$. Then the mean lifetime of a system for the general δ-shock model with the recovery function $\delta(t, y) = \delta_0 + \sigma_1 t + \sigma_2 y$ is given by*

$$\begin{aligned} E(L) &= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)} \int_0^\infty \frac{\Gamma_\nu(k, (\alpha+\lambda t)l)}{(\alpha+\lambda t)^{k-\nu}} dt \\ &\quad + \frac{\gamma\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)} \sum_{n=1}^{\infty} \int_{g^{n-1}(\delta_0, \underbrace{0, 0, \ldots, 0}_{n\,-\,1\ times})}^{\infty} \left(\frac{\Gamma_\nu(k+n, (\alpha+\lambda t)l)}{n!(\alpha+\lambda t)^{k+n-\nu}} \right) \left(\frac{\lambda}{\xi} \right)^n \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} v(t, n) dt, \end{aligned}$$

provided the expectation exists; here $v(t, n)$ is the same as in Theorem 4.1. □

Below we give three corollaries which are obtained from Theorem 4.4. In the first corollary, we give a nice relation between the mean lifetime of the system and the expected number of shocks before the failure of the system.

Corollary 4.9 *Assume that $\nu = 0$, $\alpha = k$ and $k \to \infty$ (i.e., the shocks occur according to the HPP with intensity λ) in Theorem 4.4. Then, we have*

$$E(L) = \frac{1}{\lambda} E(M) = \frac{1}{\lambda} + \frac{1}{\lambda} \sum_{m=1}^{\infty} \Omega_m,$$

where Ω_m is the same as in Corollary 4.4.

Corollary 4.10 *Assume that $\sigma_2 \to 0$ in Theorem 4.4. Then, for $k > 1$,*

$$E(L) = \frac{\alpha^{k-\nu}}{\lambda\Gamma_\nu(k, \alpha l)} \sum_{n=0}^{\infty} \left(\frac{\gamma}{\xi} \right)^n \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \frac{\Gamma_\nu\left(k-1, \left(\alpha + \lambda\delta_0 \left(\frac{(1+\sigma_1)^n - 1}{\sigma_1}\right)\right) l\right)}{\left(\alpha + \lambda\delta_0 \left(\frac{(1+\sigma_1)^n - 1}{\sigma_1}\right)\right)^{k-\nu-1}}. \quad □$$

Two special cases of the above corollary are as follows. When $\lambda = 1/b,\ \nu = 0,\ k = \beta\ (> 1)$ and $\alpha = 1$ (i.e., the shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$), we have

$$E(L) = \left(\frac{b}{\beta-1} \right) \sum_{n=0}^{\infty} \left(\frac{\gamma}{\xi} \right)^n \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \left[\frac{b}{b + \lambda\delta_0 \left(\frac{(1+\sigma_1)^n - 1}{\sigma_1}\right)} \right]^{\beta-1}.$$

Further, when $\nu = 0,\ \alpha = k$ and $k \to \infty$ (i.e., the shocks occur according to the HPP with intensity λ), we have

$$E(L) = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{\gamma}{\xi} \right)^n \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \exp\left\{ -\lambda\delta_0 \left(\frac{(1+\sigma_1)^n - 1}{\sigma_1} \right) \right\}.$$

Corollary 4.11 *Assume that $\sigma_1 \to 0$ in Theorem 4.4. Then, for $k \to \infty$, we have*

$$\begin{aligned} E(L) &= \left(\frac{\alpha}{\lambda}\right)\left(\lim_{k\to\infty}\frac{\Gamma_\nu(k-1,\alpha l)}{\Gamma_\nu(k,\alpha l)}\right) \\ &\quad + \lim_{k\to\infty}\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\left(\frac{1}{\lambda\xi}\right)^n\frac{\Gamma_\nu(k-n,(\alpha+n\lambda\delta_0+i\gamma\sigma_2\lambda)l)}{(\alpha+n\lambda\delta_0+i\gamma\sigma_2\lambda)^{k-\nu-n}}\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}, \end{aligned}$$

provided the above expression has a finite value. □

An important special case of Corollary 4.11 is as follows. When $\nu = 0$, $\alpha = k$ and $k \to \infty$ (i.e., the shocks occur according to the HPP with intensity λ), we have

$$E(L) = \frac{1}{\lambda} + \frac{1}{\lambda}\sum_{n=1}^{\infty}\left(\frac{\gamma\exp\{-\lambda n\delta_0\}}{\xi^n\lambda^{n-1}\sigma_2^{n-1}}\right)(1-\exp\{-\lambda\sigma_2\gamma\})^{n-1}.$$

In the next theorem, we prove the same result, as in Corollary 4.9, under the assumption that shocks' magnitudes follow the exponential distribution. Here, we also establish an useful relationship between the mean lifetime of the system and the expected number of shocks that the system may experience before its failure.

Theorem 4.5 *Let shocks occur according to the HPP with intensity $\lambda > 0$. Assume that $Y_i \sim \exp(\theta)$, $i \in \mathbb{N}$, for some $\theta > 0$. Then the mean lifetime of a system for the general δ-shock model with the recovery function $\delta(t, y) = \delta_0 + \sigma_1 t + \sigma_2 y$ is given by*

$$E(L) = \frac{1}{\lambda}E(M) = \frac{1}{\lambda} + \frac{1}{\lambda}\sum_{m=1}^{\infty}\Delta_m,$$

where Δ_m is the same as in Theorem 4.3. □

Now, we illustrate the result given in Corollary 4.9. We assume the relevant parameters as follows: $\lambda = 0.5$ and $\xi = 9$. In Figure 1, we plot $E(L)$ against $\sigma_1 \in [0, 1]$, for fixed $\sigma_2 = 0.01$ and $\gamma = 8$. In Figure 2, we plot $E(L)$ against $\sigma_2 \in [0, 1]$, for fixed $\sigma_1 = 0.01$ and $\gamma = 8$. In Figure 3, we plot $E(L)$ against $\gamma \in (0, \xi]$, for fixed $\sigma_1 = 0.1$ and $\sigma_2 = 0.01$. Figures 1 and 2 show that $E(L)$ decreases as σ_1 or σ_2 increases, whereas the reverse scenario is observed in Figure 3.

5 Application: Optimal replacement policy

In this section, we consider the optimal replacement policy N^* for a system with the lifetime described by the general δ-shock model (the linear recovery function). The system is replaced by a new one when it experiences N^* failures. Imperfect repairs are performed between replacements. Lam and Zhang [21] have studied the δ-shock maintenance model for a repairable and deteriorating system by assuming that shocks occur according to the HPP. Later, Tang and Lam [35] have generalized the same problem by considering the renewal shock process with the Weibull or the gamma-distributed inter-arrival times. Further, Eryilmaz [10] have considered the same maintenance model for the Pólya process of shocks and have compared it with the case when shocks occur according to a renewal process with the Pareto distributed inter-arrival times. In what follows, as an example, we study the optimal replacement policy for the developed general δ-shock model with the linear recovery time and the HPP process of shocks. We consider two different distributions of shocks' magnitudes, namely, the exponential and the uniform

