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Fast Chaos Expansions of Diffusive and Sub-Diffusive Processes in Orbital Mechanics

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This paper considers the case in which the dynamics of an object in orbit is subject to a random process that can be modelled as a generic Continuous Time Random-Walk (CTRW). This model can describe the situation in which an orbiting object is subject to random, non destructive, collisions. In the limit of an infinite number of collisions the process can converge to a standard Weiner process if the impacts occur at time increments that are stationary with finite variance. In this case it can be shown that the dynamics is subject to normal diffusion. However, impact can occur more slowly with time increments drawn from a heavy tailed distribution, in this case the dynamics is subject to a sub-diffusion process. The paper discusses how polynomial expansions with dynamic re-initialisation can be used to study the evolution of the dynamical system subject to a CTRW. A simple indicator of diffusion is then derived from the coefficients of the polynomial expansion. The same indicator is then used to study regular and chaotic motions in a dynamical system with uncertain model parameters. A few simple illustrative examples will complete the paper.
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1. Introduction

In the past decade uncertainty quantification has gain popularity in the field of astrodynamics in order to propagate and quantify uncertainty in initial conditions and system parameters. Techniques like non-intrusive Polynomial Chaos Expansions, Stochastic Collocation, High Dimensional Model Representations and polynomial algebras have been applied to problems of re-entry, collision prediction and disposal trajectories, just to name a few.

While the quantification of parametric uncertainty, including uncertainty in initial conditions, has seen a flourishing of papers and methods, the quantification of uncertainty introduced by stochastic processes has received less attention. The treatment of process noise is common in both state estimation and control, normally with some underlying assumptions on the nature of the noise. In celestial mechanics the study of stochastic process has been introduced to describe interesting phenomena, like forces induced by a cloud having a density which fluctuates stochastically, as it is observed in the zodiacal dust around the sun [1]. The study of the model by Sharma and Parthasarathy [1] led to the formulation of the stochastic version of Gauss planetary equations [2]. In the same work it was shown that in these models the energy is not a strong integral of motion and the system can diffuse.

Manzi and Vasile [3] presented a treatment of some stochastic processes in quasi-integrable systems in orbital mechanics and showed diffusion under certain conditions. In [3], it is also shown that polynomial expansions can be used to derive a quantification of chaotic dynamics. In particular intrusive polynomial chaos expansions can be used to easily quantify second order statistical moments and cummulants which are closely related to the divergence of the realisations of the initial conditions.

In this paper we consider a simple model of random-walk that can be used to describe the effect of random, non-destructive, collisions in orbit. Random-walks in celestial mechanics were previously studied by Belbruno [4] to model the long term effect of collisions on the motion of objects at the stable Lagrange points of the Earth-Sun system.

The goal of this paper is to quantify the effect of a random-walk on the evolution of the state of the system, using polynomial expansions. It will be shown that this representation allows for a simple quantification of the uncertainty associated to a trajectory that evolves under the effect of random impulses. A simple index of diffusion is then derived from the coefficients of the polynomial expansion. When the time and magnitude of the impulses can be modelled with distributions with finite variance, the relation between the coefficients of the polynomial expansion and the statistical moments of the state of the system can be used to show normal diffusion. On the other hand for collision time increments coming from a heavy tailed distributions like the Pareto one, the system displays subdiffusivity, in agreement with Alves et al. [5]. It is argued that if objects in orbit were subject to collisions with fragments at a uniform rate or a standard Poison rate, one should observe normal diffusion of the orbits. On the other hand if collisions happened at a slower rate following a heavy-tailed distribution one should observe a subdiffusion of the orbits with characteristic exponent $\alpha < 1$ dependent on the nature of the distribution of the time increments.

The paper is structured as follows. After a brief introduction to the concept of diffusion and subdisffusion, with some results on diffusive processes in celestial mechanics and astrodynamics, section 3 will recall the definition of intrusive polynomial chaos expansions and the concept of dynamic re-initialisation. Section 4 will then focus on the use of polynomial expansions to represent a particular model of CTRW and will show how to use dynamic polynomial chaos expansions to quantify the associated uncertainty in the state variables. Section 5 will present some illustrative examples with particular attention to the case of impacts in Earth orbit and how this problem relates to some existing evolutionary models. Finally section 6 will discuss the use of polynomial expansions to study the evolution of a dynamical system with uncertain model parameters.

2. Normal and Anomalous Diffusion

Consider the following Langevin type of equation:

$$\begin{aligned}\dot{v} &= f(r) + \phi(r, v, t)\xi(t) \\ \dot{r} &= v\end{aligned} \tag{1}$$

where $\xi(t) = \frac{dW_t}{dt}$ is white noise with expected value $\mathbb{E}(\xi(t)) = 0$ and autocorrelation $\mathbb{E}(\xi(t)\xi(s)) = \delta(t-s)$. If $f(r) = 0$ and $\phi = D$, the time integral of the first of Eq. (1) reduces to the simple:

$$v = DW_t \tag{2}$$

The mean and the variance of both quantities in Eq. (2) gives:

$$\mathbb{E}(v) = 0 \quad \mathbb{E}(v^2) = Dt \tag{3}$$

Thus in this case the velocity is subject to a diffusion process with diffusion constant D. This is equivalent to a Brownian motion where a particle is subject to continues and independent impacts. When $f(r)$ is the force of gravity, first Sharma and Parthasarathy [1] and later Pierret [2], Cresson et al. [6] demonstrated that if the dynamics is affected by a Weiner type of random process an orbiting object is subject to diffusion as a consequence the total mechanical energy E is not an integral of motion, either strong or weak. This diffusion term can be directly derived from Itô's lemma as presented in [3]. In fact, one has:

$$\dot{E} = \langle \dot{v}, v\rangle - \langle f(r) + \phi(r, v, t)\xi(t), v\rangle \tag{4}$$

