Delay-dependent Asymptotic Stability of Highly Nonlinear Stochastic Differential Delay Equations Driven by G-Brownian Motion

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Abstract

Based on the classical probability, the stability of stochastic differential delay equations (SDDEs) whose coefficients are growing at most linearly has been investigated intensively. Moreover, the delay-dependent stability of highly nonlinear hybrid stochastic differential equations (SDEs) has also been studied recently. In this paper, using the nonlinear expectation theory, we first explore the delay-dependent criteria on the asymptotic stability for a class of highly nonlinear SDDEs driven by G-Brownian motion (G-SDDEs). Then, the (weak) quasi-sure stability of solutions to G-SDDEs is developed. Finally, an example is analyzed by the φ-max-mean algorithm to illustrate our theoretical results.

Keywords: Nonlinear expectation; Highly nonlinear; G-SDDE; asymptotic stability; Lyapunov functional

1. Introduction

A sublinear expectation (see, e.g., [47, 48, 50]) can be represented as the upper expectation of a subset of linear expectations. In most cases, this subset is often treated as an uncertain model of probabilities. Peng introduced the G-expectation theory (G-framework) (see, e.g., [47] and the references therein) in 2006, where the notion of G-Brownian motion (GBm) and the corresponding stochastic calculus of Itô's type were established. For more further details on GBm we refer the reader to Denis *et al.* [5], Fei [8], Fei and Fei [15], Hu and Peng [26], Li and Peng [31], Peng and Zhang [52], Soner et al. [56] and Zhang [64], etc.

So far, there is a large amount of literature on the problem of asset pricing and financial decisions under model uncertainty. Chen and Epstein [1] put forward to the model of an intertemporal recursive utility, where risk and ambiguity are differentiated, but uncertainty is only a mean uncertainty without a volatility uncertainty. The model of the optimal consumption and portfolio with ambiguity are also investigated in Fei [13, 14]. Epstein and Ji [6, 7] generalized the Chen-Epstein model and maintained a separation between risk aversion and intertemporal substitution. We know that equivalence of priors is an optional assumption in Gilboa and Schmeidler [19]. Apart from very recent developments, the stochastic calculus presumes a probability space framework. However, from an economics perspective, the assumption of equivalence seems far from innocuous. Informally, if her environment is complex, how could the decision-maker come to be certain of which scenarios regarding future asset prices and rates of return, for example, are possible? In particular, ambiguity about volatility implies ambiguity about which scenarios are possible, at least in a continuous time setting. A large amount of literature has argued that the stochastic time varying volatility is important for understanding features of asset returns, and particularly empirical regularities in the derivative markets.

The classical SDEs driven by Brownian motion do not take an ambiguous factor into consideration. Thus, in some complex environments, these equations are too restrictive for describing some phenomena. In fact, by taking

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into account uncertainties including probabilistic and Kightian ones [1], many real uncertain systems have been characterized by Peng's sublinear expectation framework. Recently, under uncertainty, a kind of SDEs driven by GBm (G-SDEs) has been investigated. In Gao [18], the pathwise properties and homeomorphic flows for G-SDEs were studied. Lin [35] gave an analysis on some properties of G-SDEs. Hu *et al.* [27] discussed backward G-SDEs. Lin [36] explored G-SDEs with reflecting boundary. In Luo and Wang [39], G-SDEs and ordinary differential equations were compared. Moreover, the strong Markov property for G-SDEs was analyzed in Hu *et al.* [28]. Fei and Fei [9] proved the consistency of least squares estimation to the parameter for G-SDEs. Fei [8] investigated optimality principles of stochastic control with an application to optimal consumption and portfolio in the framework of G-SDEs.

The stability of the classical SDEs is an important topic in the study of stochastic systems. Influenced by Lyapunov [40] and LaSalle [29] on the stability of nonlinear systems, Hasminskii [20] first studied the stability of the linear Itô SDEs. Since then, the stability analysis for SDEs has been done by many researchers. The stabilization and destabilization of hybrid systems of SDEs have been explored by many researchers, such as Mao et al. [42]. In Luo and Liu [38], the stability of infinite dimensional stochastic evolution equations with memory and Markovian jumps was investigated.

Since a system described by G-SDEs provides a characterization of the real world with both randomness and ambiguity, it is necessary to investigate its stability. Li et al. [32] investigated the delay feedback stabilisation of G-SDEs. In Li and Yan [33], the stability of delayed Hopfield neural networks under a sublinear expectation framework was discussed. And Li *et al.* [34] investigated stabilization of multi-weights stochastic complex networks with time-varying delay and GBm via aperiodically intermittent adaptive control. The boundedness and stability analysis for impulsive G-SDEs were studied by Xu *et al* [58]. Moreover, in Ren *et al.* [53], Yin *et al.* [61], Yin *et al.* [63], they studied stabilization of G-SDEs with feedback control based on discrete-time state observation from differential perspectives while Yin and Cao [62] analyzed stability of large-scale G-SDEs by a decomposition approach. The exponential stability for G-SDEs was discussed by Zhang and Chen [65] where the quasi-sure analysis was used.

However, in many real systems, such as science, industry, economics and finance, we will run into time lag. Differential delay equations (DDEs) have been used to model such time-lag systems. Hence, the researchers have studied the stability of DDEs for more than 50 years. In 1980's, the SDDEs were developed in order to model real systems which are subjected to external noises. Since then, in the study of SDDEs including hybrid SDDEs the stability analysis has been one of the most significant topics (see, e.g., [30, 37, 41]).

Most of the existing delay-dependent stability criteria are for the (hybrid) SDDEs whose coefficients are either linear or nonlinear but bounded by linear functions (or, satisfy the linear growth condition). But, there are a large number of highly nonlinear systems in practice, such as Ait-Sahalia interest rate model in financial engineering (see, e.g., Deng *et al.* [4]). That was why Hu *et al.* [21, 22] initiated the investigation on the stability of the highly nonlinear hybrid SDDEs driven by Brownian motion. Based on their results, Fei *et al.* [16] further established the delay-dependent stability criterion, and more results can be found in [10, 12, 17, 44, 54, 55]. Recently, Mei *et al.* [45] worked on exponential stabilization by delay feedback control for highly nonlinear hybrid stochastic functional differential equations with infinite delay, in the meantime Mei *et al.* [46] explored feedback control for highly nonlinear neutral SDDEs with Markovian switching.

Under the sublinear expectation, the stability of G-SDDEs has been discussed. For example, Fei and Fei [15] investigated the quasi-sure exponential stability by G-Lyapunov functional method. Fei *et al.* [11] studied the stability of highly nonlinear G-SDDEs while Deng *et al.* [3] analyzed the stability equivalence between G-SDDEs and the corresponding numerical solutions by the Euler-Maruyama method. Yao and Zong [60] explored delay-dependent stability of a class of stochastic delay systems driven by GBm. Zhu and Huang [66] implemented a stability analysis for a class of stochastic delay nonlinear systems with GBm. Different from above work, in this paper we will study the dependent stability criteria of G-SDDEs, where the coefficients of G-SDDEs are highly nonlinear (namely, without the linear growth condition). Our results extend those developed in Fei *et al.* [16] for the classical SDDEs to G-SDDEs.

We discuss an example to motivate our aims in this paper more clearly. Consider the following highly nonlinear G-SDDE

$$dX(t) = [-3X^3(t) - X(t-\delta(t))]dt + 0.5X^2(t-\delta(t))dB(t), \tag{1.1}$$

where $\delta(t) \geq 0$ is a time lag, $X(t) \in \mathbb{R}$ is the state, $B(t)$ is a scalar GBm.

For system (1.1), if the time delay $\delta(t) = 0.01$, the computer simulation shows it is asymptotically stable (see Fig. 1). If the time-delay is large, say $\delta(t) = 3$, the computer simulation shows that G-SDDE (1.1) is unstable (see

Fig. 2). In other words, whether the underlying system is stable or not depends on the size of the time-lag. On the other hand, both drift and diffusion coefficients of G-SDDE affect more significantly the stability of the system due to super-linearity. To the best of our knowledge, there is no delay dependent criterion which can be applied to G-SDDE to derive a sufficient bound on the time-delay $\delta(t)$ such that it is stable. Hence, the aim of this paper is to establish the delay dependent criteria of stability of highly nonlinear G-SDDEs.

The main contributions of this paper are presented as follows:

- A new criterion on delay-dependent stability of highly nonlinear G-SDDEs is proposed for the first time.
- New techniques are developed to establish the delay-dependent criteria of stability. Since, $\hat{\mathbb{E}}[-X] = -\hat{\mathbb{E}}[X]$ for random variable X generally does NOT hold under sublinear expectation, we overcome the difficulties caused by it. A generalized Birkhold-Davis-Gundy inequality (see Lemma 2.4) is established to deal with highly nonlinear G-SDDEs.
- The stochastic calculus on nonlinear expectations is applied to cope with the stability of the systems with ambiguity, such as non-additive probability.

The arrangement of the paper is as follows. In Section 2, we give preliminaries on sublinear expectations and GBm. Furthermore, we formulate the properties of GBms and G-martingales. Next, in Section 3, the quasi-sure exponential stability of the solutions to G-SDDE is studied. We give an illustrative example in Section 4, where we use the φ-max-mean algorithm. Finally, Section 5 concludes this paper.

2. Notations and Preliminaries

In this section, we first give the notion of sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, where Ω is a given state set and $\mathcal{H}$ a linear space of real valued functions defined on Ω. The space $\mathcal{H}$ can be considered as the space of random variables. The following concepts come from Peng [50].

Definition 2.1. *A sublinear expectation $\hat{\mathbb{E}}$ is a functional $\hat{\mathbb{E}}$: $\mathcal{H} \to \mathbb{R}$ satisfying*
(i) Monotonicity: $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$ if $X \geq Y$;
(ii) Constant preserving: $\hat{\mathbb{E}}[c] = c$;
(iii) Sub-additivity: For each $X, Y \in \mathcal{H}$, $\hat{\mathbb{E}}[X+Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;
(iv) Positivity homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ for $\lambda \geq 0$.

A sublinear lower expectation is defined by $\mathcal{E}[X] := -\hat{\mathbb{E}}[-X]$ for a random variable X.

Definition 2.2. *Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space. $X(t) = (X_1(t), \cdots, X_d(t)), t \geq 0$ is called a d-dimensional stochastic process if for each $t \geq 0$, $X_i(t), i = 1, \cdots, d$ is a random vector in $\mathcal{H}$.*

The one-dimensional process $(B(t))_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a GBm if the following properties are satisfied:
(i) $B_0(\omega) = 0$;
(ii) for each $t, s \geq 0$, the increment $B(t+s)-B(t)$ is $N(\{0\}\times s[\underline{\sigma}^2, \overline{\sigma}^2])$-distributed and is independent from $(B(t_1), B(t_2), \cdots, B(t_n))$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$, here $0 < \underline{\sigma}^2 \leq \overline{\sigma}^2 < \infty$.