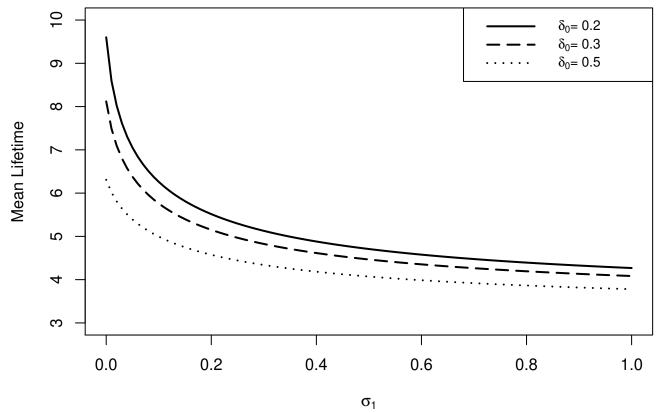

Figure 1: Mean lifetime $w.r.t.$ σ_1

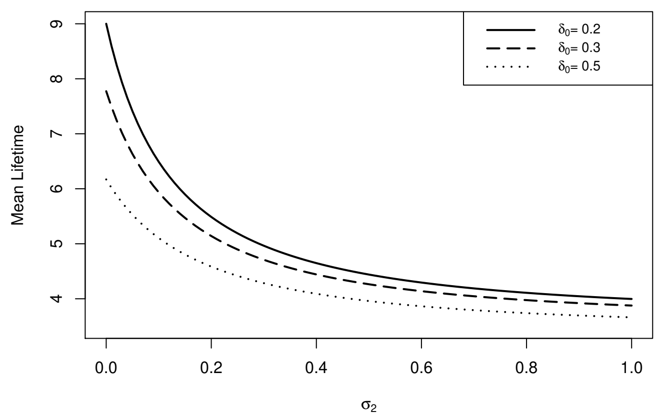

Figure 2: Mean lifetime $w.r.t.$ σ_2

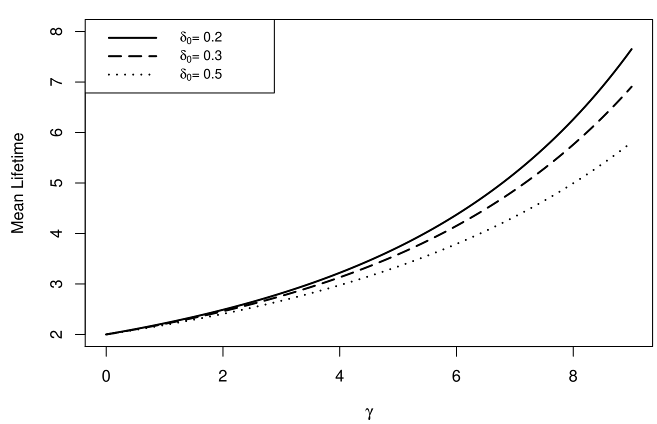

Figure 3: Mean lifetime $w.r.t.$ γ

distributions.

Below we give a list of assumptions that are similar to those considered in Lam and Zhang [21], Tang and Lam [35], and Eryilmaz [10].

Assumptions:

1. A new system is put into operation at time $t = 0$ and is repaired instantly once it is failed. The system is replaced by a new identical one after the N-th failure is observed.

2. The system is subject to external shocks that occur according to the HPP with the intensity λ (> 0).

3. After the n-th repair, the recovery function is given by $\delta_n : [0, \infty) \times [0, \infty) \to [\kappa^n \delta_0, \infty)$ such that $\delta_n(t, y) = \kappa^n \delta(t, y)$, $n = 1, 2, \ldots$; here $\kappa \geq 1$ and $\delta(t, y) = \delta_0 + \sigma_1 t + \sigma_2 y$, $\delta_0 \geq 0$, $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$. Moreover, the system is not affected by any external shocks during the repair time.

4. Let R_i be the repair time of the system after the i-th failure, $i = 1, 2, \ldots$. Then the sequence $\{R_1, R_2, \ldots\}$ forms an increasing geometric process such that $E(R_n) = \frac{\mu}{r^{n-1}}$, $n = 1, 2, \ldots$.

5. The repair cost is c; the reward rate is c_r when the system is operating. The replacement cost has two parts: the basic replacement cost is B_c, whereas the other one is proportional to the replacement time Z with rate c_p. Further, we assume that $E(Z) = \rho$.

6. The HPP, the geometric process and the replacement time Z are independent.

Now, we discuss the replacement policy for the following two cases.
Case-1: $Y_j \sim U[0, \xi]$, for all $j \in \mathbb{N}$, where $\xi > 0$;
Case-2: $Y_j \sim \exp(\theta)$, for all $j \in \mathbb{N}$, where $\theta > 0$.
Let $L_1^{(i)}$ denote the random operating time of the system to the first failure for Case-i, $i = 1, 2$. Further, let $L_n^{(i)}$ be the random time elapsed since the $(n-1)$th repair to n-th failure for Case-i, $i = 1, 2$, $n = 2, 3, \ldots$. Let W_i be a random length of a cycle under replacement policy N for Case-i, $i = 1, 2$. Then

$$W_i = \sum_{n=1}^{N} L_n^{(i)} + \sum_{n=1}^{N-1} R_n + Z.$$

From Corollary 4.9, we have

$$\begin{aligned} E(L_n^{(1)}) &= \frac{1}{\lambda}\left[1 + \sum_{m=1}^{\infty}\left(\frac{\gamma \exp\left\{-\lambda\delta_0\left(\frac{(1+\kappa^{n-1}\sigma_1)^m - 1}{\sigma_1}\right)\right\}}{\xi^m (1+\kappa^{n-1}\sigma_1)^{(m-1)^2} \lambda^{m-1} \left(\kappa^{n-1}\sigma_2\right)^{m-1}}\right)\right. \\ &\qquad \left. \times \left\{\prod_{i=1}^{m-1}\left(1 - \exp\{-\lambda\kappa^{n-1}\sigma_2(1+\kappa^{n-1}\sigma_1)^{i-1}\gamma\}\right)\right\}\right] \end{aligned} \tag{5.1}$$

and, from Theorem 4.5, we have

$$E(L_n^{(2)}) = \frac{1}{\lambda}\left[1 + \sum_{m=1}^{\infty} \theta^{m-1}\left(\frac{(1 - \exp\{-\theta\gamma\})}{(1+\kappa^{n-1}\sigma_1)^{\frac{m(m-1)}{2}}}\right) \exp\left\{-\lambda\delta_0\left(\frac{(1+\kappa^{n-1}\sigma_1)^m - 1}{\sigma_1}\right)\right\} \epsilon_m\right], \tag{5.2}$$

where

$$\epsilon_m = \prod_{i=1}^{m-1}\left(\frac{1 - \exp\left\{-\gamma(\lambda\kappa^{n-1}\sigma_2(1+\kappa^{n-1}\sigma_1)^{m-1-i} + \theta)\right\}}{\lambda\kappa^{n-1}\sigma_2(1+\kappa^{n-1}\sigma_1)^{m-1-i} + \theta}\right), \; m = 1, 2, \ldots.$$

Table 1: Values of $C_1(N)$

N	$C_1(N)$	N	$C_1(N)$	N	$C_1(N)$	N	$C_1(N)$
1	6.403121	11	5.367308	21	5.856849	34	5.987039
2	5.709264	12	5.438418	22	5.879559	38	5.994079
3	5.349527	13	5.506007	23	5.898941	42	5.997328
4	5.169131	14	5.568817	24	5.915413	46	5.998806
5	5.093363	15	5.626171	25	5.929358	50	5.999470
6	**5.081326**	16	5.677809	26	5.941125	60	5.999932
7	5.108563	17	5.723764	27	5.951022	70	5.999992
8	5.159521	18	5.764263	28	5.959326	80	5.999999
9	5.223903	19	5.799656	29	5.966274	90	6.000000
10	5.294744	20	5.830367	30	5.972076	100	6.000000

Therefore,

$$E(W_i) = \sum_{n=1}^{N} E(L_n^{(i)}) + \sum_{n=1}^{N-1} \frac{\mu}{r^{n-1}} + \rho, \quad i = 1, 2.$$

Further, the expected cost of a cycle for Case-i, $i = 1, 2$, is given by

$$E\left\{ c \sum_{n=1}^{N-1} R_n - c_r \sum_{n=1}^{N} L_n^{(i)} + B_c + c_p Z \right\} = c \sum_{n=1}^{N-1} \frac{\mu}{r^{n-1}} - c_r \sum_{n=1}^{N} E(L_n^{(i)}) + B_c + c_p \rho.$$

Then the average cost, denoted by $C_i(N)$, for Case-i, $i = 1, 2$, can be calculated as

$$\begin{aligned} C_i(N) &= \frac{\text{Expected cost incurred in a cycle}}{\text{Expected length of a cycle}} \\ &= \frac{c \sum_{n=1}^{N-1} \frac{\mu}{r^{n-1}} - c_r \sum_{n=1}^{N} E(L_n^{(i)}) + B_c + c_p \rho}{\sum_{n=1}^{N} E(L_n^{(i)}) + \sum_{n=1}^{N-1} \frac{\mu}{r^{n-1}} + \rho}, \end{aligned}$$

where $E(L_n^{(1)})$ and $E(L_n^{(2)})$ are the same as in (5.1) and (5.2), respectively.