From differentiation rules in Itô stochastic calculus we have:

$$2v\ \mathrm{d}v = \mathrm{d}(v^2) - (\mathrm{d}v)^2 \tag{5}$$

$$\mathrm{d}(U(r)) = f(r)\mathrm{d}r + \frac{1}{2}(\mathrm{d}^2U(r)/\mathrm{d}r^2)(\mathrm{d}r)^2 \tag{6}$$

where $f(r) = \mathrm{d}U(r)/\mathrm{d}r$. If we make use of the relationship:

$$\mathrm{d}r = v\ \mathrm{d}t \tag{7}$$

we have:

$$\begin{aligned}&\mathrm{d}\left(\frac{1}{2}v^2 - U(r)\right) = \\ &\frac{1}{2}\left((\mathrm{d}v)^2 + (\mathrm{d}^2U(r)/\mathrm{d}r^2)(\mathrm{d}r)^2\right) + v\mathrm{d}W_t\epsilon\phi(r, v)\end{aligned} \tag{8}$$

and by substituting the expressions for $\mathrm{d}r$ and $\mathrm{d}v$:

$$\begin{aligned}\mathrm{d}\mathcal{E} = \tfrac{1}{2}\Big(&(f(r)\mathrm{d}t + \mathrm{d}W_t\epsilon\phi(r, v))^2 + \\ &(\mathrm{d}^2U(r)/\mathrm{d}r^2)(v\mathrm{d}t)^2\Big) + v\mathrm{d}W_t\epsilon\phi(r, v)\end{aligned} \tag{9}$$

where we called $\mathcal{E} = \frac{1}{2}v^2 - U(r)$ the usual total mechanical energy. If we retain only first order terms:

$$\mathrm{d}\mathcal{E} = \frac{1}{2}\mathrm{d}W_t^2\epsilon^2\phi(r, v)^2 + v\mathrm{d}W_t\epsilon\,\phi(r, v) \tag{10}$$

but for Wiener processes $\mathrm{d}W_t^2 = \mathrm{d}t$ thus:

$$\mathrm{d}\mathcal{E} = \frac{1}{2}\epsilon^2\phi(r, v)^2\mathrm{d}t + v\mathrm{d}W_t\epsilon\phi(r, v) \tag{11}$$

We can, therefore, conclude that the classical energy $\mathcal{E}$ remains bounded if the integral of the square of the function $\phi(r, v)$ and the integral of $v\,\mathrm{d}W_t\epsilon\phi(r, v)$ are bounded $\forall t$.

If the random process in Eq. (1) is not a white noise then other forms of diffusion are possible. In particular, in the case $f(r) = 0$ the mean square variation of the velocity becomes $\mathrm{E}(v^2) = Dt^\alpha$, where the exponent defines the type of diffusion. If $\alpha = 1$ one has a normal diffusion, for $\alpha < 1$ one has sub-diffusion and for $\alpha > 1$ super-diffusion.

2.1 Random-Walk Model

Weiner process is a particular type of random-walk with stationary and independent increments with finite variance. According to Donsker's theorem, a Weiner process can be derived as the limit case of the sum:

$$S_t = \sum_{1\le i\le \lfloor nt\rfloor} Z_i \tag{12}$$

for $n \to \infty$ where Z_i are independent and equally distributed increments with zero mean and variance 1.

Consider now the case in which the dynamics is subject to a sequence of random impulses:

$$\begin{aligned}\dot{r} &= v \\ \dot{v} &= f(r) + \textstyle\sum_i^N \delta(t-\tau_i)Z_i\end{aligned} \tag{13}$$

where τ_i and Z_i are independent random variables with marginal distributions $p(\tau_i)$ and $p(Z_i)$ respectively. For

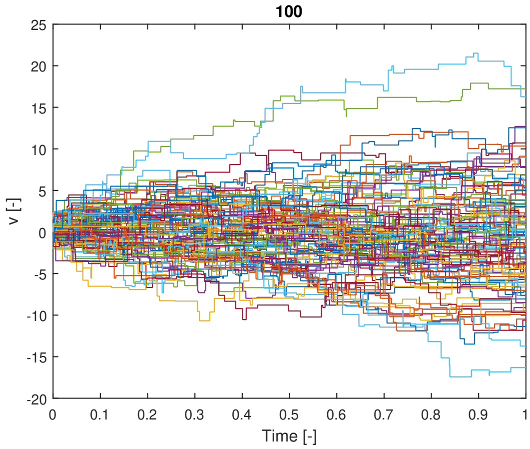

Fig. 1: Random walk with 100 impulses.

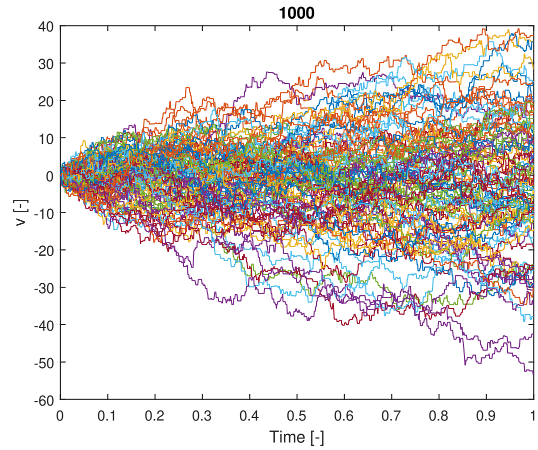

Fig. 2: Random walk with 1000 impulses.

the case in which τ_i are deterministic variables and Z_i are independent and equally distributed we have a standard random walk that converges to the Weiner process in (3). Likewise the time increments follow a Poison distribution the stochastic process in (13) converges to Weiner process for Z_i finite and $N \to \infty$. For a generic distribution $p(\tau_i)$, the second equation in (13) represents a Continuous Time Random Walk (CTRW). In the following we will consider only the separable case in which the displacements Z_i and the waiting times τ_i are independent of each other. In particular, when the distribution $p(\tau_i)$ is modelled with a power law $k_t \tau_i^{-\beta-1}$ (like a Pareto distribution), with $\beta > 0$ we can have the following situations:

- for $0 < \beta \leq 1$ the system displays subdiffusion
- for $\beta > 1$ the system can display either normal diffusion or superdiffusion depending on the increments. If the increments have infinite variance then the system superdiffuses while if the variance is finite we have normal diffusion.

