On degradation-based imperfect repair and induced generalized renewal processes

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Abstract. In this paper, we suggest and justify a new, basic approach to modelling the imperfect repair and the corresponding imperfect repair processes for items with observable degradation. We consider monotone processes of degradation with independent increments. Imperfect repair reduces degradation of an item on failure to some intermediate level. To define the state of an item after this imperfect repair, the random virtual age is introduced. Some stochastic properties describing the corresponding remaining lifetime are considered. The generalized renewal process based on the suggested notion of imperfect repair is described and some of its properties are studied. An alternative approach that considers the imperfect repair process defined in the degradation scale is outlined.

Keywords: imperfect repair; virtual age; renewal processes; remaining lifetime; stochastic comparisons; gamma process

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1. Introduction

Renewal processes, apart from their mathematical attractiveness, in practice, present an adequate operational description for the repairable systems when repair is *perfect.* In reality, this assumption does not hold due to various reasons such as variable environment, aging of spare parts, quality of repair facilities, etc. Therefore, in reliability applications, modeling of *imperfect* repair/maintenance was addressed in numerous publications. One of the most popular imperfect repair models (age reduction) is based on the notion of *virtual age* (Kijima, 1989; Doyen and Gaudoin, 2004; Finklestein, 2008). As the main goal of this paper is to discuss an alternative approach to the virtual age concept reported in the literature on imperfect repair/maintenance, we first describe the latter on the level required for the presentation to follow.

Let an item with the lifetime T, the Cdf $F(t)$, the survival function $\bar{F}(t)$, the pdf $f(t)$ and the failure rate $\lambda(t)$ be incepted into operation at t=0. Assume that $\lambda(t)$ is strictly increasing, then its value at t uniquely defines the chronological age of an item, i.e., $\lambda^{-1}(\lambda(t)) = t$.

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An item fails at $t^* \in (0,\infty)$, which is a realization of T and the repair action is then initiated. The perfect repair reduces an item's age to 0, whereas the minimal repair (Barlow and Proschan, 1975) does not change the age and the distribution of the remaining lifetime. The Brown-Prochan imperfect repair model (1983) combines these two types of repair, i.e., with a given probability the repair is perfect and with the complementary one, it is minimal (see also Badía and Berrade (2006, 2007, 2009)). Other theoretical option is when repair reduces the age of the item at failure to some intermediate level τ , i.e., $0<\tau<t^*$ (age reduction), whereas the corresponding remaining lifetime is defined in terms of survival functions as

$$\bar{F}(t\,|\,\tau)=\frac{\bar{F}(t+\tau)}{\bar{F}(\tau)}=\exp\left\{-\int_0^t \lambda(\tau+x)dx\right\}. \tag{1}$$

Relationship (1) means that the 'shape' of the failure rate after this type of imperfect repair remains the same and the function is just 'shifted' on τ in its the argument. The well-known Kijima's models for imperfect repair processes are based on this assumption and consider linear age reduction at each imperfect repair (Kijima, 1989, Finkelstein, 1989). See also Doyen and Gaudoin (2004), Finkelstein (2007), Tanwar *et al* (2014). Dijoux et al (2016), Doyen et al (2017), Zhao et al (2019) to name a few. Note that the repair in (1) can be also interpreted (and it will be important for us in what follows) as: *the failed item is replaced by the statistically identical one that was operating and did not fail in* $[0,\tau]$.

Perfect repair is usually realized in practice by the replacement of the failed item with the new one, minimal repair is also well-justified when, e.g., a small part of a large failed system is repaired/replaced. However, general reduction of age to some intermediate level modeled by (1) does not have this practical, justified meaning. It can be considered as a plausible, formal model, which possesses tractable properties and also can be successfully used for statistical inference (see, e.g., Levitin and Lisniansky (2000), Dijoux *et al* (2016), de Toledo *et al* (2015) to name a few). In order for the virtual age concept that arises in the problem of imperfect repair/maintenance to be sound and practically justified, there should be a clear description of the corresponding repair operations that result in (and conform with) relationship (1). In our opinion, this is not the case for most repairable systems except for a specific case of minimal repair. For instance, the failure rate of the "cold" standby system with n exponentially distributed i.i.d. lifetimes of components with failure rates λ is given by

$$\lambda(t)=\frac{\lambda^n t^{n-1}}{(n-1)!\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}}.$$

The imperfect repair of the failed system means that only $1<k<n$ components are replaced ($k=1$ is the 'minimal repair' and $k=n$ is the perfect repair). Obviously, after this type of repair, the shape of the failure rate changes and one cannot discuss this setting in terms of the model (1). Similar conclusion can be made for the 'continuous variant' of this setting when degradation of an item is modelled by a gamma process. The failure occurs on reaching the predetermined level of degradation, whereas the imperfect repair in this case results in decreasing this level to some intermediate value. Other numerous examples can also support this claim

In this paper, we will discuss a new approach to the imperfect repair modeling based on the suggested notion of virtual age that, in our opinion, does not possess the above described deficiency of the model based on (1). It will be more specific in terms of the class of lifetime distributions that describe degradation of repairable items. This is the price for the more informative description of the remaining lifetime than in the black-box scenario (1). We will consider the internal degradation of items due to some internal processes of 'tear and wear'. Thus, when we reduce the wear/degradation, the 'clock of the wear process' is also reset according to the model to be defined. The external degradation processes (e.g., shocks) does not usually possess the property of setting the clock back (as they influence but not influenced by item's operation) and should be considered differently.

It should be noted that some models dealing with reduction of the current degradation mostly during preventive maintenance (PM) actions were discussed in the literature (see, e.g., Zhao et al (2019) and Berenguer et al (2003), to name a few). For instance Kahle (2019) discusses linear deterioration reduction and compares it with the corresponding age reduction during the PM for a specific Wiener process of degradation. However, as mentioned above, our reasoning is different from that reported in the literature, as it employs the new and effective notion of virtual age.

The paper is organized as follows. In section 2, we describe the model. Section 3 gives some examples of degradation processes. Section 4 provides relevant stochastic comparisons. In Section 5, the corresponding repair process is described. Section 6 considers the repair process in the degradation scale. Finally, some remarks are given in Section 7.

2. Degradation-based imperfect repair

Let a failure of an item with a lifetime $T \geq 0$ occurs when its degradation induced by the internal wear and tear processes exceeds the deterministic threshold level. For instance, when the production rate of a production system is deteriorating, the threshold can be easily set as some unacceptable level. Thus, it can be not necessarily a failure as such but rather an undesirable condition/state. In case of a random threshold (see later), it is usually an 'ordinary' failure of an item.

Assume that the observable (continuously monitored or only at failure) internal deterioration process $\{W_t, t \geq 0\}, W_0 = 0$, has independent increments and is characterized by the monotonically increasing sample paths. Then

$$P(T > t) \equiv \bar{F}(t, w) = P(W_t \leq w) \tag{2}$$

is the survival function of the time to failure when the failure is defined as reaching/crossing the level *w*.