In Tables 1 and 2, we tabulated the values of $C_1(N)$ and $C_2(N)$, for different values of N. We assume the relevant parameters as follows: $\kappa = 1.06, \mu = 10, r = 0.80, c = 6, c_r = 3, B_c = 100, \delta_0 = 0.5, \rho = 30, \sigma_1 = 0.2, \sigma_2 = 0.1, c_p = 4, \lambda = 0.40, \gamma = 1$ and $\xi = 5$, $\theta = 0.4$. Note that, for both cases, the average magnitudes of shocks are the same (2.5). Tables 1 and 2 indicate that the optimal average replacement costs are $C_1(6) = 5.081326$ and $C_2(6) = 4.948404$. Although these are different, the optimal replacement policies, for both cases, are the same and are given by $N^* = 6$. Hence, for both cases, the systems should be replaced immediately after the 6-th failure.

6 Concluding remark

In this paper, we introduce a new shock model which is a generalization of the classical δ-shock model. In the classical δ-shock model, the recovery time of a system from the damage of a shock is assumed as constant. This assumption of constant δ is indeed not true in many real-life scenarios. For example, any technical system ages over time and hence, it needs more recovery time as time progresses. Further, the

Table 2: Values of $C_2(N)$

N	$C_2(N)$	N	$C_2(N)$	N	$C_2(N)$	N	$C_2(N)$
1	6.294425	11	5.289444	21	5.841683	34	5.985941
2	5.567181	12	5.370442	22	5.867000	38	5.993613
3	5.198814	13	5.447119	23	5.888572	42	5.997133
4	5.020134	14	5.518161	24	5.906876	46	5.998725
5	4.951095	15	5.582876	25	5.922347	50	5.999437
6	**4.948404**	16	5.641023	26	5.935379	60	5.999929
7	4.986342	17	5.692676	27	5.946324	70	5.999991
8	5.048593	18	5.738118	28	5.955491	80	5.999999
9	5.124352	19	5.777767	29	5.963150	90	6.000000
10	5.206297	20	5.812114	30	5.969535	100	6.000000

recovery time also depends on the magnitude of a shock. Naturally, if the magnitude of a shock is large (resp. small), then the corresponding damage is also large (resp. small) and consequently, the system needs more (resp. less) time to restore its original state. Thus, in this paper, we introduce the general δ-shock model where δ depends on both the arrival times and the magnitudes of shocks. Moreover, this model contains the time-dependent δ-shock model (i.e., the recovery time is an increasing function of calender time), the magnitude-dependent δ-shock model (i.e., the recovery time is an increasing function of the magnitude of a shock) and the classical δ-shock model (i.e., the constant recovery time) as the particular cases. Further, we also assume a more general shock process (namely, the PGPP) which contains the GPP, the Pólya process, the NHPP and the HPP as the particular cases. For the defined model, we study some important reliability indices (namely, survival function, mean lifetime, distribution for a fatal shock) and discuss some stochastic comparisons results. Finally, as an important application of the proposed model, we discuss the relevant optimal replacement policy.

The proposed model may further be generalized in different directions. A potential problem in this context could be the study of the general δ-shock model for a multi-component system with multiple failure types. Further, different mixed shock models (i.e., the combinations of the general δ-shock model with other basic models) may also be considered.

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Appendix

Proof of Theorem 3.1: We have

$$\bar{F}_L(t) = P(L > t) \quad = \quad P(L > t, N(t) = 0) + \sum_{n=1}^{\infty} P(L > t, N(t) = n). \tag{A1}$$

From Lemma 3.1, we have

$$P(L > t, N(t) = 0) = \frac{\alpha^{k-\nu}\Gamma_\nu(k, (\alpha + \Lambda(t))l)}{\Gamma_\nu(k, \alpha l)(\alpha + \Lambda(t))^{k-\nu}}. \tag{A2}$$

Further, for $n \in \mathbb{N}$, we have

$$\begin{aligned}
&P(L > t|N(t) = n) \\
&= P(T_1 > \delta_0, Y_1 \leq \gamma, T_2 > T_1 + \delta(T_1, Y_1), \ldots, T_n > T_{n-1} + \delta(T_{n-1}, Y_{n-1}), Y_n < \gamma|N(t) = n) \\
&= P(T_1 > \delta_0, Y_1 \leq \gamma, T_2 > g(T_1, Y_1), Y_2 \leq \gamma, \ldots, T_n > g(T_{n-1}, Y_{n-1}), Y_n < \gamma|N(t) = n).
\end{aligned}$$

Note that the system survives n shocks in $[0, t)$ provided $t > T_n > g(T_{n-1}, Y_{n-1}) > g^2(T_{n-2}, Y_{n-2}, Y_{n-1})$ $> \cdots > g^{n-1}(\delta_0, Y_1, Y_2, \ldots, Y_{n-1}) \geq g^{n-1}(\delta_0, \underbrace{0, 0, \ldots, 0}_{n-1 \text{ times}})$. This implies that, if $t \leq g^{n-1}(\delta_0, \underbrace{0, 0, \ldots, 0}_{n-1 \text{ times}})$, then the probability of the event "the system survives n shocks till time t" is equal to zero, i.e.,

$$P(L > t|N(t) = n) = 0, \text{ for } t \leq g^{n-1}(\delta_0, \underbrace{0, 0, \ldots, 0}_{n-1 \text{ times}}).$$

For $1 \le n \le K_0(t)$, we have

$$
\begin{aligned}
& P(L > t | N(t) = n) \\
= & \underbrace{\overbrace{\int\int\cdots\int}^{n \text{ times}}}_{A_0(t,n)} P(T_1 > \delta_0, Y_1 \le \gamma, \ldots, T_n > g(T_{n-1}, Y_{n-1}), Y_n < \gamma | N(t) = n, Y_1 = y_1, \ldots, Y_n = y_n) \\
& \times f_{Y_1,Y_2,\ldots,Y_n}(y_1, y_2, \ldots, y_n | N(t) = n) dy_1 \ldots dy_n \\
= & \underbrace{\overbrace{\int\int\cdots\int}^{n \text{ times}}}_{A_0(t,n)} P(T_1 > \delta_0, T_2 > g(T_1, y_1), \ldots, T_n > g(T_{n-1}, y_{n-1}) | N(t) = n) \\
& \times f_{Y_1,Y_2,\ldots,Y_n}(y_1, y_2, \ldots, y_n) dy_1 \ldots dy_n \\
= & \underbrace{\overbrace{\int\int\cdots\int}^{n \text{ times}}}_{A_0(t,n)} P(T_1 > \delta_0, T_2 > g(T_1, y_1), \ldots, T_n > g(T_{n-1}, y_{n-1}) | N(t) = n) \\
& \times \prod_{i=1}^{n} f_{Y_1}(y_i) dy_1 \ldots dy_n, \qquad \text{(A3)}
\end{aligned}
$$

where the second equality follows from Assumptions 2 and 3, and the third equality holds due to Assumption 4. Now, for $0 \le y_i \le \gamma$, $1 \le i \le n$, consider

$$
\begin{aligned}
& P(T_1 > \delta_0, T_2 > g(T_1, y_1), \ldots, T_n > g(T_{n-1}, y_{n-1}) | N(t) = n) \\
& = \int_{g^{n-1}(\delta_0, y_1, y_2, \ldots, y_{n-1})}^{t} \int_{g^{n-2}(\delta_0, y_1, y_2, \ldots, y_{n-2})}^{h(t_n, y_{n-1})} \cdots \int_{g(\delta_0, y_1)}^{h(t_3, y_2)} \int_{\delta_0}^{h(t_2, y_1)} f_{(T_1, \cdots, T_{N(t)} | N(t))}(t_1, \cdots, t_n | n) \\
& \quad \times dt_1 dt_2 \ldots dt_n \\
& = \int_{g^{n-1}(\delta_0, y_1, y_2, \ldots, y_{n-1})}^{t} \int_{g^{n-2}(\delta_0, y_1, y_2, \ldots, y_{n-2})}^{h(t_n, y_{n-1})} \cdots \int_{g(\delta_0, y_1)}^{h(t_3, y_2)} \int_{\delta_0}^{h(t_2, y_1)} \left(n! \prod_{i=1}^{n} \frac{\lambda(t_i)}{\Lambda(t)} \right) dt_1 \ldots dt_n,
\end{aligned}
$$

where the last equality follows from Lemma 3.1(b). On using the above expression in (A3), we get

$$
P(L > t | N(t) = n) = \frac{n!}{(\Lambda(t))^n} u(t, n),
$$

which, by Lemma 3.1(a), gives

$$
\begin{aligned}
P(L > t, N(t) = n) &= P(L > t | N(t) = n) P(N(t) = n) \\
&= \frac{\alpha^{k-\nu}}{(\alpha + \Lambda(t))^{k+n-\nu}} \frac{\Gamma_\nu(k + n, (\alpha + \Lambda(t))l)}{\Gamma_\nu(k, \alpha l)} u(t, n). \qquad \text{(A4)}
\end{aligned}
$$