We now introduce the notions of Itô integral with respect to one dimensional GBm with the function $G(\alpha) := \frac{1}{2}\hat{\mathbb{E}}[\alpha B(1)^2] = \frac{1}{2}(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-)$, where $\hat{\mathbb{E}}[B(1)^2] = \overline{\sigma}^2, \mathcal{E}[B(1)^2] = -\hat{\mathbb{E}}[-B(1)^2] = \underline{\sigma}^2$, $0 < \underline{\sigma} \leq \overline{\sigma} < \infty$.

In the rest of this paper, we use the notation $\Omega = C_0(\mathbb{R}_+)$ for the space of all $\mathbb{R}_+$-valued continuous paths $(\omega_t)_{t\geq 0}$ with zero initial value, equipped with the distance

$$\rho(\omega^{(1)}, \omega^{(2)}) = \sum_{k=1}^{\infty} 2^{-k}[(\max_{t\in[0,k]} |\omega_t^{(1)} - \omega_t^{(2)}|^2) \wedge 1].$$

For each fixed t, we set $\Omega_t := \{\omega_{\cdot\wedge t} : \omega \in \Omega\}$. Now consider the canonical process $B_t(\omega) = \omega_t, t \in [0, \infty)$, $\forall\ \omega \in \Omega$.

Let $p > 0$ be fixed. We consider the following type of simple processes: for a given partition $\pi_T = (t_0, \cdots, t_N)$ of $[0, T]$, where T can take ∞, we get

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_{k+1})}(t),$$

where $\xi_k \in L_G^p(\Omega_{t_k}; \mathbb{R}^d), k = 0, 1, \cdots, N-1$ are given. The collection of these processes is denoted by $M_G^{p,0}(0, T)$. We denote by $M_G^p(0, T)$ the completion of $M_G^{p,0}(0, T)$ with the norm

$$\|\eta\|_{M_G^p(0,T)} := \left\{ \hat{\mathbb{E}} \int_0^T |\eta(t)|^p dt \right\}^{1/p} < \infty.$$

Definition 2.3. *For each $\eta \in M_G^{2,0}(0, T)$ with $\eta_t(\omega) = \sum_{k=0}^{N-1} \mathbf{1}_{[t_k, t_{k+1})}(t)$, we define*

$$I(\eta) = \int_0^T \eta_t(\omega) dB_t := \sum_{k=0}^{N-1} \xi_k (B_{t_{k+1}} - B_{t_k}).$$

The above mapping $I : M_G^{2,0}(0, T) \mapsto L_G^2(\Omega_T)$ is a continuous linear mapping and thus be continuously extended to $I : M_G^2(0, T) \mapsto L_G^2(\Omega_T)$).

Moreover, for each $\eta \in M_G^2(0, T)$, we define the stochastic integral

$$\int_0^T \eta_t dB_t := I(\eta).$$

In what follows, similar to the setting of Peng *et al.* [51], our ambiguous probability measure family $\mathcal{P}$ representing sublinar expectation $\hat{\mathbb{E}}$ is defined as follows

$$\mathcal{P} := \left\{ P^{\sigma_\cdot} = P_0 \circ (Y^{\sigma_\cdot})^{-1}; (\sigma_t)_{t\geq 0} \text{ is an } (\Omega_t)_{t\geq 0}\text{-progressively measurable process taking in } [\underline{\sigma}, \overline{\sigma}] \right\}, \tag{2.1}$$

where P_0 is a Wiener measure on a given canonical probability space $(\Omega, \mathcal{B}(\Omega), P_0)$, $P^{\sigma_\cdot}(A) = P_0(Y^{\sigma_\cdot} \in A)$ for each event $A \in \mathcal{B}(\Omega)$, and $Y_t^{\sigma_\cdot} = \int_0^t \sigma_s dw(s)$ P_0-a.s. where $(w(t))_{t\geq 0}$ is a standard Brownian motion under $(\Omega, \mathcal{B}(\Omega), P_0)$. Obviously,

$$B(\cdot) := \{ Y^{\sigma_\cdot}(\cdot),\ (\sigma_t)_{t\geq 0} \text{ is an } (\Omega_t)_{t\geq 0}\text{-prograssively measurable process taking in } [\underline{\sigma}, \overline{\sigma}] \}$$

is GBm under the probability measure family $\mathcal{P}$. Thus, $dB(t) = \sigma_t dw(t),\ P_0$-a.s.

For $\mathcal{P}$ corresponding to sublinear expectation $\hat{\mathbb{E}}$, we now define G-upper capacity $\mathbb{V}(\cdot)$ and G-lower capacity $\mathcal{V}(\cdot)$ associated to $\hat{\mathbb{E}}$ by

$$\mathbb{V}(A) = \sup_{P \in \mathcal{P}} P(A),\ \forall\, A \in \mathcal{B}(\Omega),$$
$$\mathcal{V}(A) = \inf_{P \in \mathcal{P}} P(A),\ \forall\, A \in \mathcal{B}(\Omega).$$

Thus a property is called to hold quasi surely (q.s.) if there exists a polar set D with $\mathbb{V}(D) = 0$ such that it holds for each $\omega \in D^c$. Moreover, a property is called to hold $\mathbb{V}$-quasi surely ($\mathbb{V}$-q.s.) if there exists a probability measure $P \in \mathcal{P}$ such that it holds almost surely under P.

Moreover, under probability measure $P \in \mathcal{P}$, the process $< B(t) >$ is a quadratic variation of P-martingale $B(t)$, and fulfils

$$\underline{\sigma}^2 dt \leq d < B(t) > \leq \overline{\sigma}^2 dt, \text{ q.s.} \tag{2.2}$$

The following Burkholder-Davis-Gundy inequality provides the explicit bounds relative to those in Gao [18, Theorems 2.1-2.2].

Lemma 2.4. *(Burkholder-Davis-Gundy inequality) Let $p > 0$ and $\zeta = \{\zeta(s), s \in [0, T]\} \in M_G^p(0, T)$. Then, for all $t \in [0, T]$,*

$$\hat{\mathbb{E}} \sup_{s \le u \le t} \Big| \int_s^u \zeta(v) d < B(v) > \Big|^p$$
$$\le (t-s)^{p-1} \overline{\sigma}^{2p} \hat{\mathbb{E}} \Big(\int_s^t |\zeta(v)|^2 dv \Big)^{p/2}, \text{ for } p \ge 1,$$
$$\hat{\mathbb{E}} \sup_{s \le u \le t} \Big| \int_s^u \zeta(v) dB(v) \Big|^p \le C_p \overline{\sigma}^p \hat{\mathbb{E}} \Big(\int_s^t |\zeta(v)|^2 dv \Big)^{p/2}, \text{ for } p > 0,$$

where the constant C_p is defined as follows

$$C_p = \Big(\frac{32}{p}\Big)^{p/2}, \quad \text{if } 0 < p < 2;$$
$$C_p = \Big(\frac{p^{p+1}}{2(p-1)^{p-1}}\Big)^{p/2}, \quad \text{if } p \ge 2.$$

Proof. For $p \ge 1$, by the Hölder inequality and (2.2), we get

$$\hat{\mathbb{E}} \sup_{s \le u \le t} \Big| \int_s^u \zeta(v) d < B(v) > \Big|^p$$
$$= \sup_{P \in \mathcal{P}} E^P \sup_{s \le u \le t} \Big| \int_s^u \zeta(v) d < B(v) > \Big|^p$$
$$\le (t-s)^{p-1} \overline{\sigma}^{2p} \sup_P E^P \Big(\int_s^t |\zeta(v)|^2 dv \Big)^{p/2}$$
$$= (t-s)^{p-1} \overline{\sigma}^{2p} \hat{\mathbb{E}} \Big(\int_s^t |\zeta(v)|^2 dv \Big)^{p/2}.$$

From the classical Burkholder-Davis-Gundy inequality (see, e.g., Mao and Yuan [43, Theorem 2.13 on page 70]), we get

$$\hat{\mathbb{E}} \sup_{s \le u \le t} \Big| \int_s^u \zeta(v) dB(v) \Big|^p$$
$$= \sup_{P \in \mathcal{P}} E^P \sup_{s \le u \le t} \Big| \int_s^u \zeta(v) dB(v) \Big|^p$$
$$\le C_p \sup_{P \in \mathcal{P}} E^P \Big(\int_s^t |\zeta(v)|^2 d < B(v) > \Big)^{p/2}$$
$$\le C_p \overline{\sigma}^p \hat{\mathbb{E}} \Big(\int_s^t |\zeta(v)|^2 dv \Big)^{p/2}.$$

Thus, the proof is complete. □

Finally, in this section, we give the definitions of (weak) quasi-surely stability and moment stability as follows.

Definition 2.5. (i) *For $p > 0$, the trivial solution of SDE (3.1) is called to asymptotically stable in p-th moment if*

$$\lim_{t \to \infty} \hat{\mathbb{E}}(|X(t; \eta)|^p) = 0$$

for all η in (3.2). Especially, $p = 2$, it is called to be asymptotically stable in mean square.

(ii) *For $p > 0$, the trivial solution of SDE (3.1) is called to weak asymptotically stable in p-th moment if*

$$\lim_{t \to \infty} E^P(|X(t; \eta)|^p) = 0 \quad \forall P \in \mathcal{P}$$

for all η in (3.2).

(iii) *Moreover, it is called to be quasi-surely asymptotically stable or asymptotically stable with lower-capacity* 1 *denoted by* $\lim_{t\to\infty} x(t;\eta) = 0$ *q.s. if*

$$\mathcal{V}\{\lim_{t\to\infty} x(t;\eta) = 0\} = 1$$

for all η in (3.2). And it is called to be weak quasi-surely asymptotically stable or asymptotically stable with upper-capacity 1 *denoted by* $\lim_{t\to\infty} x(t;\eta) = 0$ $\mathbb{V}$*-q.s. if*

$$\mathbb{V}\{\lim_{t\to\infty} x(t;\eta) = 0\} = 1$$

for all η in (3.2).

3. Delay-Dependent Asymptotic Stability of *G*-SDDEs

For the convenience of presentation, all processes take values in $\mathbb{R}^d$. Let $\mathbb{R}_+ = [0,\infty)$. If A is a subset of Ω, denote by I_A its indicator function. Let $(\Omega, \mathcal{H}, \{\Omega_t\}_{t\geq 0}, \hat{\mathbb{E}}, \mathbb{V})$ be a generalized sublinear expectation space. Let $(B(t))_{t\geq 0}$ be one dimensional GBm defined on the sublinear expectation space.

