Stabilization of Nonlinear Hybrid Stochastic Delay Systems by Feedback Control Based on Discrete-time State and Mode Observations

Jianqiu Lu[1], **Yuyuan Li**[2*], **Xuerong Mao**[3], **Jiafeng Pan**[3]

[1]Department of Statistics,
Donghua University, Shanghai 201620, China.
[2] College of Information Sciences and Technology,
Donghua Univerisity, Shanghai 201620, China.
[3] Department of Mathematics and Statistics,
University of Strathclyde, Glasgow G1 1XH, U.K.

Abstract

This paper is concerned with the stabilization problem for nonlinear stochastic delay systems with Markovian switching by feedback control based on discrete-time state and mode observations. By constructing an efficient Lyapunov functional, we establish the sufficient stabilization criteria not only in the sense of exponential stability (both the mean square stability and the almost sure stability) but also in other sense–that of H_∞ stability and asymptotic stability. Meanwhile, the upper bound on the duration τ between two consecutive state and mode observations is obtained. Numerical examples are provided to demonstrate the effectiveness of our theoretical results.

Key words: hybrid stochastic delay systems, stabilization, feedback control, discrete-time state and mode, Lyapunov functional.

1 Introduction

As an important class of hybrid systems, hybrid stochastic differential equations (SDEs) (also known as SDEs with Markovian switching) have been widely employed as models to delineate many practical systems that have variable structures subject to random abrupt changes, which may result from unpredictable situations such as components failures or repairs, sudden environment disturbances, unexpected transformation of subsystem interconnections, and so on [1–8]. Accordingly, considerable attention has been given to the stochastic systems with Markovian switching. Great research efforts are focused on the automatic control of such systems, with subsequent emphasis being placed on the

*Corresponding author. E-mail:yuyuan@126.com

analysis of stability, and a great number of remarkable results have been reported, see for instance [9–14].

It is well known that time delay exists inevitably in various practical dynamic systems including population systems, manufacturing, telecommunication and network control systems due to the limitation of transmission or switching speeds. And very often, it has an unstable effect and leads to poor performance of control systems. For example, time delay can destabilize the evolution of predator-prey systems [15]. It is therefore of great significance to take the time delay information into account and investigate the stability and stabilization problems of hybrid stochastic systems with time delays. Many efforts have been devoted to these topics and lots of related literature has been published [16–26].

Now that we have known the stability of stochastic systems may encounter degradation, or even become unstable due to time delays, Markovian switching or some other factors. It is vital to investigate how these factors act on stochastic systems and it is more meaningful and urgent to study how to make the unstable stochastic delay systems with Markovian switching return to be stable, which is involved with the stabilization of hybrid stochastic delay systems. To realize the stabilization of a system, scholars have proposed different control schemes [27–33], of which feedback control is a universal and effective strategy. However, it has to be pointed out that the feedback controllers in most of the pioneering works on stabilization for stochastic systems are based on continuous observations of the state, which is expensive and sometimes not possible as the observations are always of discrete time in practice. Therefore, Mao [34] initiated a novel feedback control based on discrete-time state observations to stabilize continuous-time hybrid stochastic differential equations. Since then, this new design has attracted increasing interests of researchers. Mao et al. [35] provided us with a better bound on the duration between two consecutive state observations. You et al. [36] weakened the global Lipschitz assumption on coefficients and further considered the asymptotic stabilization of nonlinear hybrid stochastic systems. While Yang et al. [37] and Wu et al. [38] applied this new feedback control to the synchronization of stochastic neural networks and stabilization of stochastic coupled systems, respectively. Furthermore, taking time delay into account, Zhao et al. [39] extended the theory in [34] to discuss the stabilization of hybrid stochastic functional differential equations by discrete-time feedback control and based on Razumikhin technique, Li et al. [40] investigated discrete-state-feedback stabilization of hybrid stochastic systems with time-varying delay.

It is worth noting that the feedback controls in [34–40] are based on discrete-time observations of the state but they still depend on continuous-time observations of the mode. Of course this is perfectly fine if the mode of the system is fully observable at no cost. However, the mode is not obvious in many real-world situations and it costs to identify the current mode of a hybrid stochastic system. Also, Geromel and Gabriel [41] emphasized the necessity to design the feedback control based on discrete-time observations of both state and mode from the numerical point of view when studying the state feedback sampled-data control design for Markov jump systems. So it is necessary and reasonable that we identify the mode at discrete times when we make observations for the state. Having this point in mind, we have once designed a feedback controller $u(x(\delta(t)), r(\delta(t)))$, where $\delta(t) = [t/\tau]\tau$ for $t \geq 0$ and $\tau > 0$, which is based on the discrete-time observations of both state and mode, to stabilize linear hybrid stochastic systems [42]. But employing this developed controller, which is more practical and costs less, to solve the problem of stabilization for the more general nonlinear hybrid stochastic delay systems still remains an important, meaningful and challenging problem.

Motivated by the above discussion, our principal aim is to explore the stabilization for nonlinear hybrid stochastic delay systems by feedback control based on discrete-time observations of both state and mode. The primary contributions of our main findings lie in:

1. We use a more practical and cheaper controller, which is based on the discrete-time observations of both state and mode, to deal with the stabilization for a class of more general nonlinear hybrid stochastic delay systems. Furthermore, the upper bound of the duration between two consecutive observations is gotten.

2. An effective Lyapunov functional is constructed to achieve our goal. When trying to use the controller $u(x(\delta(t)), r(\delta(t)))$ to stabilize the more general nonlinear hybrid stochastic delay systems, the analysis becomes much more complicated due to the difficulties arisen from the discrete-time Markov chain $r([t/\tau]\tau)$, nonlinearity and delay and the methods in [39, 40, 42] can not work well. Fortunately, we come up with the Lyapunov functional method.

3. The sufficient conditions, which ensure the stabilization of nonlinear hybrid stochastic delay systems in the sense of exponential stability, H_∞ stability and asymptotic stability, are established.

The rest of this paper is organized as follows. Section 2 covers some necessary preliminaries and the problem formulation. Section 3 devotes to establishing some sufficient criteria that guarantee the stabilization of nonlinear hybrid stochastic delay systems. Subsequently, the theoretical findings are illustrated by numerical examples in Section 4. Finally, Section 5 ends this study and concisely summarizes the main conclusions of this paper.

2 Preliminaries and problem formulation

2.1 Preliminaries

We first introduce some basic notations. If $x \in R^n$, then $|x|$ is its Euclidean norm. For a matrix A, we let $|A| = \sqrt{\text{trace}(A^T A)}$ be its trace norm and $\|A\| = \max\{|Ax| : |x| = 1\}$ be the operator norm. For a symmetric matrix A, denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalue, respectively. Given two symmetric matrices A and B, $A > (<, \geq, \leq) B$ means that $A - B$ is positive definite (negative definite, positive semidefinite, negative semidefinite, respectively). Let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R}^n)$ denote the family of continuous functions φ from $[-\tau, 0]$ to R^n with the norm $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$. If both a, b are real numbers, then $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous with $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets) and $w(t) = (w_1(t), \cdots, w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$ represent a right-continuous Markov chain on the probability space, which is assumed to be independent of the Brownian motion $w(\cdot)$ and take values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. Denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ the family of all bounded, $\mathcal{F}_0$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables.

Next, let us introduce a useful lemma, which will play an important role in coping with the discrete-time Markov chain. For its explanation and proof details, we refer the reader to [42, 43].

Lemma 2.1 *For any $t \geq 0$, $v > 0$ and $i \in S$, we have*

$$\mathbb{P}(r(s) \neq i \quad for \quad some \quad s \in [t, t+v]|r(t) = i) \leq 1 - e^{-\bar{\gamma}v},$$

in which

$$\bar{\gamma} = \max_{i\in S}(-\gamma_{ii}).$$

2.2 Problem formulation

Consider an n-dimensional unstable hybrid stochastic delay differential equation (SDDE)

$$dx(t) = f(x(t), x(t-h), r(t), t)dt + g(x(t), x(t-h), r(t), t)dw(t) \tag{2.1}$$

on $t \geq 0$, with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-h, 0]; R^n)$ and $r(0) = r_0 \in S$, where $h > 0$ stands for the time delay, and $f : R^n \times R^n \times S \times R_+ \to R^n$, $g : R^n \times R^n \times S \times R_+ \to R^{n\times m}$. Now our aim is to design a feedback control $u(x(\delta_t), r(\delta_t), t)$ so that the controlled hybrid SDDE

$$dx(t) = \big(f(x(t), x(t-h), r(t), t) + u(x(\delta_t), r(\delta_t), t)\big)dt + g(x(t), x(t-h), r(t), t)dw(t) \tag{2.2}$$

will become stable in some certain sense, where $u : R^n \times S \times R_+ \to R^n$ and $\delta_t = [t/\tau]\tau$, in which $[t/\tau]$ is the integer part of t/τ and hence $\tau > 0$ means the duration between two consecutive observations.

We can observe that the feedback control $u(x(\delta_t), r(\delta_t), t)$ is designed based on the discrete-time state observations $x(0), x(\tau), x(2\tau), \cdots$ and discrete-time mode observations $r(0), r(\tau), r(2\tau), \cdots$ as well, though the given hybrid SDDE (2.1) is of continuous-time.

In this study, we impose the following conditions on the coefficients and the controller function.