Two examples of CTRW with $N = 100$ and $N = 1000$ respectively can be seen in Figs. 1 and 2. In this case the time increments are uniformly distributed and the jump magnitude is normally distributed. These two examples show that although the mean value of the velocity starts at 0 and remains nearly 0, the velocity is diffusing as some paths grow or decrease indefinitely.

2.2 *Fractional Brownian Motion Model*

An alternative way to model a stochastic process that leads to subdisffusion is Fractional Brownian Motion. Fractional Brownian motion B_t is a generalisation of normal Brownian motion to the case in which the mean square displacement is proportional to Dt^α with α different from 1. In particular fractional Brownian motion is generally defined in terms of the Hurst index $H \in (0, 1)$[7], such that the mean is zero and the covariance is $< B_t(t), B_t(s) >= \frac{1}{2}(t^{2H} + s^{2H} - |s - t|^{2H})$ with t and s two instants of time. For a Hurst index equal to 1/2 we have normal diffusivity while for $H < 1/2$ one has subdiffusivity.

Since fractional Brownian is time continuous Dzhaparidze and van Zanten [7] proposed the following series expansion:

$$B_t = \sum_{n=1}^{\infty} \frac{\sin x_n t}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1 - \cos y_n t}{y_n} Y_n \qquad (14)$$

with X_n and Y_n two Gaussian processes with variances respectively equal to $2c_H^2 x_n^{-2H} J_{1-H}^{-2}(x_n)$ and $2c_H^2 y_n^{-2H} J_{-H}^{-2}(y_n)$, J_{1-H} and J_{-H} the Bessel functions of the first kind of order $-H$ and $1 - H$and x_n and y_n their real positive zeros.

3. Dynamic Polynomial Expansion

Polynomial Chaos is a computational spectral method, originated from the works of Wiener [8], in which the functional of the state with respect to a set of uncertain parameters is decomposed into a set of orthogonal polynomials. In this section we will briefly introduce the concept of *dynamic* generalized Polynomial Chaos, recently proposed by Ozen and Bal [9]. The idea is to partition the integration domain in sub-intervals and develop a new PC expansion on each subinterval. As the random process on each subinteval is independent of the ones in other subintervals the independence of the uncertain variables that is required for the PC expansion is preserved. A similar restarting schemes have been also employed for deterministic dynamical systems with stochastic parameters [10–13].

The method works as follows: given a vector of uncertain parameters $\mathbf{z}$, leading to a joint probability distribution, it is possible to build a basis of polynomials, orthogonal with respect to such probability distribution, of a given degree

$$\Phi_i = \Phi_i(\mathbf{z}) \tag{15}$$

The three terms recursion relation [14] can be used to create stabilised univariate orthogonal polynomials:

$$\Phi_{n+1}(q) = \Phi_n(q)(q - A_n) - \Phi_{n-1}(q)B_n, \tag{16a}$$

$$A_n = \frac{\mathbb{E}[q\Phi_n^2]}{\mathbb{E}[\Phi_n^2]}, \tag{16b}$$

$$B_n = \frac{\mathbb{E}[\Phi_n^2]}{\mathbb{E}[\Phi_{n-1}^2]} \tag{16c}$$

It is then possible to build orthogonal multivariate polynomials using tensor product rules [15], when the distributions are stochastically independent (the case of dependent variables is discussed later on in this sections). As polynomials are orthogonal with respect to the inner product defined by the expectation operator with respect to the associated probability distribution, the following holds:

$$\begin{aligned}\langle\Phi_j, \Phi_k\rangle &= \int_\Omega \Phi_j(\mathbf{z})\Phi_k(\mathbf{z})p(\mathbf{z})\mathrm{d}\mathbf{z} \\ &= \mathbb{E}[\Phi_j, \Phi_k] \neq 0 \Leftrightarrow j = k\end{aligned} \tag{17}$$

and the constant, non-trivial terms can be easily computed. Using the set of orthogonal polynomials in Eq. (16), we can write a solution, in which the randomness of the process and its time-dependence are explicitly separated, by means of:

$$\mathbf{x}(\mathbf{z}, t) = \sum_{i=1}^{n} \mathbf{c}_i(t)\Phi_i(\mathbf{z}) \tag{18}$$

with

$$n = \binom{K+N}{N} \tag{19}$$

In Eq. (19), K is the number of random variables characterising the dynamics, and N is the order of the polynomial basis used to spectrally decompose the state distribution of the system: the *curse of dimensionality* arises from the fact that the number of coefficients of the PCE scales exponentially with K.

Once the solution is expressed as in Eq. (18), the differential equation Eq. (1) becomes:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n} c_i^{x_n}(t)\Phi_i(\mathbf{z}) &= \sum_{i=1}^{n} \dot{c}_i^{x_n}(t)\Phi_i(\mathbf{z}) \\ &= f(\sum_{i=1}^{n} c_i^{x_n}(t)\Phi_i(\mathbf{z})))\end{aligned} \tag{20}$$

One can now exploit the orthogonality of the polynomials and project the equation on the space defined by the polynomial basis $\Phi_k(\mathbf{z})$:

$$\begin{aligned}&\int_\Omega \left(\sum_{i=1}^{n} \dot{c}_i^{x_n}(t)\Phi_i(\mathbf{z})\right)\Phi_k(\mathbf{z})p(\mathbf{z})\mathrm{d}\mathbf{z} = \dot{c}_k^{x_n}(t)\langle\Phi_k, \Phi_k\rangle = \\ &= \int_\Omega f\left(\sum_{i=1}^{n} c_i^{x_n}(t)\Phi_i(\mathbf{z})\right)p(\mathbf{z})\mathrm{d}\mathbf{z}\end{aligned} \tag{21}$$

The initial value of the time-dependent part of the state is obtained by means of:

$$\begin{aligned}&\sum_{i=1}^{n} c_i^{x_n}(0)\Phi_i(\mathbf{z}) = x_{n0} \implies \\ &\sum_{i=1}^{n} c_i^{x_n}(0)\langle\Phi_i, \Phi_k\rangle = \langle x_{n0}, \Phi_k\rangle \implies \\ &c_k^{x_n}(0) = \frac{1}{\langle\Phi_k, \Phi_k\rangle}\int_\Omega x_{n0}\Phi_k(\mathbf{z})p(\mathbf{z})\mathrm{d}\mathbf{z}\end{aligned} \tag{22}$$

In the general case the integrals cannot be solved analytically and a high-dimensional numerical integration scheme is required. Sparse grids [16] can be sued to reduce the computational cost.

