Physics of Fluids ARTICLE scitation.org/journal/phf

Unsteady coating flow on a rotating cylinder in the presence of an irrotational airflow with circulation

Cite as: Phys. Fluids **34**, 043105 (2022); doi: 10.1063/5.0084472
Submitted: 6 January 2022 · Accepted: 11 March 2022 ·
Published Online: 6 April 2022

View Online Export Citation CrossMark

Andrew J. Mitchell,[a] Brian R. Duffy,[b] and Stephen K. Wilson[c]

AFFILIATIONS

Department of Mathematics and Statistics, University of Strathclyde, Livingstone Tower, 26 Richmond Street, Glasgow G1 1XH, United Kingdom

[a]andrew.j.mitchell@strath.ac.uk
[b]Author to whom correspondence should be addressed: b.r.duffy@strath.ac.uk
[c]s.k.wilson@strath.ac.uk

ABSTRACT

Unsteady two-dimensional coating flow of a thin film of a viscous fluid on the outside of a uniformly rotating horizontal circular cylinder in the presence of a steady two-dimensional irrotational airflow with circulation is considered. The analysis of this problem by Newell and Viljoen [Phys. Fluids **31**(3), 034106 (2019)], who sought to generalize the work of Hinch and Kelmanson [Proc. R. Soc. London, Ser. A **459**(2033), 1193–1213 (2003)] to include the effect of the airflow, is revisited. In contrast with the claim of Newell and Viljoen that the flow is conditionally unstable (in the sense that the solution for the film thickness grows without bound for certain values of the physical parameters), it is shown that, in fact, the film remains unconditionally stable in the presence of the airflow.

I. INTRODUCTION

Since the publication of the seminal papers by Moffatt[1] and Pukhnachev[2] in 1977, coating flow and rimming flow (i.e., flow of a film of fluid on the outside and inside, respectively, of a rotating solid horizontal cylinder) have come to be regarded as paradigm problems in the study of free-surface flows of viscous fluids. A rather large literature has grown up concerning these flows, including the study of discontinuous "shock" solutions in rimming flow by Johnson,[3] the pioneering numerical investigation of coating flow by Hansen and Kelmanson,[4] the study of "curtain" solutions, which are unbounded at the top and the bottom of the cylinder, by Duffy and Wilson,[5] the study of the effect of capillarity in rimming flow by Ashmore *et al.*,[6] studies of the large-time dynamics of unsteady coating flow by Hinch and Kelmanson,[7] Hinch *et al.*,[8] Kelmanson,[9] and Groh and Kelmanson,[10] the numerical investigations of two- and three-dimensional coating flow by Evans *et al.*,[11,12] the bifurcation analysis of coating flow by Lin *et al.*,[13] and the discovery of new branches of steady solutions in coating and rimming flow by Lopes *et al.*[14] In addition, there have been many extensions to the basic problems, including the studies of the effect of a uniform azimuthal shear stress on the free surface of the film by Black[15] and Villegas-Díaz *et al.*,[16] the studies of coating flow on elliptical cylinders by Hunt[17] and Li *et al.*,[18] the study of thermoviscous effects in coating and rimming flow by Leslie *et al.*,[19] the study of coating flow on patterned cylinders by Li *et al.*,[20] the studies of unsteady and steady coating flow in the presence of an irrotational airflow with circulation by Newell and Viljoen[21] and Mitchell *et al.*,[22] respectively, the study of a "thick-film" model for coating flow by Wray and Cimpeanu,[23] and the study of coating flow of a film laden with colloidal particles in the presence of solvent evaporation by Parrish and Kumar.[24]

Studies of the stability of coating and/or rimming flow in various parameter regimes have been undertaken. Hosoi and Mahadevan[25] found numerically that two-dimensional solutions for rimming flow can be stable even to three-dimensional perturbations in the presence of capillarity and weak inertia. Peterson *et al.*[26] determined regions of parameter space for which unsteady solutions of the Stokes equation for coating flow approach steady states at large times. O'Brien[27] confirmed the conclusion of Benjamin *et al.*[28] that subcritical solutions for rimming flow (that is, those for which the mass of fluid is less than a critical maximum value) are neutrally stable to small two-dimensional perturbations, and O'Brien[29] showed subsequently that weak capillarity can render these solutions stable. Villegas-Díaz *et al.*[16] used kinematic-wave theory to show that subcritical solutions are stable and that solutions with a shock on the rising side of the cylinder are stable if the shock is in the lower quadrant but unstable if it is in the upper quadrant. Benilov *et al.*[30] found that the linearized equations for rimming flow have non-harmonic solutions that develop singularities in finite time, and Benilov *et al.*[31] showed subsequently that the inclusion of capillarity precludes these singular solutions and renders most of the eigenmodes stable. Groh and Kelmanson[32] revealed previously undiscovered contributions to capillary decay and gravitational drift in coating flow. Pougatch and Frigaard[33] showed that in rimming flow capillarity may stabilize some modes but destabilize others.

In the present work, we are concerned with the flow of a thin film of a viscous fluid on a moving substrate in the presence of an airflow. As Newell and Viljoen[21] describe, just such a situation arises in the operation of a novel rotary pesticide applicator for crops, comprising a rotating cylinder covered in a film of fluid (pesticide) that brushes the undersides of leaves of plants as it is moved through the foliage. Similar situations include the jet-wiping (or air-knife) coating process in which impinging jets of air are used to control the thickness of a film of fluid (see, for example, Mendez *et al.*[34] and Barreiro-Villaverde *et al.*[35]) and the interaction between the airflow within and the film of oil on the inside of the outer shaft of the bearing chamber in a rapidly rotating aeroengine, which is a key element of the overall performance of the engine (see, for example, Farrall *et al.*,[36] Noakes *et al.*,[37] and Williams *et al.*[38])

In the present study, we model the air as inviscid, so that it exerts a non-uniform pressure but no shear stress on the film. Specifically, we investigate unsteady two-dimensional coating flow of a thin film of a viscous fluid on the outside of a uniformly rotating solid horizontal circular cylinder in the presence of a steady two-dimensional irrotational airflow with circulation. Three of the above-mentioned papers are particularly relevant to the present study, namely, those by Hinch and Kelmanson,[7] Newell and Viljoen,[21] and Mitchell *et al.*[22] Very recently, Mitchell *et al.*[22] investigated the steady version of the present problem in the absence of capillarity. They classified the possible steady solutions that can occur and proved (by a straightforward generalization of the argument of O'Brien[27]) that subcritical solutions remain neutrally stable to small two-dimensional perturbations in the presence of the airflow. Earlier, Hinch and Kelmanson[7] constructed the asymptotic solution for unsteady coating flow in the absence of an airflow in the case in which the effects of gravity and of capillarity are weak compared with those of viscous shear; in particular, they showed that at very large times the solution decays to a steady state in which the thickness of the film exhibits a gravity-induced phase lag relative to the solid cylinder. More recently, Newell and Viljoen[21] sought to generalize the work of Hinch and Kelmanson[7] to include the effect of the airflow; specifically, they sought to obtain the corresponding asymptotic solution to the present problem and found that it is conditionally unstable (in the sense that it grows without bound at large times for certain values of the physical parameters). However, the work of Newell and Viljoen[21] is compromised by a number of unfortunate errors, and so in the present study we revisit their analysis. In particular, we shall show that, in fact, the film remains unconditionally stable in the presence of the airflow.