Assumption 2.2 *Assume that the coefficients f and g are locally Lipschitz continuous and obey linear growth condition. That is, for each integer $k \geq 1$, there exists a positive constant L_k such that for those $x, y, \bar{x}, \bar{y} \in R^n$ with $|x|\vee|y|\vee|\bar{x}|\vee|\bar{y}| \leq k$ and $(i, t) \in S\times R_+$, one has*

$$|f(x, y, i, t) - f(\bar{x}, \bar{y}, i, t)| \vee |g(x, y, i, t) - g(\bar{x}, \bar{y}, i, t)| \leq L_k(|x - \bar{x}| + |y - \bar{y}|) \tag{2.3}$$

and there is a constant $L > 0$ such that

$$|f(x, y, i, t)| \vee |g(x, y, i, t)| \leq L(|x| + |y|) \tag{2.4}$$

holds for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$.

Therefore, it is easy to obtain that for all $(i, t) \in S \times R_+$,

$$f(0, 0, i, t) = 0, \qquad g(0, 0, i, t) = 0. \tag{2.5}$$

Assumption 2.3 *There exists a positive constant K such that*

$$|u(x,i,t) - u(y,i,t)| \le K|x-y| \tag{2.6}$$

for all $(x,y,i,t) \in R^n \times R^n \times S \times R_+$. Moreover, for all $(i,t) \in S \times R_+$,

$$u(0,i,t) = 0 \tag{2.7}$$

Also, we can see that Assumption 2.3 implies the following linear growth condition on the controller function:

$$|u(x,i,t)| \le K|x| \tag{2.8}$$

for all $(x,i,t) \in R^n \times S \times R_+$.

From the above assumptions, it follows that system (2.2) has a unique solution $x(t)$ with initial value $x_0 = \varphi \in C^b_{\mathcal{F}_0}([-\tau^*, 0]; R^n)$, $r(0) = r_0 \in S$ such that $\mathbb{E}(\sup_{-\tau^* \le t < \infty} |x(t)|^2) < \infty$ (see e.g. [6, 42]), where $\tau^* = h \vee \tau$. And it is known that $x(t) \equiv 0$ is the trivial solution of system (2.2).

3 Stabilization analysis for hybrid SDDE

In this section, we aim to establish sufficient criteria to ensure the stabilization for system (2.1) in different senses. To realize our purpose, let's first construct the Lyapunov functional on the segments $\hat{x}_t := \{x(t+s) : -2\tau^* \le s \le 0\}$ and $\hat{r}_t := \{r(t+s) : -2\tau^* \le s \le 0\}$ for $t \ge 0$, where $\tau^* = h \vee \tau$. For $\hat{x}_t$ and $\hat{r}_t$ to be well defined for $0 \le t < 2\tau^*$, we set $x(s) = \bar{\varphi} \in C^b_{\mathcal{F}_0}([-2\tau^*, 0]; R^n)$, $r(s) = r_0$ for $-2\tau^* \le s \le 0$. The Lyapunov functional used in this paper will be of the form

$$\begin{aligned} V(\hat{x}_t, \hat{r}_t, t) =& U(x(t), r(t), t) + \int_{t-h}^{t} x(s)^T P(r(s)) x(s) ds \\ &+ \theta \int_{t-\tau}^{t} \int_{s}^{t} \Big[\tau |f(x(v), x(v-h), r(v), v) + u(x(\delta_v), r(\delta_v), v)|^2 \\ &+ |g(x(v), x(v-h), r(v), v)|^2 \Big] dv ds \end{aligned} \tag{3.1}$$

for $t \ge 0$, where $P(r(s)) := P_i$ are all symmetric positive-definite matrices and θ is a positive number to be determined later. We set

$$f(x,y,i,s) = f(x,y,i,0), \quad g(x,y,i,s) = g(x,y,i,0), \quad u(x,i,s) = u(x,i,0)$$

for $(x,y,i,s) \in R^n \times R^n \times S \times [-2\tau^*, 0)$ and $(x,i,s) \in R^n \times S \times [-2\tau^*, 0)$. We also require $U \in C^{2,1}(R^n \times S \times R_+; R_+)$, the family of non-negative functions $U(x,i,t)$ defined on $(x,i,t) \in R^n \times S \times R_+$ which are continuously twice differentiable in x and once in t, and define an operator $\mathcal{L}U : R^n \times R^n \times S \times R_+ \to R$ by

$$\begin{aligned} \mathcal{L}U(x,y,i,t) =& U_t(x,i,t) + U_x(x,i,t)[f(x,y,i,t) + u(x,i,t)] \\ &+ \frac{1}{2} \text{trace}[g^T(x,y,i,t) U_{xx}(x,i,t) g(x,y,i,t)] \\ &+ \sum_{j=1}^{N} \gamma_{ij} U(x,j,t), \end{aligned} \tag{3.2}$$

where $U_t(x,i,t) = \frac{\partial U(x,i,t)}{\partial t}$, $U_x(x,i,t) = \left(\frac{\partial U(x,i,t)}{\partial x_1}, \cdots, \frac{\partial U(x,i,t)}{\partial x_n} \right)$, $U_{xx}(x,i,t) = \left(\frac{\partial^2 U(x,i,t)}{\partial x_i \partial x_j} \right)_{n \times n}$.

To develop our theory, let us put forward an assumption on U.

Assumption 3.1 *Assume that there is a function $U \in C^{2,1}(R^n \times S \times R_+; R_+)$ and three positive numbers λ_1, λ_2 and λ_3 such that*

$$\mathcal{L}U(x, y, i, t) + \lambda_1 |U_x(x, i, t)|^2 \leq -\lambda_2 |x|^2 + \lambda_3 |y|^2 \tag{3.3}$$

for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$ and $(x, i, t) \in R^n \times S \times R_+$.

3.1 Asymptotic stabilization

Theorem 3.2 *Let Assumptions 2.2, 2.3 and 3.1 hold. Assume that there exist positive-definite symmetric matrices $P_i (i \in S)$ such that $\lambda_2 > \lambda_{PM} := \max_{i \in S} \lambda_{\max}(P_i)$ and $\lambda_3 \leq \lambda_{Pm} := \min_{i \in S} \lambda_{\min}(P_i)$. Set*

$$\theta = \frac{2K^2}{\lambda_1}\big(1 + 8(1 - e^{-\frac{\bar{\gamma}}{4K}})\big). \tag{3.4}$$

If $\tau > 0$ is sufficiently small for

$$\lambda_2 > \frac{4K^2}{\lambda_1}(1 - e^{-\bar{\gamma}\tau}) + \theta\tau(4\tau + 2)L^2 + 4\theta\tau^2 K^2 + \lambda_{PM},$$
$$\lambda_3 \leq \lambda_{Pm} - \theta\tau(4\tau + 2)L^2 \quad \text{and} \quad \tau \leq \frac{1}{4K}, \tag{3.5}$$

then the controlled system (2.2) is H_∞-stable in the sense that

$$\int_0^\infty \mathbb{E}|x(s)|^2 ds < \infty. \tag{3.6}$$

for every initial data $x_0 = \bar{\varphi} \in C^b_{\mathcal{F}_0}([-2\tau^*, 0]; R^n)$ and $r_0 \in S$.

Proof. For any given $x_0 = \bar{\varphi} \in C^b_{\mathcal{F}_0}([-2\tau^*, 0]; R^n)$ and $r_0 \in S$, regarding the solution $x(t)$ of equation (2.2) as an Itô process and applying the generalized Itô formula (see e.g. [6]) to $U(x(t), r(t), t)$, we can get

$$\begin{aligned}
&dU(x(t), r(t), t) \\
=&\Big(U_t(x(t), r(t), t) + U_x(x(t), r(t), t)[f(x(t), x(t-h), r(t), t) + u(x(\delta_t), r(\delta_t), t)] \\
&+ \frac{1}{2}\text{trace}[g^T(x(t), x(t-h), r(t), t) U_{xx}(x(t), r(t), t) g(x(t), x(t-h), r(t), t)] \\
&+ \sum_{j=1}^N \gamma_{r(t),j} U(x(t), j, t)\Big) dt + dM(t).
\end{aligned}$$

On the other hand, the fundamental theory of calculus shows

$$\begin{aligned}
&d\Big(\int_{t-\tau}^t \int_s^t \Big[\tau|f(x(v), x(v-h), r(v), v) + u(x(\delta_v), r(\delta_v), v)|^2 + |g(x(v), x(v-h), r(v), v)|^2\Big] dv ds\Big) \\
=&\Big(\tau\Big[\tau|f(x(t), x(t-h), r(t), t) + u(x(\delta_t), r(\delta_t), t)|^2 + |g(x(t), x(t-h), r(t), t)|^2\Big] \\
&- \int_{t-\tau}^t \Big[\tau|f(x(s), x(s-h), r(s), s) + u(x(\delta_s), r(\delta_s), s)|^2 + |g(x(s), x(s-h), r(s), s)|^2\Big] ds\Big) dt.
\end{aligned}$$

Applying the generalized Itô formula to the Lyapunov functional defined by (3.1) and combining the above two equalities, we have

$$dV(\hat{x}_t, \hat{r}_t, t) = LV(\hat{x}_t, \hat{r}_t, t)dt + dM(t) \tag{3.7}$$

for $t \geq 0$, where $M(t)$ is a continuous martingale with $M(0) = 0$ and

$$\begin{aligned}
&LV(\hat{x}_t, \hat{r}_t, t)\\
=&U_t(x(t), r(t), t) + U_x(x(t), r(t), t)[f(x(t), x(t-h), r(t), t) + u(x(\delta_t), r(\delta_t), t)]\\
&+ \frac{1}{2}\text{trace}[g^T(x(t), x(t-h), r(t), t)U_{xx}(x(t), r(t), t)g(x(t), x(t-h), r(t), t)]\\
&+ \sum_{j=1}^{N} \gamma_{r(t),j}U(x(t), j, t) + x(t)^T P(r(t))x(t) - x(t-h)^T P(r(t-h))x(t-h)\\
&+ \theta\tau\Big[\tau|f(x(t), x(t-h), r(t), t) + u(x(\delta_t), r(\delta_t), t)|^2 + |g(x(t), x(t-h), r(t), t)|^2\Big]\\
&- \theta\int_{t-\tau}^{t} \Big[\tau|f(x(s), x(s-h), r(s), s) + u(x(\delta_s), r(\delta_s), s)|^2 + |g(x(s), x(s-h), r(s), s)|^2\Big]ds.
\end{aligned} \tag{3.8}$$

