Stabilisation of Hybrid System with Different Structures by Feedback Control Based on Discrete-Time State Observations*

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Abstract

This paper considers a class of hybrid stochastic differential equations (SDEs) with different structures in different modes. In some modes, the coefficients of the SDEs satisfy the linear growth condition, while in the other modes, the coefficients are highly nonlinear. These systems are often unstable. This paper aims to design a discrete-time feedback control which is only put in a part of modes where the coefficients are highly nonlinear, such that these systems become stable. The stabilities concerned include H_∞-stability and exponential stability in the moment, as well as almost sure stability. Finally, an example is given to illustrate these results.

Keywords. hybrid SDEs, discrete-time state observations, feedback control, exponential stability, asymptotic stability.

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*This work is entirely theoretical and the results can be reproduced using the methods described in this paper.

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1 Introduction

It is well recognized that practical systems are rife with different sources of noises such that these stochastic systems involve both continuous dynamics and discrete events. It has also been well recognized that such systems provide more realistic models for many applications in biology, mathematical finance, wireless communication, etc. Due to the interaction between continuous dynamics and discrete events, these systems are often described by SDEs with Markovian switching which are also called as hybrid SDEs. We refer the reader to [1–8] and references therein for more details of these systems and their applications.

This paper will consider a hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t), \tag{1.1}$$

where $r(t)$ is a Markov chain taking values in a finite space $S = \{1, 2, ..., N\}$, $B(t)$ is an m-dimensional Brownian motion, the mappings $f : \mathbb{R}^n \times S \times \mathbb{R}_+ \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times S \times R_+ \to \mathbb{R}^{n\times m}$ are Borel measurable. The system (1.1) may be unstable, and the instability of stochastic systems could be verified by applying the known criteria [5, Theorem 3.23, page82] and [9, Theorem 3.5, page123] or computer simulations. To make this system become stable, the common method is to find a feedback control $u(x(t), r(t), t)$ such that the controlled system

$$dx(t) = [f(x(t), r(t), t) + u(x(t), r(t), t)]dt + g(x(t), r(t), t)dB(t)$$

becomes stable. Note that $u(x(t), r(t), t)$ is a continuous-time feedback, which relies on the continuous observations of the state $x(t)$. This continuous-time feedback control has been discussed widely (see e.g., Ji and Chizeck [10], Mao and his coauthors [9, 11–14]). However, since states can be observed only at discrete times $0, \tau, 2\tau, \cdots$, it will be more realistic to consider the discrete-time control. This feedback control problem can be described as the following controlled system

$$dx(t) = [f(x(t), r(t), t) + u(x([t/\tau]\tau), r(t), t)]dt + g(x(t), r(t), t)dB(t), \tag{1.2}$$

where $\tau > 0$ is a constant that stands for the duration between two consecutive state observations, and $[t/\tau]$ denotes the integer part of t/τ.

For a given stochastic system which may be unstable, Mao [15] proved that if the continuous-time feedback control function can stabilise this unstable system with exponential speed in the sense of mean square, then the corresponding discrete-time feedback control can also play the same role for sufficiently small observation time τ and Mao also gave estimation of τ. By using the Lyapunov functionals method, You et al. [16] investigated the exponential stability, H_∞ stability as well as the asymptotic stability and gave a better estimation for τ. Fei et al. [17] made a significant progress by showing that a class of highly nonlinear hybrid SDEs can be stabilised by discrete-time feedback control and they also analysed the H_∞ stability and almost sure stability for the controlled system. The theory of discrete-time feedback control has also been developed widely, for example, [18–21].

For the feedback control based on discrete-time observations, the state needs to be observed only at discrete times. On the other hand, the classical continuous-time feedback control, the state needs to be observed continuously for all time. Not mentioning that it is impossible to observe state continuously in practice, we see the time spent on the state observations at discrete times is less than the continuous-time case. This indicates the control cost can be reduced by the feedback control based on discrete-time observations. Often the system is observable only when it operates in some modes but not all. [22] examined the effects of different structures in different modes and derived the boundedness and the stability of the hybrid systems. Motivated by these observations, this paper considers a class of hybrid SDEs with different structures in different modes. And the main aim is to design a discrete-time feedback control for only a part of modes to make this hybrid system become stable in the senses of H_∞ stability and almost sure stability. This can also reduce cost further. Most of the existing papers consider the discrete-time feedback control for all modes. There are a few papers where some special examples are discussed using the control for a part of modes (see, e.g., Example 6.2 in [17] and Song et al. [23]), but these special examples have the same structure in all modes.

The rest of this paper is organised as follows. Section 2 gives the necessary notation and some assumptions. In section 3, we show the original system has a unique solution and a uniform bound in qth moment under the assumptions of section 2. We further show in this section that the controlled system has also a unique solution and preserves the uniform bound in qth moment as long as the control function satisfies the Lipschitz condition. In section 4, we give some rules on the control function and prove that the controlled system is H_∞ stable, asymptotically stable as well as exponentially stable under these rules. Finally, an example is given to illustrate these results.

2 Notation and assumptions

Let us introduce some notations and definitions that will be used. Let $\{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\}$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing, right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), ..., B_m(t))^T$ be an m-dimensional Brownian motion defined on this probability space. Let $r(t), t \geq 0$, be a continuous time Markov chain on this probability space taking values in a finite state space $S = \{1, 2, ..., N\}$ with the generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if} \quad i \neq j \\ 1 + \gamma_{ij}\Delta + o(\Delta), & \text{if} \quad i = j \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} > 0$ is the transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. Assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. Denote by $|x|$ the Euclidean norm for $x \in \mathbb{R}^n$. If A is a nonsingular matrix, its inverse is denoted by A^{-1}. For a matrix A, denote its trace norm by $|A| = \sqrt{trace(A^TA)}$. Let $R_+ = [0, \infty)$ and $\tau > 0$. If both a and b are real numbers, then $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. For a vector or matrix A, $A > 0$ means all elements of A are positive. Throughout the paper, C denotes a generic positive constant, whose value may change for different usage. Similarly, C_α denotes the generic positive constant depending on parameter α. In this paper, the nonsingular M-matrix plays an important role. We present some conditions equivalent to nonsingular M-matrix, and refer the reader to [24] for more details on this topic.

Lemma 2.1. *The following statements are equivalent:*

(1) $A = (a_{ij})_{N\times N}$ *is a nonsingular M-matrix,*

(2) A^{-1} *exists and its elements are all nonnegative,*

(3) There exists $x > 0$ *in* $\mathbb{R}^N$ *such that* $Ax > 0$.

Assumption 2.1. *Assume for each integer* $h \geq 1$, *there exists a positive constant* k_h *such that*

$$|f(x,i,t) - f(\bar{x},i,t)| \vee |g(x,i,t) - g(\bar{x},i,t)| \leq k_h(x - \bar{x})$$

for all $x, \bar{x} \in \mathbb{R}^n$ *with* $|x| \vee |\bar{x}| \leq h$ *and all* $(i,t) \in S \times R_+$. *Without loss of any generality, assume that the state space* $S = S_1 \cup S_2$ *with* $S_1 = \{1, ..., N_1\}$ *and* $S_2 = \{N_1+1, ..., N\}$, *where* $1 \leq N_1 < N$. *Assume there exist positive constants* $K, q_1 > 1, q_2 \geq 1$ *such that for* $(x,i,t) \in \mathbb{R}^n \times S_1 \times R_+$

$$|f(x,i,t)| \leq K|x|, \quad |g(x,i,t)| \leq K|x|, \tag{2.1}$$

and for $(x,i,t) \in \mathbb{R}^n \times S_2 \times R_+$,

$$|f(x,i,t)| \leq K(|x| + |x|^{q_1}), \quad |g(x,i,t)| \leq K(|x| + |x|^{q_2}). \tag{2.2}$$

Remark 2.2. Assumption 2.1 implies that in any S_1-mode, the coefficients of the underlying system (1.1) satisfy the linear growth condition. In any S_2-mode, coefficients of the underlying system (1.1) are highly nonlinear. Conditions (2.1) and (2.2) imply $f(0,i,t) \equiv 0$ and $g(0,i,t) \equiv 0$, which are required for the stability purpose.

Remark 2.3. (2.2) shows that in any S_2-mode, the coefficients of the underlying system (1.1) are controlled by polynomial functions with highest orders q_1 and q_2. By the Young inequality, $|x|^{q_i}, i = 1,2$, together with $|x|$, yield that f and g can include the forms of $\sum_{i=1}^n x^{\theta_i}$ with $\theta_i \in [1, q_1]$ and $\theta_i \in [1, q_2]$ respectively. If the linear term is removed, f and g cannot include the polynomial with order less than q_1 and q_2, respectively.

Assumption 2.2. *Let* $q, p > 2$ *with*

$$\begin{cases} q \geq (p + q_1 - 1) \vee (2(q_1 \vee q_2)), \\ p \geq (q_1 + 1) \vee (2q_2 - q_1 + 1). \end{cases}$$

Moreover, for each $i \in S_1$, *there exists constant* $\bar{\beta}_i \in R$ *such that*

$$x^T f(x,i,t) + \frac{q+p-3}{2}|g(x,i,t)|^2 \leq \bar{\beta}_i |x|^2, \tag{2.3}$$

and

$$\mathcal{D} := -(q+p-2)\mathrm{diag}(\bar{\beta}_1, ..., \bar{\beta}_{\mathrm{N_1}}) - (\gamma_{\mathrm{ij}})_{\mathrm{i,j}\in \mathrm{S_1}} \tag{2.4}$$

is a nonsingular M-matrix.

For each $i \in S_2$, *there exist constants* α, $\epsilon_i \in R_+$ *such that*

$$x^T f(x,i,t) + \frac{q-1}{2}|g(x,i,t)|^2 \leq -\alpha|x|^p + \epsilon_i |x|^2. \tag{2.5}$$

3 Rules for feedback control

To avoid confusion, let $z(t)$ denote the solution to the underlying system (1.1), and $x(t)$ denote the solution to the controlled system (1.2). Rewrite (1.1) as follows:

$$dz(t) = f(z(t), r(t), t)dt + g(z(t), r(t), t)dB(t), \quad z(0) = z_0 \in \mathbb{R}^n. \tag{3.1}$$

Note that (2.3) and (2.5) implies that (3.1) satisfies the monotonic condition. This, together with the local Lipschitz condition in Assumption 2.1, guarantees that the underlying system (3.1) has a unique global solution $z(t)$ and this solution satisfies $\mathbb{E}|z(t)|^q < C_t$ (see [1, Corollary 3.21, Page 97]). But in the following, we will show this solution $z(t)$ can be bounded uniformly in qth moment under Assumptions 2.1 and 2.2. For this purpose, we let

$$(\theta_1, \theta_2, ..., \theta_{N_1})^T = \mathcal{D}^{-1}(1, ..., 1)^T \tag{3.2}$$

and $\bar{\beta}_{\max} = \max_{i\in S_1} \bar{\beta}_i$, $\epsilon_{\max} = \max_{i\in S_2} \epsilon_i$, $\theta_{\max} = \max_{i\in S_1} \theta_i$ and $\theta_{\min} = \min_{i\in S_1} \theta_i$.