Let $f, g, h : \mathbb{R}^{2d} \times \mathbb{R}_+ \to \mathbb{R}^d$ be Borel measurable functions. For $\tau > 0$, let $\delta(t) \in [0,\tau], t \geq 0$ satisfy $d\delta(t)/dt = \dot{\delta}(t) \leq \bar{\delta} < 1$. Consider one dimensional highly nonlinear variable delay SDE driven by GBm (*G*-SDDE)

$$\begin{aligned} dX(t) =& f(X(t), X(t-\delta(t)), t)dt \\ &+ g(X(t), X(t-\delta(t)), t)d < B(t) > + h(X(t), X(t-\delta(t)), t)dB(t) \end{aligned} \tag{3.1}$$

on $t \geq 0$ with nonrandom initial data

$$\{X(t) : -\tau \leq t \leq 0\} = \eta \in C([-\tau, 0]; \mathbb{R}^d). \tag{3.2}$$

The uniqueness of solutions to SDEs driven by GBm has been proved under the coefficients satisfying non-Lipschitzian conditions, where the coefficients are often bounded by a linear function (see, e.g., Lin [35]). In this paper, however, the coefficients of *G*-SDDE (3.1) cannot be bounded by a linear function. The coefficients in (3.1) are called highly nonlinear in terms of Fei *et al.* [16] and Hu *et al.* [21], the corresponding equations (3.1) are called highly nonlinear *G*-SDDEs. In Fei *et al.* [11], the existence and uniqueness of solutions to *G*-SDDEs (3.1) are proved, while the stability and boundedness of solutions to *G*-SDDE (3.1) are investigated as well. We will consider highly nonlinear *G*-SDDEs which, in general, do not satisfy the linear growth condition in this paper. Therefore, we impose the polynomial growth condition, instead of the linear growth condition. Let us provide these conditions as an assumption for our aim.

Assumption 3.1. *Assume that for any $b > 0$, there exists a positive constant K_b such that*

$$\begin{aligned} &|f(x,y,t) - f(\bar{x},\bar{y},t)| \vee |g(x,y,t) - g(\bar{x},\bar{y},t)| \vee |h(x,y,t) - h(\bar{x},\bar{y},t)| \\ &\leq K_b(|x-\bar{x}| + |y-\bar{y}|) \end{aligned}$$

for all $x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$ with $|x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \leq b$ and all $t \in \mathbb{R}_+$. Assume moreover that there exist three constants $K > 0$, q_1, q_2 such that

$$\begin{aligned} &|f(x,y,t)| \vee |g(x,y,t)| \leq K(1 + |x|^{q_1} + |y|^{q_1}), \\ &|h(x,y,t)| \leq K(1 + |x|^{q_2} + |y|^{q_2}) \end{aligned} \tag{3.3}$$

for all $x, y \in \mathbb{R}^d, t \in \mathbb{R}_+$.

The condition (3.3) with $q_1 = q_2 = 1$ is the familiar linear growth condition. But, we emphasise that we are here interested in highly nonlinear G-SDDEs which mean $q_1 > 1$ or $q_2 > 1$. The condition (3.3) is referred as the polynomial growth condition. It is known that Assumption 3.1 only guarantees that the G-SDDE (3.1) with the initial data (3.2) has a unique maximal solution, which may explode to infinity at a finite time. To avoid such a possible explosion, we need to impose an additional condition by Lyapunov functions. To this end, we need more notation.

We denote $C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$ as the family of non-negative functions $U(x,t)$ defined on $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$ which are continuously twice differentiable in x and once in t. Now we can state another assumption.

Assumption 3.2. *Let $H(\cdot) \in C(\mathbb{R}^d \times [-\tau,\infty); \mathbb{R}_+)$. Assume that there exists a function $\bar{U} \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$, nonnegative constants c_1, c_2, c_3 and $q = 2(q_1 \vee q_2\ )$ such that*

$$c_3 < c_2(1-\bar{\delta}), |x|^q \le \bar{U}(x,t) \le H(x,t) \tag{3.4}$$

for $\forall\ (x,t) \in \mathbb{R}^d \times \mathbb{R}_+$, and

$$\begin{aligned}\mathbb{L}\bar{U}(x,y,t) :=& \bar{U}_t(x,t) + < \bar{U}_x(x,t), f(x,y,t) > \\ &+ G(< \bar{U}_x(x,t), 2g(x,y,t) > + < \bar{U}_{xx}(x,t)h(x,y,t), h(x,y,t) >) \\ \le& c_1 - c_2 H(x,t) + c_3 H(y, t-\delta(t))\end{aligned} \tag{3.5}$$

for all $x, y \in \mathbb{R}^d, t \in \mathbb{R}_+$. Here,

$$\bar{U}_t(x,t) = \frac{\partial \bar{U}(x,t)}{\partial t},\ \bar{U}_x(x,t) = \Big(\frac{\partial \bar{U}(x,t)}{\partial x_i}\Big)_{d\times 1},\ \bar{U}_{xx}(x,t) = \Big(\frac{\partial^2 \bar{U}(x,t)}{\partial x_i \partial x_j}\Big)_{d\times d}.$$

The following result gives the boundedness of the solution to G-SDDE (3.1) (see, e.g., Fei *et al.* [11, Theorem 5.2]).

Theorem 3.3. *Under Assumptions 3.1 and 3.2, the highly nonlinear G-SDDE (3.1) with initial data (3.2) has unique global solution satisfying*

$$\sup_{-\tau \le t < \infty} \hat{\mathbb{E}}|X(t)|^q < \infty.$$

Next, we will use the method of Lyapunov functionals to investigate the delay-dependent asymptotic stability. We define two segments $\bar{X}(t) := \{X(t+s) : -2\tau \le s \le 0\}$ for $t \ge 0$. For $\bar{X}(t)$ to be well defined for $0 \le t < 2\tau$, we set $X(s) = \eta(-\tau)$ for $s \in [-2\tau, -\tau)$. We construct the Lyapunov functional as follows

$$\begin{aligned}&V(\bar{X}(t), t) = U(X(t), t) \\ &+ \theta \int_{-\tau}^{0} \int_{t+s}^{t} \Big[\tau |f(X(u), X(u-\delta(u)), u)|^2 \\ &\quad + \tau \overline{\sigma}^4 |g(X(u), X(u-\delta(u), u))|^2 + \overline{\sigma}^2 |h(X(u), X(u-\delta(u), u))|^2\Big] du ds\end{aligned}$$

for $t \ge 0$, where $U \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$ such that

$$\lim_{|x|\to\infty} [\inf_{t \in \mathbb{R}_+} U(x,t)] = \infty,$$

and θ is a positive number to be determined later while we set

$$f(x,s) = f(x,0),\ g(x,s) = g(x,0),\ h(x,s) = h(x,0)$$

for all $x \in \mathbb{R}^d, s \in [-2\tau, \infty)$. Applying Itô's formula for GBm (see, e.g., [50]) to $U(X(t), t)$, we get, for $t \ge 0$, quasi-surely,

$$\begin{aligned}
&dU(X(t),t)\\
&=\Big(U_t(X(t),t)+<U_x(X(t),t),f(X(t),X(t-\delta(t)),t)>\Big)dt\\
&\quad+\Big(<U_x(X(t),t),g(X(t),X(t-\delta(t)),t)>\\
&\quad+\frac{1}{2}<U_{xx}(X(t),t)h(X(t),X(t-\delta(t)),t),h(X(t),X(t-\delta(t)),t)>\Big)d<B(t)>\\
&\quad+<U_x(X(t),t),h(X(t),X(t-\delta(t)),t)>dB(t)\\
&\le \mathbb{L}U(X(t),X(t-\delta(t)),t)dt+<U_x(X(t),t),h(X(t),X(t-\delta(t)),t)>dB(t)
\end{aligned}$$

by [15, Proposition 2.5]. Rearranging terms gives

$$\begin{aligned}
&dU(X(t),t)\\
&\quad\le\Big(<U_x(X(t),t),f(X(t),X(t-\delta(t)),t)-f(X(t),X(t),t)>\\
&\quad+\mathcal{L}U(X(t),X(t-\delta(t)),t)\Big)dt+<U_x(X(t),t),h(X(t),X(t-\delta(t)),t)>dB(t),
\end{aligned}$$

where the function $\mathcal{L}U:\mathbb{R}^2\times C([-\delta(t),0];\mathbb{R})\times\mathbb{R}_+\to\mathbb{R}$ is defined by

$$\begin{aligned}
\mathcal{L}U(x,y,t)&=U_t(x,t)+<U_x(x,t),f(x,x,t)>\\
&\quad+G(<U_x(x,t),2g(x,y,t)>+<U_{xx}(x,t)h(x,y,t),h(x,y,t)>).
\end{aligned}\tag{3.6}$$

Moreover, the fundamental theory of calculus shows

$$\begin{aligned}
&d\Big(\int_{-\tau}^{0}\int_{t+s}^{t}\Big[\tau|f(X(u),X(u-\delta(u)),u)|^2\\
&+\tau\overline{\sigma}^4|g(X(u),X(u-\delta(u)),u)|^2+\overline{\sigma}^2|h(X(u),X(t-\delta(u)),u)|^2\Big]duds\Big)\\
&=\Big(\tau\Psi(t)-\int_{t-\tau}^{t}\Psi(u)du\Big)dt,
\end{aligned}$$

where $\Psi(t):=\tau|f(X(t),X(t-\delta(t)),t)|^2+\tau\overline{\sigma}^4|g(X(t),X(t-\delta(t)),t)|^2+\overline{\sigma}^2|h(X(t),X(t-\delta(t)),t)|^2$.

Lemma 3.4. *With the notations above, $V(\bar{X}(t),t)$ is G-Itô process on $t\ge 0$ with its Itô differential*

$$dV(\bar{X}(t),t)\le LV(\bar{X}(t),t)dt+dM(t)\quad q.s.,$$

where $M(t)$ is a G-continuous martingale with $M(0)=0$ and

$$\begin{aligned}
&LV(\bar{X}(t),t)=\\
&\quad U_x(X(t),t)[f((X(t),X(t-\delta(t)),t)-f(X(t),X(t),t)]\\
&\quad+\mathcal{L}U(X(t),X(t-\delta(t)),t)+\theta\tau\Psi(t)-\theta\int_{t-\tau}^{t}\Psi(u)du.
\end{aligned}$$

To study the delay-dependent asymptotic stability of the G-SDDE (3.1), we need to impose two new assumptions.

