Optimal preventive switching of components in degrading systems

Maxim Finkelstein[a,b], Ji Hwan Cha[1,c] and Amy Langston[d]

[a] Department of Mathematical Statistics, University of the Free State, Bloemfontein, South Africa
E-mail: FinkelM@ufs.ac.za

[b]Department of Management Science, University of Strathclyde, Glasgow, UK

[c]Department of Statistics, Ewha Womans University, Seoul, 120-750, Rep. of Korea.
e-mail: jhcha@ewha.ac.kr

[d]Department of Statistics, Rhodes University, South Africa
e-mail: a.langston@ru.ac.za

Abstract. In practice, at many instances, it is important to maintain the failure-free performance of components in a standby system, as each sudden failure of an operating component can result in a failure of a system, e.g., due to imperfect or/and 'non-instantaneous' switching on failure and related adverse effects. Therefore, the scheduled preventive switching/replacement to the standby component that can be executed without these consequences is one of the effective methods for increasing reliability characteristics of such systems, especially in the safety-critical applications. In this paper, the corresponding optimal strategy for switching is described and justified for the cold standby system of two aging components with degradation modeled by the counting Poisson and gamma processes. An inspection is carried out at some optimally predetermined time and based on the observed degradation switching is performed after the optimally obtained delay. Detailed numerical examples illustrate our findings.

Keywords: Standby systems; optimal switching; Poisson process; gamma process; mission success probability

Acronyms

PS/PR	preventive switching/preventive replacement
MSP	mission success probability
VM	virtual machine
Cdf	cumulative distribution function
PM	preventive maintenance
IFR	increasing failure rate
HPP	homogeneous Poisson process
NHPP	nonhomogeneous Poisson process

[1] Corresponding Author

1. Introduction

The most conventional type of redundancy in practice is the structural redundancy on the elements' or on the system's level. When an operating element fails, switching to the operable redundant one that was kept in the cold, warm, or hot standby mode is executed.

In this paper, we consider cold standby systems performing missions of the fixed duration *T*. At many instances, a failure of an operating element is unacceptable, as it results in a direct mission failure or substantial economic losses. This is relevant, e.g., for many safety-critical and important missions. Some examples of these systems will be given below. In order to extend lifetimes of degrading systems with respect to sudden failures of components and, therefore, to increase the mission success probability, the preventive switching/replacement (PS) of an operating element by the available standby one can be executed in practice.

The PS for standby systems was addressed in numerous publications (see, e.g., [1, 2] for some general settings). In [3], the reusable PS strategy was considered when after replacement, the operable component can be used as a standby in the future. The influence of the lifetime distribution parameters on the optimal PS policy was analyzed in [4]. In [5], the postponed PS was considered in combination with the imperfect repair. In [6], the PR policy was modeled and optimized for a standby system undergoing the preventive and corrective replacements as well as periodic backups. In [7], the PS and system inspection policies were investigated for systems with degrading standby elements. Periodic switching policies for the cold-standby two-unit system were discussed in the recent paper [8], where the concept of virtual age was used for the units in the cold-standby 'recovering' after the active operation. The reusable strategy for heterogeneous items was also studied in [9]. Maximization of the component's sequencing for cold-standby systems with imperfect switching has been addressed in [10]. Some other relevant properties of the cold-standby systems can be found, e.g., in references [11-14].

Most of the papers on the PS management in the described context were dealing with degradation of elements that was modeled by the increasing failure rates and no additional information was available (the black-box scenario). In [15] the standby component was already switched into operation before the failure of the active one, whereas in [16-17] the probability of a first failure of a component in the standby system was maximized by implementing the optimal switching strategy. However, in practice, deterioration processes or its proxies can be observed for components, at least, at inspections and, therefore, can provide additional, important information for obtaining optimal PS times for achieving the components' failure free performance in a system. (Some analogy can be found in the field of the condition-based maintenance (see e.g., [18-20]). Therefore, the *main contribution* of our paper can be formulated as:

Distinct from the studies of models for preventive switching/replacement reported in the literature, in this paper, we are considering maximization of the mission success probability (MSP) in standby systems with observable degradation. An inspection is carried out on the active component at the optimally obtained time and, depending on the observed degradation and using the developed optimal procedure, the time of actual switching to the standby component is found. We also show that the suggested approach outperforms the black-box scenario when the time of switching is obtained without information on degradation.