II. PROBLEM FORMULATION

We consider unsteady two-dimensional coating flow of a thin film of incompressible viscous fluid of constant density ρ and viscosity μ on a solid horizontal circular cylinder of radius a rotating anticlockwise with uniform angular speed $\Omega\,(>0)$ in the presence of a steady two-dimensional airflow, as sketched in Fig. 1. Specifically, we take the air to be undergoing steady two-dimensional irrotational flow with uniform horizontal velocity U_∞ from left to right and pressure p_∞ in the far field and a circulation κ (measured anticlockwise) around the cylinder. In particular, we assume that since the film is thin, the airflow is unaffected by the presence of the film.

For the airflow around the cylinder to be even approximately irrotational, it is necessary that the Reynolds number based on the circumferential speed of the cylinder, $\rho_\mathrm{a} a^2\Omega/\mu_\mathrm{a} \gg 1$, where μ_a is the viscosity of the air, is large, and that the boundary layer that forms in the air remains attached to the cylinder. As Mitchell *et al.*[22] describe, the conclusions of a body of analytical and numerical studies of high-Reynolds-number flow around a rotating cylinder without a film of viscous fluid (notably the work by Glauert,[39] Moore,[40] Kang *et al.*,[41] Stojkovic *et al.*,[42] Mittal and Kumar,[43] and Aljure *et al.*[44]) are that the rotation of the cylinder tends to suppress the separation of the boundary layer, that when also the circumferential speed of the cylinder is large compared with the speed of the far-field airflow, $a\Omega \gg U_\infty$, the boundary layer does indeed remain attached all the way around the cylinder, and that the circulation then takes the value $\kappa = 2\pi a^2\Omega$. The question of what effect the presence of the thin film of viscous fluid may have on the suppression of boundary-layer separation remains open.

Referred to polar coordinates r–θ with origin on the axis of the cylinder and with θ measured from the horizontal, the pressure in the film, denoted by $p = p(\theta, t)$, where t denotes time, is given by

$$p = p_\infty + \frac{\sigma}{a} - \frac{\sigma}{a^2}\left(\frac{\partial^2 h}{\partial\theta^2} + h\right) + \frac{\rho_\mathrm{a}}{2}\left[U_\infty^2 - \left(2U_\infty \sin\theta - \frac{\kappa}{2\pi a}\right)^2\right], \tag{1}$$

where $h = h(\theta, t)$ is the thickness of the film, σ is the constant coefficient of surface tension, and ρ_a is the constant density of the air. The

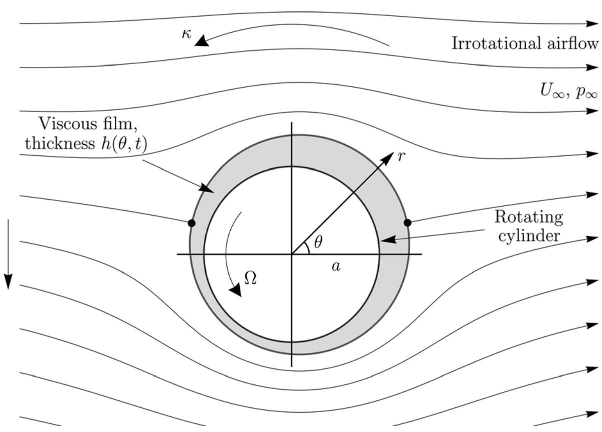

FIG. 1. Definition sketch: two-dimensional coating flow of a thin film of incompressible viscous fluid of thickness $h(\theta, t)$ on a solid horizontal circular cylinder of radius a rotating with uniform angular speed Ω in the presence of a steady two-dimensional irrotational flow of air with uniform horizontal velocity U_∞ and pressure p_∞ in the far field and a circulation κ around the cylinder. The streamlines and stagnation points of the airflow (the latter marked with dots) are sketched in a case with $0 < \kappa < 4\pi a U_\infty$.

azimuthal volume flux of fluid per unit axial length in the film, denoted by $Q = Q(\theta, t)$, is given by

$$Q = a\Omega h - \frac{h^3}{3\mu}\left(\rho g \cos\theta + \frac{1}{a}\frac{\partial p}{\partial \theta}\right), \tag{2}$$

where g is the magnitude of the acceleration due to gravity, and the statement of conservation of mass in the film gives the evolution equation for h,

$$\frac{\partial h}{\partial t} + \frac{1}{a}\frac{\partial Q}{\partial \theta} = 0. \tag{3}$$

We non-dimensionalize t with Ω^{-1}, h with the (constant) average film thickness $\bar{h}$, given by

$$\bar{h} = \frac{1}{2\pi}\int_0^{2\pi} h(\theta, t)\,\mathrm{d}\theta, \tag{4}$$

$p - p_\infty - \sigma/a$ with $\rho g a$, and Q with $a\Omega\bar{h}$. Then the pressure (1) becomes

$$p = -\frac{\alpha}{\gamma}\left(\frac{\partial^2 h}{\partial \theta^2} + h\right) + \frac{F^2}{2} - \frac{1}{2}(2F\sin\theta - K)^2, \tag{5}$$

the flux (2) becomes

$$Q = h - \gamma h^3\left(\cos\theta + \frac{\partial p}{\partial \theta}\right), \tag{6}$$

that is,

$$Q = h - \gamma h^3 \cos\theta + \alpha h^3 \frac{\partial}{\partial \theta}\left(\frac{\partial^2 h}{\partial \theta^2} + h\right) - 2\gamma F h^3 (K\cos\theta - F\sin 2\theta), \tag{7}$$

and, hence, the governing evolution equation (3) takes the form

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial \theta}\left[h - \gamma h^3 \cos\theta + \alpha h^3 \frac{\partial}{\partial \theta}\left(\frac{\partial^2 h}{\partial \theta^2} + h\right) - 2\gamma F h^3 (K\cos\theta - F\sin 2\theta)\right] = 0, \tag{8}$$

where α, γ, F, and K, which are defined by

$$\alpha = \frac{\sigma}{3\Omega\mu a}\left(\frac{\bar{h}}{a}\right)^3, \quad \gamma = \frac{\rho g a}{3\Omega\mu}\left(\frac{\bar{h}}{a}\right)^2, \quad F = \left(\frac{\rho_a U_\infty^2}{\rho g a}\right)^{\frac{1}{2}}, \quad K = \frac{\kappa}{2\pi a}\left(\frac{\rho_a}{\rho g a}\right)^{\frac{1}{2}}, \tag{9}$$

are nondimensional measures of surface tension, acceleration due to gravity, the speed of the far-field airflow, and the circulation of the airflow, respectively. Note that $\alpha \geq 0$, $\gamma \geq 0$, and $F \geq 0$, but that K has the same sign as κ, which may be positive, negative, or zero (corresponding to anticlockwise, clockwise, and no circulation, respectively). The evolution equation (8) is to be solved subject to periodicity conditions on h and its derivatives, and an initial condition specifying $h(\theta, 0)$.

In the special case of no far-field airflow, $F = 0$, the parameter K appears in (5) only in a constant contribution $-K^2/2$ to p, and does not appear in (7) or (8), showing that the evolution of the film is unaffected by a purely circulatory airflow.