Assumption 3.5. *Assume that there are functions $U\in C^{2,1}(\mathbb{R}\times\mathbb{R}_+;\mathbb{R}_+)$, $U_1\in C(\mathbb{R}\times[-\tau,\infty);\mathbb{R}_+)$, and positive numbers α_1,α_2 and β_k $(k=1,2,3,4)$ such that*

$$\alpha_2<\alpha_1(1-\bar{\delta})\tag{3.7}$$

and

$$\begin{aligned}
&\mathcal{L}U(x,y,t)+\beta_1|U_x(x,t)|^2\\
&+\beta_2|f(x,y,t)|^2+\beta_3|g(x,y,t)|^2+\beta_4|h(X(t),X(t-\delta(t)),t)|^2\\
&\le-\alpha_1U_1(x,t)+\alpha_2U_1(y,t-\delta(t))
\end{aligned}\tag{3.8}$$

for all $x\in\mathbb{R}^d, t\in\mathbb{R}_+$.

Assumption 3.6. *Assume that there exists a positive number ϖ such that*

$$|f(x,x,t) - f(x,y,t)| \le \varpi |x-y|$$

for all $x \in \mathbb{R}, t \in [-2\tau, \infty)$.

Theorem 3.7. *Let Assumptions 3.1, 3.2, 3.5 and 3.6 hold, where $h(\cdot)$ is a deterministic function. Assume also that*

$$\tau \le \sqrt{\frac{4\beta_1\beta_2}{3\varpi^2}} \bigwedge \sqrt{\frac{4\beta_1\beta_3}{3\varpi^2\overline{\sigma}^4}} \bigwedge \frac{4\beta_1\beta_4}{3\varpi^2\overline{\sigma}^2}. \tag{3.9}$$

Then for any given initial data (3.2), the solution of the G-SDDE (3.1) has the properties that

$$\hat{\mathbb{E}} \int_0^\infty U_1(X(t),t)dt < \infty \tag{3.10}$$

and

$$\sup_{0\le t<\infty} \hat{\mathbb{E}} U(X(t),t) < \infty. \tag{3.11}$$

Proof. Fix the initial data $\eta \in C([-\tau,0];\mathbb{R})$ arbitrarily. Let $k_0 > 0$ be a sufficiently large integer such that $\|\eta\| := \sup_{-\tau\le s\le 0}|\eta(s)| < k_0$. For each integer $k > k_0$, define (Ω_t)-stopping time

$$\nu_k = \inf\{t \ge 0 : |x(t)| \ge k\}.$$

It is easy to see that ν_k is increasing as $k \to \infty$ and $\lim_{k\to\infty} \nu_k = \infty$ q.s. By Itô's formula for GBm we obtain from Lemma 3.4 that

$$\begin{aligned} &V(\bar{X}(t\wedge\nu_k), t\wedge\nu_k) \\ &\le V(\bar{X}(0),0) + \int_0^{t\wedge\nu_k} LV(\bar{X}(s),s)ds + M(t\wedge\nu_k) \end{aligned} \tag{3.12}$$

for any $t \ge 0$ and $k \ge k_0$. Let $\theta = 3\varpi^2/(4\beta_1)$. By Assumption 3.6 and Cauchy-Schwartz inequality, it is easy to see that

$$\begin{aligned} &U_x(X(t),t)[f(X(t),X(t),t) - f(X(t),X(t-\delta(t)),t)] \\ &\le \beta_1|U_x(X(t),t)|^2 + \frac{\varpi^2}{4\beta_1}|X(t) - X(t-\delta(t))|^2. \end{aligned} \tag{3.13}$$

By condition (3.9), we also have

$$\theta\tau^2 \le \beta_2,\ \theta\overline{\sigma}^4\tau^2 \le \beta_3,\ \theta\overline{\sigma}^2\tau \le \beta_4.$$

It then follows from Lemma 3.4 that

$$\begin{aligned} &LV(\bar{X}(s),s) \le \mathcal{L}U(X(s),X(s-\delta(t)),s) + \beta_1|U_x(X(s),s)|^2 \\ &+\beta_2|f(X(s),X(s-\delta(s)),s)|^2 \\ &+\beta_3|g(X(s),X(s-\delta(s)),s)|^2 + \beta_4|h(X(s),X(s-\delta(s)),s)|^2 \\ &+\frac{\varpi^2}{4\beta_1}|X(s) - X(s-\delta(s))|^2 \\ &-\frac{3\varpi^2}{4\beta_1}\int_{s-\delta(s)}^s \Psi(u)du. \end{aligned}$$

By Assumption 3.5, we then have

$$\begin{aligned}
&LV(\bar{X}(s), s) \\
&\leq -\alpha_1 U_1(X(s), s) + \alpha_2 U_1(X(s-\delta(s)), s-\delta(s)) \\
&+ \frac{\varpi^2}{4\beta_1}\int_{-\delta(s)}^{0} |X(s) - X(s+u)|^2 du \\
&- \frac{3\varpi^2}{4\beta_1}\int_{s-\delta(s)}^{s} \Psi(u) du.
\end{aligned}$$

Substituting this into (3.12) implies

$$V(\bar{X}(t\wedge\nu_k), t\wedge\nu_k) \leq V(\bar{X}(0), 0) + I_1 + I_2 + M(t\wedge\nu_k), \tag{3.14}$$

where

$$\begin{aligned}
I_1 &= \int_0^{t\wedge\nu_k} [-\alpha_1 U_1(X(s), s) + \alpha_2 U_1(X(s-\delta(s)), s-\delta(s))] ds, \\
I_2 &= \frac{\varpi^2}{4\beta_1}\Big[\int_0^{t\wedge\nu_k} [|X(s) - X(s-\delta(s))|^2 - 3\int_{s-\delta(s)}^{s} \Psi(u) du] ds\Big].
\end{aligned}$$

We notice the following fact

$$\begin{aligned}
&\int_0^{\nu_k\wedge t} U_1(X(s-\delta(s)), s-\delta(s)) ds \\
&\leq \frac{1}{1-\bar{\delta}}\int_{-\tau}^{0} U_1(\eta(s), s) ds + \frac{1}{1-\bar{\delta}}\int_0^{\nu_k\wedge t} U_1(x(s), s) ds.
\end{aligned}$$

Thus we get

$$I_1 \leq \frac{\alpha_2}{1-\bar{\delta}}\int_{-\tau}^{0} U_1(\eta(u), u) du - \frac{\bar{\alpha}}{1-\bar{\delta}}\int_0^{t\wedge\nu_k} U_1(X(s), s) ds, \tag{3.15}$$

where $\bar{\alpha} = (1-\bar{\delta})\alpha_1 - \alpha_2 > 0$ by Assumption 3.5. Substituting this into (3.14) yields

$$\frac{\bar{\alpha}}{1-\bar{\delta}}\int_0^{t\wedge\nu_k} U_1(X(s), s) ds \leq C_1 + I_2 + M(t\wedge\nu_k), \tag{3.16}$$

where C_1 is a constant defined by

$$C_1 = V(\bar{X}(0), 0) + \frac{\alpha_2}{1-\bar{\delta}}\int_{-\tau}^{0} U_1(\eta(s), s) ds.$$

Taking the upper expectation in (3.16), then setting $k \to \infty$, we get

$$\frac{\bar{\alpha}}{1-\bar{\delta}}\hat{\mathbb{E}}\int_0^t U_1(X(s), s) ds \leq C_1 + \bar{I}_2, \tag{3.17}$$

where

$$\bar{I}_2 = \frac{\varpi^2}{4\beta_1}\hat{\mathbb{E}}\Big[\int_0^t [|X(s) - X(s-\delta(s))|^2 - 3\int_{s-\delta(s)}^{s} \Psi(u) du] ds\Big].$$

For $t \in [0, \tau]$, we have

$$\begin{aligned}
\bar{I}_2 &\leq \frac{\varpi^2}{2\beta_1}\int_0^{\tau} (\hat{\mathbb{E}}|X(s)|^2 + \hat{\mathbb{E}}|X(s-\delta(s))|^2) ds \\
&\leq \frac{\tau\varpi^2}{\beta_1}\Big(\sup_{-\tau\leq u\leq\tau} \hat{\mathbb{E}}|X(u)|^2\Big).
\end{aligned} \tag{3.18}$$

For $t > \tau$, we have

$$\begin{aligned}\bar{I}_2 \leq &\frac{\tau\varpi^2}{\beta_1}\Big(\sup_{-\tau\leq u\leq\tau}\hat{\mathbb{E}}|X(u)|^2\Big)\\ &+\frac{\varpi^2}{4\beta_1}\hat{\mathbb{E}}\Big[\int_\tau^t[|X(s)-X(s-\delta(s))|^2-3\int_{s-\delta(s)}^s\Psi(u)du]ds\Big].\end{aligned} \tag{3.19}$$

Noting

$$\begin{aligned}&|X(s)-X(s-\delta(s))|\\ &=\Big|\int_{s-\delta(s)}^s f(X(u),X(u-\delta(u)),u)du\\ &+\int_{s-\delta(s)}^s g(X(u),X(u-\delta(u)),u)d<B(u)>+\int_{s-\delta(t)}^s h(X(u),X(u-\delta(u)),u)dB(u)\Big|,\end{aligned}$$

by [15, Proposition 2.5] and Cauchy-Schwartz inequality, we have

$$\begin{aligned}&|X(s)-X(s-\delta(s))|^2\\ &\leq 3\int_{s-\delta(s)}^s\tau|f(X(u),X(u-\delta(u)),u)|^2du\\ &+3\overline{\sigma}^4\int_{s-\delta(s)}^s\tau|g(X(u),X(u-\delta(u)),u)|^2du\\ &+3\Big(\int_{s-\delta(s)}^s h(X(u),X(u-\delta(u)),u)dB(u)\Big)^2\quad\text{q.s.}\end{aligned}$$

Noting (2.2) and the definition of the ambiguous probability family $\mathcal{P}$, we obtain

$$\begin{aligned}&\hat{\mathbb{E}}\int_\tau^t[|X(s)-X(s-\delta(s))|^2-3\int_{s-\delta(s)}^s\Psi(u)du]ds\\ &\leq\hat{\mathbb{E}}\Big[3\int_\tau^t\Big(\int_{s-\delta(s)}^s h(X(u),X(u-\delta(u)),u)dB(u)\Big)^2ds\\ &\quad-3\overline{\sigma}^2\int_\tau^t\int_{s-\delta(s)}^s|h(X(u),X(u-\delta(u)),u)|^2duds\Big]\\ &=3\sup_{P\in\mathcal{P}}E^P\Big[\int_\tau^t\Big(\int_{s-\delta(s)}^s h(X(u),X(u-\delta(u)),u)dB(u)\Big)^2ds\\ &\quad-\overline{\sigma}^2\int_\tau^t\int_{s-\delta(s)}^s|h(X(u),X(u-\delta(u)),u)|^2duds\Big]\\ &=3\sup_{P\in\mathcal{P}}\Big[\int_\tau^t E^P\Big(\int_{s-\delta(s)}^s h(X(u),X(u-\delta(u)),u)dB(u)\Big)^2ds\\ &\quad-\overline{\sigma}^2\int_\tau^t\int_{s-\delta(s)}^s E^P|h(X(u),X(u-\delta(u)),u)|^2duds\Big]\\ &\leq 3\sup_{P\in\mathcal{P}}\Big[\int_\tau^t E^P\int_{s-\delta(s)}^s|h(X(u),X(u-\delta(u)),u)|^2d<B(u)>ds\\ &\quad-\overline{\sigma}^2\int_\tau^t\int_{s-\delta(s)}^s|h(X(u),X(u-\delta(u)),u)|^2duds\Big]=0.\end{aligned}$$

Moreover, from (3.18) and (3.19) we get

$$\bar{I}_2\leq\frac{\tau\varpi^2}{\beta_1}\Big(\sup_{-\tau\leq u\leq\tau}\hat{\mathbb{E}}|X(u)|^2\Big). \tag{3.20}$$

Substituting (3.20) into (3.17), together with (3.9), yields

$$\frac{\bar{\alpha}}{1-\bar{\delta}}\hat{\mathbb{E}}\int_0^t U_1(X(s),s)ds \le C_1 + \frac{4\beta_3}{3\overline{\sigma}^2}\sup_{-\tau\le u\le\tau}\hat{\mathbb{E}}|X(u)|^2 := C_2.$$

Letting $t\to\infty$ gives

$$\hat{\mathbb{E}}\int_0^\infty U_1(X(s),s)ds \le \frac{(1-\bar{\delta})C_2}{\bar{\alpha}},$$

which shows (3.10).