Remark 1. Generalization of the following to the case of non-monotone stochastic processes (e.g., Wiener process with positive drift) can be also considered based on distributions of the first passage times for specific processes. Moreover, processes with dependent increments can be also discussed. However, here we want to emphasize the suggested novel approach, which can be better illustrated by a simpler case of monotone processes with independent increments.

Perfect degradation-wise repair brings degradation to 0. Obviously, there is no minimal repair in this case. *Define* the imperfect repair as the repair that reduces degradation *w* to some intermediate level $\tilde{w}$, $0 < \tilde{w} < w$. See Figure 1 for the change in the degradation level of the item under the imperfect repair.

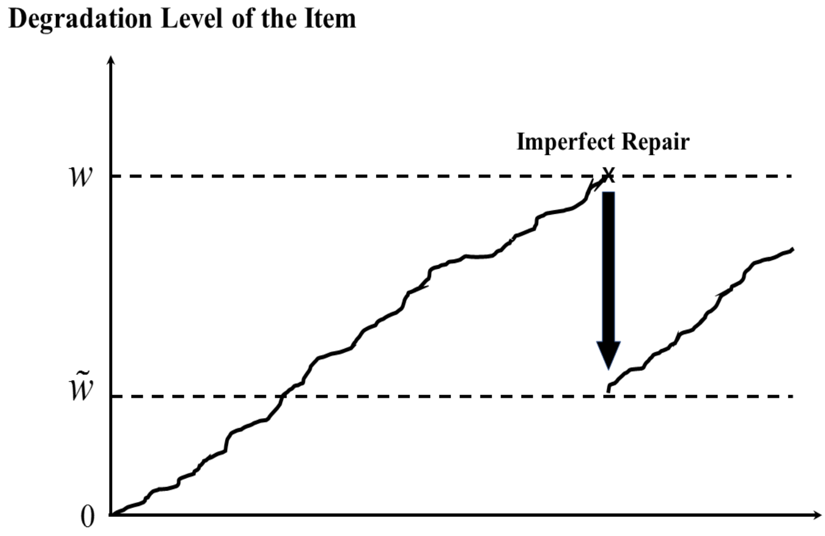

Figure 1. The change in the degradation level of the item under the imperfect repair

This is equivalent mathematically to reducing the threshold to $w-\tilde{w}$, and returning degradation to zero level in the homogeneous case (see below and Section 5). Both interpretations of this imperfect repair can exist. However, the initial one is better justified in practice due to the observed degradation level after the imperfect repair of this kind.

The remaining lifetime (or the residual lifetime) is an important characteristic in reliability (Salehi et al., 2012; Belzunce et al. 2008; Hazra et al. 2018). We will consider now two cases:

a. The process $\{W_t, t \geq 0\}$ is *homogeneous.* In this case, due to the property of independent increments, the state of an item after repair is completely described by the decreased wear $\tilde{w}$, as the remaining lifetime of an item T_r is, obviously, completely defined by the Cdf

$$F_{\tilde{w}}(t) \equiv F(t, w-\tilde{w}), \tag{3}$$

where $F(t,w) \equiv 1-\bar{F}(t,w)$.

b. The process $\{W_t, t \geq 0\}$ is non-homogeneous. The following question arises first: what age of an item that have started operation at t =0 corresponds to $\tilde{w}$? The remaining lifetime should obviously depend on it, whereas in (3), this age was irrelevant due to homogeneity of the process. Although it has a different meaning than in imperfect repair Kijima-type models, we prefer to call it also the *virtual age* and denote by $T_V(\tilde{w})$. Using (2), we define it as a *random variable* with the Cdf $F(t,\tilde{w})$. Thus, $T_V(\tilde{w})$ describes the time that is needed for the process that starts at $t=0$ to accumulate wear $\tilde{w}$. Then the pair $\{\tilde{w}, T_V(\tilde{w})\}$ completely and unambiguously defines the state of an item, whereas the corresponding remaining lifetime is defined by the following survival function

$$\bar{F}_{\tilde{w}}(t) = \int_0^\infty P(W_{x+t} - W_x \leq w-\tilde{w}) f(x,\tilde{w})dx, \tag{4}$$

where $f(x,\tilde{w}) = \frac{\partial}{\partial x} F(x,\tilde{w})$. In the non-homogeneous case, the failure after the imperfect repair occurs when the degradation process, starting from the virtual age, exceeds the threshold $w-\tilde{w}$. Thus, similar to the homogeneous case, the repair has reduced the threshold from w to $w-\tilde{w}$ for the degradation process starting from the virtual age (in each realization).

It should be noted that here we also imply some assumptions that we think to be reasonable for describing the model. However, in our opinion, they are much more practically justified than those of the 'black-box' imperfect repair model (1), as they are based on the real observed degradation of an item.

Definition 1. *The degradation-based imperfect repair (DIR) is the operation that decreases the wear w at failure (corrective maintenance) or at preventive maintenance of an item to the value* $\tilde{w}$, $0<\tilde{w}<w$.

For the homogeneous degradation process, the remaining lifetime after this operation is described by the distribution (3).

For the nonhomogeneous degradation process $\{W_t, t \geq 0\}$*, the remaining lifetime is described by (4), where* $f(t,\tilde{w})$ *is the pdf of the virtual age* $T_V(\tilde{w})$*, i.e., of the random time that is needed for a statistically identical item that starts at* $t=0$ *to accumulate wear* $\tilde{w}$*. The Cdf of* $T_V(\tilde{w})$ *is given by* $F(t,\tilde{w})$ *defined by (2).*

For the homogeneous case, Definition 1 does not use any additional assumptions, as (3) unambiguously defines the remaining lifetime, and, therefore, the model is completely justified. In fact, we do not need the virtual age (although it, obviously, formally exists) for this case. In the nonhomogeneous case, the remaining lifetime is defined via the virtual age of an item that is statistically identical to the initial one and had accumulated wear $\tilde{w}$. The similar assumption, as already mentioned in the Introduction, characterizes (1), but in that case, it is much less informative, as age τ of an operating item can correspond to any level of deterioration w^*, $0<w^*<w$. One can argue that there can exist a certain dependence on the past for an item, as the wear at failure for this specific item is decreased, whereas we define the remaining lifetime for the statistically identical item that had gained the same wear $\tilde{w}$. However, as we, in general, do not know (or cannot effectively model) the mechanism of repair except for the observed reduced wear, *the imposed assumption* seems reasonable and not, in fact, restrictive for modelling on this general level. See the last paragraph of "Concluding remarks and discussion" for more detail. On the other hand, for specific deterioration processes, e.g., for the homogeneous gamma process, one can adjust the introduced notion of virtual age to the case when e.g., the corresponding shape parameter is increased after the imperfect repair, modelling more intensive degradation afterwards.