Equation (8) is invariant under the transformation

$$\theta \to \theta + \pi, \quad K \to -\left(K + \frac{1}{F}\right) \quad (F > 0), \tag{10}$$

showing that if a free-surface profile $h(\theta, t)$ is a solution of (8) corresponding to a given value of the circulation K, then the phase-shifted profile $h(\theta + \pi, t)$ is a solution corresponding to the circulation $-[K + (1/F)]$. Moreover, although Eq. (8) involves all four of the parameters α, F, K, and γ, for $F \neq 0$, it may be reduced to the form

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial \theta}\left[h - \hat{\gamma}\beta h^3 \cos\theta + \alpha h^3 \frac{\partial}{\partial \theta}\left(\frac{\partial^2 h}{\partial \theta^2} + h\right) + \frac{1}{2}\hat{\gamma} h^3 \sin 2\theta\right] = 0, \tag{11}$$

involving only α and two new parameters β and $\hat{\gamma}$ defined by

$$\beta = \frac{1 + 2KF}{4F^2}, \quad \hat{\gamma} = 4F^2\gamma, \tag{12}$$

which may be regarded as reduced measures of K and γ, respectively [and we note that the transformation of K in (10) corresponds simply to $\beta \to -\beta$]. Despite this reduction in the number of parameters, it is usually more convenient to use the evolution equation in its original form (8), because (11) obscures somewhat the dependence of the film flow on the original parameters F and K of the airflow; in particular, (8) makes the comparison between our analysis and that of Hinch and Kelmanson[7] in the case $F = K = 0$ more transparent. However, the reduced form (11) is useful when the dependence of results on α is being discussed.

In the absence of the airflow, $F = K = 0$, Eq. (8) reduces to Eq. (3.14) of Pukhnachev,[2] whose parameters μ and χ are related to α and γ by $\mu = 3\gamma$ and $\chi = 3\alpha$. Moreover, Eqs. (5)–(8) reduce to the leading-order versions of Eqs. (35), (52), and (54), respectively, of Evans *et al.*,[11] whose parameters Bo and U_Ω are related to α and γ by $\mathrm{Bo} = \gamma\bar{h}/(\alpha a)$ and $U_\Omega = 1/(3\gamma)$, and to Eq. (2.13) of Lopes *et al.*,[14] whose parameters Ω_λ and Bo_λ are related to α and γ by $\Omega_\lambda = 1/(3\gamma)$ and $Bo_\lambda = \gamma/\alpha$. Note that the terms in the pressure p in Eqs. (2.2) and (2.3) of Hinch and Kelmanson[7] have the opposite signs from those in (2) and (1), but this is of no consequence because these differences in sign cancel out in their evolution equation (2.4) [which is identical to the present (8) with $F = K = 0$] on which all of their subsequent analysis is based. All of the results in the present work obtained from (8) agree with those of Hinch and Kelmanson[7] in the absence of the airflow.

For flow in the presence of the airflow, $F \neq 0$, Eqs. (1) and (2) reduce to Eqs. (2.1) and (3.3), respectively, of Mitchell *et al.*[22] in the case of steady flow in the absence of capillarity, $\alpha = 0$.

Newell and Viljoen[21] investigated the particular case $\kappa = 2\pi a^2\Omega$, while allowing U_∞ to take values in the range $-a\Omega \leq U_\infty \leq a\Omega$. Note, however, that, irrespective of whether or not a particular choice of κ is made, there are two free parameters associated with the airflow, namely, F and K in the present notation, or, correspondingly, the parameters w and φ in the notation of Newell and Viljoen,[21] which are related to F and K by $w = F/K$ and $\varphi = 2K^2$; in the case, $\kappa = 2\pi a^2\Omega$,

$$w = \frac{F}{K} = \frac{U_\infty}{a\Omega}, \quad \varphi = 2K^2 = \frac{2\rho_a a\Omega^2}{\rho g} \ (\geq 0) \tag{13}$$

in the present notation. (Newell and Viljoen[21] omitted the factor 2 from the definition of their parameter φ, but this appears to be simply a typographical error.)

As Mitchell *et al.*[22] describe, Newell and Viljoen[21] (evidently following Hinch and Kelmanson[7]) have the opposite signs on p in their Eqs. (5) and (6) from those in the present (2) and (1), but unfortunately, unlike for Hinch and Kelmanson,[7] these differences in sign do *not* cancel out in their evolution equation (7) [i.e., their version of the present Eq. (8)], leading to their (7) having the incorrect sign on the term due to the airflow, i.e., the term involving their parameter φ. However, as we shall show in what follows, obtaining the correct description of the behavior of the film is *not* simply a matter of reversing the sign of the term due to the airflow in the analysis of Newell and Viljoen.[21]

III. EVOLUTION OF THE FILM THICKNESS

In this section, we revisit the asymptotic analysis of Newell and Viljoen,[21] who, as we have already described, sought to extend the analysis of Hinch and Kelmanson[7] to include the effect of the airflow.

Hinch and Kelmanson[7] considered the case $\gamma \ll 1$ and $\alpha \ll 1$ (in the absence of the airflow, $F = K = 0$) and showed that the film evolves on four different timescales, and they posited a two-timescale expansion of the film thickness to reveal the structure of this evolution. Specifically, Hinch and Kelmanson[7] analyzed the evolution of an initially uniform film [so that $h(\theta, 0) = 1$] in the regime

$$\gamma^2 \ll \alpha \ll \gamma \ll 1, \tag{14}$$

by seeking a solution of (8) with $F = 0$ for h as an expansion in powers of γ,

$$h = 1 + \gamma\psi_1 + \gamma^2\psi_2 + \gamma^3\psi_3 + O(\gamma^4), \tag{15}$$

where the $\psi_i = \psi_i(\theta, t_0, t_1)$ depend on the two timescales $t_0 = t$ and $t_1 = \gamma^2 t$, so that $\partial\psi_i/\partial t = \partial\psi_i/\partial t_0 + \gamma^2\partial\psi_i/\partial t_1$, and the asymptotic solution (15) is uniformly valid only for $t < O(\gamma^{-2})$. Note that the ψ_i satisfy the initial conditions $\psi_i(\theta, 0, 0) = 0$.

Based on the observation that the film thickness h in the absence of the airflow is only weakly dependent on α, Hinch and Kelmanson[7] make the point that, despite the formal restrictions on α in (14), in practice their analysis is valid for α as large as $O(1)$. For the present purposes, the main result of Hinch and Kelmanson[7] is that the growth rate of the film thickness, which we denote by s_1, is given by

$$s_1 = -\frac{81\alpha}{1 + 144\alpha^2} \leq 0, \tag{16}$$

which, for $\alpha > 0$, is clearly negative, showing that the solution obtained is stable in the sense that it decays to a steady state at very large times and, for $\alpha = 0$, is clearly zero, showing that the solution is neutrally stable in the absence of capillarity. We adopt the same approach to the problem in the presence of the airflow by analyzing the evolution equation (8) for the film with $F \neq 0$, and, in particular, we obtain the corresponding expression for the growth rate s_1, to understand how the presence of the airflow affects the evolution of the film. On the assumption that in the presence of the airflow the film thickness h is again only weakly dependent on α, we follow Hinch and Kelmanson[7] and Newell and Viljoen[21] and allow α to be as large as $O(1)$.

Some of the algebraic manipulations involved in obtaining the analytical results presented in the next three subsections are rather lengthy, and so as a check on the accuracy of our results we implemented the entire calculation in two independent ways using the symbolic computational systems Maple[45] and Mathematica.[46] Two simple but important checks on the validity of the present analysis are that in the absence of the airflow, $F = K = 0$, the results obtained reduce to those of Hinch and Kelmanson,[7] and that under the transformation (10) the free-surface profile $h(\theta, t)$ becomes the phase-shifted profile $h(\theta + \pi, t)$.