In addition, we deduce easily from (3.14) that

$$\hat{\mathbb{E}}U\Big(X(t\wedge\nu_k),t\wedge\nu_k\Big) \le C_1 + \hat{\mathbb{E}}I_1 + \hat{\mathbb{E}}I_2.$$

Letting $k\to\infty$, by (3.15) and (3.20) we get

$$\hat{\mathbb{E}}U(X(t),t) \le C_1 + \frac{\alpha_2}{1-\bar{\delta}}\int_{-\tau}^0 U_1(\eta(u),u)du + \frac{\tau\varpi^2}{\beta_1}\Big(\sup_{-\tau\le u\le\tau}\hat{\mathbb{E}}|X(u)|^2\Big) < \infty,$$

which shows (3.11). Therefore, the proof is complete. □

Corollary 3.8. *Let the conditions of Theorem 3.7 hold. If there exists a pair of positive constants c and p such that*

$$c|x|^p \le U_1(x,t), \quad \forall\,(x,t)\in\mathbb{R}\times\mathbb{R}_+,$$

then for any given initial data (3.2), the solution of G-SDDE (3.1) satisfies

$$\hat{\mathbb{E}}\Big[\int_0^\infty |X(t)|^p dt\Big] < \infty. \tag{3.21}$$

This corollary follows from Theorem 3.7 obviously. Generally, it does not follow from (3.21) that $\lim_{t\to\infty}\mathbb{E}|X(t)|^p = 0$. But, the following proposition provides a slightly weaker claim.

Theorem 3.9. *Let the conditions of Corollary 3.8 hold. If, moreover,*

$$p\ge 2 \quad \text{and} \quad (p+q_1-1)\vee(p+q_2-1)\le q,$$

then the solution of G-SDDE (3.1) satisfies
(i) $\hat{\mathbb{E}}|X(t)|^p$ *is uniformly continuous in t on* $\mathbb{R}_+$*;*
(ii) *for each* $P\in\mathcal{P}$*,*

$$\lim_{t\to\infty} E^P|X(t)|^p = 0$$

for any initial data (3.2). That is, the solution X(t) of G-SDDE (3.1) is weak asymptotically stable.

Proof. Fix the initial data (3.2) arbitrarily. For any $0\le t_1<t_2<\infty$, by the Itô formula for GBm, we get

$$\begin{aligned} d|X(t)|^p &= p|X(t)|^{p-2}\langle X(t), f(X(t),X(t-\delta(t)),t)\rangle dt \\ &\quad + \Big(\frac{1}{2}p(p-2)|X(t)|^{p-4}|\langle X(t),h(t)\rangle|^2 \\ &\quad + \frac{1}{2}p|X(t)|^{p-2}|h(t)|^2 \\ &\quad + p|X(t)|^{p-2}\langle X(t), g(X(t),X(t-\delta(t)),t)\rangle\Big)d\langle B(t)\rangle \\ &\quad + p|X(t)|^{p-2}\langle X(t),h(t)\rangle dB(t). \end{aligned}$$

By Peng [50, Lemma 3.4.3], along with the polynomial growth condition (3.3), for each $P \in \mathcal{P}$ we get

$$
\begin{aligned}
&\left|E^P|X(t_2)|^p - E^P|X(t_1)|^p\right| \\
&\le \hat{\mathbb{E}} \int_{t_1}^{t_2} p|X(t)|^{p-1}|f(X(t), X(t-\delta(t)), t)|dt \\
&\quad + \hat{\mathbb{E}} \int_{t_1}^{t_2} \Big(\frac{1}{2}p(p-1)|X(t)|^{p-2}|h(t)|^2 + p|X(t)|^{p-1}|g(X(t), X(t-\delta(t)), t)|\Big) d < B(t) > \\
&\le \hat{\mathbb{E}} \int_{t_1}^{t_2} \Big(pK|X(t)|^{p-1}[1 + |X(t)|^{q_1} + |X(t-\delta(t))|^{q_1}]dt \\
&\quad + \overline{\sigma}^2 \hat{\mathbb{E}} \int_{t_1}^{t_2} \Big(\frac{1}{2}p(p-1)K^2|X(t)|^{p-2} + pK|X(t)|^{p-1}[1 + |X(t)|^{q_2} + |X(t-\delta(t))|^{q_2}]\Big)dt.
\end{aligned}
$$

By inequalities,

$$
\begin{aligned}
&|X(t)|^{p-1}|X(t-\delta(t))|^{q_1} \le |X(t)|^{p+q_1-1} + |X(t-\delta(t))|^{p+q_1-1}, \\
&|X(t)|^{p-1} \le 1 + |X(t)|^q,
\end{aligned}
$$

etc., and noting that for any $1 \le \bar{p} \le q$, by Theorem 3.3 we have

$$
\hat{\mathbb{E}}|X(t-\delta(t))|^{\bar{p}} \le 1 + \sup_{-\tau \le s < \infty} \hat{\mathbb{E}}|X(s)|^q < \infty,
$$

we can obtain, by the sub-additivity of sublinear expectation,

$$
\left|E^P|X(t_2)|^p - E^P|X(t_1)|^p\right| \le C_3(t_2 - t_1), \ \ P \in \mathcal{P},
$$

where

$$
\begin{aligned}
C_3 = &(4pK(1+\overline{\sigma}^2) \\
&+ \frac{1}{2}p(p-1)K^2\overline{\sigma}^2)(1 + \sup_{-\tau \le s < \infty} \hat{\mathbb{E}}|X(s)|^q) < \infty.
\end{aligned}
$$

Here, C_3 is independent of $P \in \mathcal{P}$. Thus, we show $E^P|X(t)|^p$ is uniformly continuous on $\mathbb{R}_+$ for each $P \in \mathcal{P}$. On the other hand, we have

$$
E^P \int_0^\infty |X(s)|^p ds < C, \ \forall \ P \in \mathcal{P}
$$

for some positive constant C. Thus, there is a sequence $\{t_l\}_{l=1}^\infty$ in $\mathbb{R}_+$ such that $E^P|X(t_l)|^p \to 0$. Moreover, we get $\lim_{t\to\infty} E^P|X(t)|^p = 0$ since $E^P|X(t)|^p$ is uniformly continuous on $\mathbb{R}_+$ for each $P \in \mathcal{P}$. Thus, the proof is complete. □

Theorem 3.10. *Let the conditions of Theorem 3.7 hold. Assume also that there are positive constants p and c such that*

$$
c|x|^p \le U(x,t), \quad \forall \ (x,t) \in \mathbb{R}^d \times \mathbb{R}_+. \tag{3.22}
$$

Moreover assume there exists a function $W : \mathbb{R} \to \mathbb{R}_+$ such that

$$
W(x) = 0 \text{ if and only if } x = 0
$$

and

$$
W(x) \le U_1(x,t), \quad \forall \ (x,t) \in \mathbb{R}^d \times \mathbb{R}_+.
$$

Then for any given initial data (3.2), the solution $X(\cdot)$ to Eq. (3.1) is weak quasi-surely asymptotically stable, i.e.,

$$
\lim_{t\to\infty} X(t) = 0 \quad \mathbb{V}\text{-}q.s.
$$

Proof. Let $X(\cdot)$ be the solution to Eq. (3.1) with initial data η defined in (3.2). Since the conditions in Theorem 3.7 hold, we can show that

$$C_4 := \hat{\mathbb{E}}\Big[\int_0^\infty W(X(t))dt\Big] < \infty, \tag{3.23}$$

which implies

$$\int_0^\infty W(X(t))dt < \infty \quad \text{q.s.} \tag{3.24}$$

In fact, if (3.24) is false, then there is a set A with $\mathbb{V}(A) > 0$ such that $(\int_0^\infty W(X(t))dt)I_A(\omega) = \infty$. Thus, from (3.23), we deduce

$$\infty = \hat{\mathbb{E}}\Big[I_A(\omega)\int_0^\infty W(X(t))dt\Big] \le \hat{\mathbb{E}}\Big[\int_0^\infty W(X(t))dt\Big] < \infty,$$

which contradicts (3.23). Therefore, (3.24) holds.