The way to choose $\tilde{w}$ depends on the specific application in practice and there can be many settings, where the developed general model can be applied. One of the possibilities that can be applied in practice is to set as $\tilde{w}=qw, 0<q<1$. Then, q (distinct from the age-reduction models) has a clear 'physical' meaning.

3. Supplementary examples

3.1. Poisson process of degradation. As the simplest but meaningful reliability example, assume that stochastic degradation is defined by a step function modelled by the corresponding homogeneous Poisson process with rate λ. Thus, the survival function of the waiting time until the M-th event in this process is

$$\bar{F}(t, M) = \exp\{-\lambda t\} \sum_{i=0}^{M-1} \frac{(\lambda t)^i}{i!},$$

which describes reliability of the 1 out of M cold standby system with the i.i.d., exponentially distributed components.

In what follows, we provide some practical background and examples for the cold standby systems. Redundancy is a commonly used technique to ensure high reliability of various systems. In a cold standby system, only a minimum number of components are kept in operation and the standby ones are switched to the fully operational mode only when the operating components fail. Since the cold standby components are not exposed to operational stresses, they are not subject to failures in the standby mode and, therefore, the systems with this mode of components are more reliable than those with the hot standby or the warm standby modes. The cold standby mode of elements is widely used in practice especially in the mission-oriented, autonomous and safety-critical systems (spaceships, submarines, nuclear power stations, etc).

The spare parts sufficiency can be also modelled via the Poisson degradation process, whereas the spare parts replenishment is executed when the number of available components reaches the predetermined level. In this way, we can speak in terms of *imperfect repair* of a system that consists of the main component and several identical spare parts. A similar slightly modified modeling can be applied to the *imperfect software debugging* when the bugs in the software during operation occur in accordance with the Poisson process (Finkelstein (2008)).

As the 1 out of M cold standby system described above is composed of M statistically identical components, the model parameter λ can be estimated just by estimating the parameter of the i.i.d. exponentially distributed components having the failure rate λ. The detailed estimation procedures can be found, e.g., in Meeker and Escobar (2014).

The failure of a system occurs with the M-th failure of components and degradation is completely defined by the number of failed components m and can be easily observed in practice. Thus, in accordance with our approach, the virtual age of a system, which degradation after the imperfect repair has been reduced from M to $m < M$, is a *random variable* with the Cdf $F(t,m)$ and the pdf $f(t,m)$, whereas the remaining lifetime distribution is, obviously, given by

$$F(t, M-m) = 1 - \exp\{-\lambda t\} \sum_{i=0}^{M-m-1} \frac{(\lambda t)^i}{i!} \tag{5}$$

and does not depend on this virtual age.

Consider now the nonhomogeneous Poisson process (NHPP) with rate $\lambda(t)$ and degradation described by the number of occurred events with the threshold M and the reduced wear m. Thus, the pdf of the corresponding virtual age in this case is

$$f(x,m) = \lambda(x)\exp\{-\Lambda(x)\}\frac{(\Lambda(x))^{m-1}}{(m-1)!},$$

where $\Lambda(t) = \int_0^t \lambda(u)du$. Then, in accordance with (4), the survival function describing the remaining lifetime is

$$\bar{F}_m(t) = \int_0^\infty P(W_{x+t} - W_x < M - m) f(x,m)dx$$

where

$$P(W_{x+t} - W_x < M - m) = \exp\{-\Lambda(x,t)\}\sum_{i=0}^{M-m-1}\frac{(\Lambda(x,t))^i}{i!};\ \Lambda(x,t) = \int_x^{x+t}\lambda(u)du .$$

3.2. Gamma process of degradation. In the same manner, we can illustrate the introduced notion of virtual age for the gamma process of degradation. The gamma process is also widely used in the literature for modeling ‘continuous’ degradation (the fatigue in materials, cracks growth, structural engineering etc.). More specifically, the gamma process can be applied to modeling the degradation processes in aircraft landing gear brakes, in boiler heat exchangers, in GaAs lasers, etc.

In the homogeneous case,

$$\bar{F}(t,w) = 1 - \frac{\Gamma(\alpha t, \lambda w)}{\Gamma(\alpha t)}, \tag{6}$$

where $\Gamma(a) = \int_0^\infty z^{a-1}\exp\{-z\}dz\,; \Gamma(a,x) = \int_x^\infty z^{a-1}\exp\{-z\}dz$, $\alpha > 0, \lambda > 0$ are the shape and scale parameters, respectively. The corresponding remaining lifetime is defined by

$$\bar{F}(t, w - \tilde{w}) = 1 - \frac{\Gamma(\alpha t, \lambda(w - \tilde{w}))}{\Gamma(\alpha t)},$$

whereas for the nonhomogeneous gamma process with the non-linear shape function $\alpha(t)$, the corresponding survival function is defined by (4) and (6) and

$$P(W_{x+t} - W_x \le w - \tilde{w}) = 1 - \frac{\Gamma((\alpha(t+x) - \alpha(x)), \lambda(w - \tilde{w}))}{\Gamma(\alpha(t+x) - \alpha(x))}.$$

Observe that

$$F(x,\tilde{w}) = 1 - \int_0^{\tilde{w}} \frac{1}{\Gamma(\alpha(x))}\lambda^{\alpha(x)}u^{\alpha(x)-1}\exp(-\lambda u)\ \mathrm{du},$$

$$f(x,\tilde{w}) = \frac{\partial}{\partial x}\left(1 - \int_0^{\tilde{w}} \frac{1}{\Gamma(\alpha(x))}\lambda^{\alpha(x)}u^{\alpha(x)-1}\exp(-\lambda u)\ \mathrm{du}\right)$$

$$= \frac{A(x,\tilde{w})}{\left(\int\limits_0^\infty s^{\alpha(x)-1} \exp(-s)\ \mathrm{ds} \right)^2},$$

where

$$A(x,\tilde{w}) = \int\limits_0^{\tilde{w}} \left[-\left(\ln\lambda + \ln u\right)\ \alpha'(x)\lambda^{\alpha(x)} u^{\alpha(x)-1} \exp(-\lambda u) \left(\int\limits_0^\infty s^{\alpha(x)-1} \exp(-s)\ \mathrm{ds} \right) \right.$$

$$\left. + \lambda^{\alpha(x)} u^{\alpha(x)-1} \exp(-\lambda u) \left(\int\limits_0^\infty \alpha'(x)\ (\ln s)\ \mathrm{s}^{\alpha(x)-1} \exp(-s)\ \mathrm{ds} \right) \right] \mathrm{d}u\,.$$

Then, combining all of these, the corresponding remaining lifetime is given by

$$\bar{F}_{\tilde{w}}(t) = \int\limits_0^\infty P(W_{x+t} - W_x \le w - \tilde{w}) f(x,\tilde{w}) dx\,.$$

The two most common methods of parameter estimation (in reliability framework) for gamma processes, namely, the maximum likelihood and the method of moments, are discussed in detail in van Noortwijk (2009), along with the Bayesian analysis when the scale parameter of the gamma process is assumed to have an inverted gamma distribution as a prior. See also, e.g., Dufresne et al. (1991) and Wang (2009).