A. Solution for ψ_1

Substituting (15) into (8) yields

$$\mathcal{L}\psi_1 = -(1 + 2KF)s_{1,0} - 4F^2 c_{2,0} \tag{17}$$

at $O(\gamma)$, where the linear operator $\mathcal{L}$, introduced by Hinch and Kelmanson,[7] is defined by

$$\mathcal{L} = \frac{\partial}{\partial t_0} + \frac{\partial}{\partial \theta} + \alpha\left(\frac{\partial^4}{\partial\theta^4} + \frac{\partial^2}{\partial\theta^2}\right), \tag{18}$$

and $c_{i,j}$ and $s_{i,j}$ are defined by $c_{i,j} = \cos(i\theta - jt_0)$ and $s_{i,j} = \sin(i\theta - jt_0)$. As Hinch and Kelmanson[7] describe, 2π-periodic solutions $\Psi(\theta, t_0, t_1)$ of the equation $\mathcal{L}\Psi = 0$ are of the form

$$\Psi(\theta, t_0, t_1) = \sum_{n=1}^{\infty}\left[A_n(t_1)c_{n,n} + B_n(t_1)s_{n,n}\right]\exp\left[-n^2(n^2 - 1)\alpha t_0\right], \tag{19}$$

where $A_n(t_1)$ and $B_n(t_1)$ are arbitrary functions of t_1, showing that the nth harmonics $c_{n,n}$ and $s_{n,n}$ ($n = 2, 3, 4, \ldots$) decay exponentially quickly on the fast timescale αt_0, but that the fundamental ($n = 1$) modes $c_{1,1}$ and $s_{1,1}$ can decay only slowly via the functions $A_1(t_1)$ and $B_1(t_1)$ on the timescale $t_1 = \gamma^2 t_0$.

The solution of (17) is of the form

$$\psi_1 = \sum_{n=1}^{\infty}\left[A_{1n}(t_1)c_{n,n} + B_{1n}(t_1)s_{n,n}\right]\exp\left[-n^2(n^2 - 1)\alpha t_0\right] + (1 + 2KF)c_{1,0} - \frac{2F^2}{1 + 36\alpha^2}(s_{2,0} + 6\alpha c_{2,0}), \tag{20}$$

where the $A_{1n}(t_1)$ and $B_{1n}(t_1)$ are required by the initial condition $\psi_1(\theta, 0, 0) = 0$ to satisfy

$$\begin{gathered} A_{11}(0) = -(1 + 2KF), \quad B_{11}(0) = 0, \\ A_{12}(0) = \frac{12F^2\alpha}{1 + 36\alpha^2}, \quad B_{12}(0) = \frac{2F^2}{1 + 36\alpha^2}, \\ A_{1n}(0) = 0, \quad B_{1n}(0) = 0 \quad \text{for} \quad n = 3, 4, 5, \ldots. \end{gathered} \tag{21}$$

Since the contributions from the higher modes ($n \geq 2$) are non-negligible only for small times, we may replace $A_{1n}(t_1)$ and $B_{1n}(t_1)$ with $A_{1n}(0)$ and $B_{1n}(0)$ for $n \geq 2$ (but not for $n = 1$). Thus, (20) becomes, with the subscripts omitted from A_{11} and B_{11} for clarity,

$$\psi_1 = A(t_1)c_{1,1} + B(t_1)s_{1,1} + (1 + 2KF)c_{1,0} - \frac{2F^2}{1 + 36\alpha^2}\left[(s_{2,0} + 6\alpha c_{2,0}) - (s_{2,2} + 6\alpha c_{2,2})\exp(-12\alpha t_0)\right], \tag{22}$$

where, from (21), $A(t_1)$ and $B(t_1)$ satisfy $A(0) = -(1 + 2KF)$ and $B(0) = 0$. In Subsection III C, we shall determine A and B by considering secular terms at $O(\gamma^3)$.

Note that the solution for ψ_1 given by (22) is in agreement with the solution for ψ_1 given by Eqs. (9) and (10) of Newell and Viljoen;[21] also in the absence of the airflow, $F = K = 0$, it reduces to the solution for ψ_1 given by Eq. (3.3) of Hinch and Kelmanson.[7]

Since modes with $n \geq 2$ decay exponentially quickly on the timescale t_0, henceforth we follow Hinch and Kelmanson[7] and Newell and Viljoen[21] in dropping them when we consider higher-order terms in γ.

B. Solution for ψ_2

At $O(\gamma^2)$, Eq. (8) gives

$$\begin{aligned}\mathcal{L}\psi_2 = &-\frac{3F^2\{[A(t_1)-6\alpha B(t_1)]c_{1,-1}-[6\alpha A(t_1)+B(t_1)]s_{1,-1}\}}{1+36\alpha^2}-\frac{6F^2(1+2KF)(c_{1,0}-6\alpha s_{1,0})}{1+36\alpha^2}-3(1+2KF)^2 s_{2,0}\\ &+3(1+2KF)\left[B(t_1)c_{2,1}-A(t_1)s_{2,1}\right]-\frac{18F^2(1+2KF)[c_{3,0}-6\alpha s_{3,0}]}{1+36\alpha^2}-\frac{9F^2\{[A(t_1)+6\alpha B(t_1)]c_{3,1}-[6\alpha A(t_1)-B(t_1)]s_{3,1}\}}{1+36\alpha^2}\\ &+\frac{24F^4\left[12\alpha c_{4,0}+(1-36\alpha^2)s_{4,0}\right]}{(1+36\alpha^2)^2}. \end{aligned} \tag{23}$$

The solution of (23) may be written as a sum $\psi_2 = \psi_{2s} + \psi_{2u}$ of a steady part ψ_{2s} and an unsteady part ψ_{2u}, where

$$\begin{aligned}\psi_{2s} = &-\frac{6F^2(1+2KF)(6\alpha c_{1,0}+s_{1,0})}{1+36\alpha^2}+\frac{3(1+2KF)^2(c_{2,0}-6\alpha s_{2,0})}{2(1+36\alpha^2)}-\frac{6F^2(1+2KF)\left[30\alpha c_{3,0}+(1-144\alpha^2)s_{3,0}\right]}{(1+36\alpha^2)(1+576\alpha^2)}\\ &-\frac{6F^4\left[(1-756\alpha^2)c_{4,0}-72\alpha(1-30\alpha^2)s_{4,0}\right]}{(1+36\alpha^2)^2(1+3600\alpha^2)}\end{aligned} \tag{24}$$

and

$$\begin{aligned}\psi_{2u} = &-\frac{3F^2\{[6\alpha A(t_1)+B(t_1)]c_{1,-1}+[A(t_1)-6\alpha B(t_1)]s_{1,-1}\}}{2(1+36\alpha^2)}+\frac{3(1+2KF)\{[A(t_1)+12\alpha B(t_1)]c_{2,1}-[12\alpha A(t_1)-B(t_1)]s_{2,1}\}}{1+144\alpha^2}\\ &-\frac{9F^2\{\left[42\alpha A(t_1)-(1-216\alpha^2)B(t_1)\right]c_{3,1}+\left[(1-216\alpha^2)A(t_1)+42\alpha B(t_1)\right]s_{3,1}\}}{2(1+36\alpha^2)(1+1296\alpha^2)}. \end{aligned} \tag{25}$$

Note that the solution for ψ_2 given by Eqs. (24) and (25) does *not* agree with that given by Eqs. (11) and (12) of Newell and Viljoen,[21] and moreover that, as previously pointed out, obtaining the correct solution for ψ_2 is not simply a matter of reversing the sign of the term due to the airflow in their solution. However, in the absence of the airflow, $F = K = 0$, both of these solutions reduce to that given by Eq. (3.5) of Hinch and Kelmanson,[7] as they should.