Optimal switching of components to be discussed in this paper can be also considered in a broader context of lifetimes extension in various applications. For instance, lifetime extensions by means of the finite number of preventive maintenance (PM) actions for systems with a relatively long lifecycle (see, e.g., [21-22], for the basic PM models) were studied in [23-24]. Failures of these systems during operation can result in catastrophic events and, therefore, the lifetime extension (increasing the *expected lifetime* without failure in the operational mode) is an important tool for improving reliability in practice.

As switching of an operating component to a cold standby in the context of the current paper can be also considered as the corresponding PM (replacement), we will briefly discuss this interpretation showing that the expected costs for maintenance in the fixed interval of time are minimal if maintenance is performed at the time suggested by our optimal strategy. Thus, the PM setting provides another *practically important application* of our methodology (see Remark1 in Section 2 and Remark 3 in Section 5). Note that, at many instances, e.g., for the offshore units the time of PM should be carefully planned, as it can be very costly to get the PM crew to the cite, whereas an inspection can be performed remotely. The proposed method decreases the overall costs of the PM operations. We plan to consider this topic in a more generality in the future research. Some recent relevant papers with respect to this application (although not in the framework of our approach) are [25-30]. Specifically, in [30], the authors consider the risk-based adaptive planning of inspections and maintenance in structural systems using stochastic optimization.

We provide now a practical, cyber-security example that illustrates the setting discussed in our paper. Consider n=2 virtual machines (VMs) performing a data processing task in a cloud computing environment. Only operating VM has access to the sensitive data. The time needed to complete the task is fixed. During this time the operating VM experiences hackers' attacks. If an attack on the operating VM succeeds, hackers get access to the data through this VM and corrupt the data. The probability of an attack's success increases with the number of attacks (or with time), as in each attack hackers gain some information about the system protection. This describes the corresponding deterioration, which can be measured as a function of the number of attacks (or the time of exposure to continuous attack). The mission fails if the data is corrupted. To increase the mission success probability the user transfers the task execution to the standby VM (having larger level of protection) if deterioration of operating VM reaches some level. Each cyber-attack leave traces and the inspection reveals traces of attacks and, therefore, the level of protection degradation. If this level exceeds a threshold, the standby VM is activated in optimally obtained time. The standby VM protection has other codes and the attacker should start the attack from the start. Degradation can be understood in the sense that an attacker usually uses brute force enumeration algorithms to find the keys to protection codes. The more time the attack lasts (or the more attacks have been launched) the larger is the probability of the attack's success.

Finally, our paper assumes a perfect switching mechanism. Generalizations to a non-perfect switching can be worth looking at while considering specific applications. For instance, in the recent paper [31] (that can be also used as a survey on conventional switching strategies in standby systems), it was shown that the MSP can be increased if the standby component is activated before the failure of the main one and both of them remain active after that. Obviously, the MSP can only decrease (as compared with the case without pre-activation) if switching is perfect.

The paper is organized as follows. In Section 2, we briefly recall the black-box scenario for switching of components in the standby system executing a mission of the fixed duration and add some relevant remarks for the discussion to follow. Supplementary Section 3 defines two degradation processes and relevant relationships, i.e., the Poisson and the gamma process. Section

4 discusses the basics of the suggested approach, whereas Section 5 describes the corresponding optimization procedures. Final remarks are given in Section 6.

2. The black-box scenario

Consider a cold standby system of two i.i.d. components with Cdfs $F(t)$ and the failure rate $\lambda(t)$ Assume that $\lambda(t)$ is increasing, thus $F(t)$ belongs to the IFR class, which indirectly describes the overall degradation processes in components. Let T denote the fixed duration of a mission or a task to be executed.