Theorem 3.1. *Under Assumptions 2.1, 2.2, the solution $z(t)$ of the system (3.1) satisfies*

$$\sup_{0\le t<\infty} \mathbb{E}|z(t)|^q < C. \tag{3.3}$$

Proof. For any $\mathcal{C} > 0$, define a Lyapunov function $U : \mathbb{R}^n \times S \to R_+$ by

$$U(z, i) = \begin{cases} |z|^q + \mathcal{C}\theta_i|z|^{q+p-2} & \text{if} \quad i \in S_1, \\ |z|^q & \text{if} \quad i \in S_2, \end{cases} \tag{3.4}$$

and introduce $LU : \mathbb{R}^n \times S \times R_+ \to R$ by

$$LU(z, i, t) = U_z(z, i)f(z, i, t) + \frac{1}{2}trace[g^T(z, i, t)U_{zz}(z, i)g(z, i, t)] + \sum_{j\in S} \gamma_{ij}U(z, j),$$

where

$$U_z(z, i) = \Big(\frac{\partial U(z, i)}{\partial z_1}, ..., \frac{\partial U(z, i)}{\partial z_n}\Big) \quad \text{and} \quad U_{zz}(z, i) = \Big[\frac{\partial^2 U(z, i)}{\partial z_k \partial z_l}\Big]_{n\times n}.$$

For any given $i \in S_1$, we have

$$\begin{aligned} LU(z, i, t) \quad = \quad & q|z|^{q-2}z^T f(z, i, t) + \frac{q}{2}|z|^{q-2}|g(z, i, t)|^2 + \frac{q(q-2)}{2}|z|^{q-4}|z^T g(z, i, t)|^2 \\ & +(q + p - 2)\mathcal{C}\theta_i|z|^{q+p-4}z^T f(z, i, t) + \frac{q + p - 2}{2}\mathcal{C}\theta_i|z|^{q+p-4}|g(z, i, t)|^2 \\ & +\frac{(q + p - 2)(q + p - 4)}{2}\mathcal{C}\theta_i|z|^{q+p-6}|z^T g(z, i, t)|^2 + \mathcal{C}\sum_{j\in S_1} \gamma_{ij}\theta_j|z|^{q+p-2}. \end{aligned}$$

Note that $|z^T g(z, i, t)|^2 \le |z|^2|g(z, i, t)|^2$. By (2.3), (2.4) and (3.2), we get for $i \in S_1$

$$LU(z, i, t) \quad \le \quad q|z|^{q-2}\left(z^T f(z, i, t) + \frac{q-1}{2}|g(z, i, t)|^2\right)$$

$$
\begin{aligned}
&+\mathcal{C}(q+p-2)\theta_i|z|^{q+p-4}\left(z^Tf(z,i,t)+\frac{q+p-3}{2}|g(z,i,t)|^2\right)\\
&+\mathcal{C}\sum_{j\in S_1}\gamma_{ij}\theta_j|z|^{q+p-2}\\
\leq\ & q\bar{\beta}_i|z|^q+(q+p-2)\mathcal{C}\bar{\beta}_i\theta_i|z|^{q+p-2}+\mathcal{C}\sum_{j\in S_1}\gamma_{ij}\theta_j|z|^{q+p-2}\\
=\ & q\bar{\beta}_i|z|^q-\mathcal{C}|z|^{q+p-2}\\
=\ & -\Big(\frac{1}{2\theta_i}\Big)(\mathcal{C}\theta_i|z|^{q+p-2}+|z|^q)-\Big(\frac{\mathcal{C}}{2}\Big)|z|^{q+p-2}+\Big(\frac{1}{2\theta_i}+q\bar{\beta}_i\Big)|z|^q\\
\leq\ & -\Big(\frac{1}{2\theta_{\max}}\Big)U(z,i)+c_1, \qquad (3.5)
\end{aligned}
$$

where

$$
c_1:=\sup_{u\geq 0}\left\{-\Big(\frac{\mathcal{C}}{2}\Big)|u|^{q+p-2}+\Big(\frac{1}{2\theta_{\min}}+q\bar{\beta}_{\max}\Big)|u|^q\right\}<\infty. \qquad (3.6)
$$

Similarly, it follows from (2.5) that for $i\in S_2$

$$
\begin{aligned}
LU(z,i,t)\ &\leq\ q|z|^{q-2}\left(z^Tf(z,i,t)+\frac{q-1}{2}|g(z,i,t)|^2\right)+\mathcal{C}\sum_{j\in S_1}\gamma_{ij}\theta_j|z|^{q+p-2}\\
&\leq\ q|z|^{q-2}(-\alpha|z|^p+\epsilon_i|z|^2)+\mathcal{C}\sum_{j\in S_1}\gamma_{ij}\theta_j|z|^{q+p-2}\\
&=\ -\Big(q\alpha-\mathcal{C}\sum_{j\in S_1}\gamma_{ij}\theta_j\Big)|z|^{q+p-2}+q\epsilon_i|z|^q.
\end{aligned}
$$

Since $\alpha>0$, we can always choose $\mathcal{C}$ sufficiently small such that

$$
\alpha>\frac{\mathcal{C}\sum_{j\in S_1}\gamma_{ij}\theta_j}{q},\quad \forall i\in S_2. \qquad (3.7)
$$

This implies that for $i\in S_2$

$$
LU(z,i,t)\leq-\Big(\frac{1}{2\theta_{\max}}\Big)|z|^q+c_2=-\Big(\frac{1}{2\theta_{\max}}\Big)U(z,i)+c_2, \qquad (3.8)
$$

where

$$
c_2:=\sup_{u\geq 0}\left\{-\Big(q\alpha-\mathcal{C}\sum_{j\in S_1}\gamma_{ij}\theta_j\Big)|u|^{q+p-2}+\Big(q\epsilon_{\max}+\frac{1}{2\theta_{\max}}\Big)|u|^q\right\}<\infty.
$$

Combining (3.5) with (3.8), for all $i\in S$ we have

$$
LU(z,i,t)\leq-\Big(\frac{1}{2\theta_{\max}}\Big)U(z,i)+c_1\vee c_2.
$$

Recalling the definition of $U(z,i)$, we have $|z|^q\leq U(z,i)$. Therefore, by [1, Theorem 5.2, Page157], the required result (3.3) follows. □

But Eq. (3.1) may be unstable. The main aim of this section is to design a feedback control $u(x([t/\tau]\tau),r(t),t)$, based on discrete-time state observations, such that the controlled system

$$
dx(t)=[f(x(t),r(t),t)+u(x(\delta_t),r(t),t)]dt+g(x(t),r(t),t)dB(t) \qquad (3.9)
$$

is stable, where $\delta_t:=[t/\tau]\tau$ and the control function $u:\mathbb{R}^n\times S\times R_+\to\mathbb{R}^n$ is Borel measurable.

Rule 3.2. For $i \in S_2$, the control function satisfies the Lipschitz condition, i.e.,

$$|u(x,i,t) - u(y,i,t)| \le \kappa |x-y|$$

with some $\kappa > 0$ and $u(x,i,t) = 0$ for $i \in S_1$. For the stability purpose, assume also that $u(0,i,t) \equiv 0$ for all $i \in S_2$.

Theorem 3.1 shows that the qth moment of the underlying system (3.1) is bounded uniformly. The following theorem shows that the controlled system (3.9) preserves this property under Rule 3.2.

Theorem 3.3. *Let Assumptions 2.1, 2.2 and Rule 3.2 hold. Then for any initial value* $x(0) = x_0 \in \mathbb{R}^n$*, the controlled system* (3.9) *has a unique global solution and*

$$\sup_{0 \le t < \infty} \mathbb{E}|x(t)|^q < C. \tag{3.10}$$

Proof. In fact, the controlled system (3.9) is a hybrid stochastic differential delay equation (SDDE) with a bounded variable delay. Define a function $\pi : R_+ \to [0,\tau]$ by

$$\pi(t) = t - k\tau \quad \text{for} \quad k\tau \le t < (k+1)\tau, \quad k = 0,1,2,\cdots.$$

Rewrite the controlled system (3.9) as

$$dx(t) = [f(x(t),r(t),t) + u(x(t-\pi(t)),r(t),t)]dt + g(x(t),r(t),t)dB(t), \qquad t \ge 0.$$

Recalling the definition of U, applying the generalized Itô formula (see [1, Lemma 1.9, page 49]) gives

$$dU(x(t),r(t)) = \bar{L}U(x(t),x(t-\pi(t)),r(t),t)dt + dM(t), \quad t \ge 0,$$

where $M(t)$ is a continuous local martingale with $M(0) = 0$ and the function $\bar{L}U : \mathbb{R}^n \times \mathbb{R}^n \times S \times R_+ \to R$ is given by

$$\bar{L}U(x,y,i,t) = U_x(x,i)[f(x,i,t) + u(y,i,t)] + \frac{1}{2}trace[g^T(x,i,t)U_{xx}(x,i)g(x,i,t)] + \sum_{j \in S} \gamma_{ij} U(x,j).$$

We divide the remaining proof into two steps.
Step 1: existence and uniqueness. For any given $i \in S_1$, we have

$$\begin{aligned}
\bar{L}U(x,y,i,t) \;&=\; q|x|^{q-2}x^T[f(x,i,t)+u(y,i,t)] + \frac{q}{2}|x|^{q-2}|g(x,i,t)|^2 + \frac{q(q-2)}{2}|x|^{q-4}|x^T g(x,i,t)|^2 \\
&\quad + (q+p-2)\mathcal{C}\theta_i |x|^{q+p-4}x^T[f(x,i,t)+u(y,i,t)] + \frac{q+p-2}{2}\mathcal{C}\theta_i|x|^{q+p-4}|g(x,i,t)|^2 \\
&\quad + \frac{(q+p-2)(q+p-4)}{2}\mathcal{C}\theta_i|x|^{q+p-6}|x^T g(x,i,t)|^2 + \mathcal{C}\sum_{j \in S_1}\gamma_{ij}\theta_j|x|^{q+p-2}.
\end{aligned}$$

In the same way as (3.5), we can show that for $i \in S_1$

$$\bar{L}U(x,y,i,t) \;\le\; q|x|^{q-2}\Big(x^T f(x,i,t) + \frac{q-1}{2}|g(x,i,t)|^2\Big)$$

$$
\begin{aligned}
&\quad +(q+p-2)\mathcal{C}\theta_i|x|^{q+p-4}\Big(x^T f(x,i,t)+\frac{q+p-3}{2}|g(x,i,t)|^2\Big)+\mathcal{C}\sum_{j\in S_1}\gamma_{ij}\theta_j|x|^{q+p-2}\\
&\le \; -\Big(\frac{1}{2\theta_{\max}}\Big)U(x,i)+c_1, \qquad (3.11)
\end{aligned}
$$

where we have used the fact that $u(x,i,t)=0, i\in S_1$ and c_1 is from (3.6). Similarly, for each $i\in S_2$, we calculate

$$
\begin{aligned}
\bar{L}U(x,y,i,t) \;&=\; q|x|^{q-2}x^T[f(x,i,t)+u(y,i,t)]+\frac{q}{2}|x|^{q-2}|g(x,i,t)|^2\\
&\quad +\frac{q(q-2)}{2}|x|^{q-4}|x^T g(x,i,t)|^2+\mathcal{C}\sum_{j\in S_1}\gamma_{ij}\theta_j|x|^{q+p-2}.
\end{aligned}
$$

Note that $|u(x,i,t)|\le\kappa|x|$. By (2.5), we have for $i\in S_2$

$$
\begin{aligned}
\bar{L}U(x,y,i,t) \;&\le\; q|x|^{q-2}\Big(x^T[f(x,i,t)+u(y,i,t)]+\frac{q-1}{2}|g(x,i,t)|^2\Big)+\mathcal{C}\sum_{j\in S_1}\gamma_{ij}\theta_j|x|^{q+p-2}\\
&\le\; -q\alpha|x|^{q+p-2}+q\epsilon_i|x|^q+\mathcal{C}\sum_{j\in S_1}\gamma_{ij}\theta_j|x|^{q+p-2}+q|x|^{q-1}|u(y,i,t)|\\
&\le\; \Big(-q\alpha+\mathcal{C}\sum_{j\in S_1}\gamma_{ij}\theta_j\Big)|x|^{q+p-2}+q\epsilon_i|x|^q+q\kappa|x|^{q-1}|y|.
\end{aligned}
$$