C. Solution for s_1

At $O(\gamma^3)$, Eq. (8) gives

$$\begin{aligned}\mathcal{L}\psi_3 = &-3\frac{\partial}{\partial\theta}\left\{\left(\psi_1^2+\psi_2\right)\left[\alpha\frac{\partial\psi_1}{\partial\theta}+\alpha\frac{\partial^3\psi_1}{\partial\theta^3}\right.\right.\\ &\left.\left.+2F^2\sin 2\theta-(1+2FK)\cos\theta\right]\right\}\\ &-3\alpha\frac{\partial}{\partial\theta}\left[\psi_1\left(\frac{\partial\psi_2}{\partial\theta}+\frac{\partial^3\psi_2}{\partial\theta^3}\right)\right]-\frac{\partial\psi_1}{\partial t_1},\end{aligned} \tag{26}$$

with ψ_1 and ψ_2 given by (22), (24), and (25).

The expanded form of (26), omitted for brevity, involves secular terms $s_{1,1}$ and $c_{1,1}$; setting the coefficients of these terms to zero leads to a pair of ordinary differential equations for $A(t_1)$ and $B(t_1)$, namely,

$$\frac{dA}{dt_1} = s_1 A + s_2 B, \tag{27}$$

$$\frac{dB}{dt_1} = -s_2 A + s_1 B, \tag{28}$$

where the real constants s_1 and s_2 are given by

$$s_1 = -81\alpha\left(\frac{10F^4}{(1+36\alpha^2)(1+1296\alpha^2)}+\frac{(1+2KF)^2}{1+144\alpha^2}\right)\leq 0 \tag{29}$$

and

$$s_2 = \frac{30(1+324\alpha^2)F^4}{(1+36\alpha^2)(1+1296\alpha^2)}+\frac{3(5+72\alpha^2)(1+2KF)^2}{2(1+144\alpha^2)}\geq 0. \tag{30}$$

Note that these expressions for s_1 and s_2 are invariant under the transformation of K given in (10), which provides a check on their validity. The solution of (27) and (28) subject to the initial conditions $A(0) = -(1+2KF)$ and $B(0) = 0$ is simply

$$A(t_1) = -(1+2KF)\exp(s_1t_1)\cos(s_2t_1), \tag{31}$$
$$B(t_1) = (1+2KF)\exp(s_1t_1)\sin(s_2t_1). \tag{32}$$

Equation (29) shows that for $\alpha > 0$ the growth rate of the solution, s_1, is always negative, and so we arrive at our main result, namely, that the solution (15) for h to $O(\gamma^3)$ is unconditionally stable, decaying exponentially quickly like $\exp(s_1t_1)$ to a steady state at very large times. In particular, in the limit of a slow airflow, $F \to 0$,

$$s_1 = -\frac{81\alpha}{1+144\alpha^2} - \frac{324K\alpha F}{1+144\alpha^2} + O(F^2) \to -\frac{81\alpha}{1+144\alpha^2} \le 0, \tag{33}$$

in the limit of a fast airflow, $F \to \infty$,

$$s_1 = -\frac{810\alpha F^4}{(1+36\alpha^2)(1+1296\alpha^2)} + O(F^2) \to -\infty, \tag{34}$$

in the case of no circulation, $K = 0$,

$$s_1 = -81\alpha\left(\frac{10F^4}{(1+36\alpha^2)(1+1296\alpha^2)} + \frac{1}{1+144\alpha^2}\right) \le 0, \tag{35}$$

and in the limit of strong circulation, $K \to \pm\infty$,

$$s_1 = -\frac{324\alpha F^2}{1+144\alpha^2}K^2 + O(K) \to -\infty. \tag{36}$$

Furthermore, in the limit of weak capillarity, $\alpha \to 0$,

$$s_1 = -81\left[(1+2KF)^2 + 10F^4\right]\alpha + O(\alpha^3) \to 0^-, \tag{37}$$

which is consistent with the conclusion of the limited stability analysis of Mitchell *et al.*[22] that in the absence of capillarity, $\alpha = 0$, the solution is neutrally stable (i.e., $s_1 = 0$).

Figure 2 shows plots of s_1 as a function of F for several values of K for the values of α considered by Hinch and Kelmanson[7] and Newell and Viljoen,[21] namely, $\alpha = 0.0048$ and $\alpha = 0.058$. As Fig. 2 illustrates, s_1 decreases monotonically with F when $K \ge 0$, but first increases to a negative maximum before decreasing monotonically when $K < 0$; in particular, the behavior of s_1 is consistent with the asymptotic results (33)–(37). The dashed curves in Fig. 2 show the locus of the maximum of s_1 as K varies from $-\infty$ [corresponding to the point $(F, s_1) = (0, 0)$] to 0 [corresponding to the point $(F, s_1) = (0, -81\alpha/(1+144\alpha^2))$], confirming that $s_1 \le 0$, and hence that the solution is unconditionally stable, for all values of $F(\ge 0)$ and K.

Equations (29) and (30) show that s_1 and s_2 depend on all three of the parameters α, F, and K; however, the behavior of s_1 and s_2 can be seen more clearly if (29) and (30) are instead written as equations for the scaled quantities s_1/F^4 and s_2/F^4, which depend on just the two parameters α and β,

$$\frac{s_1}{F^4} = -162\alpha\left(\frac{5}{(1+36\alpha^2)(1+1296\alpha^2)} + \frac{8\beta^2}{1+144\alpha^2}\right) \le 0 \tag{38}$$

and

$$\frac{s_2}{F^4} = \frac{30(1+324\alpha^2)}{(1+36\alpha^2)(1+1296\alpha^2)} + \frac{24(5+72\alpha^2)\beta^2}{1+144\alpha^2} \ge 0. \tag{39}$$

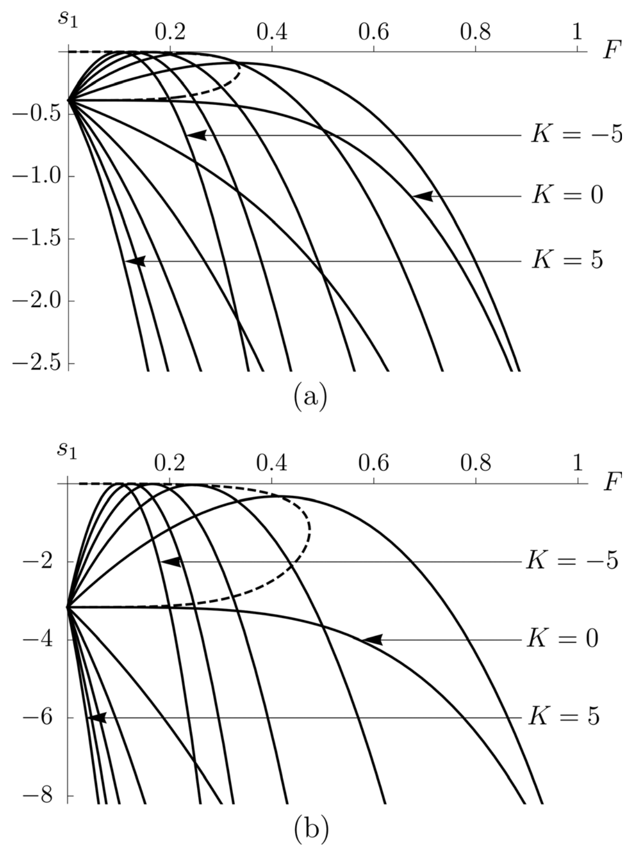

FIG. 2. Plot of the growth rate s_1 given by (29) as a function of F for $K = -5, -4, -3, \ldots, 5$ for the values of α considered by Hinch and Kelmanson[7] and Newell and Viljoen,[21] namely, (a) $\alpha = 0.0048$ and (b) $\alpha = 0.058$. The dashed curves show the locus of the maximum of s_1 as K varies from $-\infty$ [corresponding to the point $(F, s_1) = (0, 0)$] to 0 [corresponding to the point $(F, s_1) = (0, -81\alpha/(1+144\alpha^2))$].