Remark 2. As degradation is decreased to the fixed $\tilde{w}$, we are not concerned with a possible overshooting for the gamma process, i.e., achieving the value larger than w at failure, as in the case of the linear reduction is performed. Note that. the inverse-Gaussian process with continuous sample paths (no jumps) can be also considered as the specific case for our modelling.

4. Some stochastic comparisons

In this section, we will discuss some initial stochastic comparisons (mostly in the sense of the usual stochastic ordering) involving the virtual age introduced in Section 2 and the corresponding remaining lifetime. For general introductions of the concepts of stochastic comparison, see Belzunce et al. (2015). Further, more detailed studies can constitute a topic for future research in this specific direction. Note that, here $\tilde{w}$, $0 < \tilde{w} < w$ will be considered as some intermediate value of the accumulated degradation and not necessarily related to the imperfect repair model. Thus, these comparisons will be discussed in a more general setup.

For the homogeneous case, in accordance with our notation,

$$P(W_{x+t} - W_x \le w - \tilde{w}) = P(W_t \le w - \tilde{w}) = \bar{F}(t, w - \tilde{w}) \equiv \bar{F}_{\tilde{w}}(t)\,. \quad (7)$$

Thus, for all positive $w_1 \le w_2$, obviously,

$$\bar{F}(t,w_1) \le \bar{F}(t,w_2) \quad (8)$$

and thus, the remaining lifetime in (7) is decreasing in the sense of the usual stochastic ordering (Shaked and Shantikumar, 2007) as $\tilde{w}$ is increasing.

We will compare now the corresponding remaining lifetimes for the nonhomogeneous case.

Theorem 1. *Let* $W_{x+t} - W_x$ *be stochastically increasing in x in the sense of the usual stochastic order for all fixed* $t > 0$. *Then, for* $0 < w_1 \le w_2 < w$,

(i) $T_V(w_1) \le_{st} T_V(w_2)$

(ii) $\bar{F}_{w_1}(t) \ge \bar{F}_{w_2}(t)$, *for all* $t > 0$.

Proof.

Due to the definition of the virtual age $T_V(\tilde{w})$ in (2), Eq. (8) clearly means that the virtual age $T_V(\tilde{w})$ that corresponds to deterioration $\tilde{w}$ is increasing in $\tilde{w}$ in the same stochastic sense, i.e.,

$$T_V(w_1) \le_{st} T_V(w_2).$$

Observe that $\bar{F}_{\tilde{w}}(t) = E[h(T_V(\tilde{w}))]$, where $h(x) \equiv P(W_{x+t} - W_x \le w - \tilde{w})$. As $h(x)$ is a decreasing function of x and $T_V(w_1) \le_{st} T_V(w_2)$, it holds that $\bar{F}_{w_1}(t) \ge \bar{F}_{w_2}(t)$, for all $t > 0$.

■

Corollary 1. *For the nonhomogeneous Gamma process with convex* $\alpha(t)$ *and* $\lambda > 0$ *(see Section 3), and for* $0 < w_1 \le w_2 < w$,

$$\bar{F}_{w_1}(t) \ge \bar{F}_{w_2}(t), \text{ for all } t > 0.$$

Proof.

The fact that $W_{x_1+t} - W_{x_1} \le_{st} W_{x_2+t} - W_{x_2}$, for $x_1 < x_2$, follows from Müller and Stoyan (2002). Thus, the assumption in Theorem 1 is satisfied.

■

Let us compare now the remaining lifetimes for two items characterized by the degradation processes, $\{W_{1,t}, t \ge 0\}, \{W_{2,t}, t \ge 0\}$ when, in both cases, the intermediate level (e.g., to which the repair reduces degradation at failure, *w*) is the same , i.e., $\tilde{w}$, $0 < \tilde{w} < w$.

Theorem 2. *Let* $W_{1,x+t} - W_{1,x} \le_{st} W_{2,x+t} - W_{2,x}$ *for all* $x \ge 0, t > 0$, *and* $W_{2,x+t} - W_{2,x}$ *is stochastically decreasing in x in the sense of the usual stochastic order for all fixed* $t > 0$. *Then,*

$$\bar{F}_{1\tilde{w}}(t) \ge \bar{F}_{2\tilde{w}}(t), \textit{ for all } t > 0,$$

where $\bar{F}_{1\tilde{w}}(t)$ *and* $\bar{F}_{2\tilde{w}}(t)$ *are the survival functions of the remaining lifetimes for the first and the second items.*

Proof.

From the assumption, $W_{1,t} \le_{st} W_{2,t}$,

$$\bar{F}_1(t, \tilde{w}) \equiv P(T_{1V}(\tilde{w}) > t) = P(W_{1,t} \le \tilde{w}) \ge P(W_{2,t} \le \tilde{w}) = P(T_{2V}(\tilde{w}) > t) \equiv \bar{F}_2(t, \tilde{w}).$$

This means that $T_{1V}(\tilde{w}) \ge_{st} T_{2V}(\tilde{w})$, where $T_{1V}(\tilde{w})$ and $T_{2V}(\tilde{w})$ are the random virtual ages for the two processes, respectively. Observe that

$$\bar{F}_{1\tilde{w}}(t) = E[P(W_{1,T_{1V}(\tilde{w})+t} - W_{1,T_{1V}(\tilde{w})} \le w - \tilde{w})]$$
$$\ge E[P(W_{2,T_{1V}(\tilde{w})+t} - W_{2,T_{1V}(\tilde{w})} \le w - \tilde{w})]$$
$$\ge E[P(W_{2,T_{2V}(\tilde{w})+t} - W_{2,T_{2V}(\tilde{w})} \le w - \tilde{w})] = \bar{F}_{2\tilde{w}}(t),$$

where the expectations, similar to Theorem 1, are with respect to the pdfs of the corresponding virtual ages. The first inequality holds due to the assumption that $W_{1,x+t} - W_{1,x} \le_{st} W_{2,x+t} - W_{2,x}$, for all $x \ge 0, t > 0$, and the second inequality holds due to the fact that $T_{1V}(\tilde{w}) \ge_{st} T_{2V}(\tilde{w})$ and $W_{2,x+t} - W_{2,x}$ is stochastically decreasing in x in the usual stochastic order sense for all fixed $t > 0$.

■

Corollary 2*. Consider two nonhomogeneous Gamma processes with* $(\alpha_1(t), \lambda_1)$ *and* $(\alpha_2(t), \lambda_2)$ *(see Section 3). Suppose that* $\alpha_1(x+t) - \alpha_1(x) \le \alpha_2(x+t) - \alpha_2(x)$ for all $x \ge 0, t > 0$, and $\lambda_1 \ge \lambda_2$, *and* $\alpha_2(t)$ *is concave. Then,* $\bar{F}_{1\tilde{w}}(t) \ge \bar{F}_{2\tilde{w}}(t)$, *for all* $t > 0$.