Finally, on using (A2) and (A4) in (A1), we get the required result. $\square$

Proof of Theorem 3.2: We have

$$\begin{aligned}
E(L) &= \int_0^\infty \bar{F}_L(t)dt \\
&= \int_0^{\delta_0} \bar{F}_L(t)dt + \int_{\delta_0}^{g(\delta_0,0)} \bar{F}_L(t)dt + \int_{g(\delta_0,0)}^{g^2(\delta_0,0,0)} \bar{F}_L(t)dt + \ldots \\
&= \int_0^{\delta_0} P(L>t, N(t)=0)dt + \int_{\delta_0}^{g(\delta_0,0)} \left[P(L>t,N(t)=0)+P(L>t,N(t)=1)\right]dt \\
&\quad + \int_{g(\delta_0,0)}^{g^2(\delta_0,0,0)} \left[P(L>t,N(t)=0)+P(L>t,N(t)=1)+P(L>t,N(t)=2)\right]dt + \ldots \\
&= \int_0^\infty P(L>t,N(t)=0)dt + \sum_{n=1}^{\infty} \int_{g^{n-1}(\delta_0,\underbrace{0,0,\ldots,0}_{n-1 \text{ times}})}^{\infty} P(L>t,N(t)=n)dt
\end{aligned}$$

On using (A2) and (A4) in the above expression, we get the required result.

Proof of Theorem 3.5: Since $\delta_1(t,y) \leq \delta_2(t,y)$, we get $g_1(t,y) \leq g_2(t,y)$, for all $(t,y) \in [0,\infty)\times[0,\infty)$. We know that both $g_1(\cdot,y)$ and $g_2(\cdot,y)$ are bijective and strictly increasing functions on $[0,\infty)$, for fixed $y \in [0,\infty)$. Let $h_1(\cdot,y)$ and $h_2(\cdot,y)$ be the inverse functions of $g_1(\cdot,y)$ and $g_2(\cdot,y)$, respectively, for fixed $y \in [0,\infty)$. Then, for fixed $y \in [0,\infty)$, the inequality "$g_1(t,y) \leq g_2(t,y)$, for all $t \in [0,\infty)$" implies $h_2(t,y) \leq h_1(t,y)$ for all $t \in [\delta_0,\infty)$. Now, from Theorem 3.1, we have

$$\bar{F}_{L_i}(t) = \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)(\alpha+\Lambda(t))^{k-\nu}} \left[\Gamma_\nu(k,(\alpha+\Lambda(t))l) + \sum_{n=1}^{K_{i0}(t)} \frac{\Gamma_\nu(k+n,(\alpha+\Lambda(t))l)}{(\alpha+\Lambda(t))^n} u_i(t,n)\right],$$

where

$$K_{i0}(t) = max\{n \geq 1 | g_i^{n-1}(\delta_0,\underbrace{0,0,\ldots,0}_{n-1 \text{ times}}) < t\},$$

and

$$\begin{aligned}
u_i(t,n) &= \overbrace{\int\int\cdots\int}^{n \text{ times}}_{\{(y_1,y_2,\ldots,y_n):\, g_i^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1}) < t,\, 0\leq y_j \leq \gamma,\, 1\leq j\leq n\}} \Bigg\{ \\
&\quad \int_{g_i^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1})}^{t} \int_{g_i^{n-2}(\delta_0,y_i,y_2,\ldots,y_{n-2})}^{h_i(t_n,y_{n-1})} \cdots \int_{g_i(\delta_0,y_1)}^{h_i(t_3,y_2)} \int_{\delta_0}^{h_i(t_2,y_1)} \left(\prod_{j=1}^{n}\lambda(t_j)\right) \\
&\quad dt_1 dt_2 \ldots dt_n \Bigg\} f_{Y_1,Y_2,\ldots,Y_n}(y_1,y_2,\ldots,y_n)dy_1dy_2\ldots dy_n, \quad i=1,2.
\end{aligned}$$

Since, for fixed $y \in [0,\infty)$, $g_1(t,y) \leq g_2(t,y)$, for all $t \in [0,\infty)$, and $h_2(t,y) \leq h_1(t,y)$ for all $t \in [\delta_0,\infty)$, we get

$$\begin{aligned}
&\int_{g_1^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1})}^{t} \cdots \int_{g_1(\delta_0,y_1)}^{h_1(t_3,y_2)} \int_{\delta_0}^{h_1(t_2,y_1)} \left(\prod_{i=1}^{n}\lambda(t_i)\right) dt_1dt_2\ldots dt_n \\
&\geq \int_{g_2^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1})}^{t} \cdots \int_{g_2(\delta_0,y_1)}^{h_2(t_3,y_2)} \int_{\delta_0}^{h_2(t_2,y_1)} \left(\prod_{i=1}^{n}\lambda(t_i)\right) dt_1dt_2\ldots dt_n,
\end{aligned}$$

which further implies that $u_1(t,n) \geq u_2(t,n)$, for all $t \in [0,\infty)$ and $n \in \mathbb{N}$. Again, $K_{10}(t) \geq K_{20}(t)$, for all $t \in [0,\infty)$. Thus, $\bar{F}_{L_2}(t) \leq \bar{F}_{L_1}(t)$, for all $t \in [0,\infty)$, and hence, the result is proved. □

Proof of Theorem 4.1: Given that the recovery function is $\delta(t,y) = \delta_0 + \sigma_1 t + \sigma_2 y$, for all $(t,y) \in [0,\infty) \times [0,\infty)$, which implies $g(t,y) = \delta_0 + (1+\sigma_1)t + \sigma_2 y$, for all $(t,y) \in [0,\infty) \times [0,\infty)$, and $h(t,y) = (t - \delta_0 - \sigma_2 y)/(1+\sigma_1)$, for all $(t,y) \in [\delta_0,\infty) \times [0,\infty)$. Consequently,

$$g^{n-1}(\delta_0, y_1, y_2, \ldots, y_{n-1}) = \delta_0 \left(\frac{(1+\sigma_1)^n - 1}{\sigma_1}\right) + \sigma_2 \left[(1+\sigma_1)^{n-2} y_1 + (1+\sigma_1)^{n-3} y_2 + \cdots + y_{n-1}\right].$$

Then

$$\begin{aligned} K_0(t) &= max\left\{ n \geq 1 | g^{n-1}(\delta_0, \underbrace{0,0,\ldots,0}_{n-1 \text{ times}}) < t \right\} \\ &= max\left\{ n \geq 1 | \delta_0 \left(\frac{(1+\sigma_1)^n - 1}{\sigma_1}\right) < t \right\} \\ &= max\left\{ n \geq 1 | n < \left(\frac{\ln\left(\frac{\delta_0 + \sigma_1 t}{\delta_0}\right)}{\ln(1+\sigma_1)}\right) \right\} \\ &= \left\lfloor \frac{\ln\left(\frac{\delta_0+\sigma_1 t}{\delta_0}\right)}{\ln(1+\sigma_1)} \right\rfloor = S_0(t). \end{aligned} \tag{A5}$$