Figure 3 shows the scaled growth rate s_1/F^4 as a function of α for a range of values of β, the collapse of the results onto curves of constant β being a consequence of the reduction from three parameters to two. In all cases, s_1/F^4 decreases with α from 0 to a minimum value before increasing monotonically to 0.

Note that the expression for s_1 given by Eq. (29) does *not* agree with that given by Eq. (13) of Newell and Viljoen,[21] and moreover that, as previously pointed out, obtaining the correct solution for s_1 is not simply a matter of reversing the sign of the term due to the airflow in their solution. However, in the absence of the airflow, $F = K = 0$, it reduces to (16), that is, to Eq. (3.10) of Hinch and Kelmanson,[7] as it should. Confirmation that the result of Newell and Viljoen[21] is erroneous comes from the fact that their expression for s_1 does not agree with Eq. (3.10) of Hinch and Kelmanson[7] when $\varphi = 0$ (due, we believe, to a double-counting of terms of the types $c_{1,1}c_{n,0}^2 + c_{1,1}s_{n,0}^2 = c_{1,1}$ and $s_{1,1}c_{n,0}^2 + s_{1,1}s_{n,0}^2 = s_{1,1}$), and so is in error even in the absence of the airflow.

D. Evolution of the free surface of the film

To $O(\gamma)$, we have $h = 1 + \gamma\psi_1$, and so to $O(\epsilon\gamma)$, where $\epsilon = \bar{h}/a \ll 1$ is the small aspect ratio of the film, the free surface has the form $r = 1 + \epsilon(1 + \gamma\psi_1)$, that is,

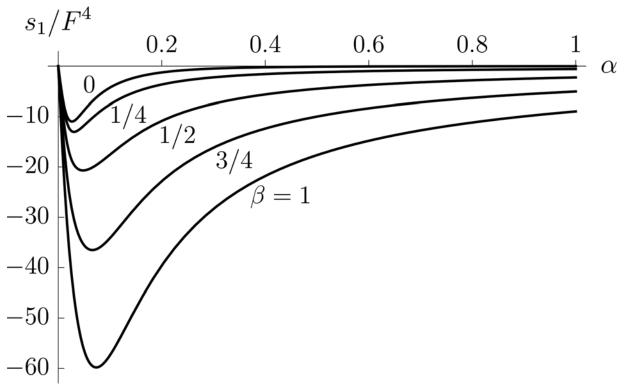

FIG. 3. Plot of the scaled growth rate s_1/F^4 given by (38) as a function of α for $\beta = 0$, 1/4, 1/2, 3/4, and 1, where $\beta = (1 + 2KF)/(4F^2)$.

$$r = 1 + \epsilon + \epsilon\gamma\Big\{(1 + 2KF)\big(c_{1,0} - \big[c_{1,1}\cos(s_2 t_1) - s_{1,1}\sin(s_2 t_1)\big]\exp(s_1 t_1)\big) - \frac{2F^2}{1 + 36\alpha^2}\big[(s_{2,0} + 6\alpha c_{2,0}) - (s_{2,2} + 6\alpha c_{2,2})\exp(-12\alpha t_0)\big]\Big\}, \tag{40}$$

where r has been scaled with a. The free surface (40) oscillates temporally; for $\alpha = 0$, the amplitude of the oscillation is finite for all t, whereas for $\alpha > 0$ the amplitude decays to zero in the limit $t \to \infty$, and the free surface approaches the steady profile

$$r = 1 + \epsilon + \epsilon\gamma\left\{(1 + 2KF)c_{1,0} - \frac{2F^2}{1 + 36\alpha^2}(s_{2,0} + 6\alpha c_{2,0})\right\}. \tag{41}$$

As Fig. 3 shows, for each β there is a unique value of α corresponding to the minimum of s_1 (i.e., a special value of the surface tension) such that (40) approaches (41) quickest: if α is larger or smaller than this (i.e., if surface tension is stronger or weaker than this special value) then the approach is slower.

As Hinch and Kelmanson[7] pointed out, in the absence of the airflow, $F = K = 0$, the free surface (40) is a circular cylinder of radius $1 + \epsilon$ whose center is offset from the axis of the solid cylinder by $\epsilon\gamma(1 + \hat{A}\cos t_0 - \hat{B}\sin t_0, \hat{A}\sin t_0 + \hat{B}\cos t_0)$ at any instant, where $\hat{A} = \hat{A}(t_1)$ and $\hat{B} = \hat{B}(t_1)$ are given by A and B in (31) and (32) with $F = 0$; thus, for $\alpha > 0$ the center of the circular free surface spirals around the point $(\epsilon\gamma, 0)$, approaching it in the limit $t \to \infty$. Moreover, rewriting the term $c_{1,1}\cos(s_2 t_1) - s_{1,1}\sin(s_2 t_1)$ in (40) with $F = 0$ in the form $\cos[\theta - (t_0 - s_2 t_1)]$ shows that the free surface lags the solid cylinder by the amount $s_2 t_1$.

In the presence of the airflow, the occurrence of the terms involving $c_{2,0}$ and $s_{2,0}$ (i.e., $\cos 2\theta$ and $\sin 2\theta$) in (40) and (41) means that the free surface, while still, of course, a cylinder, is no longer simply circular, and that it is no longer possible to identify a uniquely defined lag of the free surface relative to the solid cylinder.

Figure 4 shows examples of snapshots of free surfaces given by (40) at various times, comparing a case in the absence of the airflow, $F = K = 0$, with cases with far-field airflow $F = 1$ and positive, negative, or zero air circulation K, all plotted for $\alpha = 0$ and $\gamma = 1/15$ (the latter value being chosen for illustrative purposes). Figure 4 illustrates the effect of the airflow in both distorting the film and modifying its offset from the center of the solid cylinder. Figure 5 shows corresponding plots for $\alpha = 1/15$ (and again with $\gamma = 1/15$), illustrating the tendency of capillarity to suppress the distortion of the film. The largest time shown in Figs. 4 and 5, namely, $t_0 = 5$, was chosen simply because the free surfaces in Fig. 5 (but not those in Fig. 4) are close to their large-time asymptotic state (41) by this time and change little thereafter.

The successive parts of both Figs. 4 and 5 correspond to the values $\beta = \infty, -3/4, -1/4$, 1/4, 3/4, and 5/4, respectively, and therefore, parts (b) and (e) and parts (c) and (d) provide examples of the phase shift of π under the transformation (10), mentioned in Sec. II; for example, the value of r given by (40) at any position θ at time t in Fig. 4(b) is the same as the value of r at position $\theta + \pi$ at time t in Fig. 4(e).