3.1 Dynamic Re-initialisation

In order to reduce the number of random variables necessary to describe the process, one can decompose the time horizon into a number of sub-intervals and make use of a new PCE on each interval. In order to do so, at the end of every time segment, the initial conditions are defined as a new set of random variable (or uncertain variables) and we perform PCE with respect to the new initial state distribution, together with a new set of random parameters within that interval. The most critical step of the process is the computation of a new orthogonal basis, with respect to the new probability space. This is the case because of the correlation characterising the state components' joint distribution.

A possible way to generate orthogonal bases at each re-initialisation is via the Gram-Schmidt method [15]. For such orthogonalisation procedure, we start from a set of linearly independent functions $\{\chi_1, \chi_2, \dots\}$ which are not orthonormal with respect to the inner product defined by the expectation operator; for simplicity, we start from a monomial basis, leading to $\Phi_1 = 1.0$. For the following terms of the basis, we make use of the relation:

$$\Phi_j = \chi_j - \sum_{k=0}^{j-1} \Phi_k \frac{\langle \Phi_k, \chi_j \rangle}{\langle \Phi_k, \Phi_k \rangle} \tag{23}$$

It is, therefore, possible to perform projections and iteratively build terms in an expansion which are orthogonal to the previous ones.Another approach, faster but both numerically unstable and not easily scalable to high number of dimensions, is based on the Cholesky decomposition of the covariance matrix of the state, leveraging the relation between orthogonality and uncorrelation.

If one is not interested in the exact propagation of the distribution but interprets the polynomial expansion as a way to propagate a set of possible states then a simpler approach was proposed in [13]. This is illustrated in Fig. 3. At each time τ_i the polynomial expansion is re-initialised taking a box enclosing the realisations of the state in the previous interval of time $[\tau_{i-1}, \tau_i)$. The actual distribution of the states is then obtained by sampling the polynomial expansion. This approach does not require polynomials that are orthogonal with respect to a particular weight. In other words the weight does not need to represent a specific distribution.

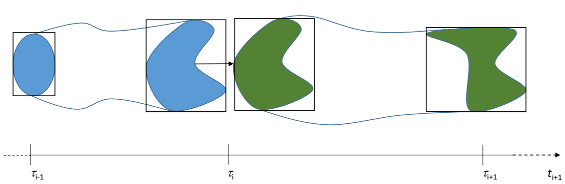

Fig. 3: Dynamic polynomial re-initialisation.

4. Representation with Polynomial Expansions

In the following the interest is not in the behaviour for a number of impulses $N \to \infty$ but in the observable behaviour for N finite and small over a time span $[0, T]$. The two models of diffusive and subdiffusive processes, using a CTRW or a Fractional Brownian Motion, can be partially represented with a polynomial expansion as it will be explained in this section.

4.1 Representation of Fractional Brownian Motion

The fractional Brownian model in (14) is an extension of the Karhunen–Loève expansion of standard Brownian motion. Thus a continues polynomial expansion in the uncertain parameters X_n and Y_n is fairly straightforward. The main problem with this representation is the number of terms in the series expansion which would lead to a relatively high number of uncertain parameters. Furthermore, from (14) one can see that it is not well suited to represent isolated jumps due to the continuous nature of the expansion.

4.2 Representation of Random Walks

Consider now the case in which we want to study the effect of a finite number of jumps. We have seen that for stationary times the system displays normal diffusion. However, subdiffusive processes are characterised by random times with infinite mean. Thus in this section we will represent with a polynomial expansion only the effect of the step length of the walker, which has finite mean and variance for both normal and subdiffusion. We also introduce the assumption that the impulse is small and produces a small perturbation of the orbit over a single orbital period. Since the Dirac delta can be written as the derivative of the Heaviside step function we write the dynamical system as:

$$\dot{r} = v \tag{24a}$$

$$\dot{v} = f(r, v) + \varepsilon \sum_{i}^{N} \dot{\theta}(t - \tau_i) Z_i \tag{24b}$$

Consider now the time intervals $[\tau_{i-1}, \tau_i)$ and $[\tau_i, \tau_{i+1})$. Over the interval $[\tau_{i-1}, \tau_i)$ the dynamics is not affected by the jump at $t = \tau_i$, while over the interval $[\tau_i, \tau_{i+1})$ the dynamics is:

$$\dot{r} = v \tag{25a}$$

$$\dot{v} = f(r, v) \tag{25b}$$

but with velocity at $t = \tau_i$ given by:

$$v(\tau_i^+) = v(\tau_i^-, Z_{i-1}) + \varepsilon Z_i \tag{26}$$

Thus, we can treat the velocity at $t = \tau_i^-$ as an uncertain variable to which we add a random variable scaled by a quantity ε. For the system in (24) over the interval $[\tau_i, \tau_{i+1})$ we can expand the position and velocity as:

$$r = \sum_k d_k(t) \Phi_k(Z_i, r_i, v_i) \tag{27}$$

$$v = \sum_k c_k(t) \Phi_k(Z_i, r_i, v_i) \tag{28}$$

where $r_i = r(\tau_i^-)$ and $v_i = v(\tau_i^-)$. We can now introduce the vector $\xi_i = [r_i, v_i]$ and write the equations for the coefficients of the polynomial expansion as:

$$\begin{aligned} \dot{c}_0 &= \int f(r,v)p(Z_i)p(\xi)dZ_i d\xi \\ \dot{c}_1 &= \int f(r,v)2Z_i p(Z_i)p(\xi)dZ_i d\xi \\ \dot{c}_2 &= \int f(r,v)2\xi p(Z_i)p(\xi)dZ_i d\xi \\ &\dots \end{aligned} \tag{29}$$