Proof.

It follows from Müller and Stoyan (2002) that the assumptions in Theorem 2 are satisfied.

■

Consider now a random failure threshold W described by the Cdf $G(w)$ and the pdf $g(w)$. It is just more convenient to write the following in terms of the distribution functions and not survival functions as above. Then the time to failure of an item is described by the following Cdf

$$F(t) = \int_0^\infty F(t,w)g(w)dw = \int_0^\infty P(W_t > w)g(w)dw = E[P(W_t > W)] = E[h(W)], \tag{9}$$

where $h(w) = P(W_t > w)$, which is a decreasing function of w. Thus, for two random thresholds ordered as $W_1 \le_{st} W_2$, that is, $G_2(w) \le G_1(w)$, we have

$$F_1(t) \equiv E[h(W_1)] \ge E[h(W_2)] \equiv F_2(t),$$

which implies the ordering of lifetimes in this case is also in the sense of the usual stochastic order.

What about the corresponding remaining lifetime for a random failure threshold? Assume that no failure had occurred in the degradation interval $[0, \hat{w})$ and we are looking at $F(t \mid \hat{w})$ - the remaining lifetime since the corresponding random time (which is, obviously, the virtual age $T_V(\hat{w})$). In the homogeneous case, $F(t \mid \hat{w})$ does not depend on $T_V(\hat{w})$, i.e.,

$$F(t \mid \hat{w}) = \int_{\hat{w}}^\infty F(t, w - \hat{w})g(w \mid \hat{w})dw = \int_{\hat{w}}^\infty P(W_t > w - \hat{w})g(w \mid \hat{w})dw, \tag{10}$$

where $g(w \mid \hat{w})$ is the pdf that corresponds to the Cdf

$$G(w \mid \hat{w}) = \frac{G(w) - G(\hat{w})}{\bar{G}(\hat{w})}, w > \hat{w}.$$

Obviously, (10) reduces to (3) for $\hat{w} = \tilde{w}$ and the degenerate $G(w)$.

On the contrary, the remaining lifetime depends on $T_V(\hat{w})$ for the nonhomogeneous case. Thus, from (4),

$$F(t \mid \hat{w}) = \int_{\hat{w}}^{\infty}\int_{0}^{\infty} P(W_{x+t} - W_x > w - \hat{w}) f(x, \hat{w}) g(w \mid \hat{w}) dx dw = E_{(W|W>\hat{w})}[h(W)],$$

where $E_{(W|W>\hat{w})}[\cdot]$ stands for the expectation with respect to the conditional distribution of $(W \mid W > \hat{w})$ and $h(w) = \int_{0}^{\infty} P(W_{x+t} - W_x > w - \hat{w}) f(x, \hat{w}) dx$, which is decreasing in w. Consider now two random failure thresholds W_1, W_2 with the corresponding Cdfs $G_1(w), G_2(w)$, respectively. It is well known, that the usual stochastic ordering of two random variables does not necessarily lead to the usual stochastic ordering of the corresponding remaining lifetimes for all values of arguments. Therefore, we assume the hazard rate ordering. Then,

$$W_1 \le_{hr} W_2 \Rightarrow G_1(w \mid \hat{w}) \ge G_2(w \mid \hat{w}), \text{ i.e., } (W_1 \mid W_1 > w) \le_{st} (W_2 \mid W_2 > w)$$

and, similar to (9), the ordering of the corresponding remaining lifetimes in the sense of the usual stochastic ordering can be obtained, i.e.,

$$F_2(t \mid \hat{w}) \le F_1(t \mid \hat{w}).$$

As this ordering holds for all realizations of the virtual age $T_V(\hat{w})$, it holds for the nonhomogeneous case as well.

5. Reduction of degradation and the imperfect repair processes

Consider now the imperfect repair processes based on the notion of imperfect repair suggested in this paper. For motivation and further discussion, we must first recall the relevant facts describing the geometric process (Lam, 1988, 2007; Pérez-Ocón and Torres-Castro, 2002; Pérez-Ocón and Montoro-Cazorla, 2004). Then the generalized renewal process of imperfect repairs that is prompted by analogies with the geometric process will be considered. Finally, we end this section with a general description of the generalized renewal process of imperfect repairs.

Let $\{T_n\}, n = 1, 2, ...$ be a collection of independent lifetimes with the Cdfs $F_n(t)$ that are called cycles and interpreted as the inter-arrival times for the corresponding generalized renewal process

$$N(t) = \sup\{n : S_n \le t\}, \quad t \ge 0; \; S_n = \sum_{i=1}^{n} T_i, \quad S_0 = 0.$$

Assume the specific form of the Cdfs, namely

$$F_n(t) = F(a^{n-1}t), n = 1, 2, ..., \tag{11}$$

where $F(t)$ is a baseline distribution. Then the defined process is called the *geometric process* (Lam, 1988, 2007). When $a > 1$,

$$F_n(t) \le F_{n+1}(t), t > 0 .$$

and, therefore, the inter-arrival times are stochastically decreasing and the process is converging ($\lim_{n\to\infty} E[S_n] < \infty$), if the mean that corresponds to $F(t)$ is finite.

Thus, the geometric process defined above presents a mathematically tractable model for imperfect repair at each cycle. Moreover, the corresponding renewal equations generalizing renewal equations for the standard renewal process can be derived and solved in terms of Laplace transforms. This process has attracted a lot of attention in reliability applications. For instance, various optimal replacement problems can be solved minimizing the long-run cost rate when preventive replacement (incepting the new system) is scheduled upon failure after the *m*-th imperfect repair of the described type.

On the other hand, from a practical point of view, it is not clear how a in (11), which is, in fact, just a scale transformation in the argument of the corresponding Cdf, relates to the real 'physical' action of repair. Similar to our discussion of (1) and Kijima-type imperfect repair processes, we can conclude that it can constitute a plausible *black-box statistical model*. However, when we have an additional information, in the form of the observed degradation, it is tempting to use the approach discussed in the previous sections of this paper. This approach can be justified as a real 'physical' operation of reducing degradation of an item and not just a statistical model.

Let us now generalize one single imperfect repair defined in Section 2 to successive imperfect corrective repairs for the degradation model studied in this paper. That is, after the first failure, the imperfect repair reduces degradation w to some intermediate level $\tilde{w}_1$, and after the second failure, the imperfect repair reduces degradation w to some intermediate level $\tilde{w}_2$, and so on.