Since $Y_i \sim U[0,\xi]$, for all $i \in \mathbb{N}$, for some $\xi > 0$, we have

$$f_{Y_1,Y_2,\ldots Y_n}(y_1, y_2, \ldots, y_n) = \prod_{i=1}^{n} f_{Y_i}(y_i) = \frac{1}{\xi^n}, \text{ for all } 0 \leq y_1, y_2, \ldots, y_n \leq \xi, n \in \mathbb{N}. \tag{A6}$$

Now, consider the following multiple-integration.

$$\int_{g^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1})}^{t} \int_{g^{n-2}(\delta_0,y_1,y_2,\ldots,y_{n-2})}^{h(t_n,y_{n-1})} \cdots \int_{g(\delta_0,y_1)}^{h(t_3,y_2)} \int_{\delta_0}^{h(t_2,y_1)} dt_1 \ldots dt_n.$$

We solve the above integration by iterative process. Let us first consider the following integration.

$$\begin{aligned} \int_{g(\delta_0,y_1)}^{h(t_3,y_2)} \int_{\delta_0}^{h(t_2,y_1)} dt_1 dt_2 &= \int_{\delta_0[1+(1+\sigma_1)]+\sigma_2 y_1}^{\frac{t_3-\sigma_2 y_2-\delta_0}{1+\sigma_1}} \int_{\delta_0}^{\frac{t_2-\sigma_2 y_1-\delta_0}{1+\sigma_1}} dt_1 dt_2 \\ &= \frac{1}{2(1+\sigma_1)^3}[t_3 - \delta_0(1+(1+\sigma_1)+(1+\sigma_1)^2) - \sigma_2(y_2+(1+\sigma_1)y_1)]^2 \\ &= \frac{1}{2(1+\sigma_1)^3}[t_3 - g^2(\delta_0,y_1,y_2)]^2. \end{aligned}$$

Similarly, we get

$$\begin{aligned} &\int_{g^2(\delta_0,y_1,y_2)}^{h(t_4,y_3)} \int_{g(\delta_0,y_1)}^{h(t_3,y_2)} \int_{\delta_0}^{h(t_2,y_1)} dt_1 dt_2 dt_3 \\ &= \int_{\delta_0[1+(1+\sigma_1)+(1+\sigma_1)^2]+\sigma_2(y_2+(1+\sigma_1)y_1)}^{\frac{t_4-\sigma_2 y_3-\delta_0}{1+\sigma_1}} \int_{\delta_0[1+(1+\sigma_1)]+\sigma_2 y_1}^{\frac{t_3-\sigma_2 y_2-\delta_0}{1+\sigma_1}} \int_{\delta_0}^{\frac{t_2-\sigma_2 y_1-\delta_0}{1+\sigma_1}} dt_1 dt_2 dt_3 \\ &= \frac{1}{2(1+\sigma_1)^3} \int_{\delta_0[1+(1+\sigma_1)+(1+\sigma_1)^2]+\sigma_2(y_2+(1+\sigma_1)y_1)}^{\frac{t_4-\sigma_2 y_3-\delta_0}{1+\sigma_1}} [t_3 - g^2(\delta_0,y_1,y_2)]^2 dt_3 \\ &= \frac{1}{3!(1+\sigma_1)^6}[t_4 - g^3(\delta_0,y_1,y_2)]^3. \end{aligned}$$

By proceeding in a similar manner, we get

$$\int_{g^{n-2}(\delta_0,y_1,y_2,\ldots,y_{n-2})}^{h(t_n,y_{n-1})}\cdots\int_{g(\delta_0,y_1)}^{h(t_3,y_2)}\int_{\delta_0}^{h(t_2,y_1)} dt_1\ldots dt_{n-1} = \frac{1}{(n-1)!(1+\sigma_1)^{\frac{n(n-1)}{2}}} \times[t_n-g^{n-1}(\delta_0,y_1,\ldots,y_{n-1})]^{n-1}.$$

Consequently,

$$\int_{g^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1})}^{t}\int_{g^{n-2}(\delta_0,y_1,y_2,\ldots,y_{n-2})}^{h(t_n,y_{n-1})}\cdots\int_{g(\delta_0,y_1)}^{h(t_3,y_2)}\int_{\delta_0}^{h(t_2,y_1)} dt_1\ldots dt_n$$
$$= \frac{1}{n!(1+\sigma_1)^{\frac{n(n-1)}{2}}}[t-g^{n-1}(\delta_0,y_1,\ldots,y_{n-1})]^n. \tag{A7}$$

On using (A6) and (A7) in the expression of $u(t,n)$, given in Theorem 3.1, we get

$$u(t,n) = \left(\frac{\gamma}{\xi^n}\right)(\lambda)^n\left(\frac{1}{n!(1+\sigma_1)^{\frac{n(n-1)}{2}}}\right)v(t,n).$$

Finally, the result follows from Theorem 3.1 by using the above equality along with (A5). □

Proof of Theorem 4.2: Let $\mathcal{X}$ be a structure random variable with the probability density given by

$$dH(\chi)=\frac{\left(\frac{b}{\omega}\right)^{\frac{\beta}{\omega}}}{\Gamma\left(\frac{\beta}{\omega}\right)}\chi^{\left(\frac{\beta}{\omega}\right)-1}\exp\left\{-\left(\frac{b}{\omega}\right)\chi\right\}d\chi. \tag{A8}$$

Given that shocks occur according to the HPGGP with the set of parameters $\{1/b, 0, \beta/\omega, 1/\omega, l\}$, which is indeed the Pólya process with the set of parameters $\left\{\frac{\beta}{\omega},\frac{b}{\omega}\right\}$. Moreover, on condition $\mathcal{X}=\chi$, the Pólya process is the same as the HPP with intensity χ (see Beichelt [3], p.130). Consequently, the inter-arrival times of the given HPGGP are i.i.d. random variables and follow the exponential distribution with parameter χ. Then, the conditional probability density function of the random vector $(X_1, X_2,\ldots,X_m)$, given $\mathcal{X}=\chi$, is given by

$$f_{X_1,X_2,\ldots,X_m|\mathcal{X}}(x_1,x_2,\ldots,x_m|\chi)=\prod_{i=1}^{m}\chi\exp\{-\chi x_i\},\quad 0<x_1,x_2,\ldots,x_m<\infty. \tag{A9}$$

Now,

$$\begin{aligned} P(M=1) &= 1-P(X_1>\delta_0, Y_1\le\gamma) && \text{(A10)}\\ &= 1-F_{Y_1}(\gamma)\int_0^\infty P(X_1>\delta_0|\mathcal{X}=\chi)dH(\chi)\\ &= 1-F_{Y_1}(\gamma)\int_0^\infty \exp\{-\chi\delta_0\}\frac{\left(\frac{b}{\omega}\right)^{\frac{\beta}{\omega}}}{\Gamma(\frac{\beta}{\omega})}\chi^{\frac{\beta}{\omega}-1}\exp\left\{-\left(\frac{b}{\omega}\right)\chi\right\}d\chi\\ &= 1-\frac{\gamma}{\xi}\left(\frac{b}{b+\omega\delta_0}\right)^{\frac{\beta}{\omega}}=\Upsilon_0-\Upsilon_1. && \text{(A11)}\end{aligned}$$

Further, for $m=2,3,\ldots$, we have

$$\begin{aligned} P(M=m) &= P(X_1>\delta_0, Y_1\le\gamma,\ldots,X_{m-1}>g(T_{m-2},Y_{m-2}),Y_{m-1}\le\gamma)\\ &\quad -P(X_1>\delta_0,Y_1\le\gamma,\ldots,X_m>g(T_{m-1},Y_{m-1}),Y_m\le\gamma)\\ &= \zeta_{m-1}-\zeta_m \text{ (say) }, && \text{(A12)}\end{aligned}$$

where

$$\zeta_1 = P(X_1 > \delta_0, Y_1 \le \gamma) = \frac{\gamma}{\xi}\left(\frac{b}{b+\omega\delta_0}\right)^{\frac{\beta}{\omega}} = \Upsilon_1. \tag{A13}$$

and

$$\begin{aligned}\zeta_m &= P(X_1 > \delta_0, Y_1 \le \gamma, X_2 > g(T_1, Y_1), \ldots, X_m > g(T_{m-1}, Y_{m-1}), Y_m \le \gamma)\\ &= \underbrace{\int_0^\gamma \int_0^\gamma \cdots \int_0^\gamma}_{m \text{ times}} \Big[P(X_1 > \delta_0, X_2 > g(T_1, y_1), \ldots, X_m > g(T_{m-1}, y_{m-1}))\\ &\quad \times f_{Y_1, Y_2, \ldots, Y_m}(y_1, y_2, \ldots, y_m)\Big] dy_1 dy_2 \ldots dy_m, \quad m = 2, 3, \ldots.\end{aligned} \tag{A14}$$