Denote by $P_1(a,T)$ the mission success probability (MSP), i.e., no components' failures in $[0,T)$ with one switching at $0<a<T$. The sub index "1" stands for "until the first failure of a component". Then, it is easy to show that $\max_a (P_1(a,T))$ is attained at $a=T/2$. Indeed,

$$P_1(a,T)=\exp\left\{-\int_0^a \lambda(u)du\right\}\exp\left\{-\int_0^{T-a}\lambda(u)du\right\}. \tag{1}$$

After differentiating the sum of integrals and equating the derivative to 0, we get the equation

$$\lambda(a)=\lambda(T-a). \tag{2}$$

which, as $\lambda(t)$ is increasing, has the unique solution $a=T/2$. Due to additivity of the integrals, additional switching does not increase the probability in (1) [16].

For further discussion of the setting with additional information on degradation, we need also to consider the case when the components are not statistically identical. In this case, the similar reasoning results in the following equation

$$\lambda_1(a)=\lambda_2(T-a), \tag{3}$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are the increasing failure rates of the first and the second (standby) components, respectively. Equation (3), under some additional assumptions (see later), also has the unique solution that maximizes the probability of survival without failures of components in $[0,T]$. The diagram describing operation of the described system is given in Fig.1.

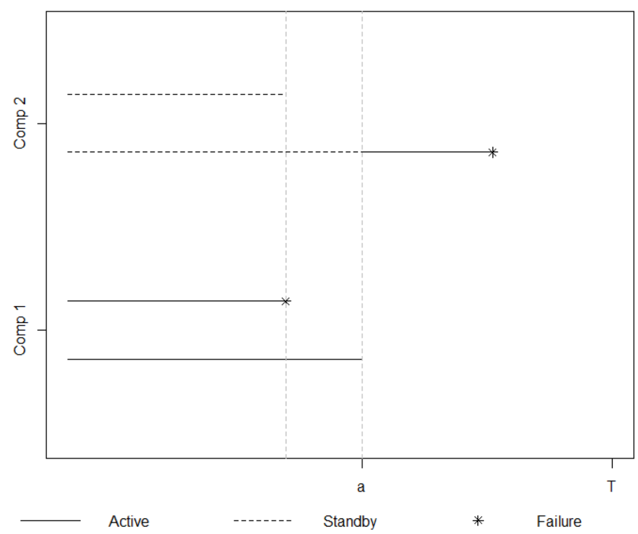

Fig.1 Operation of a system for two scenarios: failure of the active component before the planned switching (upper) and after it (lower)

For dealing with the case with additional information on degradation, the following observation is meaningful. In accordance with the foregoing, the switching for maximizing $P_1(a,T)$ in (1) is scheduled a priory at $a = T/2$. Let at time $a_i \in (0, T/2)$ the main component be operable and we want to schedule the further switching optimally to maximize the remaining survival probability of the standby system in $[a_i, T)$. In the rest of the paper, a_i will have the meaning of the inspection time for observing degradation. Thus, some *conditioning* is involved in this optimization problem. The Cdf of the remaining lifetime of the main component is

$$F_1(t) = \frac{F(a_i + t) - F(a_i)}{\bar{F}(a_i)}$$

with the corresponding failure rate $\lambda_1(t) = \lambda(t + a_i)$. From (3), denoting the time since inspection to switching by a^*,

$$\lambda(a_i + a^*) = \lambda((T - (a_i + a^*)). \tag{4}$$

This means that $a_i + a^* = T/2$, and therefore, switching should be executed again at $T/2$. Specifically, when $a_i = T/2$, we have: $a^* = 0$. Therefore, information that the component is operable at a_i does not change the optimal switching time. We will show that the situation is different when we have an additional information on the degradation process of a component, as the remaining lifetime 'could be already sufficient' for the first component to continue to operate after $a_i = T/2$!