Taking the expectation of both sides of equality (3.8) and recalling (3.2), we obtain

$$\begin{aligned}
&\mathbb{E}LV(\hat{x}_t, \hat{r}_t, t)\\
=&\mathbb{E}\mathcal{L}U(x(t), x(t-h), r(t), t)\\
&- \mathbb{E}\left(U_x(x(t), r(t), t)[u(x(t), r(t), t) - u(x(\delta_t), r(t), t) + u(x(\delta_t), r(t), t) - u(x(\delta_t), r(\delta_t), t)]\right)\\
&+ \mathbb{E}\left(x(t)^T P(r(t))x(t)\right) - \mathbb{E}\left(x(t-h)^T P(r(t-h))x(t-h)\right)\\
&+ \theta\tau\mathbb{E}\Big[\tau|f(x(t), x(t-h), r(t), t) + u(x(\delta_t), r(\delta_t), t)|^2 + |g(x(t), x(t-h), r(t), t)|^2\Big]\\
&- \theta\mathbb{E}\int_{t-\tau}^{t} \Big[\tau|f(x(s), x(s-h), r(s), s) + u(x(\delta_s), r(\delta_s), s)|^2 + |g(x(s), x(s-h), r(s), s)|^2\Big]ds.
\end{aligned} \tag{3.9}$$

Moreover, by Assumption 2.3 and Lemma 2.1, we can derive that

$$\begin{aligned}
&- \mathbb{E}\left(U_x(x(t), r(t), t)[u(x(t), r(t), t) - u(x(\delta_t), r(t), t) + u(x(\delta_t), r(t), t) - u(x(\delta_t), r(\delta_t), t)]\right)\\
\leq&\lambda_1\mathbb{E}|U_x(x(t), r(t), t)|^2\\
&+ \frac{1}{4\lambda_1}\mathbb{E}|u(x(t), r(t), t) - u(x(\delta_t), r(t), t) + u(x(\delta_t), r(t), t) - u(x(\delta_t), r(\delta_t), t)|^2\\
\leq&\lambda_1\mathbb{E}|U_x(x(t), r(t), t)|^2 + \frac{K^2}{2\lambda_1}\mathbb{E}|x(t) - x(\delta_t)|^2 + \frac{1}{2\lambda_1}\mathbb{E}|u(x(\delta_t), r(t), t) - u(x(\delta_t), r(\delta_t), t)|^2\\
\leq&\lambda_1\mathbb{E}|U_x(x(t), r(t), t)|^2 + \frac{K^2}{2\lambda_1}\mathbb{E}|x(t) - x(\delta_t)|^2 + \frac{2K^2}{\lambda_1}(1 - e^{-\bar{\gamma}\tau})\mathbb{E}|x(\delta_t)|^2\\
\leq&\lambda_1\mathbb{E}|U_x(x(t), r(t), t)|^2 + \left[\frac{K^2}{2\lambda_1} + \frac{4K^2}{\lambda_1}(1 - e^{-\bar{\gamma}\tau})\right]\mathbb{E}|x(t) - x(\delta_t)|^2 + \frac{4K^2}{\lambda_1}(1 - e^{-\bar{\gamma}\tau})\mathbb{E}|x(t)|^2.
\end{aligned} \tag{3.10}$$

In addition, according to Assumptions 2.2 and 2.3, it follows that

$$\begin{aligned}
&\theta\tau\mathbb{E}\Big[\tau|f(x(t),x(t-h),r(t),t)+u(x(\delta_t),r(\delta_t),t)|^2+|g(x(t),x(t-h),r(t),t)|^2\Big]\\
&\quad+\mathbb{E}\left(x(t)^TP(r(t))x(t)\right)-\mathbb{E}\left(x(t-h)^TP(r(t-h))x(t-h)\right)\\
&\le\big[\theta\tau(4\tau+2)L^2+\lambda_{PM}\big]\mathbb{E}|x(t)|^2+\big[\theta\tau(4\tau+2)L^2-\lambda_{Pm}\big]\mathbb{E}|x(t-h)|^2+2\theta\tau^2K^2\mathbb{E}|x(\delta_t)|^2\\
&\le\big[\theta\tau(4\tau+2)L^2+4\theta\tau^2K^2+\lambda_{PM}\big]\mathbb{E}|x(t)|^2+\big[\theta\tau(4\tau+2)L^2-\lambda_{Pm}\big]\mathbb{E}|x(t-h)|^2\\
&\quad+4\theta\tau^2K^2\mathbb{E}|x(t)-x(\delta_t)|^2.
\end{aligned}\tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9) yields

$$\begin{aligned}
&\mathbb{E}LV(\hat{x}_t,\hat{r}_t,t)\\
&\le\mathbb{E}\mathcal{L}U(x(t),x(t-h),r(t),t)+\lambda_1\mathbb{E}|U_x(x(t),r(t),t)|^2\\
&\quad+\left[\frac{4K^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau})+\theta\tau(4\tau+2)L^2+4\theta\tau^2K^2+\lambda_{PM}\right]\mathbb{E}|x(t)|^2\\
&\quad+\left[\theta\tau(4\tau+2)L^2-\lambda_{Pm}\right]\mathbb{E}|x(t-h)|^2\\
&\quad+\left[\frac{K^2}{2\lambda_1}+\frac{4K^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau})+4\theta\tau^2K^2\right]\mathbb{E}|x(t)-x(\delta_t)|^2\\
&\quad-\theta\mathbb{E}\int_{t-\tau}^{t}\Big[\tau|f(x(s),x(s-h),r(s),s)+u(x(\delta_s),r(\delta_s),s)|^2+|g(x(s),x(s-h),r(s),s)|^2\Big]ds.
\end{aligned}\tag{3.12}$$

By Assumption 3.1 and the conditions in (3.5), one finds

$$\begin{aligned}
&\mathbb{E}LV(\hat{x}_t,\hat{r}_t,t)\\
&\le-\lambda\mathbb{E}|x(t)|^2+\left[\frac{K^2}{2\lambda_1}+\frac{4K^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau})+4\theta\tau^2K^2\right]\mathbb{E}|x(t)-x(\delta_t)|^2\\
&\quad-\theta\mathbb{E}\int_{t-\tau}^{t}\Big[\tau|f(x(s),x(s-h),r(s),s)+u(x(\delta_s),r(\delta_s),s)|^2+|g(x(s),x(s-h),r(s),s)|^2\Big]ds,
\end{aligned}\tag{3.13}$$

where

$$\lambda=\lambda(\theta,\tau):=\lambda_2-\frac{4K^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau})-\theta\tau(4\tau+2)L^2-4\theta\tau^2K^2-\lambda_{PM}.$$

Considering that $t-\delta_t\le\tau$ for all $t\ge 0$, we can prove from equation (2.2) that

$$\begin{aligned}
&\mathbb{E}|x(t)-x(\delta_t)|^2\\
&\le 2\mathbb{E}\int_{t-\tau}^{t}\Big[\tau|f(x(s),x(s-h),r(s),s)+u(x(\delta_s),r(\delta_s),s)|^2+|g(x(s),x(s-h),r(s),s)|^2\Big]ds.
\end{aligned}\tag{3.14}$$

We choose

$$\theta=\frac{2K^2}{\lambda_1}\big(1+8(1-e^{-\frac{\bar{\gamma}}{4K}})\big),\qquad \tau\le\frac{1}{4K},$$

then it follows from (3.13) and (3.14) that

$$\mathbb{E}(LV(\hat{x}_t,\hat{r}_t,t))\le-\lambda\mathbb{E}|x(t)|^2.\tag{3.15}$$

Hence we can further obtain

$$\mathbb{E}(V(\hat{x}_t, \hat{r}_t, t)) \le C_1 - \lambda \int_0^t \mathbb{E}|x(s)|^2 ds \tag{3.16}$$

for $t \ge 0$, where

$$C_1 = U(\bar{\varphi}(0), r_0, 0) + h\lambda_{PM}||\bar{\varphi}||^2 + 4\theta\tau^2 \left[(2\tau+1)L^2 + 2\tau K^2\right] ||\bar{\varphi}||^2,$$

which is a positive number and we notice condition (3.5) indicates $\lambda > 0$. It thus follows from (3.16) immediately that

$$\int_0^\infty \mathbb{E}|x(s)|^2 ds \le C_1/\lambda.$$

This implies the desired assertion (3.6). □

Theorem 3.3 *Assume that all the conditions in Theorem 3.2 are satisfied. Then, the solution of the controlled system (2.2) is asymptotically stable in mean square, namely*

$$\lim_{t\to\infty} \mathbb{E}|x(t)|^2 = 0$$

for every initial data $x_0 = \bar{\varphi} \in C^b_{\mathcal{F}_0}([-2\tau^*, 0]; R^n)$ *and* $r_0 \in S$.