Set $\nu_k := \inf\{t \ge 0 : |X(t)| \ge k\}$, which is (Ω_t)-stopping time. We observe from (3.24) that

$$\lim_{t\to\infty}\inf W(X(t)) = 0 \quad \text{q.s.} \tag{3.25}$$

Moreover, in the same way as Theorem 3.7 was proved, from (3.22) we can show that

$$\hat{\mathbb{E}}|X(T \wedge \nu_k)|^p \le C, \quad \forall\, T > 0,$$

which implies, by the Chebyshev inequality for sublinear expectation (see, e.g., Chen *et al.* [2, Proposition 2.1 (2)]),

$$k^p\mathbb{V}(\nu_k \le T) \le C.$$

Letting $T \to \infty$ yields

$$k^p\mathbb{V}(\nu_k < \infty) \le C. \tag{3.26}$$

We now claim that

$$\lim_{t\to\infty} W(X(t)) = 0 \quad \mathbb{V}\text{-q.s.} \tag{3.27}$$

In fact, if this is false, then we can find a number $\varepsilon \in (0, 1/4)$ such that

$$\mathcal{V}(\widetilde{\Omega}_1) \ge 4\varepsilon, \tag{3.28}$$

where $\widetilde{\Omega}_1 = \{\limsup_{t\to\infty} W(X(t)) > 2\varepsilon\}$. Recalling (3.26), we can find an integer m sufficiently large for $\mathbb{V}(\nu_m < \infty) \le \varepsilon$. This means that

$$\mathcal{V}(\widetilde{\Omega}_2) = 1 - \mathbb{V}(\widetilde{\Omega}_2^c) \ge 1 - \varepsilon. \tag{3.29}$$

where $\widetilde{\Omega}_2 := \{|X(t)| < m \text{ for } \forall\, t \ge -\tau\}$, and $\widetilde{\Omega}_2^c$ is the complement of $\widetilde{\Omega}_2$. By (3.28) and (3.29) we get

$$\mathcal{V}(\widetilde{\Omega}_1 \cap \widetilde{\Omega}_2) \ge \mathcal{V}(\widetilde{\Omega}_1) - \mathbb{V}(\widetilde{\Omega}_1 \cap \widetilde{\Omega}_2^c) \ge 3\varepsilon. \tag{3.30}$$

Let us now define the stopped process $\zeta(t) = X(t \wedge \nu_m)$ for $t \ge -\tau$. Clearly, $\zeta(t)$ is a bounded Itô process with its differential

$$d\zeta(t) = \phi(t)dt + \psi(t)d < B(t) > +\chi(t)dB(t),$$

where

$$\phi(t) = f(X(t), X(t-\delta(t)), t)I_{[0,\nu_m)}(t),$$
$$\psi(t) = g(X(t), X(t-\delta(t)), t)I_{[0,\nu_m)}(t),$$
$$\chi(t) = h(X(t), X(t-\delta(t)), t)I_{[0,\nu_m)}(t),$$

Here f, g, h are defined by (3.1). Recalling the polynomial growth condition (3.3), we know that $\phi(\cdot), \psi(\cdot)$ and $\chi(\cdot)$ are bounded processes, say

$$|\phi(t)| \vee |\psi(t)| \vee |\chi(t)| \leq C_5 \quad \text{q.s.} \tag{3.31}$$

for all $t \geq 0$ and some $C_5 > 0$. Moreover, we also observe that $|\zeta(t)| \leq m$ for all $t \geq -\tau$. Define a sequence of stopping times

$$\rho_1 = \inf\{t \geq 0 : W(\zeta(t)) \geq 2\varepsilon\},$$
$$\rho_{2j} = \inf\{t \geq \rho_{2j-1} : W(\zeta(t)) \leq \varepsilon\}, \quad j = 1, 2, \cdots,$$
$$\rho_{2j+1} = \inf\{t \geq \rho_{2j} : W(\zeta(t)) \geq 2\varepsilon\}, \quad j = 1, 2, \cdots.$$

From (3.25) and the definition of $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$, we have

$$\widetilde{\Omega}_1 \cap \widetilde{\Omega}_2 \subset \{\nu_m = \infty\} \bigcap \Big(\cap_{j=1}^{\infty} \{\rho_j < \infty\} \Big). \tag{3.32}$$

We also note that for all $\omega \in \widetilde{\Omega}_1 \cap \widetilde{\Omega}_2$, and $j \geq 1$,

$$\begin{aligned} & W(\zeta(\rho_{2j-1})) - W(\zeta(\rho_{2j})) = \varepsilon \quad \text{and} \\ & W(\zeta(t)) \geq \varepsilon \quad \text{when } t \in [\rho_{2j-1}, \rho_{2j}]. \end{aligned} \tag{3.33}$$

Since $W(\cdot)$ is uniformly continuous in the close ball $\bar{S}_m = \{x \in \mathbb{R} : |x| \leq m\}$. We can choose $\delta = \delta(\varepsilon) > 0$ small sufficiently for which

$$|W(\zeta) - W(\bar{\zeta})| < \varepsilon, \zeta, \bar{\zeta} \in \bar{S}_m, \text{ with } \quad |\zeta - \bar{\zeta}| < \delta. \tag{3.34}$$

We emphasize that for $\omega \in \widetilde{\Omega}_1 \cap \widetilde{\Omega}_2$, if $|\zeta(\rho_{2j-1}+u) - \zeta(\rho_{2j-1})| < \delta$ for all $u \in [0, \lambda]$ and some $\lambda > 0$, then $\rho_{2j} - \rho_{2j-1} \geq \lambda$. Choose a sufficiently small positive number λ and then a sufficiently large positive integer j_0 such that

$$3C_5^2 \lambda(\lambda + \lambda C_1(2, \overline{\sigma}) + C_2(2, \overline{\sigma})) \leq \varepsilon\delta^2 \text{ and } C_4 < \varepsilon^2 \lambda j_0. \tag{3.35}$$

By (3.30) and (3.32) we can further choose a sufficiently large number T for

$$\mathcal{V}(\rho_{2j_0} \leq T) \geq 2\varepsilon. \tag{3.36}$$

In particular, if $\rho_{2j_0} \leq T$, then $|\zeta(\rho_{2j_0})| < m$, and hence $\rho_{2j_0} < \nu_m$ by the definition of $\zeta(t)$. We hence have

$$\zeta(t, \omega) = X(t, \omega) \text{ for all } 0 \leq t \leq \rho_{2j_0} \text{ and } \omega \in \{\rho_{2j_0} \leq T\}.$$

By the Burkholder-Davis-Gundy inequality under sublinear expectation (see, e.g., Lemma 2.4), we can have that, for $1 \leq j \leq j_0$,

$$\begin{aligned} & \hat{\mathbb{E}}\Big(\sup_{0 \leq t \leq \lambda} |\zeta(\rho_{2j-1} \wedge T + t) - \zeta(\rho_{2j-1} \wedge T)|^2 \Big) \\ \leq & 3\lambda \hat{\mathbb{E}} \int_{\rho_{2j-1} \wedge T}^{\rho_{2j-1} \wedge T + \lambda} |\phi(s)|^2 ds \\ & + 3C_1(\overline{\sigma})\lambda \hat{\mathbb{E}} \int_{\rho_{2j-1} \wedge T}^{\rho_{2j-1} \wedge T + \lambda} |\psi(s)|^2 ds \\ & + 3C_2(\overline{\sigma}) \int_{\rho_{2j-1} \wedge T}^{\rho_{2j-1} \wedge T + \lambda} |\chi(s)|^2 ds \\ \leq & 3C_5^2 \lambda(\lambda + \lambda C_1(\overline{\sigma}) + C_2(\overline{\sigma})), \end{aligned}$$

which, together with (3.35) and the Chebyshev inequality for sublinear expectation $\hat{\mathbb{E}}$, we can obtain that

$$\mathbb{V}\Big(\sup_{0\le t\le\lambda}|\zeta(\rho_{2j-1}\wedge T+t)-\zeta(\rho_{2j-1}\wedge T)|\ge\delta\Big)\le\varepsilon.$$

Noting that $\rho_{2j-1}\le T$ if $\rho_{2j_0}\le T$, we can derive from (3.36) and the above inequality that

$$\begin{aligned}&\mathcal{V}\Big(\{\rho_{2j_0}\le T\}\cap\Big\{\sup_{0\le t\le\lambda}|\zeta(\rho_{2j-1}+t)-\zeta(\rho_{2j-1})|<\delta\Big\}\Big)\\&=\mathcal{V}(\rho_{2j_0}\le T)\\&-\mathbb{V}\Big(\{\rho_{2j_0}\le T\}\cap\Big\{\sup_{0\le t\le\lambda}|\zeta(\rho_{2j-1}+t)-\zeta(\rho_{2j-1})|\ge\delta\Big\}\Big)\\&\ge\mathcal{V}(\rho_{2j_0}\le T)-\mathbb{V}\Big(\sup_{0\le t\le\lambda}|\zeta(\rho_{2j-1}+t)-\zeta(\rho_{2j-1})|\ge\delta\Big)\\&\ge\varepsilon.\end{aligned}$$

This, together with (3.34), implies easily that

$$\mathcal{V}\Big(\{\rho_{2j_0}\le T\}\cap\{\rho_{2j}-\rho_{2j-1}\ge\lambda\}\Big)\ge\varepsilon$$

which means, for each $P\in\mathcal{P}$,

$$P\Big(\{\rho_{2j_0}\le T\}\cap\{\rho_{2j}-\rho_{2j-1}\ge\lambda\}\Big)\ge\varepsilon. \tag{3.37}$$

By (3.33), (3.37), the sub-additivity and the Chebyshev inequality for sublinear expectation, we derive

$$\begin{aligned}C_4&\ge\hat{\mathbb{E}}\sum_{j=1}^{j_0}\Big(I_{\{\rho_{2j_0}\le T\}}\int_{\rho_{2j-1}}^{\rho_{2j}}W(X(t))dt\Big)\\&\ge\varepsilon\sup_{P\in\mathcal{P}}\sum_{j=1}^{j_0}E^P\Big(I_{\{\rho_{2j_0}\le T\}}(\rho_{2j}-\rho_{2j-1})\Big)\\&\ge\varepsilon\lambda\sum_{j=1}^{j_0}P\Big(\{\rho_{2j_0}\le T\}\cap\{\rho_{2j}-\rho_{2j-1}\ge\lambda\}\Big)\\&\ge\varepsilon^2\lambda j_0.\end{aligned}$$

This contradicts the second inequality in (3.35). Thus (3.27) must hold.