The initial conditions for the coefficients will need to account for the jump. In fact:

$$\begin{aligned} c_0(\tau_i) &= \int (v(\tau_i^-) + \varepsilon Z_i)p(Z_i)p(\xi)dZ_i d\xi \\ c_1(\tau_i) &= \int (v(\tau_i^-) + \varepsilon Z_i)2Z_i p(Z_i)p(\xi)dZ_i d\xi \\ c_2(\tau_i) &= \int (v(\tau_i^-) + \varepsilon Z_i)2\xi p(Z_i)p(\xi)dZ_i d\xi \\ &\dots \end{aligned} \tag{30}$$

Thus the initial value of c_0 is the mean of the velocity distribution pre-impulse as the impulse distribution is assumed to have zero mean. The first two coefficients c_1 and c_2 instead add the variance of the impulse to that of the velocity pre-impulse.

If the sequence of times τ_i can be pre-defined, the polynomial expansions can be easily computed by integration of the coefficients over each sub-interval $[\tau_i, \tau_{i+1})$. Note that as in [13] we do not necessarily need to propagate the exact distribution but simply the polynomial expansion and then use the polynomial expansion as a surrogate to efficiently sample the actual distribution without the need to re-propagate any trajectory.

In the case of random times with heavy tails a proper representation would require a propagation for every realisation of the random variable τ_i. On the other had the propagation is independent of the number of jumps as each subinterval is dependent only on the initial conditions at the start of the subinterval.

5. Simple Diffusive Orbital Motion

We want now to study the simple case in which an object is moving along a circular orbit and is subject to repeated impacts. The nature of each impact is not important but it is not destructive, it is in the plane of the orbit and transfers an impulse of random magnitude εZ_i with finite variance. The dynamics in this case is simply:

$$\dot{r} = v \tag{31a}$$

$$\dot{\nu} = \frac{v_t}{r} \tag{31b}$$

$$\dot{v}_r = -\frac{\mu}{r^2} + \frac{v_t^2}{r} + \varepsilon \sum_i^N \dot{\theta}(t-\tau_i)Z_{ri} \tag{31c}$$

$$\dot{v}_t = -\frac{v_t v_r}{r} + \varepsilon \sum_i^N \dot{\theta}(t-\tau_i)Z_{ti} \tag{31d}$$

5.1 Zero Order Approximation

Assuming that each impulse introduces a small variation to the orbit and the total effect of the impulses accumulates over a period much longer than the orbital period then we can have the simple zero order approximation:

$$\dot{r} = v \tag{32a}$$

$$\dot{\nu} = c \tag{32b}$$

$$\dot{v}_r = \varepsilon \sum_i^N \dot{\theta}(t-\tau_i)Z_{ri} \tag{32c}$$

$$\dot{v}_t = \varepsilon \sum_i^N \dot{\theta}(t-\tau_i)Z_{ti} \tag{32d}$$

Taking now the polynomial expansion of the velocity:

$$\begin{aligned} v_r &= c_{vr0} + \textstyle\sum_j c_{vrj}\Phi_j(Z_{ri}, Z_{ti}) \\ v_t &= c_{vt0} + \textstyle\sum_j c_{vtj}\Phi_j(Z_{ri}, Z_{ti}) \end{aligned} \tag{33}$$

and thus:

$$\begin{aligned} \dot{c}_{vr0} + \textstyle\sum_j \dot{c}_{vrj}\Phi_j(Z_{ri}, Z_{ti}) &= \varepsilon \textstyle\sum_i \dot{\theta}(t-\tau_i)Z_{ri} \\ \dot{c}_{vt0} + \textstyle\sum_j \dot{c}_{vtj}\Phi_j(Z_{ri}, Z_{ti}) &= \varepsilon \textstyle\sum_i \dot{\theta}(t-\tau_i)Z_{ti} \end{aligned} \tag{34}$$

The mean value of the right hand side is zero and for zero uncertain initial conditions, $c_{r0} = 0$ and $c_{t0} = v_t(t_0)$. If we use Chebyshev polynomials of the first kind or probabilistic Hermite polynomials the first order basis is simply Z_{ri} and Z_{ti} thus an expansion up to the first order gives:

$$\begin{aligned} c_{vri} &= \varepsilon\theta(t-\tau_i)K_{ri} \\ c_{vti} &= \varepsilon\theta(t-\tau_i)K_{ti} \end{aligned} \tag{35}$$

which gives the obvious velocity solution:

$$\begin{aligned} v_r &= \varepsilon \textstyle\sum_i \theta(t-\tau_i)Z_{ri}K_{ri} \\ v_t &= c_{t0} + \varepsilon \textstyle\sum_i \theta(t-\tau_i)Z_{ti}K_{ti} \end{aligned} \tag{36}$$

Note that the impulse term $\dot{\theta}(t-\tau_i)Z_i$ is orthogonal to all the basis except $\Phi_1(Z_i)$, thus a higher order expansion is useful to capture the nonlinearities in the gravity force terms but the impulse introduces only linear terms. We can see this problem from an energy point of view in the following form:

$$\langle \dot{v}, v \rangle = \langle \varepsilon \sum_i^N \dot{\theta}(t-\tau_i) Z_i, v \rangle \tag{37}$$

The variation of the energy is then:

$$\frac{d}{dt}\mathcal{E} = \langle \varepsilon \sum_i^N \dot{\theta}(t-\tau_i) Z_i, v \rangle \tag{38}$$

where each random impulse Z_i has two components Z_{ri} and Z_{ti}.

$$\begin{aligned} \frac{d}{dt}\mathcal{E} = \varepsilon^2 \left(\sum_i \dot{\theta}(t-\tau_i) K_{ri} Z_{ri} \right)^2 \\ + \varepsilon^2 \left(\sum_i \dot{\theta}(t-\tau_i) K_{ti} Z_{ti} \right)^2 \\ + \varepsilon c_{t0} \sum_i \dot{\theta}(t-\tau_i) K_{ti} Z_{ti} \end{aligned} \tag{39}$$

and exploiting the orthogonality of the basis, we can write the expected value:

$$\frac{d}{dt}\hat{\mathcal{E}} = \varepsilon^2 [\sum_i \dot{\theta}(t-\tau_i) K_{ri}^2 Z_{ri}^2 + \sum_i \dot{\theta}(t-\tau_i) K_{ti}^2 Z_{ti}^2] \tag{40}$$