Proof. Fix any $x_0 = \bar{\varphi} \in C^b_{\mathcal{F}_0}([-2\tau^*, 0]; R^n)$ and $r_0 \in S$. Applying the Itô formula leads to

$$\begin{aligned}\mathbb{E}(|x(t)|^2) =& |\bar{\varphi}(0)|^2 + \mathbb{E}\int_0^t \Big(2x(s)[f(x(s), x(s-h), r(s), s) + u(x(\delta_s), r(\delta_s), s)] \\ &+ |g(x(s), x(s-h), r(s), s)|^2\Big) dt\end{aligned}$$

for all $t \ge 0$. Under Assumptions 2.2 and 2.3, we can show that

$$\mathbb{E}|x(t)|^2 \le \|\bar{\varphi}\|^2 + C\int_0^t \mathbb{E}|x(s)|^2 ds + C\int_0^t \mathbb{E}|x(s-h)|^2 ds + C\int_0^t \mathbb{E}|x(s) - x(\delta_s)|^2 ds, \tag{3.17}$$

where, and in the remaining part of this paper, C denotes a positive constant that may change from line to line but its special form is of no use. For any $s \ge 0$, there is a unique integer $v \ge 0$ for $s \in [v\tau, (v+1)\tau)$. Moreover, $\delta_z = v\tau$ for $z \in [v\tau, s]$. Then we can deduce from (2.2) that

$$\begin{aligned}&x(s) - x(\delta_s) = x(s) - x(v\tau) \\ =& \int_{v\tau}^s [f(x(z), x(z-h), r(z), z) + u(x(v\tau), r(v\tau), z)]dz + \int_{v\tau}^s g(x(z), x(z-h), r(z), z)dw(z).\end{aligned}$$

By Assumptions 2.2 and 2.3, we can derive

$$\begin{aligned}&\mathbb{E}|x(s) - x(\delta_s)|^2 \\ \le& 3(\tau+1)L^2\mathbb{E}\int_{v\tau}^s (|x(z)| + |x(z-h)|)^2 dz + 3\tau^2 K^2 \mathbb{E}|x(v\tau)|^2 \\ \le& 6(\tau+1)L^2 \left(\int_{\delta_s}^s \mathbb{E}|x(z)|^2 dz + \int_{\delta_s}^s \mathbb{E}|x(z-h)|^2 dz\right) + 6\tau^2 K^2 (\mathbb{E}|x(s)|^2 + \mathbb{E}|x(s) - x(\delta_s)|^2).\end{aligned}$$

We know from condition (3.5) that $6\tau^2K^2 < 1$ and hence

$$\mathbb{E}|x(s)-x(\delta_s)|^2 \le \frac{6(\tau+1)L^2}{1-6\tau^2K^2}\left(\int_{\delta_s}^{s}\mathbb{E}|x(z)|^2dz+\int_{\delta_s}^{s}\mathbb{E}|x(z-h)|^2dz\right)+\frac{6\tau^2K^2}{1-6\tau^2K^2}\mathbb{E}|x(s)|^2. \tag{3.18}$$

Substituting this into (3.17) yields

$$\begin{aligned}\mathbb{E}|x(t)|^2 \le & \|\bar{\varphi}\|^2+C\int_0^t\mathbb{E}|x(s)|^2ds+C\int_0^t\mathbb{E}|x(s-h)|^2ds\\ &+C\int_0^t\int_{\delta_s}^{s}\left(\mathbb{E}|x(z)|^2+\mathbb{E}|x(z-h)|^2\right)dzds.\end{aligned} \tag{3.19}$$

On the other hand, it is easy to show

$$\int_0^t\mathbb{E}|x(s-h)|^2ds=\int_{-h}^{t-h}\mathbb{E}|x(z)|^2dz,$$

$$\begin{aligned}&\int_0^t\int_{\delta_s}^{s}\mathbb{E}|x(z)|^2dzds\le\int_0^t\int_{s-\tau}^{s}\mathbb{E}|x(z)|^2dzds\\ \le&\int_{-\tau}^{t}\mathbb{E}|x(z)|^2\int_z^{z+\tau}dsdz\le\tau\int_{-\tau}^{t}\mathbb{E}|x(z)|^2dz\end{aligned}$$

and

$$\begin{aligned}&\int_0^t\int_{\delta_s}^{s}\mathbb{E}|x(z-h)|^2dzds\le\int_0^t\int_{s-\tau}^{s}\mathbb{E}|x(z-h)|^2dzds\\ =&\int_0^t\int_{s-\tau-h}^{s-h}\mathbb{E}|x(y)|^2dyds\le\int_{-\tau-h}^{t-h}\mathbb{E}|x(y)|^2\int_{y+h}^{y+h+\tau}dsdy\\ \le&\tau\int_{-\tau-h}^{t-h}\mathbb{E}|x(y)|^2dy.\end{aligned}$$

Substituting these into (3.19) and applying Theorem 3.2, we can obtain

$$\mathbb{E}|x(t)|^2\le C,\quad \forall t\ge 0. \tag{3.20}$$

In addition, by means of Itô formula, we reach

$$\begin{aligned}&\mathbb{E}|x(t_2)|^2-\mathbb{E}|x(t_1)|^2\\ =&\mathbb{E}\int_{t_1}^{t_2}\left(2x(t)\left[f(x(t),x(t-h),r(t),t)+u(x(\delta_t),r(\delta_t),t)\right]+|g(x(t),x(t-h),r(t),t)|^2\right)dt\end{aligned}$$

for any $0\le t_1<t_2<\infty$. By (3.20) and Assumptions 2.2 and 2.3, we can easily show that

$$|\mathbb{E}|x(t_2)|^2-\mathbb{E}|x(t_1)|^2|\le C(t_2-t_1).$$

That is, $\mathbb{E}|x(t)|^2$ is uniformly continuous in t on R_+. It then follows from (3.6) that $\lim_{t\to\infty}\mathbb{E}|x(t)|^2=0$, as required. □

Theorem 3.4 *Under the same assumptions of Theorem 3.2, the solution of the controlled system (2.1) satisfies*

$$\lim_{t\to\infty} x(t) = 0 \quad a.s.$$

for every initial data $x_0 = \bar{\varphi} \in C^b_{\mathcal{F}_0}([-2\tau^*, 0]; R^n)$ *and* $r_0 \in S$. *That is, the controlled system (2.2) is almost surely asymptotically stable.*

Proof. We divide the proof into three steps.

Step 1. Again we fix any $x_0 = \bar{\varphi} \in C^b_{\mathcal{F}_0}([-2\tau^*, 0]; R^n)$ and $r_0 \in S$. It follows from Theorem 3.3 and the well known Fubini theorem that

$$\mathbb{E}\int_0^\infty |x(t)|^2 dt < \infty. \tag{3.21}$$

This implies

$$\int_0^\infty |x(t)|^2 dt < \infty \quad a.s.$$

We must therefore have

$$\liminf_{t\to\infty} |x(t)| = 0 \quad a.s. \tag{3.22}$$

We now claim that

$$\lim_{t\to\infty} |x(t)| = 0 \quad a.s. \tag{3.23}$$

If this is false, then

$$\mathbb{P}\Big(\limsup_{t\to\infty} |x(t)| > 0\Big) > 0$$

Hence there exist a sufficiently small positive number ε such that

$$\mathbb{P}(\Omega_1) \geq 3\varepsilon, \tag{3.24}$$

where

$$\Omega_1 = \Big\{\limsup_{t\to\infty} |x(t)| > 2\varepsilon\Big\}.$$

Step 2. Let $h > \|\bar{\varphi}\|$ be a number. Define the stopping time

$$\beta_h = \inf\{t \geq 0 : |x(t)| \geq h\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (in which $\emptyset$ denotes the empty set as usual). Then, by the Itô formula, we have

$$\begin{aligned}
&\mathbb{E}|x(t\vee\beta_h)|^2 \\
&= |\bar{\varphi}(0)|^2 + \mathbb{E}\int_0^{t\vee\beta_h} \Big(2x(s)[f(x(s), r(s), s) + u(x(\delta_s), r(s), s)] + |g(x(s), r(s), s)|^2\Big)dt
\end{aligned}$$

for all $t \geq 0$. By Assumptions 2.2 and 2.3 as well as Theorem 3.2, it is easy to show that

$$\mathbb{E}|x(t\vee\beta_h)|^2 \leq C.$$

Hence

$$h^2\mathbb{P}(\beta_h \leq t) \leq C.$$

Letting $t \to \infty$ and then choosing h sufficiently large, we get

$$\mathbb{P}(\beta_h < \infty) \le \frac{C}{h^2} \le \varepsilon.$$

This implies

$$\mathbb{P}(\Omega_2) \le \varepsilon, \tag{3.25}$$

where

$$\Omega_2 = \{|x(t)| \ge h \text{ for all } 0 \le t < \infty\}.$$

It then follows easily from (3.24) and (3.25) that

$$\mathbb{P}(\Omega_1 \setminus \Omega_2) \ge 2\varepsilon. \tag{3.26}$$

Step 3. Define a sequence of stopping times:

$$\begin{aligned} \alpha_1 &= \inf\{t \ge 0 : |x(t)|^2 \ge 2\varepsilon\}, \\ \alpha_{2i} &= \inf\{t \ge \alpha_{2i-1} : |x(t)|^2 \le \varepsilon\}, \quad i = 1, 2, \cdots, \\ \alpha_{2i+1} &= \inf\{t \ge \alpha_{2i} : |x(t)|^2 \ge 2\varepsilon\}, \quad i = 1, 2, \cdots. \end{aligned}$$