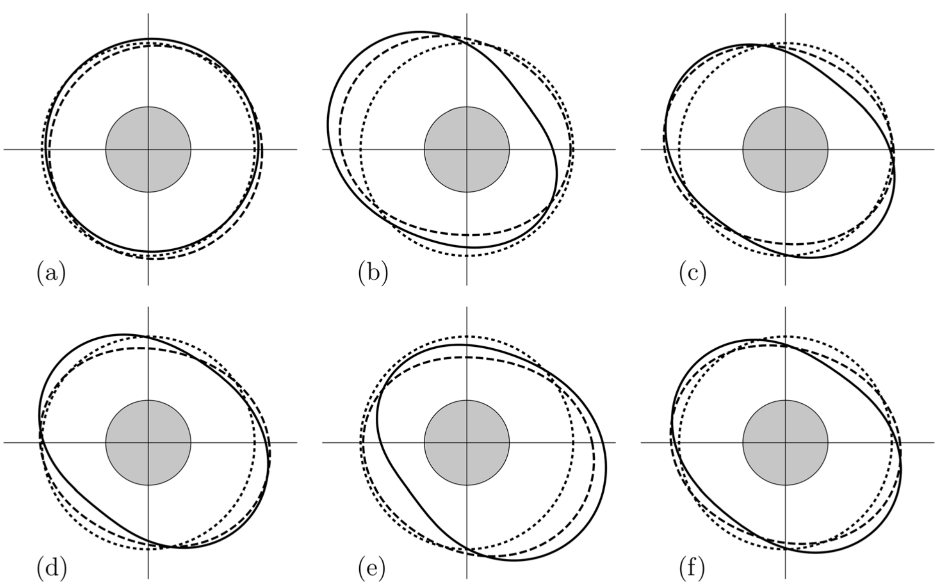

FIG. 4. Snapshots of free surfaces to $O(\epsilon\gamma)$ in the cases (a) $F = K = 0$, (b) $F = 1$, $K = -2$, (c) $F = 1$, $K = -1$, (d) $F = 1$, $K = 0$, (e) $F = 1$, $K = 1$, (f) $F = 1$, $K = 2$, with $\alpha = 0$ and $\gamma = 1/15$ in each case, at times $t_0 = 0$ (dotted curves), $t_0 = 2.5$ (dashed curves), and $t_0 = 5$ (full curves). The film thickness is exaggerated for clarity.

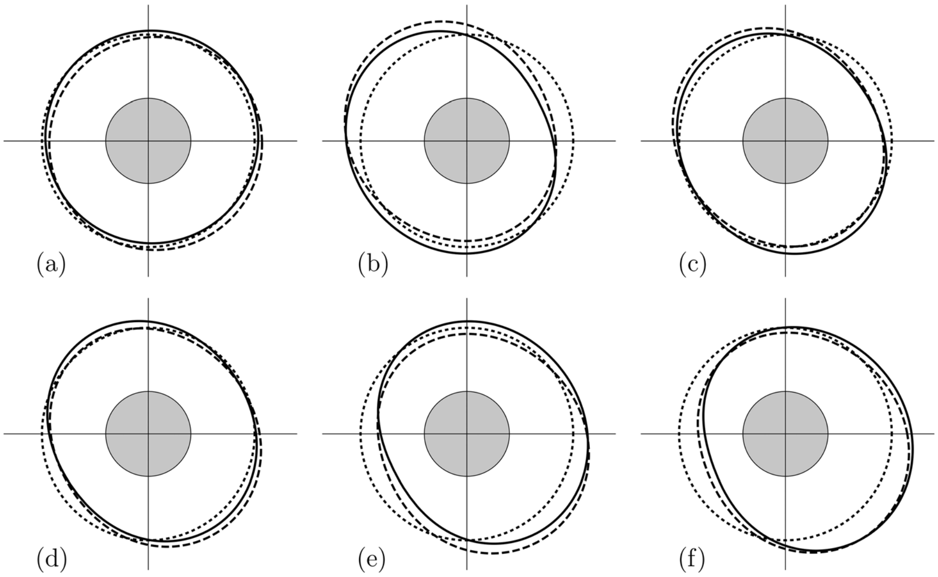

FIG. 5. Snapshots of free surfaces to $O(\epsilon\gamma)$ in the cases (a) $F = K = 0$, (b) $F=1$, $K=-2$, (c) $F=1$, $K=-1$, (d) $F=1$, $K=0$, (e) $F=1$, $K=1$, (f) $F=1$, $K=2$, with $\alpha = 1/15$ and $\gamma = 1/15$ in each case, at times $t_0 = 0$ (dotted curves), $t_0 = 2.5$ (dashed curves), and $t_0 = 5$ (full curves). The film thickness is exaggerated for clarity.

IV. CONCLUSIONS

We revisited the analysis of Newell and Viljoen[21] of unsteady two-dimensional coating flow of a thin film of a viscous fluid on the outside of a uniformly rotating solid horizontal circular cylinder in the presence of a steady two-dimensional irrotational airflow with circulation, in the case in which the effects of gravity and of capillarity are weak compared with those of viscous shear. In contrast with the claim of Newell and Viljoen[21] that the solution is unstable for certain values of the physical parameters, we found that the growth rate s_1, given by (29) in terms of the parameters F, K, and α, representing the speed of the far-field airflow, the circulation of the airflow, and surface tension, respectively, is always non-positive, and so the solution for h to $O(\gamma^3)$ is unconditionally stable.

From their study, Newell and Viljoen[21] drew five conclusions concerning the operation of the novel rotary pesticide applicator for crops described in Sec. I. Unfortunately, their erroneous prediction of instability renders their first conclusion incorrect and their third, fourth, and fifth conclusions moot; their second conclusion is correct but concerns only the case of no airflow and so provides no new information about the use of the applicator. On a more positive note, however, the successful use of the pesticide applicator depends on the film being stable—and our results show that this is always the case in the regime considered.

ACKNOWLEDGMENTS

The first author (A.J.M.) gratefully acknowledges the financial support of the United Kingdom Engineering and Physical Sciences Research Council (EPSRC) via EPSRC Doctoral Training Partnership Grant No. EP/S515632/1.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts of interest to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

REFERENCES

[1] H. K. Moffatt, "Behaviour of a viscous film on the outer surface of a rotating cylinder," J. Méc **16**(5), 651–673 (1977).
[2] V. V. Pukhnachev, "Motion of a liquid film on the surface of a rotating cylinder in a gravitational field," J. Appl. Mech. Tech. Phys. **18**(3), 344–351 (1977); available at https://link.springer.com/article/10.1007/BF00851656.
[3] R. E. Johnson, "Steady-state coating flows inside a rotating horizontal cylinder," J. Fluid Mech. **190**, 321–342 (1988).
[4] E. B. Hansen and M. A. Kelmanson, "Steady, viscous, free-surface flow on a rotating cylinder," J. Fluid Mech. **272**, 91–107 (1994).
[5] B. R. Duffy and S. K. Wilson, "Thin-film and curtain flows on the outside of a rotating horizontal cylinder," J. Fluid Mech. **394**, 29–49 (1999).
[6] J. Ashmore, A. E. Hosoi, and H. A. Stone, "The effect of surface tension on rimming flows in a partially filled rotating cylinder," J. Fluid Mech. **479**, 65–98 (2003).
[7] E. J. Hinch and M. A. Kelmanson, "On the decay and drift of free-surface perturbations in viscous thin-film flow exterior to a rotating cylinder," Proc. R. Soc. London, Ser. A **459**(2033), 1193–1213 (2003).
[8] E. J. Hinch, M. A. Kelmanson, and P. D. Metcalfe, "Shock-like free-surface perturbations in low-surface-tension, viscous, thin-film flow exterior to a rotating cylinder," Proc. R. Soc. London, Ser. A **460**(2050), 2975–2991 (2004).
[9] M. A. Kelmanson, "On inertial effects in the Moffatt–Pukhnachov coating-flow problem," J. Fluid Mech. **633**, 327–353 (2009).
[10] C. M. Groh and M. A. Kelmanson, "Inertially induced cyclic solutions in thin-film free-surface flows," J. Fluid Mech. **755**, 628–653 (2014).
[11] P. L. Evans, L. W. Schwartz, and R. V. Roy, "Steady and unsteady solutions for coating flow on a rotating horizontal cylinder: Two-dimensional theoretical and numerical modeling," Phys. Fluids **16**(8), 2742–2756 (2004).
[12] P. L. Evans, L. W. Schwartz, and R. V. Roy, "Three-dimensional solutions for coating flow on a rotating horizontal cylinder: Theory and experiment," Phys. Fluids **17**(7), 072102 (2005).
[13] T.-S. Lin, S. Rogers, D. Tseluiko, and U. Thiele, "Bifurcation analysis of the behavior of partially wetting liquids on a rotating cylinder," Phys. Fluids **28**(8), 082102 (2016).
[14] A. v. B. Lopes, U. Thiele, and A. L. Hazel, "On the multiple solutions of coating and rimming flows on rotating cylinders," J. Fluid Mech. **835**, 540–574 (2018).