Consider the *homogeneous* case, defined by the degradation reduction in (3). Assume that with each imperfect repair, the threshold w stays the same but the reduced wear in (3) is increasing with each repair, e.g., also in a geometric-type way, i.e.,

$$\tilde{w}_1 = (1-\rho)w, \tilde{w}_2 = (1-\rho^2)w,...; w - \tilde{w}_i = \rho^i w; 0 < \rho < 1. \tag{12}$$

Then, (12) results in the sequence that does not depend on the levels $\tilde{w}_i$, i.e.,

$$F(t,w) \equiv F_1(t) < F(t,\rho w) \equiv F_2(t) < F(t,\rho^2 w) \equiv F_3(t),... \tag{13}$$

Thus, the cycles of this process are stochastically decreasing. Whether it converges in the sense $\lim_{n\to\infty} E[S_n] < \infty$, as the geometric process above, depends on the underlying degradation process. Obviously, the similar to (13), ordering holds for any increasing sequence of degradation levels after consecutive repairs, i.e., $\tilde{w}_1 < \tilde{w}_2 < ...$.

The cycles $\{T_n\}, n = 1,2,...$ that are described by the Cdfs $F_i(t)$ in (13) constitute the generalized renewal process. Assume that the mean durations of the cycles μ_i are finite and decrease with *i* in such a way that the process is converging in the following sense

$$\sum_{i=1}^{\infty} E[T_i] = \sum_{i=1}^{\infty} \mu_i = b < \infty . \tag{14}$$

and denote $\sum_{i=1}^{\infty} T_i \equiv S$.

Remark 3. For illustration, consider the homogeneous Poisson process with a sufficiently large intensity λ, so that degradation levels M and m in (5) can be considered approximately as 'continuous'. Let $1/\lambda = d$ be the corresponding mean of the inter-arrival times. Then $\mu_i = \rho^{i-1} M\, d, i = 1,2...$ Thus (14), for the process (13) takes place.

As in the ordinary renewal theory, the expectation (renewal function), $H(t) \equiv E[N(t)]$ is of the main interest. By analogy (Lam, 2007; Wang and Pham, 2006), the same general equation for the renewal function $H(t)$ holds:

$$H(t) \equiv E[N(t)] = \sum_{n=1}^{\infty} F^{(n)}(t), \tag{15}$$

where $F^{(n)}(t)$ is the Cdf of S_n, $n = 1,2,...$. , $F^{(1)}(t) \equiv F_1(t)$.

We will show now that in our case this function is infinite, which looks a bit counter-intuitive, however, similar 'burstiness' of the 'renewal function' was reported in the literature for the standard geometric process in Braun et al (2005). The following result is more general than for the specific setting (13)-(14) described above.

Theorem 3. *Assume that the governing lifetime Cdf $F(t,w)$ is absolutely continuous, strictly positive and strictly increasing for all $t > 0$. Let the point process $N(t), t \ge 0$ be converging in the sense of (14).*

Then $E[N(t)]$ is infinite for all $t > 0$.

Proof. Similar to Finkelstein (2010), let $K(t)$ denote the Cdf of $\sum_{i=1}^{\infty} T_i$. It follows from (14) that there exists $\varepsilon > 0$ such that

$$K(b) \equiv \mathrm{P}\left(\sum_{i=1}^{\infty} T_i \le b\right) > \varepsilon. \tag{16}$$

As

$$\mathrm{P}\left(\sum_{i=1}^{n} T_i \le b\right) \ge \mathrm{P}\left(\sum_{i=1}^{\infty} T_i \le b\right)$$

and

$$N(b) \ge n \Leftrightarrow \sum_{i=1}^{n} T_i \le b,$$

we have

$$\mathrm{P}\big(N(b) \ge n\big) > \varepsilon, n \ge 1. \tag{17}$$

Thus, from (17),

$$H(b) = E[N(b)] = \sum_{i=1}^{\infty} \mathrm{P}(N(b) \ge i) = \infty.$$

As $H(t)$ is non-decreasing, it is also infinite for all $t \ge b$.

To show this property on $(0,\infty)$ consider first the sum

$$S = \sum_{i=1}^{\infty} T_i = T_1 + \sum_{i=2}^{\infty} T_i \,.$$

Denote the Cdf of $\sum_{i=2}^{\infty} T_i$ by $K_2(t)$ and consider the corresponding convolution:

$$K(t) = \int_0^t f_1(x) K_2(t-x) dx \,.$$

Thus, there is some $\varepsilon_2 > 0$ such that

$$K(b-\mu_1) = \int_0^{b-\mu_1} f_1(x) K_2(b-\mu_1-x) dx > \varepsilon_2 \,. \tag{18}$$

This is because $f_1(x) > 0$ for $x > 0$ (assumption of the theorem) and $\sum_{i=2}^{\infty} E[T_i] = b - \mu_1$, whereas the latter implies that, similar to (16), there exists $\delta_2 > 0$ such that

$$K_2(b-\mu_1) \equiv \Pr\left(\sum_{i=2}^{\infty} T_i \le b-\mu_1 \right) > \delta_2 \,.$$

Therefore, keeping in mind that from (15),

$$H(b-\mu_1) = \sum_{n=1}^{\infty} F^{(n)}(b-\mu_1) \,,$$

and using (18) for $t = b - \mu_1$ in the same way we used (16) for $t = b$, we arrive at

$$H(b-\mu_1) = E[N(b-\mu_1)] = \infty \,,$$

thus, increasing the interval of convergence to $[b-\mu_1, \infty)$.

Exactly in the same manner, we can perform any number of the described above steps. Thus, after the n-th reduction of the argument,

$$H\left(b - \sum_{i=1}^{n} \mu_i\right) = E\left[N\left(b - \sum_{i=1}^{n} \mu_i\right)\right] = \infty \,.$$

When $n \to \infty$, due to (14), $H(t)$ is infinite in any interval $[\vartheta, \infty)$, where ϑ as small as we wish.

■

Remark 4. For the case when $H(t) = \infty$, the age-replacement PM models based on the corresponding expectations are not well defined. However, they can be regularized in some way to become well-posed. For instance, if repair is considered to be non-instantaneous, then the mean number of renewals in the corresponding generalized alternating renewal process is finite in any finite interval. Then the PM is performed after *n* imperfect repairs and it can be defined optimally, e.g., cost-wise. Lam (2007) has considered another (increasing) geometric process for modeling the sequence of repair times, which makes sense in applications (see also, e.g., Zhang (2002)). Another obvious method is based on truncation of the geometric process when there cannot be more than $n \geq 1$ 'geometric renewals' in the process (Wang and Pham, 2006).

What happens in the non-homogeneous case? Consider the problem first in full generality. Let $\tilde{w}_i, i = 1, 2, ...$ be the increasing sequence of degradation levels after the i-th imperfect repair such that

$$\lim_{i\to\infty} \tilde{w}_i = c \leq w,$$

where $c > 0$ is a constant. Thus, with each imperfect repair its quality is decreasing. Note that, if it stays the same, i.e., $\tilde{w}_1 = \tilde{w}_2 = \tilde{w}_3 = ... = \tilde{w}$ the corresponding point process $\{T_i\}, i = 1, 2, ...,$ where, as above, T_i denote the inter-arrival times, is a 'classical' delayed renewal process with the first cycle described by the survival function $\bar{F}(t, w)$ in (2), whereas all subsequent cycles durations are described by the survival function $\bar{F}_{\tilde{w}}(t)$ in (4). On the other hand, when $c \leq w$, the process converges asymptotically as $i \to \infty$ to the renewal process with cycles durations described by the survival function $\bar{F}_c(t)$.