Recalling $e^{(-\tau/2\theta_{\max})}<1$ ($\theta_{\max}$ is positive), then we can choose $\varepsilon\in(0,1)$ sufficiently small such that

$$
e^{-\frac{1}{2\theta_{\max}}\tau}+\varepsilon\tau<1. \qquad (3.12)
$$

By Young's inequality, it follows from (3.7) that for $i\in S_2$,

$$
\begin{aligned}
\bar{L}U(x,y,i,t) \;&\le\; -\Big(q\alpha-\mathcal{C}\sum_{j\in S_1}\gamma_{ij}\theta_j\Big)|x|^{q+p-2}+q\epsilon_i|x|^q+\frac{(q-1)(q\kappa)^{q/(q-1)}}{q(q\varepsilon)^{1/(q-1)}}|x|^q+\varepsilon|y|^q\\
&\le\; c_3-\Big(\frac{1}{2\theta_{\max}}\Big)U(x,i)+\varepsilon U(y,i), \qquad (3.13)
\end{aligned}
$$

where

$$
c_3:=\sup_{u\ge 0}\Big\{-\Big(q\alpha-\mathcal{C}\sum_{j\in S_1}\gamma_{ij}\theta_j\Big)|u|^{q+p-2}+\Big(\frac{1}{2\theta_{\max}}+q\epsilon_{\max}+\frac{(q-1)(q\kappa)^{q/(q-1)}}{q(q\varepsilon)^{1/(q-1)}}\Big)|u|^q\Big\}.
$$

Therefore, combining (3.11) with (3.13), we have for $i\in S$

$$
\bar{L}U(x,y,i,t)\le c_1\vee c_3+U_1(x)+\varepsilon U_1(y),
$$

where $U_1(x)=|x|^q$. Obviously, $U_1(x)\le U(x,i)$. Consequently, by [1, Theorem 7.13, Page280], we conclude that the controlled system (3.9) has a unique global solution $x(t)$ with $\mathbb{E}|x(t)|^q<C_T$ for any $t\in[0,T]$.

Step 2: Uniform bound. Similarly, by virtue of (3.11) and (3.13), we derive for $i\in S$

$$
\bar{L}U(x,y,i,t)\le c-aU(x,i)+\varepsilon U(y,i)1_{\{i\in S_2\}}, \qquad (3.14)
$$

where $c := c_1 \vee c_3$, $a := 1/(2\theta_{\max})$. Then, by (3.12), we have

$$e^{-a\tau} + \varepsilon\tau < 1. \tag{3.15}$$

For any $t \geq 0$, there is a unique integer $k \geq 0$ such that $t \in [k\tau, (k+1)\tau)$. Letting $t_k = k\tau$ for $k = 0, 1, 2, \cdots$, we have $t - \pi_t = t_k$. By the generalised Itô formula, we derive that for $t \in [t_k, t_{k+1})$

$$e^{at}\mathbb{E}U(x(t), r(t)) = e^{at_k}\mathbb{E}U(x(t_k), r(t_k)) + \mathbb{E}\int_{t_k}^{t} e^{as}[aU(x(s), r(s)) + \bar{L}U(x(s), x(s-\pi(s)), r(s), s)]ds.$$

Using (3.14) yields

$$\begin{aligned}
&e^{at}\mathbb{E}U(x(t), r(t)) \\
&\quad\leq e^{at_k}\mathbb{E}U(x(t_k), r(t_k)) + \mathbb{E}\int_{t_k}^{t} e^{as}[c + \varepsilon U(x(s-\pi(s)), r(s))]1_{\{r(s)\in S_2\}}]ds \\
&\quad= e^{at_k}\mathbb{E}U(x(t_k), r(t_k)) + \mathbb{E}\int_{t_k}^{t} e^{as}[c + \varepsilon U(x(s-\pi(s)), r(s-\pi(s)))]1_{\{r(s-\pi(s))\in S_2\}}]ds \\
&\quad\leq e^{at_k}\mathbb{E}U(x(t_k), r(t_k)) + \mathbb{E}\int_{t_k}^{t} e^{as}[c + \varepsilon U(x(s-\pi(s)), r(s-\pi(s)))]ds \\
&\quad= e^{at_k}\mathbb{E}U(x(t_k), r(t_k)) + \mathbb{E}\int_{t_k}^{t} e^{as}[c + \varepsilon U(x(t_k), r(t_k))]ds \\
&\quad\leq e^{at_k}\mathbb{E}U(x(t_k), r(t_k)) + \Big(\frac{e^{at} - e^{at_k}}{a}\Big)[c + \varepsilon\mathbb{E}U(x(t_k), r(t_k))].
\end{aligned} \tag{3.16}$$

In particular

$$e^{at_{k+1}}\mathbb{E}U(x(t_{k+1}), r(t_{k+1})) \leq e^{at_k}\mathbb{E}U(x(t_k), r(t_k)) + \Big(\frac{e^{at_{k+1}} - e^{at_k}}{a}\Big)[c + \varepsilon\mathbb{E}U(x(t_k), r(t_k))].$$

This implies

$$\begin{aligned}
\mathbb{E}U(x(t_{k+1}), r(t_{k+1})) &\leq e^{-a\tau}\mathbb{E}U(x(t_k), r(t_k)) + \Big(\frac{1 - e^{-a\tau}}{a}\Big)[c + \varepsilon\mathbb{E}U(x(t_k), r(t_k))] \\
&\leq e^{-a\tau}\mathbb{E}U(x(t_k), r(t_k)) + \tau[c + \varepsilon\mathbb{E}U(x(t_k), r(t_k))] \\
&= c\tau + (e^{-a\tau} + \varepsilon\tau)\mathbb{E}U(x(t_k), r(t_k)).
\end{aligned}$$

Consequently, using (3.15) gives

$$\begin{aligned}
\mathbb{E}U(x(t_{k+1}), r(t_{k+1})) &\leq c\tau + (e^{-a\tau} + \varepsilon\tau)[c\tau + (e^{-a\tau} + \varepsilon\tau)\mathbb{E}U(x(t_{k-1}), r(t_{k-1}))] \\
&\leq c\tau[1 + (e^{-a\tau} + \varepsilon\tau) + \cdots + (e^{-a\tau} + \varepsilon\tau)^k] + (e^{-a\tau} + \varepsilon\tau)^{k+1}U(x(0), r(0)) \\
&\leq \frac{c\tau}{1 - (e^{-a\tau} + \varepsilon\tau)} + (|x(0)|^q + \theta_{\max}|x(0)|^{q+p-2}).
\end{aligned} \tag{3.17}$$

Furthermore, it follows from (3.16) that

$$\sup_{t_k \leq t \leq t_{k+1}} [e^{at}\mathbb{E}U(x(t), r(t))] \leq e^{at_k}\mathbb{E}U(x(t_k), r(t_k)) + \Big(\frac{e^{at_{k+1}} - e^{at_k}}{a}\Big)[c + \varepsilon\mathbb{E}U(x(t_k), r(t_k))].$$

This, together with (3.4) and (3.17), implies

$$\begin{aligned}\sup_{t_k\le t\le t_{k+1}} \mathbb{E}|x(t)|^q &\le \sup_{t_k\le t\le t_{k+1}} \mathbb{E}U(x(t),r(t))\\&\le \mathbb{E}U(x(t_k),r(t_k)) + \Big(\frac{e^{a\tau}-1}{a}\Big)[c+\varepsilon\mathbb{E}U(x(t_k),r(t_k))]\\&\le c\Big(\frac{e^{a\tau}-1}{a}\Big) + [1+\frac{\varepsilon}{a}(e^{a\tau}-1)]\mathbb{E}U(x(t_k),r(t_k))\\&\le c\Big(\frac{e^{a\tau}-1}{a}\Big) + [1+\frac{\varepsilon}{a}(e^{a\tau}-1)]\Big(\frac{c\tau}{1-(e^{-a\tau}+\varepsilon\tau)} + |x(0)|^q + \theta_{\max}|x(0)|^{q+p-2}\Big).\end{aligned}$$

Since k is arbitrary, the required assertion (3.10) holds. □

4 Stabilisation

4.1 H_∞ stabilisation and asymptotic stabilisation

Let us now make a remark and introduce some constants which will be used in our new Rule 4.2 below.

Remark 4.1. By Assumption 2.2, it is obvious that

$$x^T f(x,i,t) + \frac{p+q_1-2}{2}|g(x,i,t)|^2 \le \bar\beta_i|x|^2$$

and

$$\mathcal{B}_2 := -(p+q_1-1)\mathrm{diag}(\bar\beta_1,...,\bar\beta_{N_1}) - (\gamma_{ij})_{i,j\in S_1} \tag{4.1}$$

is a nonsingular M-matrix. Similarly, by (2.3), there exist constants $\beta_i, \tilde\beta_i$ $(i\in S_1)$ with $\beta_i\le\tilde\beta_i\le\bar\beta_i$ such that for all $(x,i,t)\in\mathbb{R}^n\times S_1\times R_+$

$$\begin{cases} x^T f(x,i,t) + \frac{1}{2}|g(x,i,t)|^2 \le \beta_i|x|^2,\\ x^T f(x,i,t) + \frac{q_1}{2}|g(x,i,t)|^2 \le \tilde\beta_i|x|^2.\end{cases} \tag{4.2}$$

Recalling $u(x,i,t)=0, i\in S_1$, we hence have for all $(x,i,t)\in\mathbb{R}^n\times S_1\times R_+$

$$\begin{cases} x^T[f(x,i,t)+u(x,i,t)] + \frac{1}{2}|g(x,i,t)|^2 \le \beta_i|x|^2,\\ x^T[f(x,i,t)+u(x,i,t)] + \frac{q_1}{2}|g(x,i,t)|^2 \le \tilde\beta_i|x|^2,\\ x^T[f(x,i,t)+u(x,i,t)] + \frac{p+q_1-2}{2}|g(x,i,t)|^2 \le \bar\beta_i|x|^2.\end{cases} \tag{4.3}$$

Rule 4.2. Design a control function $u:\mathbb{R}^n\times S_2\times R_+\to\mathbb{R}^n$ such that there exist constants $\alpha_i,\tilde\alpha_i,\beta_i,\tilde\beta_i(i\in S_2)$ with $\alpha_i\ge\tilde\alpha_i\ge\alpha$ and $\beta_i\le\tilde\beta_i\le\epsilon_i$ for both

$$\begin{cases} x^T[f(x,i,t)+u(x,i,t)] + \frac{1}{2}|g(x,i,t)|^2 \le -\alpha_i|x|^p + \beta_i|x|^2,\\ x^T[f(x,i,t)+u(x,i,t)] + \frac{q_1}{2}|g(x,i,t)|^2 \le -\tilde\alpha_i|x|^p + \tilde\beta_i|x|^2\end{cases} \tag{4.4}$$

to hold for all $(x,i,t) \in \mathbb{R}^n \times S_2 \times R_+$. Moreover, the constants $\beta_i, \tilde{\beta}_i (i \in S_2)$ here along with $\beta_i, \tilde{\beta}_i (i \in S_1)$ in (4.2) make the following two M-matrices

$$\begin{cases} \mathcal{B} := -2\text{diag}(\beta_1, ..., \beta_N) - (\gamma_{ij})_{i,j\in S}, \\ \mathcal{B}_1 := -(q_1+1)\text{diag}(\tilde{\beta}_1, ..., \tilde{\beta}_N) - (\gamma_{ij})_{i,j\in S} \end{cases} \tag{4.5}$$

nonsingular.