We observe from (3.22) and the definitions of Ω_1 and Ω_2 that $\alpha_{2i} < \infty$ whenever $\alpha_{2i-1} < \infty$, and moreover,

$$\beta_h(\omega) = \infty \text{ and } \alpha_i(\omega) < \infty \text{ for all } i \ge 1 \text{ whenever } \omega \in \Omega_1 \setminus \Omega_2. \tag{3.27}$$

By (3.21), we derive

$$\begin{aligned} \infty > \mathbb{E}\int_0^\infty |x(t)|^2 dt &\ge \sum_{i=1}^\infty \mathbb{E}\Big(I_{\{\alpha_{2i-1}<\infty, \beta_h=\infty\}} \int_{\alpha_{2i-1}}^{\alpha_{2i}} |x(t)|^2 dt\Big) \\ &\ge \varepsilon \sum_{i=1}^\infty \mathbb{E}\Big(I_{\{\alpha_{2i-1}<\infty, \beta_h=\infty\}}[\alpha_{2i} - \alpha_{2i-1}]\Big). \end{aligned} \tag{3.28}$$

Let use now define

$$F(t) = f(x(t), r(t), t) + u(x(\delta_t), r(t), t) \quad \text{and} \quad G(t) = g(x(t), r(t), t)$$

for $t \ge 0$. By Assumptions 2.2 and 2.3, we see that

$$|F(t)|^2 \vee |G(t)|^2 \le K_h \quad \forall t \ge 0$$

whenever $|x(t)| \vee |x(\delta_t)| \le h$ (in particular, for $\omega \in \Omega_2$), where K_h is a positive constant. By the Hölder inequality and the Doob martingale inequality, we then derive that, for any $T > 0$,

$$\begin{aligned} &\mathbb{E}\Big(I_{\{\beta_h \vee \alpha_{2i-1}<\infty\}} \sup_{0\le t\le T} |x(\beta_h \vee (\alpha_{2i-1}+t)) - x(\beta_h \vee \alpha_{2i-1})|^2\Big) \\ \le& 2\mathbb{E}\Big(I_{\{\beta_h \vee \alpha_{2i-1}<\infty\}} \sup_{0\le t\le T} \Big|\int_{\beta_h \vee \alpha_{2i-1}}^{\beta_h \vee (\alpha_{2i-1}+t)} F(s)ds\Big|^2\Big) \\ &+ 2\mathbb{E}\Big(I_{\{\beta_h \vee \alpha_{2i-1}<\infty\}} \sup_{0\le t\le T} \Big|\int_{\beta_h \vee \alpha_{2i-1}}^{\beta_h \vee (\alpha_{2i-1}+t)} G(s)dw(s)\Big|^2\Big) \\ \le& 2T\mathbb{E}\Big(I_{\{\beta_h \vee \alpha_{2i-1}<\infty\}} \int_{\beta_h \vee \alpha_{2i-1}}^{\beta_h \vee (\alpha_{2i-1}+T)} |F(s)|^2 ds\Big) \\ &+ 8\mathbb{E}\Big(I_{\{\beta_h \vee \alpha_{2i-1}<\infty\}} \int_{\beta_h \vee \alpha_{2i-1}}^{\beta_h \vee (\alpha_{2i-1}+T)} |G(s)|^2 ds\Big) \\ \le& 2K_h T(T+4). \end{aligned} \tag{3.29}$$

Let $\theta = \varepsilon/(2h)$. It is easy to see that

$$||x|^2 - |y|^2| < \varepsilon \text{ whenever } |x-y| < \theta,\ |x| \vee |y| \le h. \tag{3.30}$$

Choose T sufficiently small for

$$\frac{2K_h T(T+4)}{\theta^2} < \varepsilon. \tag{3.31}$$

It then follows from (3.29) that

$$\begin{aligned}
&\mathbb{P}\Big(\{\beta_h \vee \alpha_{2i-1} < \infty\} \cap \Big\{\sup_{0\le t\le T} |x(\beta_h \vee (\alpha_{2i-1}+t)) - x(\beta_h \vee \alpha_{2i-1})| \ge \theta\Big\}\Big)\\
&\le \frac{2K_h T(T+4)}{\theta^2} < \varepsilon.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathbb{P}\Big(\{\alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \Big\{\sup_{0\le t\le T} |x(\alpha_{2i-1}+t) - x(\alpha_{2i-1})| \ge \theta\Big\}\Big)\\
=&\mathbb{P}\Big(\{\beta_h \vee \alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \Big\{\sup_{0\le t\le T} |x(\beta_h \vee (\alpha_{2i-1}+t)) - x(\beta_h \vee \alpha_{2i-1})| \ge \theta\Big\}\Big)\\
\le&\mathbb{P}\Big(\{\beta_h \vee \alpha_{2i-1} < \infty\} \cap \Big\{\sup_{0\le t\le T} |x(\beta_h \vee (\alpha_{2i-1}+t)) - x(\beta_h \vee \alpha_{2i-1})| \ge \theta\Big\}\Big)\\
\le&\varepsilon.
\end{aligned}$$

Using (3.26) and (3.27), we then have

$$\begin{aligned}
&\mathbb{P}\Big(\{\alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \Big\{\sup_{0\le t\le T} |x(\alpha_{2i-1}+t) - x(\alpha_{2i-1})| < \theta\Big\}\Big)\\
=&\mathbb{P}(\{\alpha_{2i-1} < \infty, \beta_h = \infty\})\\
-&\mathbb{P}\Big(\{\alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \Big\{\sup_{0\le t\le T} |x(\alpha_{2i-1}+t) - x(\alpha_{2i-1})| \ge \theta\Big\}\Big)\\
\ge&2\varepsilon - \varepsilon = \varepsilon.
\end{aligned}$$

By (3.30), we get

$$\begin{aligned}
&\mathbb{P}\Big(\{\alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \Big\{\sup_{0\le t\le T} ||x(\alpha_{2i-1}+t)|^2 - |x(\alpha_{2i-1})|^2| < \varepsilon\Big\}\Big)\\
\ge&\mathbb{P}\Big(\{\alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \Big\{\sup_{0\le t\le T} |x(\alpha_{2i-1}+t) - x(\alpha_{2i-1})| < \theta\Big\}\Big)\\
\ge&\varepsilon.
\end{aligned} \tag{3.32}$$

Set

$$\hat{\Omega}_i = \Big\{\sup_{0\le t\le T} ||x(\alpha_{2i-1}+t)|^2 - |x(\alpha_{2i-1})|^2| < \varepsilon\Big\}.$$

Note that

$$\alpha_{2i}(\omega) - \alpha_{2i-1}(\omega) \ge T \quad \text{ if } \omega \in \{\alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \hat{\Omega}_i.$$

Using (3.28) and (3.32), we finally derive that

$$\begin{aligned}\infty > \varepsilon &\sum_{i=1}^{\infty} \mathbb{E}\Big(I_{\{\alpha_{2i-1}<\infty,\beta_h=\infty\}}[\alpha_{2i}-\alpha_{2i-1}]\Big)\\ \geq \varepsilon &\sum_{i=1}^{\infty} \mathbb{E}\Big(I_{\{\alpha_{2i-1}<\infty,\beta_h=\infty\}\cap\hat{\Omega}_i}[\alpha_{2i}-\alpha_{2i-1}]\Big)\\ \geq \varepsilon T &\sum_{i=1}^{\infty} \mathbb{P}\Big(\{\alpha_{2i-1}<\infty,\beta_h=\infty\}\cap\hat{\Omega}_i\Big)\\ \geq \varepsilon T &\sum_{i=1}^{\infty} \varepsilon = \infty, \end{aligned}\tag{3.33}$$

which is a contradiction. Hence, (3.23) must hold. The proof is complete. □

3.2 Exponential stabilization

We have just discussed the asymptotic stabilization for system (2.1) by feedback control based on discrete-time state and mode observations. To reveal the rate at which the solution tends to zero, we will further investigate the exponential stabilization for system (2.1) by discrete-time feedback control. Before establishing our main result, we need to impose another condition.

Assumption 3.5 *Assume that there is a pair of positive numbers c_1 and c_2 such that*

$$c_1|x|^2 \leq U(x,i,t) \leq c_2|x|^2$$

for all $(x,i,t) \in R^n \times S \times R_+$.