When the levels after imperfect repairs are different, in accordance with the virtual age model proposed in this paper, the cycles have survival functions

$$\bar{F}_1(t) = \bar{F}(t, w),$$

$$\bar{F}_{i+1}(t) = \bar{F}_{\tilde{w}_i}(t) = \int_0^\infty P(W_{x+t} - W_x \leq w - \tilde{w}_i) f(x, \tilde{w}_i) dx, i \geq 1,$$

where $f(x, \tilde{w}_i) = \frac{\partial}{\partial x} F(x, \tilde{w}_i)$ is the density describing the virtual age on reaching the level of degradation $\tilde{w}_i$.

However, the most important thing to emphasize here is that the resulting generalized renewal process is the process with *independent cycles*, which dramatically differs from the processes of imperfect repair induced by the Kijima-type modeling that have dependent cycles.

It immediately follows from Theorem 1 that

$$\bar{F}_{i+1}(t) < \bar{F}_i(t), i = 1, 2, ... \tag{19}$$

in this case means that the cycles of the defined generalized renewal process are stochastically decreasing.

Let (12) hold, but, of course, we do not have (13) now that holds only for the homogeneous case. Assume, as in Theorem 1, that the increments $W_{x+t} - W_x$ are stochastically increasing in x in the sense of the usual stochastic ordering. It follows from (2) and (4) that (setting x=0 in $W_{x+t} - W_x$)

$$\bar{F}_{i+1}(t) = \int_0^\infty P(W_{x+t} - W_x \leq w - \tilde{w}_i) f(x, \tilde{w}_i) dx \leq \bar{F}(t, w - \tilde{w}_i). \tag{20}$$

Therefore, we can define the majorizing renewal-type process with the corresponding arrival times that are larger then $\{T_{i,n}\}, i = 1, 2, ...$ Therefore, its renewal function is smaller than that of the original one. However, if we choose now in (20) the degradation levels according to

(12), then under assumptions of Theorem 3, the majorizing renewal-type process will have infinite $H(t)$ for all $t > 0$ and so does the original imperfect repair process for the nonhomogeneous process of degradation.

The latter part was described for the degradation levels defined by the specific 'geometric form' (12), whereas the general result for the corresponding generalized renewal process is given in (19). As was discussed during definition of the imperfect repair at the end of Section 2, the choice of degradation levels $\tilde{w}_i, i = 1, 2, ...$ or the corresponding model for that depends on the specific application where imperfect repairs are implemented. This choice can depend on the quality of repair facilities, resources, time, etc. The corresponding specific models can be developed for relevant applications in the future.

6. Degradation scale

As degradation can be monitored and is monotone, we can consider the suggested imperfect repair model and the corresponding repair processes not in the usual time scale, but in the *degradation scale*. In this short section, we just outline the suggested approach that can be possibly developed in the further research for a more general model.

Let an item start operating at $t = 0$. Assume that the failure on the first cycle occurs on reaching the random degradation level W_1 described by the Cdf $G(w)$ and the pdf $g(w)$ with support in $[0, \infty)$. Thus, W_1 can be generally considered as a 'lifetime' in this case. The first failure (and the instantaneous imperfect repair) occur at w_1, which is a realization of W_1. Assume that the imperfect repair decreases this degradation to $q(w_1)$, where $q(w)$ is an increasing continuous function and $0 \le q(w) \le w$. The second cycle of the process starts with the degradation level $\psi_1 = q(w_1)$ and the cycle duration W_2 (remaining degradation) has the Cdf $G^*(w \mid q(w_1)) = (G(w + q(w_1)) - G(q(w_1))) / \bar{G}(q(w_1))$. No virtual age of any kind, no additional assumptions, just as described above.

Let $W_2 = w_2$. Thus, the value of degradation just before the second repair is $q(w_1) + w_2$ and it is $\psi_2 = q(q(w_1) + w_2)$ just after the second repair. In a similar manner, recursively,

$$\psi_n = q(\psi_{n-1} + w_n), n \ge 1, \psi_0 = 0 \qquad (21)$$

or, equivalently, for the corresponding random values

$$\Psi_n = q(\Psi_{n-1} + W_n), n \ge 1, \psi_0 = 0,$$

When $q(w) = qw, 0 \le q \le 1$, e.g., (21) reduces to

$$\psi_n = q^n w_1 + q^{n-1} w_2 + ... + q w_n = \sum_{i=0}^{n-1} q^{n-i} w_{i+1},$$

where, $w_i, i \ge 1$, are realizations of inter-arrival times W_i in the point process of imperfect repairs, defined by the 'remaining degradation concept'.

Thus, the procedure is similar to that in Kijima 2 model (Kijima, 1989) when we consider this process in the *t*-scale. However, as mentioned, the approach in the *w*-scale is much better justified. By analogy, it can be also shown that when $n \to \infty$ the cycles of this process are asymptotically identically distributed (Finkelstein, 2008) in the *w*-scale. For the linear *q(x)*, the described procedure is similar to the Arithmetic Age Reduction model in Doyen and Gaudoin (2004) and numerous aftermath papers. The corresponding PM problems can be also considered, however, for that we must go back to the time scale or make additional assumptions. This can be performed for the specific problems to be formulated and considered in the future research.

7. Concluding remarks and discussion

We suggest and justify a new, basic approach to modelling imperfect repair that differs from the conventional virtual age models reported in the literature. We consider monotone processes of degradation with independent increments.

In order to define the state of an item after the imperfect repair that reduces its degradation upon failure, a notion of a random degradation-based virtual age is introduced. The corresponding remaining lifetime of an item after the imperfect repair is defined only by the reduced degradation (homogeneous degradation processes) and additionally, by the random virtual age (non-homogeneous processes).

We obtain some stochastic comparisons for remaining lifetimes with different thresholds and different underlying processes of degradation. The homogeneous and non-homogeneous gamma processes are chosen for illustration. Another practically sound option would be the inverse Gaussian process.

The geometric-type reduction of wear/threshold at each cycle of the corresponding generalized renewal-type process is considered and the infiniteness of the analogue of the renewal function in this case is proven. Finally, an alternative approach that considers the imperfect repair process defined in the degradation scale is outlined.

The future research can focus on various generalizations of the developed methodology. For instance, deterioration processes with dependent increments can be considered. On the applications side, relevant imperfect preventive maintenance models can be also discussed.