Let us explain that there exist lots of such functions satisfying Rule 4.2 under Assumption 2.2. For example, take $\bar{u}(x,i,t) = -Lx$ for $i \in S_2$ and $\bar{u}(x,i,t) = 0$ for $i \in S_1$, where L is a positive constant, then

$$x^T \bar{u}(x,i,t) = -L|x|^2, \quad \forall\ (x,i,t) \in \mathbb{R}^n \times S_2 \times R_+.$$

By (2.5), we have for $(x,i,t) \in \mathbb{R}^n \times S_2 \times R_+$

$$x^T[f(x,i,t) + \bar{u}(x,i,t)] + \frac{1}{2}|g(x,i,t)|^2 \le -\alpha|x|^p + (\epsilon_i - L)|x|^2,$$
$$x^T[f(x,i,t) + \bar{u}(x,i,t)] + \frac{q_1}{2}|g(x,i,t)|^2 \le -\alpha|x|^p + (\epsilon_i - L)|x|^2.$$

In addition, since $\mathcal{D}$ is a nonsingular M-matrix and $\beta_i \le \tilde{\beta}_i \le \bar{\beta}_i$, there exists a constant L large enough such that the following matrices

$$\mathcal{B} := -2\text{diag}(\beta_1, ..., \beta_{N_1}, \epsilon_{N_1+1} - L, ..., \epsilon_N - L) - (\gamma_{ij})_{i,j\in S}$$
$$\mathcal{B}_1 := -(q_1+1)\text{diag}(\tilde{\beta}_1, ..., \tilde{\beta}_{N_1}, \epsilon_{N_1+1} - L, ..., \epsilon_N - L) - (\gamma_{ij})_{i,j\in S}$$

are nonsingular M-matrix. That is, the control function $\bar{u}$ satisfies Rule 4.2. For convenience, let $b := \max_{i\in S_1} b_i$, b_i be the ith row sum of $\mathcal{B}_2^{-1}$, and

$$\begin{cases} (\eta_1, \eta_2, ..., \eta_N)^T = \mathcal{B}^{-1}(1, \cdots, 1)^T, \\ (\tilde{\eta}_1, \tilde{\eta}_2, \cdots, \tilde{\eta}_N)^T = \mathcal{B}_1^{-1}(1, \cdots, 1)^T, \\ D = \dfrac{\min_{i\in S_2}(q_1+1)\tilde{\alpha}_i\tilde{\eta}_i}{\max_{i\in S_2}(\sum_{j\in S_1}\gamma_{ij})b + 1}, \\ (\bar{\eta}_1, \bar{\eta}_2, ..., \bar{\eta}_{N_1})^T = D\mathcal{B}_2^{-1}(1, \cdots, 1)^T, \end{cases} \tag{4.6}$$

since $\mathcal{B}$, $\mathcal{B}_1$ are nonsingular M-matrices, then all $\eta_i, \tilde{\eta}_i$ are positive. Note that $\tilde{\alpha}_i \ge \alpha > 0$ for $i \in S_2$, $D > 0$. Moreover, $\mathcal{B}_2$ is a nonsingular M-matrix, all $\bar{\eta}_i$ are also positive. Define

$$V(x,i) = \eta_i|x|^2 + \tilde{\eta}_i|x|^{q_1+1} + \bar{\eta}_i|x|^{p+q_1-1}1_{\{i\in S_1\}} \tag{4.7}$$

and

$$L_1V(x,i,t) = V_x(x,i)[f(x,i,t) + u(x,i,t)] + \frac{1}{2}\text{trace}[g^T(x,i,t)V_{xx}(x,i)g(x,i,t)] + \sum_{j\in S}\gamma_{ij}V(x,j). \tag{4.8}$$

Lemma 4.3. *Let Assumptions 2.1,2.2 and Rules 3.2, 4.2 hold. Then there exist positive constants ρ_0–ρ_3 such that for each $i \in S$*

$$L_1V(x,i,t) + \rho_1(2\eta_i|x| + (q_1+1)\tilde{\eta}_i|x|^{q_1})^2 + \rho_2|f(x,i,t)|^2 + \rho_3|g(x,i,t)|^2 \le -\rho_0|x|^2 - |x|^{q_1+1}.$$

Proof. Let

$$W(x,i,t) = L_1V(x,i,t) + \rho_1(2\eta_i|x| + (q_1+1)\tilde{\eta}_i|x|^{q_1})^2 + \rho_2|f(x,i,t)|^2 + \rho_3|g(x,i,t)|^2. \tag{4.9}$$

Hence it suffices to prove that

$$W(x,i,t) \le -\rho_0|x|^2 - |x|^{q_1+1}. \tag{4.10}$$

We divide the following proof into two steps.
Step 1: Estimation of L_1V. It is easy to see from (4.8) and (4.7) that

$$\begin{aligned}
L_1V(x,i,t) \;=\;& 2\eta_i\big[x^T[f(x,i,t)+u(x,i,t)] + \frac{1}{2}|g(x,i,t)|^2\big] \\
&+(q_1+1)\tilde{\eta}_i|x|^{q_1-1}x^T[f(x,i,t)+u(x,i,t)] + \frac{q_1+1}{2}\tilde{\eta}_i|x|^{q_1-1}|g(x,i,t)|^2 \\
&+\frac{(q_1+1)(q_1-1)}{2}\tilde{\eta}_i|x|^{q_1-3}|x^Tg(x,i,t)|^2 \\
&+\Big\{(p+q_1-1)\bar{\eta}_i|x|^{p+q_1-3}x^T[f(x,i,t)+u(x,i,t)] + \frac{p+q_1-1}{2}\bar{\eta}_i|x|^{p+q_1-3}|g(x,i,t)|^2 \\
&+\frac{(p+q_1-1)(p+q_1-3)}{2}\bar{\eta}_i|x|^{p+q_1-5}|x^Tg(x,i,t)|^2\Big\}1_{\{i\in S_1\}} \\
&+\sum_{j\in S}\gamma_{ij}\eta_j|x|^2 + \sum_{j\in S}\gamma_{ij}\tilde{\eta}_j|x|^{q_1+1} + \sum_{j\in S_1}\gamma_{ij}\bar{\eta}_j|x|^{p+q_1-1} \\
\le\;& 2\eta_i[x^T[f(x,i,t)+u(x,i,t)] + \frac{1}{2}|g(x,i,t)|^2] \\
&+(q_1+1)\tilde{\eta}_i|x|^{q_1-1}\big[x^T[f(x,i,t)+u(x,i,t)] + \frac{q_1}{2}|g(x,i,t)|^2\big] \\
&+\Big\{(p+q_1-1)\bar{\eta}_i|x|^{p+q_1-3}\big[x^T[f(x,i,t)+u(x,i,t)] + \frac{p+q_1-2}{2}|g(x,i,t)|^2\big]\Big\}1_{\{i\in S_1\}} \\
&+\sum_{j\in S}\gamma_{ij}\eta_j|x|^2 + \sum_{j\in S}\gamma_{ij}\tilde{\eta}_j|x|^{q_1+1} + \sum_{j\in S_1}\gamma_{ij}\bar{\eta}_j|x|^{p+q_1-1}.
\end{aligned}$$

By (4.3) and (4.4), we have

$$\begin{aligned}
L_1V(x,i,t) \;\le\;& 2\eta_i[(\beta_i|x|^2)1_{\{i\in S_1\}} + (\beta_i|x|^2 - \alpha_i|x|^p)1_{\{i\in S_2\}}] \\
&+\Big\{(q_1+1)\tilde{\eta}_i|x|^{q_1-1}\big[(\tilde{\beta}_i|x|^2)1_{\{i\in S_1\}} + (-\tilde{\alpha}_i|x|^p + \tilde{\beta}_i|x|^2)1_{\{i\in S_2\}}\big]\Big\} \\
&+\Big\{(p+q_1-1)\bar{\eta}_i|x|^{p+q_1-3}\big[\bar{\beta}_i|x|^2\big]\Big\}1_{\{i\in S_1\}} \\
&+\sum_{j\in S}\gamma_{ij}\eta_j|x|^2 + \sum_{j\in S}\gamma_{ij}\tilde{\eta}_j|x|^{q_1+1} + \sum_{j\in S_1}\gamma_{ij}\bar{\eta}_j|x|^{p+q_1-1} \\
=\;& \Big\{2\eta_i\beta_i|x|^2 + \sum_{j\in S}\gamma_{ij}\eta_j|x|^2 + (q_1+1)\tilde{\eta}_i\tilde{\beta}_i|x|^{q_1+1} + \sum_{j\in S}\gamma_{ij}\tilde{\eta}_j|x|^{q_1+1} \\
&+(p+q_1-1)\bar{\eta}_i\bar{\beta}_i|x|^{p+q_1-1} + \sum_{j\in S_1}\gamma_{ij}\bar{\eta}_j|x|^{p+q_1-1}\Big\}1_{\{i\in S_1\}} \\
&+\Big\{2\eta_i\beta_i|x|^2 + \sum_{j\in S}\gamma_{ij}\eta_j|x|^2 - 2\alpha_i\eta_i|x|^p - (q_1+1)\tilde{\alpha}_i\tilde{\eta}_i|x|^{p+q_1-1}
\end{aligned}$$

$$+(q_1+1)\tilde{\eta}_i\tilde{\beta}_i|x|^{q_1+1}+\sum_{j\in S}\gamma_{ij}\tilde{\eta}_j|x|^{q_1+1}+\sum_{j\in S_1}\gamma_{ij}\bar{\eta}_j|x|^{p+q_1-1}\Big\}1_{\{i\in S_2\}}.$$

By virtue of (4.1), (4.5) and (4.6), we obtain

$$\begin{cases} 2\eta_i\beta_i+\sum_{j\in S}\gamma_{ij}\eta_j=-1 & \text{for } i\in S,\\ (q_1+1)\tilde{\eta}_i\tilde{\beta}_i+\sum_{j\in S}\gamma_{ij}\tilde{\eta}_j=-1 & \text{for } i\in S,\\ (p+q_1-1)\bar{\eta}_i\bar{\beta}_i+\sum_{j\in S_1}\gamma_{ij}\bar{\eta}_j=-D & \text{for } i\in S_1. \end{cases}$$

We hence have

$$\begin{aligned} L_1V(x,i,t) \;\le\; & \Big\{-|x|^2-|x|^{q_1+1}-D|x|^{p+q_1-1}\Big\}1_{\{i\in S_1\}} \\ & +\Big\{\sum_{j\in S_1}\gamma_{ij}\bar{\eta}_j|x|^{p+q_1-1}-|x|^2-2\alpha_i\eta_i|x|^p-(q_1+1)\tilde{\alpha}_i\tilde{\eta}_i|x|^{p+q_1-1}-|x|^{q_1+1}\Big\}1_{\{i\in S_2\}}. \end{aligned}$$

By (4.6), we have

$$\tilde{\alpha}_i\ge\Big(\frac{b\sum_{j\in S_1}\gamma_{ij}+1}{(q_1+1)\tilde{\eta}_i}\Big)D\quad\forall\, i\in S_2.$$

Noting that $\bar{\eta}_i\le Db$ for $i\in S_1$, then

$$\tilde{\alpha}_i\ge\frac{\sum_{j\in S_1}\gamma_{ij}\bar{\eta}_j+D}{(q_1+1)\tilde{\eta}_i}\quad\forall\, i\in S_2,$$

which implies

$$-(q_1+1)\tilde{\alpha}_i\tilde{\eta}_i+\sum_{j\in S_1}\gamma_{ij}\bar{\eta}_j\le -D\quad\forall\, i\in S_2.$$