Theorem 3.6 *Let Assumptions 2.2, 2.3, 3.1, 3.5 and Lemma 2.1 hold and recall that*

$$\theta = \frac{2K^2}{\lambda_1}\big(1+8(1-e^{-\frac{\bar{\gamma}}{4K}})\big)$$

and

$$\lambda = \lambda_2 - \frac{4K^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau}) - \theta\tau(4\tau+2)L^2 - 4\theta\tau^2K^2 - \lambda_{PM}.$$

If $\tau > 0$ is sufficiently small for (3.5) to hold, then the solution of the controlled system (2.2) satisfies

$$\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}|x(t)|^2) \leq -\gamma \tag{3.34}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) \leq -\frac{\gamma}{2} \quad a.s. \tag{3.35}$$

for every initial data $x_0 = \bar{\varphi} \in C^b_{\mathcal{F}_0}([-2\tau^*,0];R^n)$ and $r_0 \in S$, where $\gamma > 0$ is the unique root to the following equation

$$2\tau\gamma e^{2\tau\gamma}(H_1+\tau H_3) + 2\tau\gamma e^{(2\tau+h)\gamma}(H_2+\tau H_3) + \gamma(c_2+h\lambda_{PM}) = \lambda, \tag{3.36}$$

where

$$H_1 = 4\theta\tau^2(L^2 + K^2) + 2\theta\tau L^2 + \frac{24\theta\tau^4 K^4}{1 - 6\tau^2 K^2}, \quad H_2 = 2\theta\tau L^2(2\tau + 1),$$

$$H_3 = \frac{24\theta\tau^2(\tau + 1)K^2 L^2}{1 - 6\tau^2 K^2}.$$

Proof. According to the generalized Itô formula, we have

$$\mathbb{E}\big[e^{\gamma t} V(\hat{x}_t, \hat{r}_t, t)\big] = V(\hat{x}_0, \hat{r}_0, 0) + \mathbb{E}\int_0^t e^{\gamma z}[\gamma V(\hat{x}_z, \hat{r}_z, z) + LV(\hat{x}_z, \hat{r}_z, z)]dz$$

for $t \geq 0$. Recalling the definition of the Lyapunov functional (3.1), by using (3.15), (3.16) and Assumption 3.5, we can derive that

$$c_1 e^{\gamma t}\mathbb{E}|x(t)|^2 \leq C_1 + \int_0^t e^{\gamma z}[\gamma \mathbb{E}(V(\hat{x}_z, \hat{r}_z, z)) - \lambda \mathbb{E}|x(z)|^2]dz. \tag{3.37}$$

Define

$$\begin{aligned}\bar{V}(\hat{x}_t, \hat{r}_t, t) = &\int_{t-h}^t x(s)^T P(r(s)) x(s) ds \\ &+ \theta \int_{t-\tau}^t \int_s^t \Big[\tau |f(x(v), x(v-h), r(v), v) + u(x(\delta_v), r(\delta_v), v)|^2 \\ &+ |g(x(v), x(v-h), r(v), v)|^2\Big] dv ds.\end{aligned} \tag{3.38}$$

Then by (3.1) and Assumption 3.5, we can obtain

$$\mathbb{E}(V(\hat{x}_z, \hat{r}_z, z)) \leq c_2 \mathbb{E}|x(z)|^2 + \mathbb{E}(\bar{V}(\hat{x}_z, \hat{r}_z, z)). \tag{3.39}$$

Moreover, it follows from Assumptions 2.2 and 2.3 that

$$\begin{aligned}\mathbb{E}(\bar{V}(\hat{x}_z, \hat{r}_z, z)) \leq & h\lambda_{PM}\mathbb{E}|x(z)|^2 \\ &+ \theta\tau \int_{z-\tau}^z \Big[\left(4\tau(L^2 + K^2) + 2L^2\right)\mathbb{E}|x(v)|^2 + 2L^2(2\tau+1)\mathbb{E}|x(v-h)|^2 \\ &+ 4\tau K^2 \mathbb{E}|x(v) - x(\delta_v)|^2\Big] dv.\end{aligned} \tag{3.40}$$

By Theorem 3.2, it is easy to know that $\mathbb{E}(\bar{V}(\hat{x}_z, \hat{r}_z, z))$ is bounded on $z \in [0, 2\tau^*]$. For $z \geq 2\tau^*$, by (3.18), we get

$$\begin{aligned}\mathbb{E}(\bar{V}(\hat{x}_z, \hat{r}_z, z)) \leq & h\lambda_{PM}\mathbb{E}|x(z)|^2 + H_1 \int_{z-\tau}^z \mathbb{E}|x(v)|^2 dv + H_2 \int_{z-\tau}^z \mathbb{E}|x(v-h)|^2 dv \\ &+ H_3 \int_{z-\tau}^z \int_{\delta_v}^v (\mathbb{E}|x(y)|^2 + \mathbb{E}|x(y-h)|^2) dy dv,\end{aligned} \tag{3.41}$$

where H_1, H_2 and H_3 have been defined in (3.36). However,

$$\begin{aligned}&\int_{z-\tau}^z \int_{\delta_v}^v (\mathbb{E}|x(y)|^2 + \mathbb{E}|x(y-h)|^2) dy dv \leq \int_{z-\tau}^z \int_{v-\tau}^v (\mathbb{E}|x(y)|^2 + \mathbb{E}|x(y-h)|^2) dy dv \\ \leq & \tau \int_{z-2\tau}^z (\mathbb{E}|x(y)|^2 + \mathbb{E}|x(y-h)|^2) dy = \tau \int_{z-2\tau}^z \mathbb{E}|x(y)|^2 dy + \tau \int_{z-2\tau}^z \mathbb{E}|x(y-h)|^2) dy.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\mathbb{E}(\bar{V}(\hat{x}_z,\hat{r}_z,z)) \leq & h\lambda_{PM}\mathbb{E}|x(z)|^2+(H_1+\tau H_3)\int_{z-2\tau}^{z}\mathbb{E}|x(y)|^2dy \\ & +(H_2+\tau H_3)\int_{z-2\tau}^{z}\mathbb{E}|x(y-h)|^2dy. \end{aligned} \tag{3.42}$$

Substituting this into (3.39) and then putting the resulting inequality further into (3.37), we can find that, for $t \geq 2\tau^*$,

$$\begin{aligned} c_1e^{\gamma t}\mathbb{E}|x(t)|^2 \leq & C+\gamma(H_1+\tau H_3)\int_{2\tau}^{t}e^{\gamma z}\left(\int_{z-2\tau}^{z}\mathbb{E}|x(y)|^2dy\right)dz \\ & +\gamma(H_2+\tau H_3)\int_{2\tau}^{t}e^{\gamma z}\left(\int_{z-2\tau}^{z}\mathbb{E}|x(y-h)|^2dy\right)dz \\ & -(\lambda-\gamma(c_2+h\lambda_{PM}))\int_0^t e^{\gamma z}\mathbb{E}|x(z)|^2dz, \end{aligned} \tag{3.43}$$

where C is still a positive constant, whose special form is of no use. But

$$\begin{aligned} \int_{2\tau}^{t}e^{\gamma z}\left(\int_{z-2\tau}^{z}\mathbb{E}|x(y)|^2dy\right)dz \leq & \int_0^t \mathbb{E}|x(y)|^2\left(\int_y^{y+2\tau}e^{\gamma z}dz\right)dy \\ \leq & 2\tau e^{2\tau\gamma}\int_0^t e^{\gamma y}\mathbb{E}|x(y)|^2dy \end{aligned}$$

and

$$\begin{aligned} & \int_{2\tau}^{t}e^{\gamma z}\left(\int_{z-2\tau}^{z}\mathbb{E}|x(y-h)|^2dy\right)dz \leq \int_0^t \mathbb{E}|x(y-h)|^2\left(\int_y^{y+2\tau}e^{\gamma z}dz\right)dy \\ \leq & 2\tau e^{2\tau\gamma}\int_0^t e^{\gamma y}\mathbb{E}|x(y-h)|^2dy \leq 2\tau e^{(2\tau+h)\gamma}\int_{-h}^{t-h}e^{\gamma y}\mathbb{E}|x(y)|^2dy \\ < & C+2\tau e^{(2\tau+h)\gamma}\int_0^t e^{\gamma y}\mathbb{E}|x(y)|^2dy. \end{aligned}$$

Substituting this into (3.43) leads to

$$\begin{aligned} c_1e^{\gamma t}\mathbb{E}|x(t)|^2 \leq & C+\Big(2\tau\gamma e^{2\tau\gamma}(H_1+\tau H_3)+2\tau\gamma e^{(2\tau+h)\gamma}(H_2+\tau H_3) \\ & +\gamma(c_2+h\lambda_{PM})-\lambda\Big)\int_0^t e^{\gamma z}\mathbb{E}|x(z)|^2dz. \end{aligned}$$

Recalling (3.36), we see

$$c_1e^{\gamma t}\mathbb{E}|x(t)|^2 \leq C, \quad \forall t \geq 2\tau^*. \tag{3.44}$$

Immediately, the assertion (3.34) follows. Finally, we can obtain assertion (3.35) from (3.44) by [6, Theorem 8.8 on page 309]. The proof is therefore complete. □

3.3 Corollaries

Assumption 3.7 *Assume that there is a function $U \in C^{2,1}(R^n \times S \times R_+; R_+)$ and three positive numbers λ_4, λ_5 and λ_6 such that*

$$\mathcal{L}U(x,y,i,t) \leq -\lambda_4|x|^2+\lambda_5|y|^2 \tag{3.45}$$

and

$$|U_x(x,i,t)| \le \lambda_6 |x| \tag{3.46}$$

for all $(x,y,i,t) \in R^n \times R^n \times S \times R_+$ *and* $(x,i,t) \in R^n \times S \times R_+$.

Under this condition, if we choose a positive number $\lambda_1 < \lambda_4/\lambda_6^2$, then

$$\mathcal{L}U(x,y,i,t) + \lambda_1 |U_x(x,i,t)|^2 \le -(\lambda_4 - \lambda_1\lambda_6^2)|x|^2 + \lambda_5|y|^2. \tag{3.47}$$

If we set $\lambda_2 = \lambda_4 - \lambda_1\lambda_6^2$, it reaches the desired condition (3.3). That is to say, we have shown that Assumption 3.7 implies Assumption 3.1. The following corollary therefore follows.

Corollary 3.8 *All the theorems in Sections 3 and 4 hold if Assumption 3.1 is replaced by Assumption 3.7.*

In practice, the quadratic functions are widely used to be the Lyapunov functions. That is, we use $U(x,i,t) = x^T Q_i x$, where Q_i's are all symmetric positive-definite $n \times n$ matrices. In this case, Assumption 3.5 holds automatically with $c_1 = \min_{i\in S} \lambda_{\min}(Q_i)$ and $c_2 = \max_{i\in S} \lambda_{\max}(Q_i)$. Moreover, condition (3.46) holds as well with $\lambda_6 = 2\max_{i\in S} \|Q_i\|$. So all we need is to find Q_i's for (3.45) to hold. This gives us the following another assumption.