Remark 1. For another practically sound interpretation of our setup, assume that we have a standby system of aging components that is executing a task/mission of a fixed duration *T*. In order to increase the mission success probability one maintenance action is allowed (the case of more than one maintenance actions can be also considered, but it is much more cumbersome). It can be either corrective upon failure with the cost C_f or preventive with the cost C_r, which, in fact, is the cost of perfect repair (replacement or overhaul). Similar to the conventional PM models, $C_f = C_r + C_a$, where C_a are additional costs associated with the sudden failure of a system. We want to minimize the expected *maintenance* costs (either on failure of the first active component or on PM at $a \in (0,T)$, whichever comes first). Note that, here we are not considering the costs related to the failure of a mission when a system fails in $[0,T]$ after the maintenance of any kind. So, when to perform the PM that will minimize the expected costs? From our reasoning in this section, it follows immediately that the PM should be performed at $a = T/2$, as this strategy minimizes the probability of a failure of a system that have started operation at $t=0$ and, therefore, of the expected maintenance costs. This is because the PM cost is smaller than that of the corrective maintenance.

3. Degradation processes

Assume that the observable internal stochastic deterioration process in a component $\{W_t, t \geq 0\}, W_0 = 0,$ is characterized by the *independent increments* and has the monotonically increasing sample paths. Then the survival function for the lifetime of a component *L* can be, obviously written as

$$P(L > t) \equiv \bar{F}(t,w) = P(W_t \leq w) \tag{5}$$

when a failure is defined as reaching/crossing the threshold *w.*

3.1. Poisson process of degradation. The survival function that describes the time to the *M*-th event (threshold) in the homogeneous Poisson process (HPP) with rate λ is given, e.g., in [32]:

$$\bar{F}(t,M) = \exp\{-\lambda t\} \sum_{i=0}^{M-1} \frac{(\lambda t)^i}{i!}. \tag{6}$$

Thus, (6) can be interpreted as the survival function of a component with M i.i.d. elements (1 out of *M* cold standby) with the failure rate λ. Thus, degradation in a component of our standby system is completely defined by the number of failed elements *m* and can be easily observed in practice. In this case, the remaining lifetime of a component is defined by the following Cdf

$$F(t,M-m) = 1 - \exp\{-\lambda t\} \sum_{i=0}^{M-m-1} \frac{(\lambda t)^i}{i!}. \tag{7}$$

Indeed, as m is the number of failed elements, then M-m is the number of remaining operable components Thus, the system in the future operation, can experience at most M-m-1 failures. Otherwise, it fails.

Let now degradation be modeled by the counting nonhomogeneous Poisson process (NHPP) with rate $\lambda(t)$ (with current degradation *m* and the threshold *M* events). Then, when defining the remaining lifetime, we must also consider the corresponding pdf (occurrence of the m-th event), i.e.,

$$f(x,m)=\lambda(x)\exp\{-\Lambda(x)\}\frac{(\Lambda(x))^{m-1}}{(m-1)!}, \tag{8}$$

where $\Lambda(t)=\int_0^t \lambda(u)du$. Then, the survival function describing the remaining lifetime is [28]

$$\bar{F}_{M-m}(t)=\int_0^{\infty} P(W_{x+t}-W_x<M-m)f(x,m)dx, \tag{9}$$

where

$$P(W_{x+t}-W_x<M-m)=\exp\{-\Lambda(x,t)\}\sum_{i=0}^{M-m-1}\frac{(\Lambda(x,t))^i}{i!};\ \Lambda(x,t)=\int_x^{x+t}\lambda(u)du .$$

3.2. Gamma process of degradation. The gamma process is widely used in the literature for modeling 'continuous' degradation (the fatigue in materials, cracks growth, structural engineering etc.). It is well-known (see, e.g., references [33-34]) that the nonhomogeneous gamma process $\{W_t, t\geq 0\}$ with parameters $\alpha(t),\lambda$ ($\lambda>0$, ($\alpha(t)$ is a positive increasing function) is described by the following pdf of W_t at each time instant t

$$f(y,t)=\frac{1}{\Gamma(\alpha(t))}\lambda^{\alpha(t)}y^{\alpha(t)-1}\exp(-\lambda y),\quad y\geq 0 . \tag{10}$$

where $\Gamma(a)=\int_0^{\infty} z^{a-1}\exp\{-z\}dz$.