Thus even for a zero mean series of random impacts the system diffuses. To be noted that expression (40) has to be understood as the ensemble of all the realisations of the uncertain variable Z. Each individual trajectory can follow a path that decreases or increases the velocity, as it can be seen in Figs. 1 and 2. Note also that if the impulse is dependent on the state, for example on the velocity, then the distribution of Z does not necessarily have zero mean but can modelled with a skewed distribution. In this case the variation of the energy has the additional component $\varepsilon c_{t0} \sum_i \dot{\theta}(t-\tau_i) K_{ti} Z_{ti}$. The type of diffusion depends on the distribution of the impact time τ_i. In fact from (36) we have that:

$$< v(t)^2 >= \sum_i \theta(t-\tau_i)^2 K \tag{41}$$

for uniform and stationary times τ_i this leads to:

$$< v(t)^2 >= Kt \tag{42}$$

with the diffusion constant K that depends on the variance of the process Z_i and for a finite variance process K is finite as expected. If the velocity v is expanded in polynomial series then one can obtain the simple result:

$$< v^2 >=< v, v >=< (\sum_{i=1} c_i \phi_i)^2 >= \sum_{i=1} c_i^2 = Kt \tag{43}$$

Thus for a simple diffusion process the sum of the square of the coefficients of the polynomial expansion is proportional to the time t. Note that the evolution of the coefficients can be derived in other ways, for example via an algebra on the space of the polynomials. As long as the state can be expressed as an expansion in orthogonal polynomials one can derive Eq. (43). For anomalous diffusion one can expect that the following relationship is possible:

$$\sum_{i=1} c_i^2 \approx K t^{\alpha} \tag{44}$$

and one can try to derive the diffusion coefficient K and exponent α by a fit:

$$\underset{\alpha, b}{\operatorname{argmin}} \left| \log \left(\sum_{i=1} c_i(t_j)^2 \right) - b - \alpha \log t_j \right| \tag{45}$$

with $b = \log K$. Alternatively we can write:

$$\log \left(\sum_{i=1} c_i(t)^2 + 1 \right) = b + \alpha \log t + \log \left(1 + \frac{1}{K t^{\alpha}} \right) \tag{46}$$

$$\frac{\log \left(\sum_{i=1} c_i(t)^2 + 1 \right)}{\log t} = \frac{b}{\log t} + \alpha + \frac{\log \left(1 + \frac{1}{K t^{\alpha}} \right)}{\log t} \tag{47}$$

that for large t leads to the approximation:

$$\tilde{\alpha} = \frac{\log \left(\sum_{i=1} c_i(t)^2 + 1 \right)}{\log t} \tag{48}$$

We can come to the same solution even with the dynamic polynomial expansion presented in Section 4. In fact over each subinterval $[\tau_i, \tau_{i+1})$ the solution of the coefficients depends only on the initial conditions and is:

$$\begin{aligned} v_{ri} = \varepsilon K_{ri} Z_{ri} + K_{ri-1} \xi_{ri} \\ v_{ti} = \varepsilon K_{ri} Z_{ti} + K_{ri-1} \xi_{ti} \end{aligned} \tag{49}$$

where K_{ti-1} and K_{ri-1} come from the integration of the distribution of the velocity before the impulse at time τ_i. For N jumps we have:

$$\begin{aligned} v_r = \sum_i (\varepsilon K_{ri} Z_{ri} + K_{ri-1} \xi_{ri}) \\ v_t = \sum_i (\varepsilon K_{ri} Z_{ti} + K_{ri-1} \xi_{ti}) \end{aligned} \tag{50}$$

which equivalent to Eq. (36) because the Heaviside function $\theta(t-\tau_i)$ is equal to 1 in the interval $[\tau_i, \tau_{i+1})$.

5.2 *First Order Approximation*

We consider a slightly more complex problem with a linearised orbital dynamics with respect to a reference circular orbit.

$$\delta\dot{r} = \delta v_r \tag{51a}$$

$$\delta\dot{\nu} = -\frac{v_{t0}}{r_0^2}\delta r + \frac{v_{t0}}{r_0}\delta v_t \tag{51b}$$

$$\delta\dot{v}_r = \frac{\mu}{r_0^3}\delta r + 2\frac{v_{t0}}{r_0}\delta v_t - \frac{v_{t0}^2}{r_0^2}\delta r + \varepsilon\sum_i^N \dot{\theta}(t-\tau_i)Z_{ri} \tag{51c}$$

$$\delta\dot{v}_t = -\frac{v_{r0}}{r_0}\delta v_t - \frac{v_{t0}}{r_0}\delta v_r + \frac{v_{r0}v_{t0}}{r_0^2}\delta r + \varepsilon\sum_i^N \dot{\theta}(t-\tau_i)Z_{ti} \tag{51d}$$

This system can be written in compact form as follows:

$$\delta\dot{\mathbf{x}} = A\delta\mathbf{x} + [\mathbf{0}, \varepsilon\sum_i^N \dot{\theta}(t-\tau_i)\mathbf{Z}_i] \tag{52}$$

with $\mathbf{Z}_i = [Z_{ri}, Z_{ti}]^T$. Following the idea presented in Section 4 in each subinterval $[\tau_i, \tau_{i+1})$ the dynamics is simply $\delta\dot{\mathbf{x}} = A\delta\mathbf{x}$ with initial conditions (26). If one assumes that at t_0 the initial conditions are fully deterministic then the coefficients of the polynomial expansion of the velocity on each subinterval is:

$$\begin{array}{l} \dot{\mathbf{c}}_0 = A\mathbf{c}_0 \\ \dot{\mathbf{c}}_1 = A\mathbf{c}_1 \\ \dot{\mathbf{c}}_2 = A\mathbf{c}_2 \\ \dots \\ \dot{\mathbf{c}}_k = A\mathbf{c}_k \\ \dots \end{array} \tag{53}$$