Assumption 3.9 *Assume that there are symmetric positive-definite matrices* $Q_i \in R^{n\times n}$ $(i \in S)$ *and two positive numbers* λ_4, λ_5 *such that*

$$\begin{aligned} 2x^T Q_i[f(x,y,i,t) + u(x,i,t)] &+ \mathrm{trace}[g^T(x,y,i,t) Q_i(x,i,t) g(x,y,i,t)] \\ &+ \sum_{j=1}^N \gamma_{ij} x^T Q_j x \le -\lambda_4|x|^2 + \lambda_5|y|^2, \end{aligned} \tag{3.48}$$

for all $(x,y,i,t) \in R^n \times R^n \times S \times R_+$ *and* $(x,i,t) \in R^n \times S \times R_+$.

The following corollary follows immediately from Theorem 3.6.

Corollary 3.10 *Let Assumptions 2.2, 2.3 and 3.9 hold. Set*

$$c_1 = \min_{i\in S} \lambda_{\min}(Q_i), \quad c_2 = \max_{i\in S} \lambda_{\max}(Q_i), \lambda_6 = 2\max_{i\in S} \|Q_i\|.$$

Choose $\lambda_1 < \lambda_4/\lambda_6^2$ *and then set* $\lambda_2 = \lambda_4 - \lambda_1\lambda_6^2$. *Let* $\tau > 0$ *be sufficiently small for (3.5) to hold and set*

$$\theta = \frac{2K^2}{\lambda_1}\big(1 + 8(1 - e^{-\frac{\bar\gamma}{4K}})\big)$$

and

$$\lambda = \lambda_2 - \frac{4K^2}{\lambda_1}(1 - e^{-\bar\gamma\tau}) - \theta\tau(4\tau + 2)L^2 - 4\theta\tau^2K^2 - \lambda_{PM}$$

(so $\lambda > 0$*). Then the assertions of Theorem 3.6 hold.*

4 Example

Example 4.1 We first consider an unstable linear hybrid SDDE

$$dx(t) = \big(A(r(t))x(t) + A_d(r(t))x(t-h)\big)dt + \big(B(r(t))x(t) + B_d(r(t))x(t-h)\big)dw(t) \tag{4.1}$$

on $t \geq 0$ with initial value $x_0 = \varphi \in C^b_{\mathcal{F}_0}([-h, 0]; R^n)$. Here $h = 0.1$, $w(t)$ is a scalar Brownian motion; $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with the generator

$$\Gamma = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix};$$

and the system matrices are

$$A_1 = \begin{bmatrix} 0.9 & 3.2 \\ 4.05 & -5.02 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.94 & 6.93 \\ 6.02 & 2.01 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} 0.1 & -0.2 \\ -0.05 & 0.02 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.06 & 0.07 \\ -0.02 & -0.01 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1.98 & 3.04 \\ 0 & 1.05 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2.01 & 1.04 \\ 2.08 & 2 \end{bmatrix}.$$

$$B_{d1} = \begin{bmatrix} 0.02 & -0.04 \\ 0 & 0.05 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.01 & -0.04 \\ -0.08 & 0 \end{bmatrix}.$$

The computer simulation (Figure 6.1) shows this hybrid SDDE is not mean square exponentially stable.

Let us now design a feedback control based on discrete time state and mode observations to stabilize the system. Assume that the controlled hybrid SDDE has the form

$$\begin{aligned} dx(t) = {} & [A(r(t))x(t) + A_d(r(t))x(t-h) + F(r(\delta_t))G(r(\delta_t))x(\delta_t)]dt \\ & + \big(B(r(t))x(t) + B_d(r(t))x(t-h)\big)dw(t), \end{aligned} \tag{4.2}$$

namely, our controller function has the form $u(x, i, t) = F_i G_i x$. Here, we assume that

$$G_1 = (-1.41, -1.4402), \quad G_2 = (3.1016, 1.9571),$$

and our aim is to seek for F_1 and F_2 in $R^{2\times 1}$ and then make sure τ is sufficiently small for this controlled SDDE to be exponentially stable in mean square and almost surely as well. To apply Corollary 3.10, we observe that by Assumptions 2.2 and 2.3, it is easy to know $L = 8.0406$ and $K = 11.7523$. Then we need to verify Assumption 3.9. It is easy to see the left-hand-side term of (3.48) becomes $x^T \bar{Q}_i x$ $(i = 1, 2)$, where

$$\bar{Q}_i := Q_i(A_i + A_{di} + F_i G_i) + (A_i^T + A_{di}^T + G_i^T F_i^T)Q_i + B_i^T Q_i B_i + \sum_{j=1}^{2} \gamma_{ij} Q_j.$$

Let us now choose

$$Q_1 = \begin{bmatrix} 2.5048 & 0.9239 \\ 0.9239 & 3.1738 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 5.3836 & 3.0928 \\ 3.0928 & 3.3392 \end{bmatrix}.$$

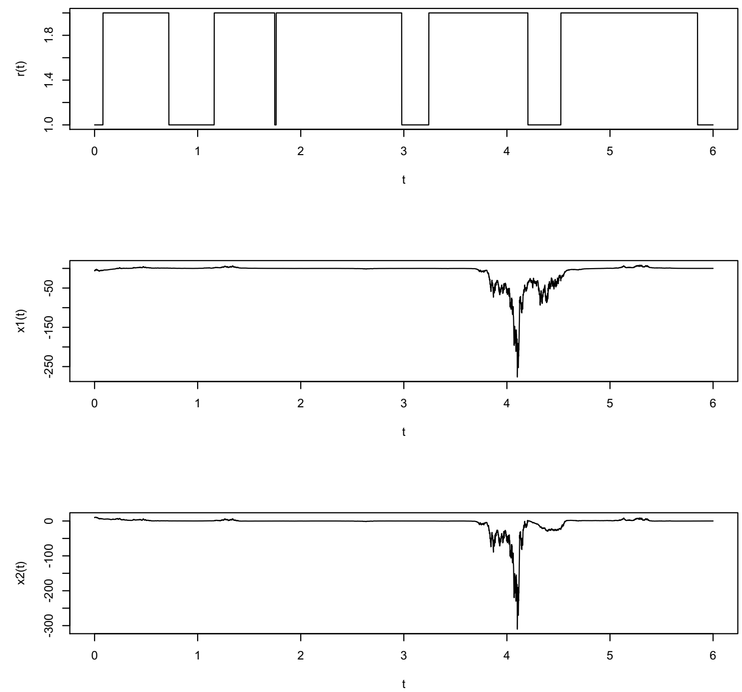

Figure 6.1: Computer simulation of the paths of $r(t)$, $x_1(t)$ and $x_2(t)$ for the hybrid SDE (4.1) using the Euler–Maruyama method with step size 10^{-6} and initial values $r(0) = 1$, $x_1(0) = -6$ and $x_2(0) = 10$.

and

$$F_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 \\ -10 \end{bmatrix}.$$

We then have

$$\bar{Q}_1 = \begin{bmatrix} -14.9558 & -4.2385 \\ -4.2385 & -35.3341 \end{bmatrix}, \quad \bar{Q}_2 = \begin{bmatrix} -53.8195 & -36.4580 \\ -36.4580 & -30.9954 \end{bmatrix}.$$

Hence, $x^T \bar{Q}_i x \leq -80.6096|x|^2$. We also know that $\lambda_3 = 0.09$. In other words, (3.48) holds with $\lambda_4 = 80.6996$. We further compute the parameters specified in Corollary 3.10: $c_1 = 1.1041$, $c_2 = 7.6187$ and $\lambda_6 = 15.2374$. Choosing $\lambda_1 = 0.3$, we then have $\lambda_2 = 10.9561$. In addition, we choose $\lambda_P M = 5.6$ and $\lambda_P m = 5.42$. Then, condition (3.5) becomes

$$10.9561 > 1841.554(1 - e^{-\tau}) + 69553.04\tau(4\tau + 2) + 594353.7\tau^2 + 5.6,$$

$$69553.04\tau(4\tau + 2) \leq 5.33 \quad \tau \leq 0.021.$$

These hold as long as $\tau < 0.0000383$. By Corollary 3.10, if we set F_i as above and make sure that $\tau < 0.0000383$, then the discrete-time-state-and-mode feedback controlled hybrid SDDE (4.2) is exponentially stable in mean square and almost surely as well. The computer simulation (Figure 6.2) supports this result clearly.