Therefore, we have

$$\begin{aligned} L_1V(x,i,t) \;\le\; & \Big\{-|x|^2-|x|^{q_1+1}-D|x|^{p+q_1-1}\Big\}1_{\{i\in S_1\}} \\ & +\Big\{-|x|^2-2\alpha_i\eta_i|x|^p-|x|^{q_1+1}-D|x|^{p+q_1-1}\Big\}1_{\{i\in S_2\}} \\ \le\; & -|x|^2-|x|^{q_1+1}-D|x|^{p+q_1-1}. \end{aligned}\tag{4.11}$$

Step 2: Proof of $W(x,i,t)\le-\rho_0|x|^2-|x|^{q_1+1}$. Substituting (4.11) into (4.9) gives

$$W(x,i,t)\le-|x|^2-|x|^{q_1+1}-D|x|^{p+q_1-1}+\rho_1(2\eta_i|x|+(q_1+1)\tilde{\eta}_i|x|^{q_1})^2+\rho_2|f(x,i,t)|^2+\rho_3|g(x,i,t)|^2.$$

By Assumption 2.1, we have for $i\in S$

$$|f(x,i,t)|\le K(|x|+|x|^{q_1})\quad\text{and}\quad|g(x,i,t)|\le K(|x|+|x|^{q_2}).$$

By $(a+b)^2\le2(a^2+b^2)$, we arrive at

$$W(x,i,t)\;\le\;-|x|^2-|x|^{q_1+1}-D|x|^{p+q_1-1}+\rho_1(8\eta_i^2|x|^2+2(q_1+1)^2\tilde{\eta}_i^2|x|^{2q_1})$$

$$+2\rho_2 K^2(|x|^2+|x|^{2q_1})+2\rho_3 K^2(|x|^2+|x|^{2q_2}).$$

In addition, by Assumption 2.2, we have $2(q_1\vee q_2)\le p+q_1-1$, and hence $(|x|^{2q_1}\vee|x|^{2q_2})\le |x|^2+|x|^{p+q_1-1}$. Therefore, we have

$$\begin{aligned} W(x,i,t) \le\; & -\big[1-(8\rho_1\eta_i^2+2\rho_1(q_1+1)^2\tilde{\eta}_i^2+4\rho_2K^2+4\rho_3K^2)\big]|x|^2-|x|^{q_1+1} \\ & -\big[D-(2\rho_1(q_1+1)^2\tilde{\eta}_i^2+2\rho_2K^2+2\rho_3K^2)\big]|x|^{p+q_1-1}. \end{aligned}$$

we can choose appropriate positive constants $\rho_j(1\le j\le 3)$ sufficiently small such that

$$\begin{aligned} &8\rho_1\max_{i\in S}\eta_i^2+2\rho_1(q_1+1)^2\max_{i\in S}\tilde{\eta}_i^2+4(\rho_2+\rho_3)K^2\le 0.5, \\ &2\rho_1(q_1+1)^2\max_{i\in S}\tilde{\eta}_i^2+2(\rho_2+\rho_3)K^2\le D. \end{aligned} \tag{4.12}$$

We then have

$$W(x,i,t)\le -0.5|x|^2-|x|^{q_1+1}.$$

Let $\rho_0\in(0,0.5]$, then (4.10) holds and the proof is complete. □

Theorem 4.4. *Let conditions of Lemma 4.3 hold. Assume further*

$$\tau<\frac{\sqrt{\rho_0\rho_1}}{2\kappa^2}\quad\textit{and}\quad \tau\le\frac{\sqrt{\rho_1\rho_2}}{\sqrt{2}\kappa}\wedge\frac{\rho_1\rho_3}{\kappa^2}\wedge\frac{1}{4\kappa}, \tag{4.13}$$

where ρ_0-ρ_3 come from Lemma 4.3, then the solution of the controlled system (3.9) satisfies

$$\int_0^\infty \mathbb{E}|x(t)|^{\bar{q}}dt<C,\quad \forall\ \bar{q}\in[2,q_1+1], \tag{4.14}$$

and

$$\lim_{t\to\infty}\mathbb{E}|x(t)|^{\bar{q}}=0,\ \forall\ \bar{q}\in[2,q). \tag{4.15}$$

Proof. We divide this proof into three steps.

Step 1: Define two segment processes $\hat{x}_t=\{x(t+s):-2\tau\le s\le 0\}$ and $\hat{r}_t=\{r(t+s):-2\tau\le s\le 0\}$ for $t\ge 0$. To ensure that $\hat{x}_t$ and $\hat{r}_t$ are well defined on $0\le t\le 2\tau$, we set $x(s)=x(0)$ and $r(s)=r(0)$ for $s\in[-2\tau,0)$. Let

$$\widehat{V}(\hat{x}_t,\hat{r}_t,t)=V(x(t),r(t))+J(t),\quad t\ge 0, \tag{4.16}$$

where V comes from (4.7) and

$$J(t):=\psi\int_{-\tau}^0\int_{t+s}^t[\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2+|g(x(v),r(v),v)|^2]dvds,$$

where ψ is a positive constant to be determined later. And, we set $f(x,i,v)=f(x,i,0)$, $u(x,i,v)=u(x,i,0)$, $g(x,i,v)=g(x,i,0)$ for $(x,i,v)\in\mathbb{R}^n\times S\times[-2\tau,0]$. We claim that $\widehat{V}(\hat{x}_t,\hat{r}_t,t)$ is an Itô process on $t\ge 0$. In fact, by the generalised Itô formula, for $t\ge 0$ we have

$$dV(x(t),r(t))=\mathcal{L}V(x(t),x(\delta_t),r(t),t)dt+dM(t), \tag{4.17}$$

where $M(t)$ is a continuous local martingale with $M(0) = 0$ and $\mathcal{L}V : \mathbb{R}^n \times \mathbb{R}^n \times S \times R_+ \to R$ is defined by

$$\mathcal{L}V(x, y, i, t) = V_x(x, i)[f(x, i, t) + u(y, i, t)] + \frac{1}{2}trace[g^T(x, i, t)V_{xx}(x, i)g(x, i, t)] + \sum_{j\in S}\gamma_{ij}V(x, j).$$

We compute

$$\begin{aligned} \mathcal{L}V(x, y, i, t) \;\; = \;\; & 2\eta_i\big[x^T[f(x, i, t) + u(y, i, t)] + \frac{1}{2}|g(x, i, t)|^2\big] \\ & +(q_1+1)\tilde{\eta}_i|x|^{q_1-1}x^T[f(x, i, t) + u(y, i, t)] + \frac{q_1+1}{2}\tilde{\eta}_i|x|^{q_1-1}|g(x, i, t)|^2 \\ & +\frac{(q_1+1)(q_1-1)}{2}\tilde{\eta}_i|x|^{q_1-3}|x^T g(x, i, t)|^2 \\ & +\Big\{(p+q_1-1)\bar{\eta}_i|x|^{p+q_1-3}x^T[f(x, i, t) + u(y, i, t)] \\ & +\frac{p+q_1-1}{2}\bar{\eta}_i|x|^{p+q_1-3}|g(x, i, t)|^2 \\ & +\frac{(p+q_1-1)(p+q_1-3)}{2}\bar{\eta}_i|x|^{p+q_1-5}|x^T g(x, i, t)|^2\Big\}1_{\{i\in S_1\}} \\ & +\sum_{j\in S}\gamma_{ij}\eta_j|x|^2 + \sum_{j\in S}\gamma_{ij}\tilde{\eta}_j|x|^{q_1+1} + \sum_{j\in S_1}\gamma_{ij}\bar{\eta}_j|x|^{q_1+p-1}. \end{aligned} \tag{4.18}$$

On the other hand, it is easy to see that

$$\begin{aligned} & d\Big(\psi\int_{-\tau}^{0}\int_{t+s}^{t}[\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 + |g(x(v), r(v), v)|^2]dvds\Big) \\ & = \psi\tau[\tau|f(x(t), r(t), t) + u(x(\delta_t), r(t), t)|^2 + |g(x(t), r(t), t)|^2] \\ & \quad -\psi\int_{t-\tau}^{t}[\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 + |g(x(v), r(v), v)|^2]dv. \end{aligned} \tag{4.19}$$

Substituting (4.17) and (4.19) into (4.16) yields

$$\begin{aligned} d\widehat{V}(\hat{x}_t, \hat{r}_t, t) \;\; \le \;\; & \mathcal{L}V(x(t), x(\delta_t), r(t), t)dt + dM(t) \\ & +\psi\tau[\tau|f(x(t), r(t), t) + u(x(\delta_t), r(t), t)|^2 + |g(x(t), r(t), t)|^2] \\ & -\psi\int_{t-\tau}^{t}[\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 + |g(x(v), r(v), v)|^2]dv. \end{aligned} \tag{4.20}$$

Furthermore, recalling $u(x, i, t) = 0, i \in S_1$, it follows from (4.18) and (4.8) that

$$\mathcal{L}V(x, y, i, t) \le L_1V(x, i, t) + \Big\{(2\eta_i + (q_1+1)\tilde{\eta}_i|x|^{q_1-1})x^T[u(y, i, t) - u(x, i, t)]\Big\}1_{\{i\in S_2\}}.$$

This, together with (4.20), implies that

$$d\widehat{V}(\hat{x}_t, \hat{r}_t, t) \le \mathbb{L}\widehat{V}(\hat{x}_t, \hat{r}_t, t) + dM(t), \tag{4.21}$$

where

$$\mathbb{L}\widehat{V}(\hat{x}_t, \hat{r}_t, t) \;\; = \;\; L_1V(x(t), r(t), t)$$

$$+\Big\{(2\eta_{r(t)}+(q_1+1)\tilde{\eta}_{r(t)}|x(t)|^{q_1-1})x^T(t)[u(x(\delta_t),r(t),t)-u(x(t),r(t),t)]\Big\}1_{\{r(t)\in S_2\}}$$
$$+\psi\tau[\tau|f(x(t),r(t),t)+u(x(\delta_t),r(t),t)|^2+|g(x(t),r(t),t)|^2]$$
$$-\psi\int_{t-\tau}^t[\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2+|g(x(v),r(v),v)|^2]dv. \quad (4.22)$$

Then by virtue of Theorem 3.3, Assumption 2.1, 2.2 and Rule 3.2, we have

$$\sup_{0\le t<\infty}\mathbb{E}|\mathbb{L}\widehat{V}(\hat{x}_t,\hat{r}_t,t)|<C. \quad (4.23)$$

Step 2: Let us now estimate $\mathbb{L}\widehat{V}(\hat{x}_t,\hat{r}_t,t)$. Choose $\psi=\kappa^2/\rho_1$. Recalling $|u(x,i,t)|\le\kappa|x|, i\in S_2$, we obtain

$$\begin{aligned}
&\Big\{(2\eta_{r(t)}+(q_1+1)\tilde{\eta}_{r(t)}|x(t)|^{q_1-1})x^T(t)[u(x(\delta_t),r(t),t)-u(x(t),r(t),t)]\Big\}1_{\{r(t)\in S_2\}}\\
\le\ &\Big\{(2\eta_{r(t)}|x(t)|+(q_1+1)\tilde{\eta}_{r(t)}|x(t)|^{q_1})\kappa|x(\delta_t)-x(t)|\Big\}1_{\{r(t)\in S_2\}}\\
\le\ &\Big\{\rho_1(2\eta_{r(t)}|x(t)|+(q_1+1)\tilde{\eta}_{r(t)}|x(t)|^{q_1})^2+\frac{\kappa^2}{4\rho_1}|x(\delta_t)-x(t)|^2\Big\}1_{\{r(t)\in S_2\}}.
\end{aligned}$$