if the matrix A has eigenvalues λ_j then the solution for each of the coefficients is in the form $\sum_j \mathbf{H}_{kj}e^{\lambda_{kj}(t-\tau_i)}$ where $\mathbf{H}_{kj}$ defines the initial conditions of each eigenvalue λ_j of each coefficients k at $t = \tau_i$. Thus at the first jump one has:

$$\begin{array}{l} \mathbf{c}_0 = \delta\mathbf{x}_0, \\ \sum_j \mathbf{H}_{1j} = [\mathbf{0}, \varepsilon\mathbf{e}], \\ \sum_j \mathbf{H}_{2j} = [\mathbf{0}, \varepsilon\mathbf{e}], \\ \dots \end{array} \tag{54}$$

with $\mathbf{e}$ a vector of 1's, because there is no uncertainty in the initial conditions and the only uncertainty comes from vector $\mathbf{Z}_i$. The initial conditions at the following jump are:

$$\begin{array}{l} \delta\mathbf{x}(\tau_{i+1}) = \delta\mathbf{x}_0 + \sum_j \mathbf{H}_{1j}e^{\lambda_j(\tau_{i+1}-\tau_i)}Z_{ri} + \\ \sum_j \mathbf{H}_{2j}e^{\lambda_j(\tau_{i+1}-\tau_i)}Z_{ti} + [\mathbf{0}, \varepsilon\mathbf{Z}_{i+1}], \end{array} \tag{55}$$

which contains the increment $\tau_{i+1} - \tau_i$, thus the rate of diffusion is dependent on the distribution of the τ_i and as before for uniformly distributed increments the rate of diffusion is only a function of the variance of the process Z_i.

5.3 Relation to the Space Environment

In [17] it was suggested that the collision rate in Earth orbit between operational satellites and debris can be expressed as:

$$\tau = \pi(R_{x_k} + R_{y_k})^2\rho_{x_k}\rho_{y_k}v_r V_{bin,k} \tag{56}$$

where R_{x_k} and R_{y_k} are the radii of the non-operational and operational objects, whereas $V_{bin,k}$ is the volume of the kth orbital shell, ρ_{x_k} and ρ_{y_k} are the spatial densities of inactive and active objects in the considered shell, and v_r is the relative velocity among the two colliding objects (which is assumed to be fixed at 10 km/s). The probability of an impact is given by the Poison distribution:

$$Pr(collision) = \tau e^{-\tau} \tag{57}$$

From (57) and (36) the conclusion is that we should observe a normal diffusion of the orbits of the objects that are subject to non-distructive collisions with a diffusion coefficient that is proportional to the average kinetic energy per impact.

5.4 Chaotic Diffusion

As demonstrated in [3] the time variation of the coefficients of the polynomial expansion can be an indication of regular or chaotic motion. Thus one could related the variation of the coefficients of the polynomial expansions to the rate of chaotic diffusion of the system. We illustrate this idea with a simple pendulum and forced pendulum motion that well relates to orbital mechanics.

$$\ddot{x} = (a\cos 5t - 1)\sin x \tag{58}$$

In re-writing Equation Eq. (58) as a system of first order differential equation

$$\dot{\mathbf{z}} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ (a\cos 5t - 1)\sin x \end{bmatrix} = \mathbf{g}(\mathbf{z}, a, t) \tag{59}$$

We can now ask the question what happens if the parameter a is uncertain? and for which initial conditions the motion is regular or chaotic. Differently from a normal nonlinear pendulum, the one with a uncertain presents different realisations of the trajectory for the same initial conditions.

We can now expand the state vector as $\mathbf{z} = \mathbf{c}_0 + 2\mathbf{c}_1 a + \sum_j \mathbf{c}_j\Phi_j(a)$, where Φ are Chebyshev polynomials of the second kind, and propagate the coefficients and then $\kappa = \sum_j c_{ij}^2$. Once κ is available, following the fit in (45) one can extract the diffusion coefficient K and the exponent α. We consider the parameter $a \in [2.25, 2.75]$ with uniform distribution and study the evolution of the coefficients of the polynomial expansion for different initial conditions.

For initial conditions $[x, v] = [0.1, 0.1]$ and period of integration $T = 10\pi$ we get the result in Figs. 4 where the ensemble of the realisations of the state variables for a set of realisations of a is represented. The two limit curves in yellow and purple are the $\pm 2\kappa$ around the mean of all the realisations. The corresponding κ is represented in Fig. 6 for both x and v with $\tilde{\alpha}$ in Fig. 6. We then tested a different set of initial conditions $[x, v] = [1, 0]$ and period of integration $T = 10\pi$ and we got the analogous result in Figs. 7. The corresponding κ is represented in Fig. 9 with $\tilde{\alpha}$ in Fig. 9. Note that the diffusion of the sate of the pendulum has to be understood as the diffusion of the ensemble of the trajectories due to the realisations of the parameter a.

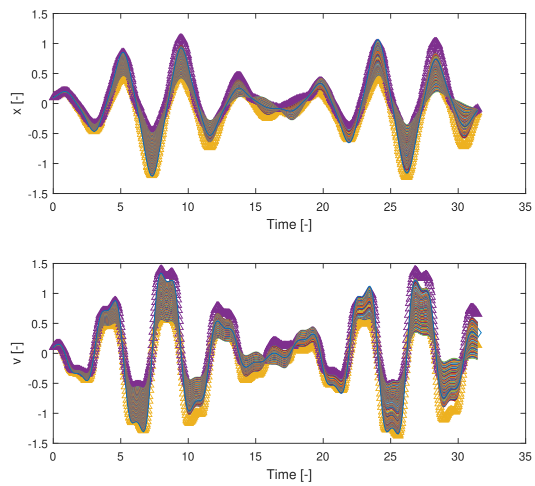

Fig. 4: State variables as a function of time for initial conditions $[0.1, 0.1]$.