[15]G. J. B. Black, "Theoretical studies of thin-film flows," M. Phil. thesis (University of Strathclyde, Glasgow, 2002).
[16]M. Villegas-Díaz, H. Power, and D. S. Riley, "On the stability of rimming flows to two-dimensional disturbances," Fluid Dyn. Res. **33**(1–2), 141–172 (2003).
[17]R. Hunt, "Numerical solution of the free-surface viscous flow on a horizontal rotating elliptical cylinder," Numer. Methods Partial Differ. Equ. **24**(4), 1094–1114 (2008).
[18]W. Li, M. S. Carvalho, and S. Kumar, "Viscous free-surface flows on rotating elliptical cylinders," Phys. Rev. Fluids **2**(9), 094005 (2017).
[19]G. A. Leslie, S. K. Wilson, and B. R. Duffy, "Thermoviscous coating and rimming flow," Q. J. Mech. Appl. Math. **65**(4), 483–511 (2012).
[20]W. Li, M. S. Carvalho, and S. Kumar, "Liquid-film coating on topographically patterned rotating cylinders," Phys. Rev. Fluids **2**(2), 024001 (2017).
[21]H. Newell and H. Viljoen, "Stability analysis of a thin film on a rotating cylinder with low airflow," Phys. Fluids **31**(3), 034106 (2019).
[22]A. J. Mitchell, B. R. Duffy, and S. K. Wilson, "Coating flow on a rotating cylinder in the presence of an irrotational airflow with circulation," J. Fluid Mech. **932**, A33 (2022).
[23]A. W. Wray and R. Cimpeanu, "Reduced-order modelling of thick inertial flows around rotating cylinders," J. Fluid Mech. **898**, A1 (2020).
[24]C. Parrish and S. Kumar, "Simultaneous liquid flow and drying on rotating cylinders," Phys. Rev. Fluids **5**(3), 034001 (2020).
[25]A. E. Hosoi and L. Mahadevan, "Axial instability of a free-surface front in a partially filled horizontal rotating cylinder," Phys. Fluids **11**(1), 97–106 (1999).
[26]R. C. Peterson, P. K. Jimack, and M. A. Kelmanson, "On the stability of viscous free-surface flow supported by a rotating cylinder," Proc. R. Soc. London, Ser. A **457**(2010), 1427–1445 (2001).
[27]S. B. G. O'Brien, "Linear stability of rimming flow," Q. Appl. Math **60**(2), 201–211 (2002).
[28]T. B. Benjamin, W. G. Pritchard, and S. J. Tavener, "Steady and unsteady flows of a highly viscous liquid inside a rotating horizontal cylinder." Report No. AM 122 (Penn State University, 1993).
[29]S. B. G. O'Brien, "A mechanism for linear stability in two-dimensional rimming flow," Q. Appl. Math **60**(2), 283–299 (2002).
[30]E. S. Benilov, S. B. G. O'Brien, and I. A. Sazonov, "A new type of instability: Explosive disturbances in a liquid film inside a rotating horizontal cylinder," J. Fluid Mech. **497**, 201–224 (2003).
[31]E. S. Benilov, N. Kopteva, and S. G. B. O'Brien, "Does surface tension stabilize liquid films inside a rotating horizontal cylinder?," Q. J. Mech. Appl. Math **58**(2), 185–200 (2005).
[32]C. M. Groh and M. A. Kelmanson, "Multiple-timescale asymptotic analysis of transient coating flows," Phys. Fluids **21**(9), 091702 (2009).
[33]K. Pougatch and I. Frigaard, "Thin film flow on the inside surface of a horizontally rotating cylinder: Steady state solutions and their stability," Phys. Fluids **23**(2), 022102 (2011).
[34]M. A. Mendez, A. Gosset, B. Scheid, M. Balabane, and J.-M. Buchlin, "Dynamics of the jet wiping process via integral models," J. Fluid Mech. **911**, A47 (2021).
[35]D. Barreiro-Villaverde, A. Gosset, and M. A. Mendez, "On the dynamics of jet wiping: Numerical simulations and modal analysis," Phys. Fluids **33**(6), 062114 (2021).
[36]M. Farrall, K. Simmons, S. Hibberd, and P. Gorse, "A numerical model for oil film flow in an aeroengine bearing chamber and comparison to experimental data," ASME J. Eng. Gas Turbines Power **128**(1), 111–117 (2006).
[37]C. J. Noakes, J. R. King, and D. S. Riley, "The effect of mass transfer on steady two-dimensional rimming flow," J. Eng. Math **71**(2), 223–236 (2011).
[38]J. Williams, S. Hibberd, H. Power, and D. S. Riley, "On the effects of mass and momentum transfer from droplets impacting on steady two-dimensional rimming flow in a horizontal cylinder," Phys. Fluids **24**(5), 053103 (2012).
[39]M. B. Glauert, "The flow past a rapidly rotating circular cylinder," Proc. R. Soc. London, Ser. A **242**, 108–115 (1957).
[40]D. W. Moore, "The flow past a rapidly rotating circular cylinder in a uniform stream," J. Fluid Mech. **2**(6), 541–550 (1957).
[41]S. Kang, H. Choi, and S. Lee, "Laminar flow past a rotating circular cylinder," Phys. Fluids **11**(11), 3312–3321 (1999).
[42]D. Stojkovic, M. Breuer, and F. Durst, "Effect of high rotation rates on the laminar flow around a circular cylinder," Phys. Fluids **14**(9), 3160–3178 (2002).
[43]S. Mittal and B. Kumar, "Flow past a rotating cylinder," J. Fluid Mech. **476**, 303–334 (2003).
[44]D. E. Aljure, I. Rodríguez, O. Lehmkuhl, C. D. Pérez-Segarra, and A. Oliva, "Influence of rotation on the flow over a cylinder at $Re = 5000$," Int. J. Heat Fluid Flow **55**, 76–90 (2015).
[45]Maple, *Mathematics-Based Software and Services for Education, Engineering, and Research* (Maple Inc., Waterloo, Ontario, Canada, 2021).
[46]Wolfram, *Mathematica, Version 12* (Wolfram Research Inc., Champaign, IL, 2021).