Now, we proceed to find the value of ζ_m. Consider

$$\begin{aligned}&P(X_1 > \delta_0, X_2 > g(T_1, y_1), \ldots, X_m > g(T_{m-1}, y_{m-1}))\\ &= \int_0^\infty P(X_1 > \delta_0, X_2 > g(T_1, y_1), \ldots, X_m > g(T_{m-1}, y_{m-1}) | \mathcal{X} = \chi) dH(\chi)\\ &= \frac{\left(\frac{b}{\omega}\right)^{\frac{\beta}{\omega}}}{\Gamma(\frac{\beta}{\omega})} \int_0^\infty P(X_1 > \delta_0, X_2 > \delta_0 + \sigma_1 X_1 + \sigma_2 y_1, \ldots, X_m > \delta_0 + \sigma_1(X_1 + \cdots + X_{m-1})\\ &\quad + \sigma_2 y_{m-1} | \mathcal{X} = \chi) \chi^{\frac{\beta}{\omega}-1} \exp\left\{-\left(\frac{b}{\omega}\right)\chi\right\} d\chi,\end{aligned} \tag{A15}$$

where the second equality follows from (A8). Again, consider

$$\begin{aligned}&P(X_1 > \delta_0, X_2 > \delta_0 + \sigma_1 X_1 + \sigma_2 y_1, \ldots, X_m > \delta_0 + \sigma_1(X_1 + \cdots + X_{m-1}) + \sigma_2 y_{m-1} | \mathcal{X} = \chi)\\ =\;& \int_{\delta_0}^\infty \int_{\delta_0 + \sigma_1 x_1 + \sigma_2 y_1}^\infty \cdots \int_{\delta_0 + \sigma_1(x_1 + \cdots + x_{m-1}) + \sigma_2 y_{m-1}}^\infty \left(\prod_{i=1}^m \chi \exp\{-\chi x_i\}\right) dx_m dx_{m-1} \ldots dx_2 dx_1\\ =\;& \frac{1}{(1+\sigma_1)^{\frac{m(m-1)}{2}}} \exp\left\{-\chi\delta_0\left[1 + (1+\sigma_1) + \cdots + (1+\sigma_1)^{m-1}\right]\right\}\\ &\times \exp\left\{-\chi\sigma_2\left[y_{m-1} + (1+\sigma_1) y_{m-2} + \cdots + (1+\sigma_1)^{m-2} y_1\right]\right\}\\ =\;& \frac{\exp\left\{-\chi\left(\delta_0\left(\frac{(1+\sigma_1)^m - 1}{\sigma_1}\right) + \sigma_2(y_{m-1} + (1+\sigma_1) y_{m-2} + \cdots + (1+\sigma_1)^{m-2} y_1)\right)\right\}}{(1+\sigma_1)^{\frac{m(m-1)}{2}}},\end{aligned} \tag{A16}$$

where the first equality follows from (A9). Then, on using (A16) in (A15), we get

$$\begin{aligned}&P(X_1 > \delta_0, X_2 > g(T_1, y_1), \ldots, X_m > g(T_{m-1}, y_{m-1}))\\ =\;& \frac{\left(\frac{b}{\omega}\right)^{\frac{\beta}{\omega}}}{\Gamma(\frac{\beta}{\omega})} \int_0^\infty \left[\frac{\exp\left\{-\chi\left(\delta_0\left(\frac{(1+\sigma_1)^m - 1}{\sigma_1}\right) + \sigma_2(y_{m-1} + (1+\sigma_1) y_{m-2} + \cdots + (1+\sigma_1)^{m-2} y_1)\right)\right\}}{(1+\sigma_1)^{\frac{m(m-1)}{2}}}\right.\\ &\left.\times\, \chi^{\frac{\beta}{\omega}-1} \exp\left\{-\left(\frac{b}{\omega}\right)\chi\right\}\right] d\chi\\ &= \frac{1}{(1+\sigma_1)^{\frac{m(m-1)}{2}}} \left(\frac{b}{\omega\delta_0\left(\frac{(1+\sigma_1)^m - 1}{\sigma_1}\right) + \omega\sigma_2 y_{m-1} + \omega\sigma_2(1+\sigma_1) y_{m-2} + \cdots + \omega\sigma_2(1+\sigma_1)^{m-2} y_1 + b}\right)^{\frac{\beta}{\omega}}.\end{aligned} \tag{A17}$$

Since $Y_i \sim U[0,\xi]$, $i \in \mathbb{N}$, for some $\xi > 0$, we have

$$f_{Y_1,Y_2,\ldots,Y_m}(y_1,y_2,\ldots,y_m) = \left(\frac{1}{\xi}\right)^m, \quad 0 < y_1,y_2,\ldots,y_m < \infty. \tag{A18}$$

On using (A17) and (A18) in (A14), we get

$$\zeta_m = \left(\frac{\gamma b^{\frac{\beta}{\omega}}}{\xi^m (1+\sigma_1)^{\frac{m(m-1)}{2}}}\right) I_m, \tag{A19}$$

where

$$\begin{aligned} I_m &= \underbrace{\int_0^\gamma \int_0^\gamma \cdots \int_0^\gamma}_{m-1 \text{ times}} \left\{ \omega\delta_0 \left(\frac{(1+\sigma_1)^m - 1}{\sigma_1}\right) + \omega\sigma_2 y_{m-1} \right. \\ &\quad \left. + \omega\sigma_2(1+\sigma_1)y_{m-2} + \cdots + \omega\sigma_2(1+\sigma_1)^{m-2} y_1 + b \right\}^{-\frac{\beta}{\omega}} dy_1 dy_2 \ldots dy_{m-1}. \end{aligned}$$

Now, we solve I_m by iterative process. Consider

$$\begin{aligned} I_2 &= \int_0^\gamma \{\omega\delta_0\,(1+(1+\sigma_1)) + \omega\sigma_2 y_1 + b\}^{\frac{-\beta}{\omega}}\, dy_1 \\ &= \left[\frac{\{\omega\delta_0(1+(1+\sigma_1)) + \omega\sigma_2\gamma + b\}^{(-\frac{\beta}{\omega}+1)} - \{\omega\delta_0(1+(1+\sigma_1)) + b\}^{(-\frac{\beta}{\omega}+1)}}{\sigma_2(\omega-\beta)}\right] \\ &= \frac{1}{\sigma_2(\omega-\beta)} \left[(-1) \left\{ \omega\delta_0(1+(1+\sigma_1)) + \sum_{s \in S^1_{A_1,0}} s + b \right\}^{(-\frac{\beta}{\omega}+1)} \right. \\ &\quad \left. + (-1)^0 \left\{ \omega\delta_0(1+(1+\sigma_1)) + \sum_{s \in S^1_{A_1,1}} s + b \right\}^{(-\frac{\beta}{\omega}+1)} \right], \end{aligned}$$

where $A_1 = \{\omega\sigma_2\gamma\}$. Similarly,

$$\begin{aligned} I_3 &= \int_0^\gamma \int_0^\gamma \{\omega\delta_0(1+(1+\sigma_1)+(1+\sigma)^2) + \omega\sigma_2 y_2 + \omega\sigma_2(1+\sigma_1)y_1 + b\}^{-\frac{\beta}{\omega}} dy_1 dy_2 \\ &= \frac{1}{\sigma_2^2(1+\sigma_1)(2\omega-\beta)(\omega-\beta)} \left[\left\{ \omega\delta_0 \left(\frac{(1+\sigma_1)^3-1}{\sigma_1}\right) + \sum_{s \in S^1_{A_2,2}} s + b \right\}^{(-\frac{\beta}{\omega}+2)} \right. \\ &\quad - \sum_{l=1}^{2} \left\{ \omega\delta_0 \left(\frac{(1+\sigma_1)^3-1}{\sigma_1}\right) + \sum_{s \in S^l_{A_2,1}} s + b \right\}^{(-\frac{\beta}{\omega}+2)} \\ &\quad \left. + \left\{ \omega\delta_0 \left(\frac{(1+\sigma_1)^3-1}{\sigma_1}\right) + \sum_{s \in S^1_{A_2,0}} s + b \right\}^{(-\frac{\beta}{\omega}+2)} \right] \end{aligned}$$