The paths of the gamma process are monotone and, therefore, e.g., for the homogeneous case, the corresponding lifetime model (reaching the failure threshold w) is defined by the following survival function

$$\bar{F}(t,w)=\int_0^{w}\frac{1}{\Gamma(\alpha t)}\lambda^{\alpha t}x^{\alpha t-1}\exp(-\lambda x)\ \mathrm{d}x=1-\frac{\Gamma(\alpha t,\lambda w)}{\Gamma(\alpha t)}, \tag{11}$$

where $\Gamma(a,x)=\int_x^{\infty} z^{a-1}\exp\{-z\}dz$ and $\alpha>0,\lambda>0$ are the shape and the scale parameters, respectively. The corresponding remaining lifetime (after reaching the degradation level $\tilde{w}<w$) is defined by

$$\bar{F}(t,w-\tilde{w})=1-\frac{\Gamma(\alpha t,\lambda(w-\tilde{w}))}{\Gamma(\alpha t)} . \tag{12}$$

4. The degradation-based scenario

In this section, we will discuss motivation for developing optimal procedures for obtaining times of inspection and switching of the next section. We will consider degradation that is modeled by the HPP with the failure threshold M. Similar general reasoning applies to the gamma process of degradation. We will demonstrate now that distinct from the black-box scenario, the remaining 'resource' of the component at time $T/2$ can be used meaning that the switching can be administrated later.

It was shown that without additional information, the switching to the standby component should be executed at the prescheduled time $a = T/2$. Moreover, as was discussed in Section 2, it can be also additionally explained in a different way using the fact that the specific form of equation (3) has the unique solution $a_i + a^* = T/2$, where $a_i \in (0, T/2]$ and the component did not fail in $[0, a_i]$). Note that, when $a_i = T/2$, the component is operable, but according to our discussion, the switching should be performed immediately although it still has some remaining resource (remaining lifetime). The situation will be different when we observe degradation.

Let degradation $m = 0,1,2,...,M-2$ be observed at $a_i = T/2$. From the properties of the Erlangen distribution it follows that both: the remaining lifetime of the first component and the standby one have failure rates that are zeroes at origin (when $m = M-1$ the failure rate of the remaining lifetime is constant and equal to λ). Thus, the optimal switching can be *postponed* in this case and performed at $T/2 + a^*$, where a^* is the unique solution obtained from (3) that reads now

$$\lambda_{M-m}(a^*) = \lambda_M(T/2 - a^*), \tag{13}$$

where $\lambda_k(t), k = 1,2,...$ denotes the failure rate of the corresponding Erlangen distribution with the 'failure threshold *k*'. Fig.1 illustrates how a^* increases when *m* decreases. Thus, depending on the observed at $T/2$ degradation m, the value a^* can be quite substantial. This important effect is neglected in the black-box scenario! If $m = M-1$, the failure rate that corresponds to the remaining lifetime of a component is constant (λ) and the switching had to be performed immediately. Indeed, when $m = M-1$, $\lambda_{M-m}(t) = \lambda > \lambda_M(t)$ for all *t*. Thus, to maximize

$$\exp\left\{-\int_0^{a^*} \lambda_{M-m}(u)du\right\} \exp\left\{-\int_0^{T/2-a^*} \lambda_M(u)du\right\},$$

we should set $a^* = 0$ in this case.