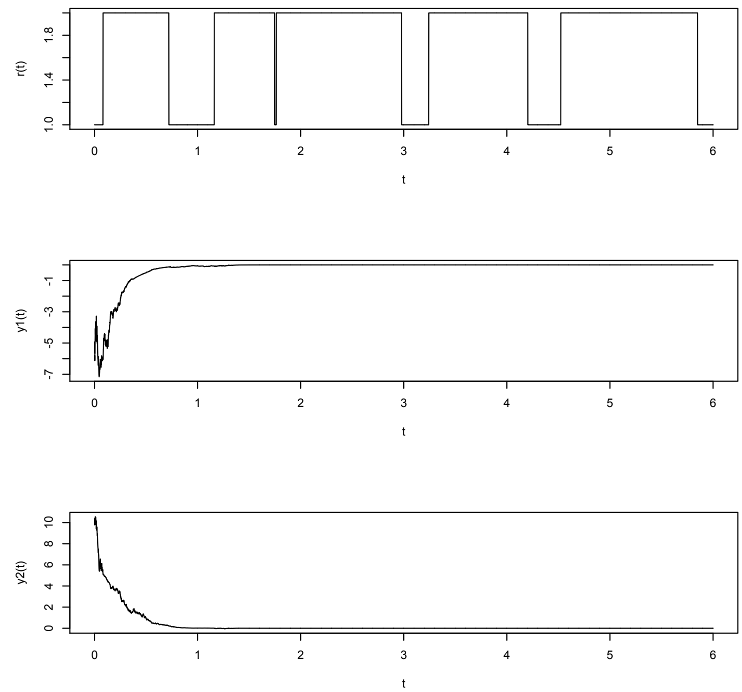

Figure 6.1: Computer simulation of the paths of $r(t)$, $x_1(t)$ and $x_2(t)$ for the hybrid SDE (4.2) using the Euler–Maruyama method with step size 10^{-6} and initial values $r(0) = 1$, $x_1(0) = -6$ and $x_2(0) = 10$.

Example 4.2 Let us now consider a nonlinear uncontrolled system (2.2). Given that its coefficients f and g satisfy the linear growth condition (2.4), we consider a linear controller function of the form $u(x, i, t) = D_i x$, where $D_i \in R^{n\times n}$ for all $i \in S$. That is, the controlled hybrid SDDE has the form

$$dx(t) = \big(f(x(t), x(t-h), r(t), t) + D_{r(\delta_t)}x(\delta_t)\big)dt + g(x(t), x(t-h), r(t), t)dw(t). \tag{4.3}$$

It is easy to observe that Assumption 2.3 holds with $K = \max_{i\in S} \|D_i\|$. Let us now establish Assumption 3.9 in order to apply Corollary 3.10. We choose $Q_i = q_i I$, where $q_i > 0$ and I is the $n \times n$ identity matrix. We estimate the left-hand-side of (3.48):

$$\begin{aligned}
&2x^T Q_i[f(x, y, i, t) + u(x, i, t)] + \text{trace}[g^T(x, y, i, t)Q_i(x, i, t)g(x, y, i, t)] + \sum_{j=1}^{N}\gamma_{ij}x^T Q_j x \\
&\le 2q_i L|x|(|x| + |y|) + 2q_i x^T D_i x + q_i L^2(|x| + |y|)^2 + \sum_{j=1}^{N}\gamma_{ij}q_j|x|^2 \\
&\le 2q_i L|x|^2 + q_i L|x|^2 + q_i L|y|^2 + 2q_i x^T D_i x + 2q_i L^2(|x|^2 + |y|^2) + \sum_{j=1}^{N}\gamma_{ij}q_j|x|^2 \\
&= x^T\Big(q_i(3L + 2L^2)I + q_i(D_i + D_i^T) + \sum_{j=1}^{N}\gamma_{ij}q_j I\Big)x + y^T(q_i(L + 2L^2))y.
\end{aligned} \tag{4.4}$$

Assume that the following linear matrix inequalities

$$q_i(3L + 2L^2)I + Y_i + Y_i^T + \sum_{j=1}^{N} \gamma_{ij} q_j I < 0 \tag{4.5}$$

have their solutions of $q_i > 0$ and $Y_i \in R^{n\times n}$ $(i \in S)$. Set $D_i = q_i^{-1} Y_i$ and

$$-\lambda_4 = \max_{i\in S} \lambda_{\max}\Big(q_i(3L + 2L^2)I + Y_i + Y_i^T + \sum_{j=1}^{N} \gamma_{ij} q_j I\Big) \tag{4.6}$$

$$\lambda_5 = \max_{i\in S} \lambda_{\max}\big(q_i(L + 2L^2)I\big). \tag{4.7}$$

We then see Assumption 3.9 is satisfied. The corresponding parameters in Corollary 3.10 becomes

$$c_1 = \min_{i\in S} q_i, \quad c_2 = \max_{i\in S} q_i, \quad \lambda_6 = 2c_2.$$

Choose $\lambda_1 < \lambda_4/\lambda_6^2$ and then set $\lambda_2 = \lambda_4 - \lambda_1\lambda_6^2$. Let $\tau > 0$ be sufficiently small for (3.5) to hold. Then, by Corollary 3.10, the controlled system (4.3) is exponentially stable in mean square.

For example, if we have the same Markov chain as that in Example 4.1, and set

$$f(t) = \begin{bmatrix} 0.2\sin x_2(t) & 1 \\ 0 & 0.5\cos x_1(t) \end{bmatrix} x(t) + \begin{bmatrix} 0.01\cos x_2(t) & 0 \\ 0.02 & 0.01\sin x_1(t) \end{bmatrix} x(t-h),$$

$$g(t) = \begin{bmatrix} 0.8\sin 2x_2(t) & 0 \\ -1 & 0.8\cos 2x_1(t) \end{bmatrix} x(t) + \begin{bmatrix} 0.01\cos 2x_2(t) & 0.03 \\ 0 & 0.01\sin 2x_1(t) \end{bmatrix} x(t-h).$$

and $h = 0.1$. Hence we observe that $L = 1.4434$. Then subsitute into the linear matrix inequalities (4.5) and get their solutions $q_1 = 1$, $q_2 = 2$,

$$Y_1 = \begin{bmatrix} -6 & 1 \\ 0 & -8 \end{bmatrix} \quad \text{and} \quad Y_2 = \begin{bmatrix} -9 & 4 \\ -2 & -10 \end{bmatrix}.$$

Then we get

$$D_1 = \begin{bmatrix} -6 & 1 \\ 0 & -8 \end{bmatrix} \quad \text{and} \quad D_2 = \begin{bmatrix} -4.5 & 2 \\ -1 & -5 \end{bmatrix}.$$

Hence $K = 8.1359$. We also observe that $\lambda_4 = 0.77$, $\lambda_5 = 11.2204$, $c_1 = 1$, $c_2 = 2$ and $\lambda_6 = 4$. Choose $\lambda_1 = 0.02$ and set $\lambda_2 = 0.45$. Let $\tau < 6.54 \times 10^{-6}$, then by Corollary 3.10,the controlled system (4.3) is exponentially stable in mean square.

5 Generalization

In this section, we will discuss a more general case. Consider an unstable hybrid SDDE

$$dx(t) = f(x(t), x(t-h(t)), r(t), t)dt + g(x(t), x(t-h(t)), r(t), t)dw(t), \tag{5.1}$$

where $t \geq 0$, $x(t) \in R^n$ is the state, $w(t) = (w_1(t), \cdots, w_m(t))^T$ is an m-dimensional Brownian motion, $r(t)$ is a continuous-time Markov chain. But h is now defined on the entire R_+, namely $h : R_+ \to [0, \bar{\tau}]$, and we assume that h is differentiable and its derivative is bounded by a constant $\bar{h} \in [0, 1)$, that is $\dot{h}(t) \leq \bar{h}$, for any t. In addition, SDDE (5.1) has initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\bar{\tau}, 0]; R^n)$ (such that $\mathbb{E}||\xi||^2 < \infty$) and $r(0) = r_0 \in S$ at time zero.

We aim to design a feedback control $u(x(\delta_t), r(\delta_t), t)$ so that the controlled hybrid SDDE

$$dx(t) = \big(f(x(t), x(t-h(t)), r(t), t) + u(x(\delta_t), r(\delta_t), t)\big)dt + g(x(t), x(t-h(t)), r(t), t)dw(t) \tag{5.2}$$

becomes H_∞-stable, asymptotically stable and exponentially stable in mean square, where $\tau > 0$, and $u : R^n \times S \times R_+ \to R^n$.

By employing the same Lyapunov functional as (3.1), all the results still hold in this paper. But Theorem (3.6) experiences changes in some coefficients. We state this result in the following theorem.

Theorem 5.1 *Let Assumptions 2.2, 2.3, 3.1, 3.5 and Lemma 2.1 hold. Let $\tau > 0$ be sufficiently small for (3.5) to hold. Recall that θ is defined as (3.4) and λ is defined as (3.12) (so $\lambda > 0$). Then the solution of the controlled system (5.2) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^2) \le -\gamma \tag{5.3}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t)|) \le -\frac{\gamma}{2} \quad a.s. \tag{5.4}$$

for every initial data $x_0 = \varphi \in C^b_{\mathcal{F}_0}([-2\tau^*, 0]; R^n)$ and $r_0 \in S$, where $\gamma > 0$ is the unique root to the following equation

$$2\tau\gamma e^{2\tau\gamma}(H_1 + \tau H_3) + \frac{2\tau e^{(2\tau+\tau^*)\gamma}}{1-\bar{h}}(H_2 + \tau H_3) + \gamma(c_2 + \bar{h}\lambda_{PM}) = \lambda, \tag{5.5}$$

$$H_1 = 4\theta\tau^2(L^2 + K^2) + 2\theta\tau L^2 + \frac{24\theta\tau^4 K^4}{1 - 6\tau^2 K^2}, \quad H_2 = 2\theta\tau L^2(2\tau + 1), \tag{5.6}$$

$$H_3 = \frac{24\theta\tau^2(\tau + 1)K^2 L^2}{1 - 6\tau^2 K^2}. \tag{5.7}$$

6 Conclusion

In this paper, we have proved the stabilization of continuous-time hybrid stochastic differential delay equations by *feedback controls based on discrete-time state and mode observations*. The stabilities here mainly referred to the H_∞ stability, mean squared asymptotic stability and mean squared exponential stability. Moreover, we also managed to build the upper bound on the duration τ between two consecutive state observations. We achieved these by employing Lyapunov functional.

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