and

$$I_4 = \int_0^\gamma \int_0^\gamma \int_0^\gamma \left\{ \omega\delta_0 \left(\frac{(1+\sigma_1)^4-1}{\sigma_1}\right) + \omega\sigma_2 y_3 + \omega\sigma_2(1+\sigma_1)y_2 + \omega\sigma_2(1+\sigma_1)^2 y_1 \right\}^{-\frac{\beta}{\omega}} dy_3 dy_2 dy_1$$

$$
\begin{aligned}
= &\ \frac{1}{\sigma_2^3(1+\sigma_1)^{1+2}(3\omega-\beta)(2\omega-\beta)(\omega-\beta)}\left[\left\{\omega\delta_0\left(\frac{(1+\sigma_1)^4-1}{\sigma_1}\right)+\sum_{s\in S^1_{A_3,3}} s+b\right\}^{(-\frac{\beta}{\omega}+3)}\right.\\
&-\sum_{l=1}^{3}\left\{\omega\delta_0\left(\frac{(1+\sigma_1)^4-1}{\sigma_1}\right)+\sum_{s\in S^l_{A_3,2}} s+b\right\}^{(-\frac{\beta}{\omega}+3)}\\
&+\sum_{l=1}^{3}\left\{\omega\delta_0\left(\frac{(1+\sigma_1)^4-1}{\sigma_1}\right)+\sum_{s\in S^l_{A_3,1}} s+b\right\}^{(-\frac{\beta}{\omega}+3)}\\
&\left.-\left\{\omega\delta_0\left(\frac{(1+\sigma_1)^4-1}{\sigma_1}\right)+\sum_{s\in S^1_{A_3,0}} s+b\right\}^{(-\frac{\beta}{\omega}+3)}\right],
\end{aligned}
$$

where $A_2 := \{\omega\sigma_2\gamma, \omega\sigma_2(1+\sigma_1)\gamma\}$ and $A_3 = \{\omega\sigma_2\gamma, \omega\sigma_2\gamma(1+\sigma_1), \omega\sigma_2\gamma(1+\sigma_1)^2\}$. By proceeding in the same line, we get

$$
\begin{aligned}
I_m = &\ \frac{1}{\sigma_2^{m-1}(1+\sigma_1)^{\frac{(m-1)(m-2)}{2}}\left(\prod_{i=1}^{m-1}\{i\omega-\beta\}\right)}\\
&\times\left[\sum_{i=0}^{m-1}(-1)^{m-1-i}\sum_{l=1}^{\binom{m-1}{i}}\left(\omega\delta_0\left(\frac{(1+\sigma_1)^m-1}{\sigma_1}\right)+\left(\sum_{s\in S^l_{A_{m-1},i}} s\right)+b\right)^{-\frac{\beta}{\omega}+m-1}\right],
\end{aligned}
$$

where $A_{m-1} = \{\omega\sigma_2\gamma, \omega\sigma_2\gamma(1+\sigma_1), \ldots, \omega\sigma_2\gamma(1+\sigma_1)^{m-2}\}$. On using the above value of I_m in (A19), we get $\zeta_m = \Upsilon_m$ and hence, the required result follows from (A11), (A12) and (A13). ☐

Proof of Theorem 4.3: Since $Y_i \sim \exp(\theta)$, for all $i \in \mathbb{N}$, for some $\theta > 0$, we have

$$
f_{Y_1,Y_2,\ldots,Y_n}(y_1,y_2,\ldots,y_m) = \prod_{i=1}^{m} f_{Y_i}(y_i) = \prod_{i=1}^{m}\theta\exp\{-\theta y_i\}, \quad 0 < y_1, y_2, \ldots, y_n < \infty, \tag{A20}
$$

and $F_{Y_m}(\gamma) = (1-\exp\{-\theta\gamma\})$ for all $m \in \mathbb{N}$. Then, from (A10), we have

$$
P(M=1) = 1-(1-\exp\{-\theta\gamma\})\exp\{-\lambda\delta_0\} = \Delta_0 - \Delta_1. \tag{A21}
$$

By proceeding in the same line as done in the proof of Theorem 4.2, we get, from (A14) and (A20), that

$$
\zeta_m = \left(\frac{(1-\exp\{-\theta\gamma\})}{(1+\sigma_1)^{\frac{m(m-1)}{2}}}\right)\exp\left\{-\lambda\delta_0\left(\frac{(1+\sigma_1)^m-1}{\sigma_1}\right)\right\}\theta^{m-1}J_m,
$$

where

$$
\begin{aligned}
J_m &= \underbrace{\int_0^\gamma\int_0^\gamma\cdots\int_0^\gamma}_{m-1\text{ times}}\prod_{i=1}^{m-1}\exp\left\{-y_i\left(\lambda\sigma_2(1+\sigma_1)^{m-1-i}+\theta\right)\right\}dy_1dy_2\ldots dy_{m-1}\\
&= \prod_{i=1}^{m-1}\left(\int_0^\gamma\exp\left\{-y_i\left(\lambda\sigma_2(1+\sigma_1)^{m-1-i}+\theta\right)\right\}dy_i\right)\\
&= \prod_{i=1}^{m-1}\left(\frac{1-\exp\left\{-\gamma(\lambda\sigma_2(1+\sigma_1)^{m-1-i}+\theta)\right\}}{\lambda\sigma_2(1+\sigma_1)^{m-1-i}+\theta}\right).
\end{aligned}
$$

On using the above value of ζ_m in (A12), we get

$$\begin{aligned} P(M=m) &= \theta^{m-2}\left(\frac{(1-\exp\{-\theta\gamma\})}{(1+\sigma_1)^{\frac{(m-1)(m-2)}{2}}}\right)\exp\left\{-\lambda\delta_0\left(\frac{(1+\sigma_1)^{m-1}-1}{\sigma_1}\right)\right\}J_{m-1} \\ &\quad -\theta^{m-1}\left(\frac{(1-\exp\{-\theta\gamma\})}{(1+\sigma_1)^{\frac{m(m-1)}{2}}}\right)\exp\left\{-\lambda\delta_0\left(\frac{(1+\sigma_1)^{m}-1}{\sigma_1}\right)\right\}J_{m} \\ &= \Delta_{m-1}-\Delta_m, \quad m=2,3,\ldots. \end{aligned} \tag{A22}$$