As we discussed In Section 2, for the homogeneous case (e.g., for Levy processes), Definition 1 does not use any additional assumptions, as (3) unambiguously defines the remaining lifetime, and, therefore, the model is completely justified. For the nonhomogeneous case, the *onset time* of the stochastic process after reduction of degradation becomes crucial and we model it by the defined virtual age $T_V(\tilde{w})$ independently from the realization of the chronological age on failure of an item. We have provided the motivation for this reasonable, in our opinion, assumption. In principle, we can assume that, as w is reduced at failure in accordance with, e.g., $\tilde{w}=qw$, the age at failure should be also reduced in accordance with (1) to $\tau=\tilde{q}t^*$. Then we can carry on with this setting as the basis for the corresponding bivariate statistical model. However, all the weaknesses of the age reduction in (1) discussed in the Introduction then come in two play. Thus, on one hand, we have the absolutely real/physical operation of reducing degradation, whereas, on the other hand, a rather vague assumption that the age should be decreased (which is true, but the model for that is not justified). This approach can possibly work for inferential matters, which can be also investigated in the future.

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References

Badia, FG, Berrade, MD. (2006). Optimum maintenance of a system under two types of failure. Int. J. Mater. Struct. Reliab, 4, 27-37.

Badia, FG, Berrade, MD. (2007). Minimal repair model for systems under both unrevealed minor failures and unrevealed catastrophic failures. In International Mathematical Forum, Vol. 2, No. 17-20, 819-834.

Badia, FG, Berrade, MD. (2009). Optimum maintenance policy of a periodically inspected system under imperfect repair. Advances in Operations Research 2009, Article ID 691203.

Barlow, RE, Proschan, F. (1975). Statistical Theory of Reliability and Life Testing. Holt, Renerhart & Winston, New York.

Belzunce, F, Riquelme, CM, Mulero J. (2015). An Introduction to Stochastic Orders. Academic Press.

Belzunce F, Nanda, AK, Ortega EM, Ruiz JM. (2008). Generalized orderings of excess lifetimes of renewal processes. Test, 17, 297-310.

Berenguer, C, Grall, A, Dieulle, L. (2003). Maintenance policy for a continuously monitored deteriorating system. Probability in Engineering and Information Sciences, 17, 235-250.

Brown M, Proschan F. (1983). Imperfect repair. Journal of Applied Probability, 20, 851-862.

Braun, WG, Li, W, and Zhao, YQ. (2005). Properties of the geometric and related processes. Naval Research Logistics, 52, 607-616.

Cha, JH, Finkelstein, M. (2018). Point Processes for Reliability Analysis. Shocks and Repairable systems. London, Springer.

deToledo, MLG, Freitas, MA, Colosimo, EA, Gilardoni, GL (2015). ARA and ARI imperfect repair models: Estimation, goodness-of-fit and reliability prediction. Reliability Engineering and System Safety, 140, 107-115.

Dijoux, Y, Fouladirad, M, Nguyen, DT. (2016). Statistical inference for imperfect maintenance models with missing data. Reliability Engineering and System Safety, 154, 84-96.

Doyen, L, Gaudoin, O. (2004). Classes of imperfect repair models based on reduction of failure intensity or virtual age. Reliability Engineering and System Safety, 84, 45–56.

Dufresne, F, Gerber, HU, Shiu, ES. (1991). Risk theory with the gamma process. ASTIN Bulletin: The Journal of the IAA, 21, 177-192.

Finkelstein, M. (1989) Perfect, minimal and imperfect repair (in Russian). Reliability and Quality Control 1989, 3, 17-21.

Finkelstein, M. (2007). On some ageing properties of general repair processes. Journal of Applied Probability, 44, 506-513.

Finkelstein, M. (2008). Failure Rate Modelling for Reliability and Risk. Springer, London.

Finkelstein, M. (2010). A note on converging geometric-type processes. Journal of Applied Probability, 47, 601-607.

Finkelstein, M, Cha, JH. (2013). Stochastic modelling for reliability (shocks, burn-in and heterogeneous populations). London, Springer.

Hazra, NK, Finkelstein M., Cha, JH. (2018). Stochastic ordering for populations of manufactured items. Test, 27, 173-196.

Kahle, W. (2019). Imperfect repair in degradation processes: a Kijima-type approach. Applied Stochastic Modelling in Business and Industry, 35, 211-220.

Kijima, M. (1989). Some results for repairable systems with general repair. Journal of Applied Probability, 26, 89-102.

Krivtsov, VV. (2007). Recent advances in theory and applications of stochastic point process models in reliability engineering. Reliability Engineering and System Safety, 92,549-551.

Lam, Y. (1988). Geometric process and the replacement problem. Acta Mathematicae Applicatae Sinica, 4, 366-382.

Lam,Y. (2007). The Geometric Process and Applications. World Scientific.

Levitin, G, Lisniansky, A. (2000). Optimization of imperfect preventive maintenance for multi-state systems. Reliability Engineering and System Safety, 67, 193-203.

Meeker, WQ, Escobar, LA. (2014). Statistical Methods for Reliability Data. John Wiley & Sons.

Müller, A, Stoyan, D. (2002). Comparison Methods for Stochastic Models and Risks. Chichester, John Wiley & Sons, Ltd.

Nguyen, DT, Dijoux, Y, Fouladirad, M. (2017). Analytical properties of an imperfect repair model and application in preventive maintenance scheduling. European Journal of Operational Research, 256, 439-453.

Pérez-Ocón R, Torres-Castro I. (2002). A reliability semi-Markov model involving geometric processes. Applied Stochastic Models in Business and Industry, 18, 157-170.

Perez-Ocon R, Montoro-Cazorla D. (2004). Transient analysis of a repairable system, using phase-type distributions and geometric processes. IEEE Transactions on Reliability, 53, 185-192.

Salehi, ET, Asadi M, Eryılmaz S. (2012). On the mean residual lifetime of consecutive k-out-of-n systems. Test, 21, 93-115.

Shaked, M, Shanthikumar, J. (2007). Stochastic orders. New York, Springer.

Tanwar, M, Rai, RN, Bolia, N. (2014). Imperfect repair modeling using Kijima type generalized renewal process. Reliability Engineering and System Safety, 124, 24-31.

van Noortwijk, JM. (2009). A survey of the application of gamma processes in maintenance. Reliability Engineering and System Safety, 94, 2-21.

Wang, X. (2009). Nonparametric estimation of the shape function in a gamma process for degradation data. Canadian Journal of Statistics, 37, 102-118.

Wang, H, and Pham, H. (2006). Reliability and Optimal Maintenance. Springer, London.

Zhao, X, Gaudoin, O, Doyen, L, Xie, M. (2019). Optimal inspection and replacement policy based on experimental degradation data with covariates. IISE Transactions, 5, 322-336.

Zhang, L. (2002). A previous term-geometric process next term repair model with good-as-new preventive repairable systems. IEEE Transactions on Reliability, R-51, 223-238. IISE Transaction, 51, 332-336.