From (4.13), we see that $2\psi\tau^2\le\rho_2$ and $\psi\tau\le\rho_3$. Noting that $u(x,i,t)=0$ for $i\in S_1$ and $|u(x,i,t)|\le\kappa|x|$ for $i\in S_2$, it follows from Lemma 4.3 and (4.22) that

$$\begin{aligned}
&\mathbb{L}\widehat{V}(\hat{x}_t,\hat{r}_t,t)\\
\le\ &L_1V(x(t),r(t),t)+\Big\{\rho_1(2\eta_{r(t)}|x(t)|+(q_1+1)\tilde{\eta}_{r(t)}|x(t)|^{q_1})^2+\frac{\kappa^2}{4\rho_1}|x(\delta_t)-x(t)|^2\Big\}1_{\{r(t)\in S_2\}}\\
&+\rho_2|f(x(t),r(t),t)|^2+\Big\{2\psi\tau^2\kappa^2|x(\delta_t)|^2\Big\}1_{\{r(t)\in S_2\}}+\rho_3|g(x(t),r(t),t)|^2\\
&-\frac{\kappa^2}{\rho_1}\int_{t-\tau}^t[\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2+|g(x(v),r(v),v)|^2]dv\\
\le\ &-\rho_0|x(t)|^2-|x(t)|^{q_1+1}+\Big\{\frac{\kappa^2}{4\rho_1}|x(\delta_t)-x(t)|^2+\frac{2\tau^2\kappa^4}{\rho_1}|x(\delta_t)|^2\Big\}1_{\{r(t)\in S_2\}}\\
&-\frac{\kappa^2}{\rho_1}\int_{t-\tau}^t[\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2+|g(x(v),r(v),v)|^2]dv.
\end{aligned}$$

In light of (4.13), we have

$$\frac{2\tau^2\kappa^4}{\rho_1}|x(\delta_t)|^2\le\frac{4\tau^2\kappa^4}{\rho_1}|x(t)|^2+\frac{\kappa^2}{4\rho_1}|x(\delta_t)-x(t)|^2.$$

We therefore have

$$\begin{aligned}
\mathbb{L}\widehat{V}(\hat{x}_t,\hat{r}_t,t)\ \le\ &-\rho_0|x(t)|^2-|x(t)|^{q_1+1}+\Big\{\frac{\kappa^2}{2\rho_1}|x(\delta_t)-x(t)|^2+\frac{4\tau^2\kappa^4}{\rho_1}|x(t)|^2\Big\}1_{\{r(t)\in S_2\}}\\
&-\frac{\kappa^2}{\rho_1}\int_{t-\tau}^t[\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2+|g(x(v),r(v),v)|^2]dv\\
\le\ &-(\rho_0-\frac{4\tau^2\kappa^4}{\rho_1})|x(t)|^2-|x(t)|^{q_1+1}+\Big\{\frac{\kappa^2}{2\rho_1}|x(\delta_t)-x(t)|^2\Big\}1_{\{r(t)\in S_2\}}
\end{aligned}$$

$$-\frac{\kappa^2}{\rho_1}\int_{t-\tau}^{t}[\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2+|g(x(v),r(v),v)|^2]dv. \quad (4.24)$$

Step 3: For any initial value x_0, choose l_0 large enough such that $|x_0|\le l_0$. For each integer $l\ge l_0$, define a stopping time

$$\zeta_l=\inf\{t\ge 0:|x(t)|\ge l\},$$

where we use the convention that $\inf\emptyset=\infty$. Theorem 3.3 indicates that ζ_l is increasing to infinity with probability 1 as $l\to\infty$. By the generalized Itô formula, it follows from (4.21) that

$$\mathbb{E}\widehat{V}(\hat{x}_{t\wedge\zeta_l},\hat{r}_{t\wedge\zeta_l},t\wedge\zeta_l)\le\widehat{V}(\hat{x}_0,\hat{r}_0,0)+\mathbb{E}\int_0^{t\wedge\zeta_l}\mathbb{L}\widehat{V}(\hat{x}_s,\hat{r}_s,s)ds,\qquad t\ge 0.$$

By virtue of (4.23), the dominated convergence theorem and the Fubini theorem, letting $l\to\infty$ yields

$$\mathbb{E}\widehat{V}(\hat{x}_t,\hat{r}_t,t)\le\widehat{V}(\hat{x}_0,\hat{r}_0,0)+\int_0^t\mathbb{E}\mathbb{L}\widehat{V}(\hat{x}_s,\hat{r}_s,s)ds. \quad (4.25)$$

By (4.24), we have

$$\begin{aligned}\mathbb{E}\mathbb{L}\widehat{V}(\hat{x}_s,\hat{r}_s,s)\ \le\ & -(\rho_0-\frac{4\tau^2\kappa^4}{\rho_1})\mathbb{E}|x(s)|^2-\mathbb{E}|x(s)|^{q_1+1}+\frac{\kappa^2}{2\rho_1}\mathbb{E}\Big\{|x(\delta_s)-x(s)|^2 1_{\{r(s)\in S_2\}}\Big\}\\ & -\frac{\kappa^2}{\rho_1}\mathbb{E}\int_{s-\tau}^{s}[\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2+|g(x(v),r(v),v)|^2]dv. \quad (4.26)\end{aligned}$$

On the other hand, by (3.9), we have

$$\begin{aligned}&\mathbb{E}\Big\{|x(\delta_s)-x(s)|^2 1_{\{r(s)\in S_2\}}\Big\}\\ \le\ & \mathbb{E}|x(\delta_s)-x(s)|^2\\ =\ & \mathbb{E}\Big|\int_{\delta_s}^{s}[f(x(v),r(v),v)+u(x(\delta_v),r(v),v)]dv+\int_{\delta_s}^{s}g(x(v),r(v),v)dB(v)\Big|^2\\ \le\ & 2\mathbb{E}\int_{\delta_s}^{s}[\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2+|g(x(v),r(v),v)|^2]dv\\ \le\ & 2\mathbb{E}\int_{s-\tau}^{s}[\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2+|g(x(v),r(v),v)|^2]dv. \quad (4.27)\end{aligned}$$

Substituting (4.26) into (4.25) yields

$$\mathbb{E}\widehat{V}(\hat{x}_t,\hat{r}_t,t))\le\widehat{V}(\hat{x}_0,\hat{r}_0,0)-\int_0^t(\rho_0-\frac{4\tau^2\kappa^4}{\rho_1})\mathbb{E}|x(s)|^2ds-\int_0^t\mathbb{E}|x(s)|^{q_1+1}ds.$$

By virtue of (4.13), $\rho_0>4\tau^2\kappa^4/\rho_1$, we have

$$\int_0^t\mathbb{E}|x(s)|^2ds<\frac{\widehat{V}(\hat{x}_0,\hat{r}_0,0)}{(\rho_0-\frac{4\tau^2\kappa^4}{\rho_1})},\qquad\int_0^t\mathbb{E}|x(s)|^{q_1+1}ds<\frac{\widehat{V}(\hat{x}_0,\hat{r}_0,0)}{(\rho_0-\frac{4\tau^2\kappa^4}{\rho_1})}.$$

This, together with $\mathbb{E}|x(s)|^{\bar{q}}\le\mathbb{E}|x(s)|^2+\mathbb{E}|x(s)|^{q_1+1}$ for any $\bar{q}\in[2,q_1+1]$, yields that

$$\int_0^\infty\mathbb{E}|x(s)|^{\bar{q}}ds<C,$$

which is the desired assertion (4.14). Furthermore, following the line to derive the Theorem 4.5 in [17], we can derive the required conclusion (4.15). □

4.2 Exponential stabilisation

In this section, we will show that the controlled system (3.9) is stable in the sense of both $L^{\bar{q}}$ ($\bar{q} \in [2, q)$) and almost sure exponential stability for proper control function u and sufficient small τ. For this purpose, let us present a lemma and introduce some constants which will be used in the Theorem 4.6 below.

Lemma 4.5. *Let Assumptions 2.1,2.2 and Rules 3.2, 4.2 hold. There exist positive constants $\bar{\rho}_0$–$\bar{\rho}_5$ with $\bar{\rho}_j \le \rho_j (j = 0, 1, 2, 3)$ and function H satisfying $\bar{\rho}_4|x|^{p+q_1-1} \le H(x) \le \bar{\rho}_5(1 + |x|^{p+q_1-1})$ such that for each $i \in S$*

$$L_1V(x, i, t) + \bar{\rho}_1(2\eta_i|x| + (q_1 + 1)\tilde{\eta}_i|x|^{q_1})^2 + \bar{\rho}_2|f(x, i, t)|^2 + \bar{\rho}_3|g(x, i, t)|^2 \le -\bar{\rho}_0|x|^2 - H(x).$$

Proof. Let

$$\bar{W}(x, i, t) = L_1V(x, i, t) + \bar{\rho}_1(2\eta_i|x| + (q_1 + 1)\tilde{\eta}_i|x|^{q_1})^2 + \bar{\rho}_2|f(x, i, t)|^2 + \bar{\rho}_3|g(x, i, t)|^2.$$

Following the line to prove Lemma 4.3 and similar to (4.12), we choose appropriate positive constants $\bar{\rho}_j$ with $\bar{\rho}_j \le \rho_j (1 \le j \le 3)$ such that

$$\begin{aligned} &8\bar{\rho}_1 \max_{i\in S} \eta_i^2 + 2\bar{\rho}_1(q_1 + 1)^2 \max_{i\in S} \tilde{\eta}_i^2 + 4(\bar{\rho}_2 + \bar{\rho}_3)K^2 \le 0.5, \\ &2\bar{\rho}_1(q_1 + 1)^2 \max_{i\in S} \tilde{\eta}_i^2 + 2(\bar{\rho}_2 + \bar{\rho}_3)K^2 \le 0.5D. \end{aligned}$$

We then have

$$\bar{W}(x, i, t) \le -0.5|x|^2 - |x|^{q_1+1} - 0.5D|x|^{p+q_1-1}.$$

Let $H(x) = |x|^{q_1+1} + 0.5D|x|^{p+q_1-1}$, $\bar{\rho}_4 \in (0, 0.5D]$, $\bar{\rho}_5 \in [0.5D + 1, \infty)$. Recalling $(q_1 + 1) \le (p + q_1 - 1)$, we have

$$\bar{\rho}_4|x|^{p+q_1-1} \le H(x) \le \bar{\rho}_5(1 + |x|^{p+q_1-1}).$$

Let $\bar{\rho}_0 = \rho_0$, Lemma 4.3 shows $\bar{W}(x, i, t) \le -\bar{\rho}_0|x|^2 - H(x)$ and the desired assertion follows. □

Theorem 4.6. *Let the conditions of Lemma 4.5 hold. Assume further*

$$\tau < \frac{\sqrt{\bar{\rho}_0\bar{\rho}_1}}{2\kappa^2} \quad and \quad \tau \le \frac{\sqrt{\bar{\rho}_1\bar{\rho}_2}}{\sqrt{2}\kappa} \wedge \frac{\bar{\rho}_1\bar{\rho}_3}{\kappa^2} \wedge \frac{1}{4\sqrt{2}\kappa}, \tag{4.28}$$

where $\bar{\rho}_0$–$\bar{\rho}_3$ come from Lemma 4.5. then the solution of the controlled system (3.9) satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log(E|x(t)|^{\bar{q}}) < 0, \ \forall \ \bar{q} \in [2, q) \tag{4.29}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t)|) < 0, \ a.s.. \tag{4.30}$$