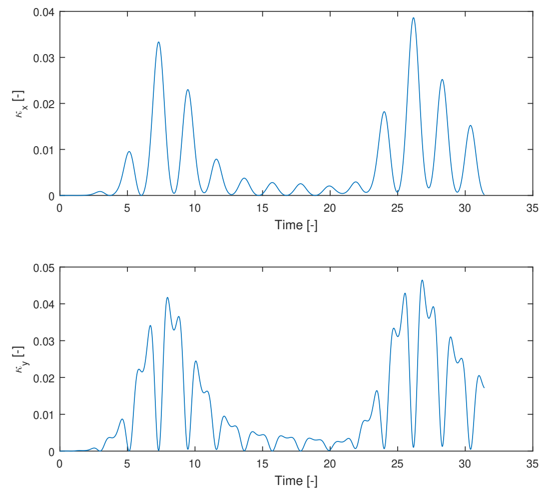

Fig. 5: κ as a function of time for initial conditions $[0.1, 0.1]$.

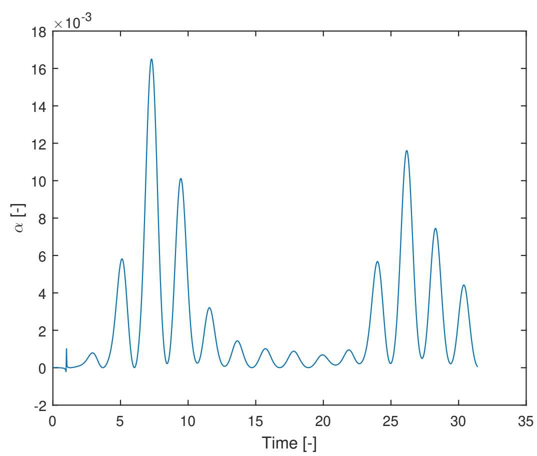

Fig. 6: $\tilde{\alpha}$ as a function of time for initial conditions $[0.1, 0.1]$.

6. Final Remarks

The paper introduced the idea of using polynomial expansions to study the diffusive character of dynamical systems in orbital mechanics. In particular it was proposed to use a Continuous Random Walk model to study the case in which an objects in space is subject to multiple, non-destructive, collisions. In this case it was shown that the orbit is subject to a diffusive process which progressively increases the energy. This is consistent with the fact that the energy is not an integral of motion in the case of stochastic perturbations even with zero mean value.

Polynomial expansions were used to model the effect of the magnitude of the impulses and it was shown how the time increments appear in the expression of the coefficients of the expansion. Thus it was suggested that studying the behaviour of the coefficients of the polynomial expansions and in particular the sum of the square of the coefficients can be used to determine the diffusion coefficient and the exponent.

Following the same principle one can then study the motion of deterministic systems that can become chaotic. In this case the coefficients of the polynomial expansion display a substantial different behaviour and the diffusion coefficient and exponent can be used to as an indication of regular and chaotic motion over a finite interval of time.

Computationally speaking the approach is limited by the fact that the polynomial expansion is only capturing the distribution of the magnitude of the impulses and not

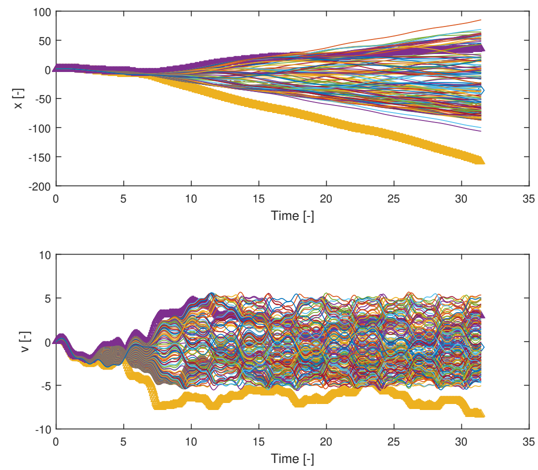

Fig. 7: State variables as a function of time for initial conditions $[1, 0]$.

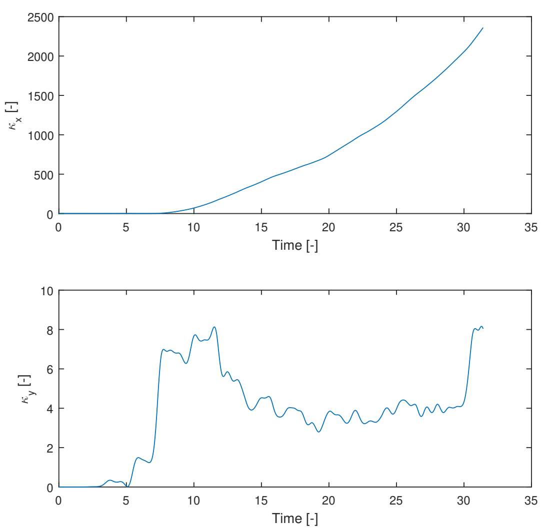

Fig. 8: κ as a function of time for initial conditions $[1, 0]$.

the time increments. Thus in a scenario in which time increments are defined by a heavy tail distribution one has to compute the polynomial expansion for multiple realisations of the increments. Future work will need to focus on a more computationally efficient quantification of the effect of the time increment as well.

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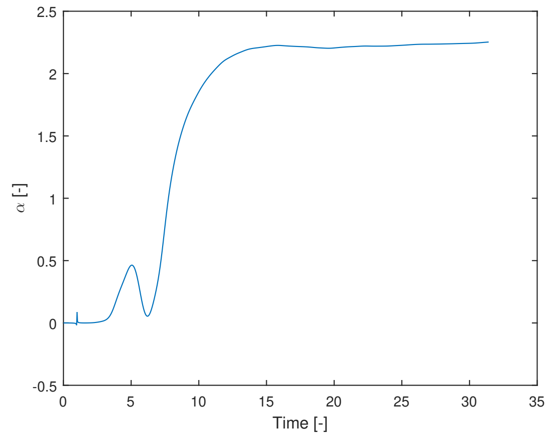

Fig. 9: $\tilde{\alpha}$ as a function of time for initial conditions $[1, 0]$.

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