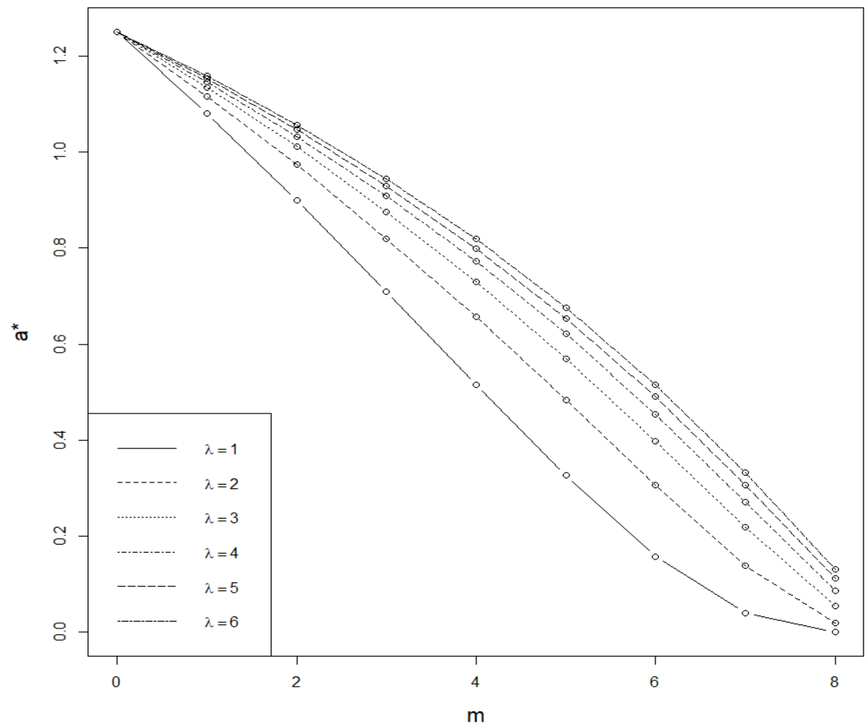

Fig.2. The relationship between a^* and the observed at $T/2$ degradation *m* for T=5, M=10 and various values of the rate λ.

Now we can go further. As in Section 2 for the black-box case, assume that at the time of inspection $a_i \in [0, T/2)$ the main component is operable and we observe degradation $m = 0,1,2,...,M-2$. Thus, we want to schedule the further switching optimally to maximize the remaining survival probability of the standby system in $[a_i, T)$, whereas optimal choice of a_i will be addressed in the next section. Note that, in the black-box scenario, the optimal switching had to be performed in any case, at $T/2$, as discussed previously. It depends now on the value of the observed degradation *m*. Equation (13) is modified in this case to

$$\lambda_{M-m}(a^*) = \lambda_M(T - (a_i + a^*)), \tag{14}$$

where the righthand side is the failure rate for the black-box scenario for the remaining (after a_i) lifetime. Thus, when m is 'small', we should expect that $a_i + a^* > T/2$, which increases the *remaining* MSP as compared with the black-box scenario, for which switching should be performed at $T/2$. On the contrary, when it is large, $a_i + a^* < T/2$ and thus the *conditionally optimal* switching should be performed earlier than the black-box one. Finally, when $m = M-1$, similar to what was discussed above, switching should be performed immediately at a_i. In any case, a^* is increasing when *m* is decreasing, which follows from (14) and the hazard rate ordering for the corresponding Erlangen distributions

$$\lambda_{m_1}(t) \geq \lambda_{m_2}(t), m_1 \leq m_2 \tag{15}$$

This fact (as well as the monotonic increase of the failure rates) is well-known (see, e.g., [17]).

Remark 2. Obviously, when degradation is continuously observed and not only at inspection, the optimal switching time is random and equal to the time when degradation of the first component reaches the level $m = M - 1$.

It was shown in [35] that the failure rate that corresponds to the survival model (11) is increasing. See also [36] for more general results. As the gamma process is the jump process with infinite number of jumps in any finite interval of time, the corresponding failure rate is not, obviously, zero at origin as for the Poisson degradation case. Therefore, the analogue of equation (14) not necessarily has a solution for a^*. In this case, the switching should be executed immediately at inspection at a_i, which corresponds to $a^* = 0$. This will be implemented in the corresponding computational procedure of the next section. Note that, Remark 2 is also not relevant in this case.

5. Optimal inspection and switching times

5.1. Poisson process of degradation

In the previous section, it was shown that when we observe degradation at some fixed $a_i \in [0, T/2)$, the optimal switching time $a_i + a^*$ maximizes the probability until the first failure of a component in the cold standby system of two i.i.d. components, $P_1(a_i, T)$. However, the survival probability should be maximized now with respect to the inspection time a_i. As follows from our discussion, this optimal solution, obviously, exists.

The following optimization procedure is proposed:

Let us first fix $a_i \in [0, T]$.

Case (i). If the first component fails before a_i, then the MSP is 0.