Proof. Let $\widehat{V}(\hat{x}_t, \hat{r}_t, t)$ be given by (4.16) with the $\psi = \kappa^2/\bar{\rho}_1$. Similar to (4.25), we have

$$e^{\varepsilon t}\mathbb{E}\widehat{V}(\hat{x}_t, \hat{r}_t, t) \le \widehat{V}(\hat{x}_0, \hat{r}_0, 0) + \int_0^t e^{\varepsilon s}\mathbb{E}\Big(\varepsilon \widehat{V}(\hat{x}_s, \hat{r}_s, s) + \mathbb{L}\widehat{V}(\hat{x}_s, \hat{r}_s, s)\Big)ds,$$

where ε is a sufficiently small positive number to be determined later. Let

$$a_1 = \min_{i\in S}\eta_i,\;\; a_2 = \max_{i\in S}\eta_i,\;\; a_3 = \max_{i\in S_1}\bar{\eta}_i, \quad a_4 = \max_{i\in S}\tilde{\eta}_i.$$

We then obtain that

$$\begin{aligned} a_1 e^{\varepsilon t}\mathbb{E}|x(t)|^2 \;\; \le \;\; & \widehat{V}(\hat{x}_0, \hat{r}_0, 0) + \frac{\varepsilon\kappa^2}{\bar{\rho}_1}\Phi_1(t) + \int_0^t e^{\varepsilon s}\Big(a_2\varepsilon\mathbb{E}|x(s)|^2 \\ & + a_3\varepsilon\mathbb{E}|x(s)|^{p+q_1-1} + a_4\varepsilon\mathbb{E}|x(s)|^{q_1+1} + \mathbb{E}\mathbb{L}\widehat{V}(\hat{x}_s, \hat{r}_s, s)\Big)ds, \end{aligned}$$

where

$$\Phi_1(t) = \mathbb{E}\int_0^t e^{\varepsilon s}\int_{-\tau}^0\int_{s+u}^s [\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 + |g(x(v), r(v), v)|^2]dvduds.$$

Noting that $|x(s)|^{q_1+1} \le |x(s)|^2 + |x(s)|^{p+q_1-1}$, we have

$$\begin{aligned} a_1 e^{\varepsilon t}\mathbb{E}|x(t)|^2 \;\; \le \;\; & \widehat{V}(\hat{x}_0, \hat{r}_0, 0) + \frac{\varepsilon\kappa^2}{\bar{\rho}_1}\Phi_1(t) + \int_0^t e^{\varepsilon s}\Big((a_2 + a_4)\varepsilon\mathbb{E}|x(s)|^2 \\ & + (a_3 + a_4)\varepsilon\mathbb{E}|x(s)|^{p+q_1-1} + \mathbb{E}\mathbb{L}\widehat{V}(\hat{x}_s, \hat{r}_s, s)\Big)ds. \end{aligned} \tag{4.31}$$

Applying Lemma 4.5 and a similar argument to derive (4.24) yields

$$\begin{aligned} \mathbb{L}\widehat{V}(\hat{x}_s, \hat{r}_s, s) \;\; \le \;\; & -\bar{\rho}_0|x(s)|^2 - H(x(s)) + \Big\{\frac{3\kappa^2}{8\bar{\rho}_1}|x(\delta_s) - x(s)|^2 + \frac{4\tau^2\kappa^4}{\bar{\rho}_1}|x(s)|^2\Big\}1_{\{r(s)\in S_2\}} \\ & - \frac{\kappa^2}{\bar{\rho}_1}\int_{s-\tau}^s [\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 + |g(x(v), r(v), v)|^2]dv \\ \le \;\; & -(\bar{\rho}_0 - \frac{4\tau^2\kappa^4}{\bar{\rho}_1})|x(s)|^2 - H(x(s)) + \Big\{\frac{3\kappa^2}{8\bar{\rho}_1}|x(\delta_s) - x(s)|^2\Big\}1_{\{r(s)\in S_2\}} \\ & - \frac{\kappa^2}{\bar{\rho}_1}\int_{s-\tau}^s [\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 + |g(x(v), r(v), v)|^2]dv. \end{aligned}$$

By virtue of (4.27), we have

$$\begin{aligned} \mathbb{E}\mathbb{L}\widehat{V}(\hat{x}_s, \hat{r}_s, s) \;\; \le \;\; & -(\bar{\rho}_0 - \frac{4\tau^2\kappa^4}{\bar{\rho}_1})\mathbb{E}|x(s)|^2 - \mathbb{E}H(x(s)) - \frac{\kappa^2}{4\bar{\rho}_1}\mathbb{E}\int_{s-\tau}^s [\tau|f(x(v), r(v), v) \\ & + u(x(\delta_v), r(v), v)|^2 + |g(x(v), r(v), v)|^2]dv. \end{aligned}$$

This, together with (4.31), implies that

$$\begin{aligned} a_1 e^{\varepsilon t}\mathbb{E}|x(t)|^2 \;\; \le \;\; & \widehat{V}(\hat{x}_0, \hat{r}_0, 0) + \frac{\varepsilon\kappa^2}{\bar{\rho}_1}\Phi_1(t) - \frac{\kappa^2}{4\bar{\rho}_1}\Phi_{11}(t) \\ & + \int_0^t e^{\varepsilon s}\Big(- (\bar{\rho}_0 - \frac{4\tau^2\kappa^4}{\bar{\rho}_1} - (a_2 + a_4)\varepsilon)\mathbb{E}|x(s)|^2 \end{aligned}$$

$$+(a_3+a_4)\varepsilon\mathbb{E}|x(s)|^{p+q_1-1}-\mathbb{E}H(x(s))\Big)ds,$$

where

$$\Phi_{11}(t)=\mathbb{E}\int_0^t e^{\varepsilon s}\int_{s-\tau}^s[\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2+|g(x(v),r(v),v)|^2]dvds.$$

It can be easily observed that

$$\begin{aligned}\Phi_1(t) &\leq \mathbb{E}\int_0^t e^{\varepsilon s}\tau\int_{s-\tau}^s[\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2+|g(x(v),r(v),v)|^2]dvds\\ &= \tau\Phi_{11}(t).\end{aligned}$$

Moreover, Lemma 4.5 gives $\bar{\rho}_4|x|^{p+q_1-1}\leq H(x)$, and hence $|x(s)|^{p+q_1-1}\leq\bar{\rho}_4^{-1}H(x(s))$. This implies

$$\begin{aligned}a_1e^{\varepsilon t}\mathbb{E}|x(t)|^2 &\leq \widehat{V}(\hat{x}_0,\hat{r}_0,0)+(\varepsilon\tau-\frac{1}{4})\frac{\kappa^2}{\bar{\rho}_1}\Phi_{11}(t)+\int_0^t e^{\varepsilon s}\Big(-[\bar{\rho}_0-\frac{4\tau^2\kappa^4}{\bar{\rho}_1}\\ &\quad -(a_2+a_4)\varepsilon]\mathbb{E}|x(s)|^2-(1-\frac{(a_3+a_4)\varepsilon}{\bar{\rho}_4})\mathbb{E}H(x(s))\Big)ds.\end{aligned}\tag{4.32}$$

We can choose a sufficiently small $\varepsilon>0$ such that

$$\varepsilon\tau\leq\frac{1}{4},\quad (a_2+a_4)\varepsilon\leq\bar{\rho}_0-\frac{4\tau^2\kappa^4}{\bar{\rho}_1},\quad \frac{(a_3+a_4)\varepsilon}{\bar{\rho}_4}\leq 1.$$

By (4.28), we have $\bar{\rho}_0>4\tau^2\kappa^4/\bar{\rho}_1$. Then it follows from (4.32) that

$$\mathbb{E}|x(t)|^2\leq(\widehat{V}(\hat{x}_0,\hat{r}_0,0)/a_1)e^{-\varepsilon t},\ \forall t\geq 0.\tag{4.33}$$

For any $\bar{q}\in(2,q)$, applying the Hölder inequality gives

$$\begin{aligned}\mathbb{E}|x(t)|^{\bar{q}} &\leq [\mathbb{E}(|x(t)|^2)]^{(q-\bar{q})/(q-2)}[\mathbb{E}|x(t)|^q])^{(\bar{q}-2)/(q-2)}\\ &\leq C^{(\bar{q}-2)/(q-2)}(\mathbb{E}(|x(t)|^2)^{(q-\bar{q})/(q-2)},\end{aligned}\tag{4.34}$$

where in the last inequality we used the result (3.10). Substituting (4.33) into (4.34) yields

$$\mathbb{E}|x(t)|^{\bar{q}}\leq C^{(\bar{q}-2)/(q-2)}(\widehat{V}(\hat{x}_0,\hat{r}_0,0)/a_1)e^{-\varepsilon t(q-\bar{q})/(q-2)}.$$

Then the desired result (4.29) follows. Finally, by a similar approach to derive the Theorem 5.4 in [17], we can obtain the assertion (4.30). This proof is completed. □

5 Example

The following example will illustrate our idea.

Example 5.1. Consider a scalar hybrid SDE

$$dx(t)=f(x(t),r(t),t)dt+g(x(t),r(t),t)dB(t),\tag{5.1}$$

where f and g are defined as follows:

$$f(x,i,t) = \begin{cases} 0.5x, & \text{if} \quad i=1 \\ 0.2x, & \text{if} \quad i=2 \\ x - x^3, & \text{if} \quad i=3 \\ 2x - 2x^3, & \text{if} \quad i=4 \end{cases} \quad \text{and} \quad g(x,i,t) = \begin{cases} 0.2x, & \text{if} \quad i=1 \\ 0.3x, & \text{if} \quad i=2 \\ 0.1|x|^{3/2}, & \text{if} \quad i=3 \\ 0.3|x|^{3/2}, & \text{if} \quad i=4, \end{cases}$$

$B(t)$ is a scalar Brownian motion, and $r(t)$ is a Markov chain with the state space $S = \{1,2,3,4\}$ and the generator

$$\Gamma = \begin{pmatrix} -8 & 1 & 4 & 3 \\ 1 & -6 & 2 & 3 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 0 & -2 \end{pmatrix}$$

Let $S_1 = \{1,2\}, S_2 = \{3,4\}$. It is straightforward to show that the Assumptions 2.1, 2.2 are satisfied for

$$p=4, q=6, q_1=3, q_2=1.5, \ \bar{\beta}_1 = 0.64, \bar{\beta}_2 = 0.515, \ \alpha = 0.9875, \epsilon_1 = 1.0125, \epsilon_2 = 2.1125, \tag{5.2}$$

and

$$\mathcal{D} = \begin{pmatrix} 2.88 & -1 \\ -1 & -1.88 \end{pmatrix} \ , \ \mathcal{D}^{-1} = \begin{pmatrix} 0.4259 & 0.2265 \\ 0.2265 & 0.6524 \end{pmatrix}.$$

By Lemma 2.1 and (3.2), $\mathcal{D}$ is a nonsingular M-matrix and $(\theta_1, \theta_2) = (0.6524, 0.8789)$. Theorem 3.1 shows the underlying system (3.1) obeys $\sup_{0 \le t < \infty} \mathbb{E}|x(t)|^6 < C$. However, the simulation with initial value $x(0) = 1$ and $r(0) = 1$ (Figure 1(a)) shows that this hybrid SDE (5.1) is not stable.

Assume that the system is observable when it operates in mode S_2. We design a feedback control relying only on mode S_2. Accordingly, we take the control function $u : R \times S \times R_+ \to R$ as follows:

$$u(x,i,t) = 0, \quad i \in S_1 \ \ \textit{and} \ \ u(x,i,t) = -3x, \quad i \in S_2. \tag{5.3}$$

Obviously, Rule 3.2 is satisfied with $\kappa = 3$. By Theorem 3.3, the controlled system

$$dx(t) = [f(x(t), r(t), t) + u(x(\delta_t), r(t), t)]dt + g(x(t), r(t), t)dB(t) \tag{5.4}$$

has a unique global solution with $\sup_{0 \le t < \infty} \mathbb{E}|x(t)|^6 < C$.