Finally, the result follows from (A21) and (A22). □

Proof of Corollary 4.10: From Theorem 4.4, we have

$$\begin{aligned} E(L) &= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\sum_{n=0}^{\infty}\int_{\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0}^{\infty}\frac{\lambda^n}{n!}\frac{\Gamma_\nu(k+n,(\alpha+\lambda t)l)}{(\alpha+\lambda t)^{k+n-\nu}}\frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}}\left(\frac{\gamma}{\xi}\right)^n \\ &\quad\times\left[t-\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0\right]^n dt \\ &= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\sum_{n=0}^{\infty}\left(\frac{\gamma}{\xi}\right)^n\frac{\lambda^n}{n!}\frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}}\int_{\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0}^{\infty}\frac{\Gamma_\nu(k+n,(\alpha+\lambda t)l)}{(\alpha+\lambda t)^{k+n-\nu}} \\ &\quad\times\left[t-\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0\right]^n dt \\ &= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\sum_{n=0}^{\infty}\left(\frac{\gamma}{\xi}\right)^n\frac{\lambda^n}{n!}\frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}}\int_{\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0}^{\infty}\int_0^{\infty}\frac{y^{n+k-1}\exp\{-(\alpha+\lambda t)y\}}{(y+l)^{\nu}} \\ &\quad\times\left[t-\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0\right]^n dydt \\ &= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\sum_{n=0}^{\infty}\left(\frac{\gamma}{\xi}\right)^n\frac{\lambda^n}{n!}\frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}}\int_0^{\infty}\frac{y^{n+k-1}\exp\{-\alpha y\}}{(y+l)^{\nu}}\left\{\int_{\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0}^{\infty}\exp\{-\lambda t y\}\right. \\ &\quad\left.\times\left[t-\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0\right]^n dt\right\}dy \\ &= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\sum_{n=0}^{\infty}\left(\frac{\gamma}{\xi}\right)^n\frac{\lambda^n}{n!}\frac{\Gamma(n+1)}{(1+\sigma_1)^{\frac{n(n-1)}{2}}}\int_0^{\infty}\frac{y^{n+k-1}\exp\{-\alpha y\}}{(y+l)^{\nu}}\frac{\exp\left\{-\lambda y\delta_0\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\right\}}{(\lambda y)^{n+1}}dy \\ &= \frac{\alpha^{k-\nu}}{\lambda\Gamma_\nu(k,\alpha l)}\sum_{n=0}^{\infty}\left(\frac{\gamma}{\xi}\right)^n\frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}}\int_0^{\infty}\frac{y^{k-2}\exp\left\{-y\left[\alpha+\lambda\delta_0\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\right]\right\}}{(y+l)^{\nu}}dy \\ &= \frac{\alpha^{k-\nu}}{\lambda\Gamma_\nu(k,\alpha l)}\sum_{n=0}^{\infty}\left(\frac{\gamma}{\xi}\right)^n\frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}}\frac{\Gamma_\nu\left(k-1,\alpha l+\lambda\delta_0 l\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\right)}{\left(\alpha+\lambda\delta_0\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\right)^{k-\nu-1}}, \end{aligned}$$

where the third and the last equalities follow from equation (2.1). Thus the result is proved. □

Proof of Corollary 4.11: From Theorem 4.4, we have

$$\begin{aligned} E(L) &= \int_0^{\infty}\frac{\alpha^{k-\nu}}{(\alpha+\lambda t)^{k-\nu}}\frac{\Gamma_\nu(k,(\alpha+\lambda t)l)}{\Gamma_\nu(k,\alpha l)}dt \\ &\quad+\frac{\alpha^{k-\nu}\gamma}{\Gamma_\nu(k,\alpha l)(\alpha+\lambda t)^{k-\nu}}\sum_{n=1}^{\infty}\int_{n\delta_0}^{\infty}\left(\frac{\lambda}{\xi}\right)^n\frac{\Gamma_\nu(k+n,(\alpha+\lambda t)l)}{n!(\alpha+\lambda t)^n}v(t,n)dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)} \int_0^\infty \int_0^\infty \frac{\exp\{-(\alpha+\lambda t)y\} y^{k-1}}{(y+l)^\nu} dy dt \\
&\quad + \frac{\alpha^{k-\nu}\gamma}{\Gamma_\nu(k,\alpha l)} \sum_{n=1}^\infty \int_{n\delta_0}^\infty \int_0^\infty \left(\frac{\lambda}{\xi}\right)^n \frac{\exp\{-(\alpha+\lambda t)y\} y^{k+n-1}}{(y+l)^\nu} \frac{v(t,n)}{n!} dy dt \\
&= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)} \int_0^\infty \frac{\exp\{-\alpha y\} y^{k-1}}{(y+l)^\nu} \left(\int_0^\infty \exp\{-\lambda t y\} dt \right) dy \\
&\quad + \frac{\alpha^{k-\nu}\gamma}{\Gamma_\nu(k,\alpha l)} \sum_{n=1}^\infty \int_0^\infty \left(\frac{\lambda}{\xi}\right)^n \frac{\exp\{-\alpha y\} y^{k+n-1}}{n!(y+l)^\nu} \left(\int_{n\delta_0}^\infty \exp\{-\lambda t y\} v(t,n) dt \right) dy \\
&= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)} \int_0^\infty \frac{\exp\{-\alpha y\} y^{k-1}}{(y+l)^\nu} \left(\int_0^\infty \exp\{-\lambda t y\} dt \right) dy \\
&\quad + \frac{\alpha^{k-\nu}\gamma}{\Gamma_\nu(k,\alpha l)} \sum_{n=1}^\infty \sum_{i=0}^{n-1} \int_0^\infty (-1)^i \binom{n-1}{i} \left(\frac{\lambda}{\xi}\right)^n \frac{\exp\{-\alpha y\} y^{k+n-1}}{n!(y+l)^\nu} \\
&\quad \times \left(\int_{n\delta_0+i\gamma\sigma_2}^\infty \frac{\exp\{-\lambda y t\} \left(t-n\delta_0-i\gamma\sigma_2\right)^{2n-1}}{(2n-1)!} dt \right) dy \\
&= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)} \int_0^\infty \frac{\exp\{-\alpha y\} y^{k-1}}{(y+l)^\nu} \left(\int_0^\infty \exp\{-\lambda t y\} dt \right) dy \\
&\quad + \frac{\alpha^{k-\nu}\gamma}{\Gamma_\nu(k,\alpha l)} \sum_{n=1}^\infty \sum_{i=0}^{n-1} \int_0^\infty (-1)^i \binom{n-1}{i} \left(\frac{\lambda}{\xi}\right)^n \frac{\exp\{-(\alpha+n\lambda\delta_0+i\gamma\sigma_2\lambda)y\} y^{k+n-1}}{n!(y+l)^\nu} \\
&\quad \times \left(\int_0^\infty \frac{\exp\{-\lambda y x\} x^{2n-1}}{(2n-1)!} dx \right) dy \\
&= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)} \int_0^\infty \frac{\exp\{-\alpha y\} y^{k-2}}{\lambda(y+l)^\nu} dy \\
&\quad + \frac{\alpha^{k-\nu}\gamma}{\Gamma_\nu(k,\alpha l)} \sum_{n=1}^\infty \sum_{i=0}^{n-1} \int_0^\infty (-1)^i \binom{n-1}{i} \left(\frac{1}{\lambda\xi}\right)^n \frac{\exp\{-(\alpha+n\lambda\delta_0+i\gamma\sigma_2\lambda)y\} y^{k-n-1}}{n!(y+l)^\nu} dy \\
&= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)} \left[\frac{\Gamma_\nu(k-1,\alpha l)}{\lambda\alpha^{k-\nu-1}} + \sum_{n=1}^\infty \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \left(\frac{1}{\lambda\xi}\right)^n \frac{\Gamma_\nu(k-n,(\alpha+n\lambda\delta_0+i\gamma\sigma_2\lambda)l)}{(\alpha+n\lambda\delta_0+i\gamma\sigma_2\lambda)^{k-\nu-n}} \right],
\end{aligned}
$$

where the second and the last equalities follows from (2.1), for $k > n$, for all $n \in \mathbb{N}$. Therefore,

$$
E(L) = \lim_{k\to\infty} \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)} \left[\frac{\Gamma_\nu(k-1,\alpha l)}{\lambda\alpha^{k-\nu-1}} + \sum_{n=1}^\infty \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \left(\frac{1}{\lambda\xi}\right)^n \frac{\Gamma_\nu(k-n,(\alpha+n\lambda\delta_0+i\gamma\sigma_2\lambda)l)}{(\alpha+n\lambda\delta_0+i\gamma\sigma_2\lambda)^{k-\nu-n}} \right]
$$

and hence, the result is proved. □

Proof of Theorem 4.5: We can write

$$L = X_1 + X_2 + \cdots + X_M,$$

where M is the random variable representing the number of shocks that have occurred before the failure of the system. As the shocks occur according to the HPP with intensity $\lambda > 0$, $\{X_1, X_2, \ldots, X_n\}$ are i.i.d. random variables with the common mean $1/\lambda$. Thus, from the Wald's equation, we get

$$E(L) = E(X_1)E(M) = \frac{1}{\lambda}E(M),$$

where $E(M)$ is the same as in Theorem 4.3. Hence, the result is proved. □