We now claim $\lim_{t\to\infty}X(t)=0$ $\mathbb{V}$-q.s. If this were not true, then

$$\varepsilon_1:=\mathcal{V}(\widetilde{\Omega}_3)>0,$$

where $\widetilde{\Omega}_3=\{\limsup_{t\to\infty}|X(t)|>0\}$. On the other hand, by (3.26), we can find a positive integer m_0 large enough for $\mathbb{V}(\nu_{m_0}<\infty)\le 0.5\varepsilon_1$. Let $\widetilde{\Omega}_4=\{\nu_{m_0}=\infty\}$. Then

$$\mathcal{V}(\widetilde{\Omega}_3\cap\widetilde{\Omega}_4)=\mathcal{V}(\widetilde{\Omega}_3)-\mathbb{V}(\widetilde{\Omega}_3\cap\widetilde{\Omega}_4^c)\ge\mathcal{V}(\widetilde{\Omega}_3)-\mathbb{V}(\widetilde{\Omega}_4^c)\ge 0.5\varepsilon_1.$$

For any $\omega\in\widetilde{\Omega}_3\cap\widetilde{\Omega}_4$, $X(t,\omega)$ is bounded on $t\in\mathbb{R}_+$. We can then find a sequence $\{t_j\}_{j\ge1}$ such that $t_j\to\infty$ and $X(t_j,\omega)\to\tilde{X}(\omega)\neq 0$ as $j\to\infty$. This, together with the continuity of W, implies

$$\lim_{j\to\infty}W(X(t_j,\omega))=W(\tilde{X}(\omega))>0,$$

which show

$$\limsup_{t\to\infty}W(X(t,\omega))>0\ \text{ for all }\omega\in\widetilde{\Omega}_3\cap\widetilde{\Omega}_4.$$

But this contradicts (3.27). We therefore have the assertion $\lim_{t\to\infty}X(t)=0$ $\mathbb{V}$-q.s. Thus, the proof is complete. □

4. Stability Example for G-SDDEs

In order to give the numerical simulation, we will estimate the pth moment of the solutions to G-SDDEs. To this end, we consider the following equation, for $\sigma_t \in [\underline{\sigma}, \overline{\sigma}], \ \forall\, t \geq 0$,

$$\begin{aligned} dX(t) =& f(X(t), X(t-\delta(t), t)dt + g(X(t), X(t-\delta(t), t)\sigma_t^2 dt \\ &+ h(X(t), X(t-\delta t), t)\sigma_t dw(t). \end{aligned} \tag{4.1}$$

Denote the solution to (4.1) by $X(t) = X(t, \sigma_t)$ being dependent on σ_t. Let E^{σ} denote the linear expectation under the corresponding probability measure P^{σ}. Define now

$$\overline{\Upsilon}(\tilde{m}) := \max_{\sigma_\ell \in [\underline{\sigma}, \overline{\sigma}], \ell = 1, \cdots, \tilde{m}} E^{\sigma_\ell}[|X(t, \sigma_\ell)|^p],$$

$$\underline{\Upsilon}(\tilde{m}) := \min_{\sigma_\ell \in [\underline{\sigma}, \overline{\sigma}], \ell = 1, \cdots, \tilde{m}} E^{\sigma_\ell}[|X(t, \sigma_\ell)|^p],$$

where $\underline{\sigma} = \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_{\tilde{m}} = \overline{\sigma}, \ell = 0, 1, \cdots, \tilde{m}$ with $\max_{\ell=1}^{\tilde{m}}(\sigma_\ell - \sigma_{\ell-1}) \to 0$.

Lemma 4.1. *Under the notations above, we have*

$$\overline{\Upsilon}(\tilde{m}) \to \sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} E^{\sigma}[|X(t, \sigma)|^p] = \hat{\mathbb{E}}[|X(t)|^p], \ \tilde{m} \to \infty, \tag{4.2}$$

$$\underline{\Upsilon}(\tilde{m}) \to \inf_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} E^{\sigma}[|X(t, \sigma)|^p] = \mathcal{E}[|X(t)|^p], \ \tilde{m} \to \infty. \tag{4.3}$$

Proof. From equation (4.1), we know that $X(t, \sigma)$ is a G-normal distribution. Since $\varphi(x) = x^p, x \geq 0, p > 0$ is a convex function, by Peng [50, Proposition 2.2.15], we get

$$\sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} E^{\sigma}[|X(t, \sigma)|^p] = \hat{\mathbb{E}}[|X(t)|^p]. \tag{4.4}$$

We easily know that function $\Lambda(\sigma) := E^{\sigma}[|X(t, \sigma|^p]$ is continuous in $\sigma \in [\underline{\sigma}, \overline{\sigma}]$. For $\forall\, \varepsilon > 0$, there exists a sufficiently enough large positive number $\tilde{N}$ such that for each $\tilde{m} > \tilde{N}$, we have $|\Lambda(\sigma') - \Lambda(\sigma_\ell)| < \varepsilon/2, \ \sigma' \in [\sigma_\ell, \sigma_{\ell+1}], \ \ell = 0, 1, \cdots, \tilde{m}$.

Besides, for above $\varepsilon > 0$, there exists $\tilde{\sigma} \in [\sigma_{\ell_0}, \sigma_{\ell_0+1}]$ for some ℓ_0 such that

$$\sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} E^{\sigma}[|X(t, \sigma)|^p] - \varepsilon/2 < E^{\tilde{\sigma}}[|X(t, \tilde{\sigma})|^p] < E^{\sigma_{\ell_0}}[|X(t, \sigma_{\ell_0})|^p] + \varepsilon/2, \tag{4.5}$$

which shows

$$0 < \sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} E^{\sigma}[|X(t, \sigma)|^p] - \overline{\Upsilon}(\tilde{m}) < \varepsilon.$$

Thus, together with (4.4), we deduce (4.2). By using the relation between lower expectation $\mathcal{E}$ and sublinear expectation $\hat{\mathbb{E}}$, we easily prove equation (4.3). Hence, the proof is complete. □

For a highly nonlinear G-SDDE (3.1) with $g(t) \equiv 0$, by the equal-spaced partition of time, it follows that

$$\begin{aligned} X(t_i) =& X(t_{i-1}) + f(X(t_{i-1}), X(t_{i-1} - [\delta(t_{i-1})/\Delta]\Delta, t_{i-1})\Delta t_i \\ &+ h(X(t_{i-1}), X(t_{i-1} - [\delta(t_{i-1})/\Delta]\Delta, t_{i-1})(B(t_i) - B(t_{i-1})) \end{aligned} \tag{4.6}$$

with initial data $\eta = \{X(t) = \eta(t), t \in [-\tau, 0]\}, \Delta = \Delta t_i = t_i - t_{i-1} = 1/N, i = 1, \cdots, N$, where N is a positive integer.

Let us introduce the simulation algorithm for GBm $(B(t))_{t \geq 0}$. Related details can be referred to Fei and Fei [9]. Now consider a random variable $\zeta = B(t_i) - B(t_{i-1}) \sim \mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2]\Delta), i = 1, \cdots, N$, we construct an experiment as follows. We take equal-step points $\sigma_k, k = 1, \cdots, m$, such that $\underline{\sigma} = \sigma_1 < \sigma_2 \cdots < \sigma_k < \cdots < \sigma_m = \overline{\sigma}$. For the kth-random sampling ($k = 1, \cdots, m$), $\zeta_i^{kj}(i = 1, \cdots, N; j = 1, \cdots, n)$ are from the classical normal distribution $\mathcal{N}(0, \sigma_k^2\Delta)$. Define $X^{kj}(t_i)$ by

$$\begin{aligned}X^{kj}(t_i) &= X^{kj}(t_{i-1})\\ &+ f(X^{kj}(t_{i-1}), X^{kj}(t_{i-1} - [\delta(t_{i-1})/\Delta]\Delta, t_{i-1})\Delta + h(X(t_{i-1}), X(t_{i-1} - [\delta(t_{i-1})/\Delta]\Delta, t_{i-1})\zeta_i^{kj}\end{aligned} \tag{4.7}$$

for $i = 1, \cdots, N; j = 1, \cdots, n; k = 1, \cdots, m$.

By Lemma 4.1, for $p > 0$ we can obtain the estimator $\hat{\bar{\mu}}_{m,n}(p)$ of $\hat{\mathbb{E}}|X(t_i)|^p$ equals to $\max_{1\le k\le m}\{\frac{1}{n}\sum_{j=1}^{n}|X^{kj}(t_i)|^p\}$, and the estimator $\hat{\underline{\mu}}_{m,n}(p)$ of $\mathcal{E}|X(t_i)|^p$ to $\min_{1\le k\le m}\{\frac{1}{n}\sum_{j=1}^{n}|X^{kj}(t_i)|^p\}, i = 1, \cdots, N$, respectively.

It is easy to find that if $\hat{\mathbb{E}}|X(t)|$ with the solution $X(t)$ to highly nonlinear G-SDDE (3.1) is stable, then the solution $X(t)$ to G-SDDE (3.1) is quasi-surely stable, i.e., $\mathcal{V}(\lim_{t\to\infty}|X(t)| = 0) = 1$. Meanwhile, if $\mathcal{E}|X(t)|$ is unstable under sublinear expectation, then the solution $X(t)$ to G-SDDE (3.1) is unstable, i.e., $\mathbb{V}(\limsup_{t\to\infty}|X(t)| \neq 0) = 1$.

Through above discussions on the simulation algorithm, we now investigate an example to illustrate our theoretic results.

Example 4.2. Let us consider the G-SDDE (1.1) with the initial data (3.2), where $f(x, y, t) = -3x^3 - y, g(x, y, t) = 0, h(t) = 0.5y^2, \underline{\sigma}^2 = 0.5, \overline{\sigma}^2 = 1$ and $\bar{\delta} = 0.1$. Consider two cases: $\delta(t) = 0.01$ and $\delta(t) = 3$ for all $t \ge 0$. In case $\delta(t) = 0.01$, let the initial data $\eta(u) = 2 + \sin(u)$ for $u \in [-0.01, 0]$, the sample paths of the solution to (1.1) are shown in Figure 1, which indicates that the G-SDDE is asymptotically stable. In the case $\delta(t) = 3$, let the initial data $\eta(u) = 2 + \sin(u)$ for $u \in [-3, 0]$, the solution to (1.1) are plotted in Figure 2, which indicates that G-SDDE (1.1) is unstable. Thereout, whether the G-SDDE (1.1) is stable or not depends on how much the time lag is.

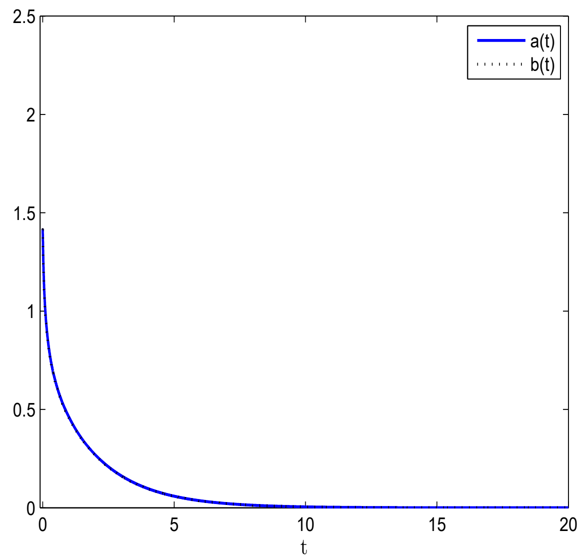

Figure 1: The computer simulation of the upper expectation $a(t) = \hat{\mathbb{E}}|X(t)|^{0.5}$ and the lower expectation $b(t) = \mathcal{E}|X(t)|^{0.5}$ of the solution to the G-SDDE (1.1) with $\delta(t) = 0.01$ using the Euler–Maruyama method with step size 10^{-3}.