We compute for $i \in S_1$,

$$x^T[f(x,i,t) + u(x,i,t)] + \frac{1}{2}|g(x,i,t)|^2 = \begin{cases} 0.52x^2 & \text{if} \ \ i=1, \\ 0.245x^2 & \text{if} \ \ i=2, \end{cases}$$

$$x^T[f(x,i,t) + u(x,i,t)] + \frac{q_1}{2}|g(x,i,t)|^2 = \begin{cases} 0.56x^2 & \text{if} \ \ i=1, \\ 0.335x^2 & \text{if} \ \ i=2, \end{cases}$$

and for $i \in S_2$,

$$x^T[f(x,i,t)+u(x,i,t)]+\frac{1}{2}|g(x,i,t)|^2=\begin{cases}-2x^2-x^4+0.5\times 0.01|x|^3 & \text{if } \ i=3,\\ -x^2-2x^4+0.5\times 0.09|x|^3 & \text{if } \ i=4.\end{cases}$$

Noting that $|x|^3 \le 0.5x^2+0.5x^4$, hence for $i \in S_2$, we have

$$x^T[f(x,i,t)+u(x,i,t)]+\frac{1}{2}|g(x,i,t)|^2=\begin{cases}-1.9975x^2-0.9975x^4 & \text{if } \ i=3,\\ -0.9775x^2-1.9775x^4 & \text{if } \ i=4.\end{cases}$$

Similarly,

$$x^T[f(x,i,t)+u(x,i,t)]+\frac{q_1}{2}|g(x,i,t)|^2=\begin{cases}-1.9925x^2-0.9925x^4 & \text{if } \ i=3,\\ -0.9325x^2-1.9325x^4 & \text{if } \ i=4.\end{cases}$$

Hence, (4.3) and (4.4) hold for

$$\begin{aligned}&\beta_1=0.52, \beta_2=0.245, \beta_3=-1.9975, \beta_4=-0.9775,\\ &\tilde\beta_1=0.56, \tilde\beta_2=0.335, \tilde\beta_3=-1.9925, \tilde\beta_4=-0.9325,\\ &\alpha_3=0.9975, \alpha_4=1.9775, \tilde\alpha_3=0.9925, \tilde\alpha_4=1.9325.\end{aligned}$$

Then we have

$$\mathcal{B}=\begin{pmatrix}6.96 & -1 & -4 & -3\\ -1 & 5.51 & -2 & -3\\ -1 & -1 & 6.995 & -1\\ -1 & -1 & 0 & 3.955\end{pmatrix} \quad \text{and} \quad \mathcal{B}_1=\begin{pmatrix}5.76 & -1 & -4 & -3\\ -1 & 4.66 & -2 & -3\\ -1 & -1 & 10.97 & -1\\ -1 & -1 & 0 & 5.73\end{pmatrix}.$$

Note that

$$\mathcal{B}^{-1}=\begin{pmatrix}0.233 & 0.1312 & 0.1707 & 0.3194\\ 0.1122 & 0.2909 & 0.1473 & 0.3430\\ 0.0618 & 0.0756 & 0.1999 & 0.1548\\ 0.0873 & 0.1067 & 0.0804 & 0.4203\end{pmatrix} \quad \text{and} \quad \mathcal{B}_1^{-1}=\begin{pmatrix}0.2536 & 0.1262 & 0.1155 & 0.2190\\ 0.1123 & 0.3109 & 0.0976 & 0.2386\\ 0.0392 & 0.0468 & 0.1140 & 0.0649\\ 0.0639 & 0.0763 & 0.0372 & 0.2544\end{pmatrix}.$$

By (5.2), we have $\bar\beta_1=0.64, \bar\beta_2=0.515$. Then we obtain

$$\mathcal{B}_2=\begin{pmatrix}4.16 & -1\\ -1 & 2.91\end{pmatrix} \quad \text{and} \quad \mathcal{B}_2^{-1}=\begin{pmatrix}0.262 & 0.09\\ 0.09 & 0.3746\end{pmatrix}.$$

By Lemma 2.1, $\mathcal{B}, \mathcal{B}_1$ and $\mathcal{B}_2$ are all nonsingular M-matrix. Then it follows from (4.6) that

$$\begin{aligned}(\eta_1,\eta_2,\eta_3,\eta_4) &= (0.8543, 0.8934, 0.4921, 0.6947),\\ (\tilde\eta_1,\tilde\eta_2,\tilde\eta_3,\tilde\eta_4) &= (0.7134, 0.7594, 0.2649, 0.4318),\\ D &= 0.5451,\end{aligned}$$

$$(\bar{\eta}_1, \bar{\eta}_2)^T \quad = \quad (0.1919, 0.2533).$$

By (4.11), $L_1 V(x, i, t) \leq -x^2 - x^4 - 0.5451x^6$. Moreover, we have

$$\begin{aligned} (2\eta_i|x| + (q_1+1)\tilde{\eta}_i|x|^{q_1})^2 &\leq 3.1927x^2 + 10.8552x^4 + 9.227x^6, \\ |f(x,i,t)|^2 &\leq 4x^2 + 8x^4 + 4x^6, \\ |g(x,i,t)|^2 &\leq 0.2025x^2 + 0.0675x^4. \end{aligned}$$

Take $\bar{\rho}_1 = 0.05$, $\bar{\rho}_2 = 0.01$, $\bar{\rho}_3 = 1$. Then we obtain

$$L_1 V(x,i,t) + \bar{\rho}_1(2\eta_i|x| + (q_1+1)\tilde{\eta}_i|x|^{q_1})^2 + \bar{\rho}_2|f(x,i,t)|^2 + \bar{\rho}_3|g(x,i,t)|^2 \leq -0.5979x^2 - H(x),$$

where $H(x) = 0.3097x^4 + 0.0437x^6$. That is, Lemma 4.5 is satisfied with $0 < \bar{\rho}_0 \leq 0.5979$, $0 < \bar{\rho}_4 \leq 0.0437$, $\bar{\rho}_5 \geq 0.3534$. Consequently, the condition (4.28) becomes $\tau < 0.0053$. By Theorems 4.6, the controlled system (5.4) with the control function (5.3) is not only exponentially stable in $L^{\bar{q}}$ for $\bar{q} \in [2, 6)$, but also almost surely provided $\tau < 0.0053$. To illustrate the stability of controlled system (5.4), we perform a simulation with $\tau = 0.005$ and initial data $x(0) = 1$ and $r(0) = 1$ to support our theoretical results, which is shown in Figure 1(b).

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References

[1] X. Mao and C. Yuan, *Stochastic Differential Equations With Markovian Switching*. London: Imperial College Press, 2006.

[2] X. Mao, N. Koroleva, and A. Rodkina, "Robust stability of uncertain stochastic differential delay equations," *Systems & Control Letters*, vol. 35, no. 5, pp. 325–336, 1998.

[3] X. Mao, "Stability of stochastic differential equations with markovian switching," *Stochastic Processes and their Applications*, vol. 79, no. 1, pp. 45–67, 1999.

[4] L. Hu, X. Mao, and Y. Shen, "Stability and boundedness of nonlinear hybrid stochastic differential delay equation," *Systems & Control Letters*, vol. 62, no. 2, pp. 178–187, 2013.

[5] G. G. Yin and C. Zhu, *Hybrid switching diffusions: Properties and applications*. New York: Springer Science & Business Media, vol. 63, 2009.

[6] G. Badowski and G. G. Yin, "Stability of hybrid dynamic systems containing singularly perturbed random processes," *IEEE Transactions on Automatic Control*, vol. 47, no. 12, pp. 2021–2032, 2002.

[7] L. Wu, X. Su, and P. Shi, "Sliding mode control with bounded l2 gain performance of markovian jump singular time-delay systems," *Automatica*, vol. 48, no. 8, pp. 1929–1933, 2012.

[8] M. Shen, C. Fei, W. Fei, and X. Mao, "Boundedness and stability of highly nonlinear hybrid neutral stochastic systems with multiple delays," *Science China Information Sciences*, vol. 62, no. 10, p. 202205, 2019.

[9] X. Mao, *Stochastic Differential Equations and Applications*. Chicester: Horwood, 2007.

[10] Y. Ji and H. J. Chizeck, "Controllability, stabilizability, and continuous-time markovian jump linear quadratic control," *IEEE Transactions on Automatic Control*, vol. 35, no. 7, pp. 777–788, 1990.

[11] X. Mao, J. Lam, and L. Huang, "Stabilisation of hybrid stochastic differential equations by delay feedback control," *Systems & Control Letters*, vol. 57, no. 11, pp. 927–935, 2008.

[12] X. Li and X. Mao, "Stabilisation of highly nonlinear hybrid stochastic differential delay equations by delay feedback control," *Automatica*, vol. 112, p. 108657, 2020.

[13] M. Shen, C. Fei, W. Fei, and X. Mao, "Stabilisation by delay feedback control for highly nonlinear neutral stochastic differential equations," *Systems & Control Letters*, vol. 137, p. 104645, 2020.

[14] C. Mei, C. Fei, W. Fei, and X. Mao, "Exponential stabilization by delay feedback control for highly nonlinear hybrid stochastic functional differential equations with infinite delay," *Nonlinear Analysis: Hybrid Systems*, vol. 40, no. 9, p. 101026, 2021.

[15] X. Mao, "Stabilization of continuous-time hybrid stochastic differential equations by discrete-time feedback control," *Automatica*, vol. 49, no. 12, pp. 3677–3681, 2013.

[16] S. You, W. Liu, J. Lu, X. Mao, and Q. Qiu, "Stabilization of hybrid systems by feedback control based on discrete-time state observations," *SIAM Journal on Control & Optimization*, vol. 53, no. 2, pp. 905–925, 2015.

[17] C. Fei, W. Fei, X. Mao, D. Xia, and L. Yan, "Stabilization of highly nonlinear hybrid systems by feedback control based on discrete-time state observations," *IEEE Transactions on Automatic Control*, vol. 65, no. 7, pp. 2899–2912, 2019.

[18] R. Dong and X. Mao, "On pth moment stabilization of hybrid systems by discrete-time feedback control," *Stochastic Analysis & Applications*, vol. 35, no. 5, pp. 803–822, 2017.

[19] J. Shao, "Stabilization of regime-switching processes by feedback control based on discrete time observations," *SIAM Journal on Control and Optimization*, vol. 55, no. 2, pp. 724–740, 2017.

[20] J. Shao and F. Xi, "Stabilization of regime-switching processes by feedback control based on discrete time observations ii: State-dependent case," *SIAM Journal on Control and Optimization*, vol. 57, no. 2, pp. 1413–1439, 2019.

[21] C. Mei, C. Fei, W. Fei, and X. Mao, "Stabilisation of highly non-linear continuous-time hybrid stochastic differential delay equations by discrete-time feedback control," *IET Control Theory and Applications*, vol. 14, no. 2, pp. 313–323, 2020.

[22] W. Fei, L. Hu, X. Mao, and M. Shen, "Structured robust stability and boundedness of nonlinear hybrid delay systems," *SIAM Journal on Control and Optimization*, vol. 56, no. 4, pp. 2662–2689, 2018.

[23] G. Song, Z. Lu, B. Zheng, and X. Mao, "Almost sure stabilization of hybrid systems by feedback control based on discrete-time observations of mode and state," *Science China Information Sciences*, vol. 61, no. 7, pp. 1–16, 2018.

[24] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences.* Philadephia: SIAM, 1994.

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(a) The computer simulation of the sample paths of the Markov chain and the solution of the Eq.(5.1) using the Euler-Maruyama method with step size 10^{-4}.

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(b) The computer simulation of the sample paths of the Markov chain and the solution of the Eq.(5.4) using the Euler-Maruyama method with step size 10^{-4}.

Figure 1: simulation