Case (ii). If the first component is operable at a_i, we observe the corresponding degradation. If $m = 0, 1, 2, ..., M - 2$, then switching is performed at $a_i + a^*(a_i, m)$, where $a^*(a_i, m)$ is the solution of (14). If $m = M - 1$, switching is performed immediately. The conditional MSP is, therefore, given by

$$\exp\left\{-\int_0^{a^*(a_i,m)} \lambda_{M-m}(u)du\right\} \exp\left\{-\int_0^{T-a_i-a^*(a_i,m)} \lambda_M(u)du\right\}, \text{ for } m = 0, 1, 2, ..., M - 2$$

and

$$\exp\left\{-\int_0^{T-a_i}\lambda_M(u)du\right\}, \text{ for } m=M-1.$$

The conditional probabilities for the degradation $m=0,1,2,...,M-1$ in $[0,a_i)$ are

$$\frac{\exp\{-\lambda a_i\}\dfrac{(\lambda a_i)^m}{m!}}{\sum_{j=0}^{M-1}\exp\{-\lambda a_i\}\dfrac{(\lambda a_i)^j}{j!}}.$$

Therefore, the conditional MSP is

$$\sum_{m=0}^{M-2}\exp\left\{-\int_0^{a^*(a_i,m)}\lambda_{M-m}(u)du\right\}\exp\left\{-\int_0^{T-a_i-a^*(a_i,m)}\lambda_M(u)du\right\}\times\frac{\exp\{-\lambda a_i\}\dfrac{(\lambda a_i)^m}{m!}}{\sum_{j=0}^{M-1}\exp\{-\lambda a_i\}\dfrac{(\lambda a_i)^j}{j!}}$$

$$+\exp\left\{-\int_0^{T-a_i}\lambda_M(u)du\right\}\times\frac{\exp\{-\lambda a_i\}\dfrac{(\lambda a_i)^{M-1}}{(M-1)!}}{\sum_{i=0}^{M-1}\exp\{-\lambda a_i\}\dfrac{(\lambda a_i)^i}{i!}},$$

whereas the corresponding unconditional MSP (which combines cases (i) and (ii)) is

$$P_1(a_i)=0\times\left[1-\sum_{j=0}^{M-1}\exp\{-\lambda a_i\}\frac{(\lambda a_i)^j}{j!}\right]$$

$$+\sum_{m=0}^{M-2}\exp\left\{-\int_0^{a^*(a_i,m)}\lambda_{M-m}(u)du\right\}\exp\left\{-\int_0^{T-a_i-a^*(a_i,m)}\lambda_M(u)du\right\}\times\exp\{-\lambda a_i\}\frac{(\lambda a_i)^m}{m!}$$

$$+\exp\left\{-\int_0^{T-a_i}\lambda_M(u)du\right\}\times\exp\{-\lambda a_i\}\frac{(\lambda a_i)^{M-1}}{(M-1)!}$$

$$=\sum_{m=0}^{M-2}\exp\left\{-\int_0^{a^*(a_i,m)}\lambda_{M-m}(u)du\right\}\exp\left\{-\int_0^{T-a_i-a^*(a_i,m)}\lambda_M(u)du\right\}\times\exp\{-\lambda a_i\}\frac{(\lambda a_i)^m}{m!}$$

$$+\exp\left\{-\int_0^{T-a_i}\lambda_M(u)du\right\}\times\exp\{-\lambda a_i\}\frac{(\lambda a_i)^{M-1}}{(M-1)!}.$$

Thus, we want to maximize $P_1(a_i)$ with respect to a_i by solving the optimization problem (16) that, obviously, has a solution $a_i{*}\in(0,T)$:

$$P_1(a_i{*})=\max_{0\le a_i\le T}P_1(a_i). \tag{16}$$

The following steps describe the implementation of the proposed procedure

1. Obtain a'priori the optimal time of switching a_i* by solving (16)

2. If the first component fails before a_i*-do nothing as it is the mission failure.

3. If the first component survives until a_i* and the observed degradation $m = M - 1$, execute switching at a_i*.

4. If the first component survives until a_i* and the observed degradation $m = 0,1,2,...,M-2$, execute switching at $a_i*+a*(a_i*, m)$, where $a*(a_i, m)$ is obtained numerically as the solution of equation (14).