We can see coefficients defined by (1.1) satisfy Assumption 3.1 with $q_1 = 3$ and $q_2 = 2$. Define $\bar{U}(x, t) = |x|^6$ for

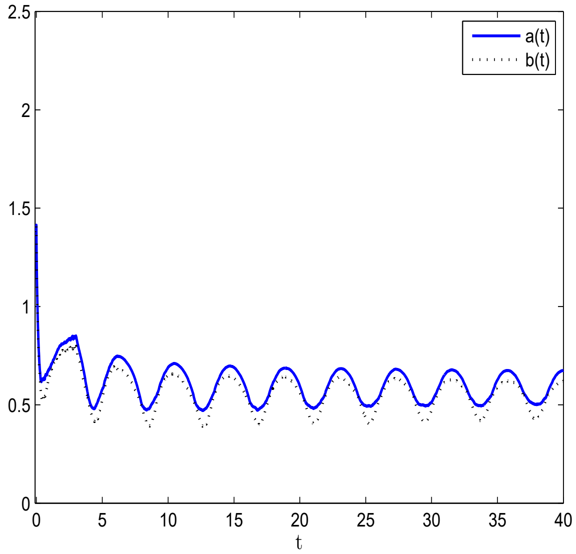

Figure 2: The computer simulation of of the upper expectation $a(t) = \hat{\mathbb{E}}|X(t)|^{0.5}$ and the lower expectation $b(t) = \mathcal{E}|X(t)|^{0.5}$ of the solution to the G-SDDE (1.1) with $\delta(t) = 3$ using the Euler–Maruyama method with step size 10^{-3}.

$(x, t) \in \mathbb{R} \times \mathbb{R}_+$. It is easy to show that

$$\begin{aligned} \mathbb{L}\bar{U}(x, y, t) &= 6x^5(-y - 3x^3) + \frac{15}{4}x^4y^4 \\ &\leq 5x^6 + y^6 - 16x^8 + 2y^8 \\ &\leq c_1 - 15.5(1 + x^8) + 2.5(1 + y^8) \end{aligned} \tag{4.8}$$

for $(x, y, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$, where

$$c_1 = \sup_{x,y\in\mathbb{R}} [13 + 5x^6 + y^6 - 0.5(x^8 + y^8)] < \infty.$$

Thus, we can set $H(x, t) = 1 + x^8$. Due to $2.5 = c_3 < (1 - \bar{\delta})c_2 = 13.95$, we know Assumption 3.2 holds. From Theorem 3.3, the solution of the G-SDDE (1.1) satisfies

$$\sup_{-\tau\leq t<\infty} \hat{\mathbb{E}}|X(t)|^6 < \infty.$$

To apply our theorems established in the previous section, we need to verify assumptions imposed there. Let us do so one by one. To verify Assumption 3.5, we define

$$U(x, t) = x^2 + x^4 \tag{4.9}$$

for $(x, t) \in \mathbb{R} \times \mathbb{R}_+$. Moreover,

$$|U_x(x, t)|^2 = 4x^2 + 16x^4 + 16x^6, \tag{4.10}$$

$$|f(x, y, t)|^2 = |y + 3x^3|^2 \leq 2y^2 + 18x^6, \tag{4.11}$$

$$|h(x,y,t)|^2 = 0.25y^4. \tag{4.12}$$

By definition (3.6), it is straightforward to show

$$\mathcal{L}U(x,y,t) = (2x+4x^3)(-3x^3-x) + \frac{1}{8}(2+12x^2)y^4$$
$$\leq -2x^2 - 10x^4 + 0.25y^4 - 11.5x^6 + y^6. \tag{4.13}$$

Set

$$\beta_1 = 0.1, \quad \beta_2 = 0.2, \quad \beta_3 = 1, \quad \beta_4 = 1. \tag{4.14}$$

By using (4.13)-(4.14), we can then show that

$$\mathcal{L}U(x,y,t) + \beta_1|U_x(x,t)|^2$$
$$+\beta_2|f(x,y,i,t)|^2 + \beta_4|h(x,y,t)|^2$$
$$\leq -1.6x^2 + 0.4y^2 - 8.4x^4 + 0.5y^4 - 6.3x^6 + y^6$$
$$\leq -6(0.2x^2 + x^4 + x^6) + 3(0.2y^2 + y^4 + y^6). \tag{4.15}$$

Let $U_1(x,t) = 0.2x^2 + x^4 + x^6, \alpha_1 = 6,\ \alpha_2 = 3$. Due to $5.4 = \alpha_1(1-\bar{\delta}) > \alpha_2 = 3$, we get condition (3.7). Note that $\varpi = 1$. Then condition (3.9) becomes

$$\delta(t) \leq 0.13.$$

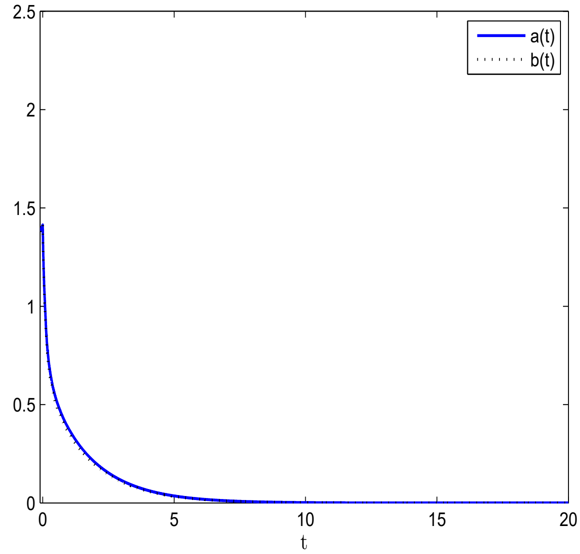

Figure 3: The computer simulation of the upper expectation $a(t) = \hat{\mathbb{E}}|X(t)|^{0.5}$ and the lower expectation $b(t) = \mathcal{E}|X(t)|^{0.5}$ of the solution to the G-SDDE (1.1) with $\delta(t) = 0.13$ using the Euler–Maruyama method with step size 10^{-3}.

By Theorem 3.7, the solution of the G-SDDE (1.1) has the properties that

$$\int_0^\infty (X^2(t) + X^4(t) + X^6(t))dt < \infty \ \ q.s.,$$
$$\hat{\mathbb{E}} \int_0^\infty (X^2(t) + X^4(t) + X^6(t))dt < \infty.$$

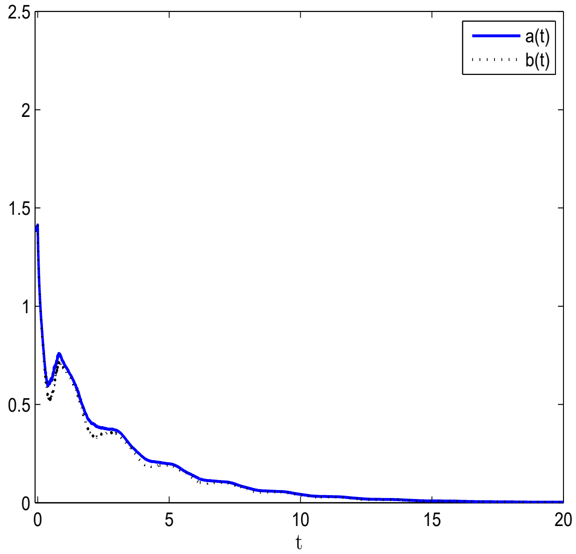

Figure 4: The computer simulation of the upper expectation $a(t) = \hat{\mathbb{E}}|X(t)|^{0.5}$ and the lower expectation $b(t) = \mathcal{E}|X(t)|^{0.5}$ of the solution to the G-SDDE (1.1) with $\delta(t) = 0.8$ using the Euler–Maruyama method with step size 10^{-3}.

Moreover, as $|X(t)|^p \leq X^2(t) + X^4(t) + X^6(t)$ for any $p \in [2, 6]$, we have

$$\hat{\mathbb{E}} \int_0^\infty |X(t)|^p dt < \infty.$$

Recalling $q_1 = 3,\ q_2 = 0$ and $q = 6$, we see that for $p = 4$, all conditions of Theorem 3.9 are satisfied so that

$$\lim_{t\to\infty} E^P |X(t)|^4 = 0,\ \forall P \in \mathcal{P}.$$

We perform a computer simulation with the time-delay $\delta(t) = 0.13$ for all $t \geq 0$ and the initial data $X(u) = 2 + \sin(u)$ for $u \in [-0.13, 0]$. The sample paths of the solution to (1.1) are plotted in Figure 3, where the simulation supports our theoretical results as well.

In addition, from Figure 4, we find the system maintains stable even if the time-delay of the system is bigger than the upper bound 0.13 computed by (3.9) in Theorem 3.7. This appearance arouses our interests to improve the results here.

5. Conclusion

In real systems, we are often faced with two kinds of uncertainties: probabilistic and Knightian ones, respectively. Under Peng's sublinear expectation framework, we can characterize the systems with ambiguity. This paper gives a description of this kind of system with delay driven by GBm. By the method of Lyapunov functional, we derive the delay-dependent stability criteria of the solutions to highly nonlinear G-SDDEs. Then, an illustrative example with its simulation algorithm is addressed in Section 4. Our work here under sublinear expectation provides a new perspective relative to the classical one for the further research on the time-delay stability of highly nonlinear G-SDDEs.

In Theorem 3.7, if $\bar{\sigma} = 1$, then the delay upper bound in (3.9) reduces to the classical case without volatility ambiguity. If $\bar{\sigma} < 1$, then the delay upper bound in (3.9) will be no less than the classical one while if $\bar{\sigma} > 1$, then the delay upper bound in (3.9) will be no more than the classical one. In Theorem 3.9, if the family $\mathcal{P}$ of uncertain probability measures includes multi-elements, we can prove the system (3.1) is weak asymptotically stable while if $\mathcal{P}$ is singleton, the stability of the system (3.1) reduces to the classical asymptotic stability. In Theorem 3.10, if the family $\mathcal{P}$ of uncertain probability measures includes multi-elements, then we can prove the system (3.1) is weak quasi-surely asymptotically stable while if $\mathcal{P}$ is singleton, then the stability of the system (3.1) reduces to the classical almost surely asymptotic stability.

In addition, Niu *et al.* [23] first put forward an adaptive neural-network-based dynamic surface control method for a class of stochastic interconnected nonlinear nonstrict-feedback systems with unmeasurable states and dead zone input. In Niu *et al.* [25], authors constructed adaptive neural output-feedback controller for a class of uncertain switched time-delay nonlinear systems with nonlower triangular structure, while Niu *et al.* [24] presented an adaptive approximation-based output-feedback tracking control scheme for a class of stochastic switching lower-triangular nonlinear systems with input saturation and unmeasurable state variables. These methods help to solve the stability for a class of nonlinear stochastic switching systems in classical probability framework. However, our present topics of the paper have two basic different points: first, our current paper is based on the framework of sublinear expectation, and second, our referred stochastic systems are highly nonlinear ones. Thus, it will be valuable and challenging to investigate the stability of our systems by applying adaptive neural-network-based dynamic surface control method.

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