The following figures illustrate the described procedure.

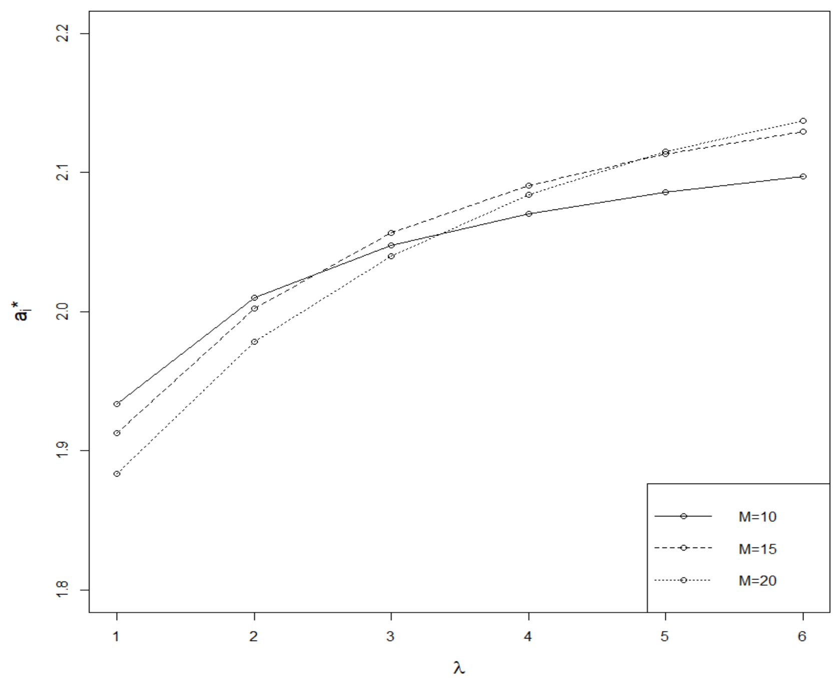

Fig.3. Optimal inspection times a_i* for T=5 and different values of M and λ.

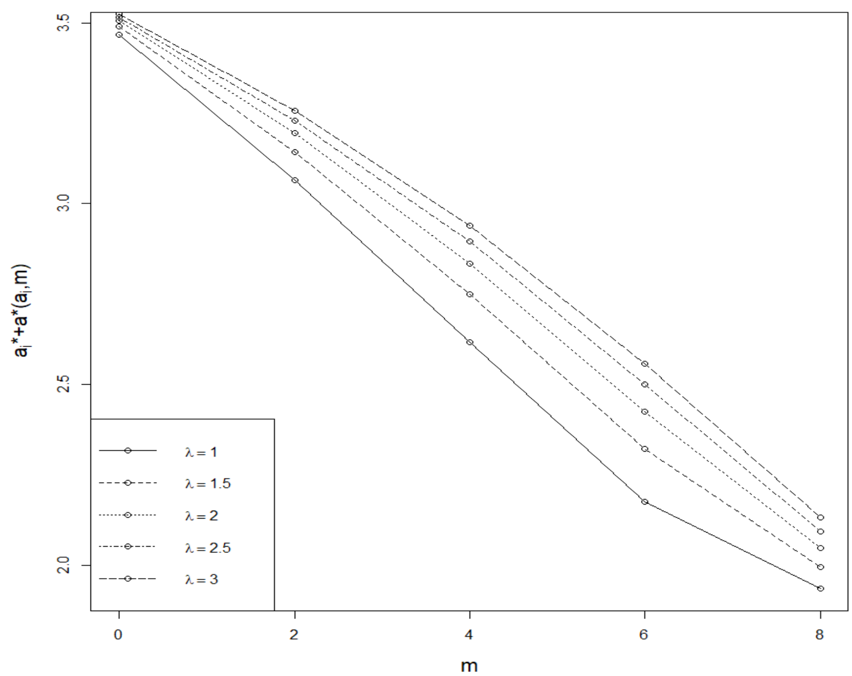

Fig. 4. Actual optimal time of switching for M=10, T=5 and different *m* and λ.

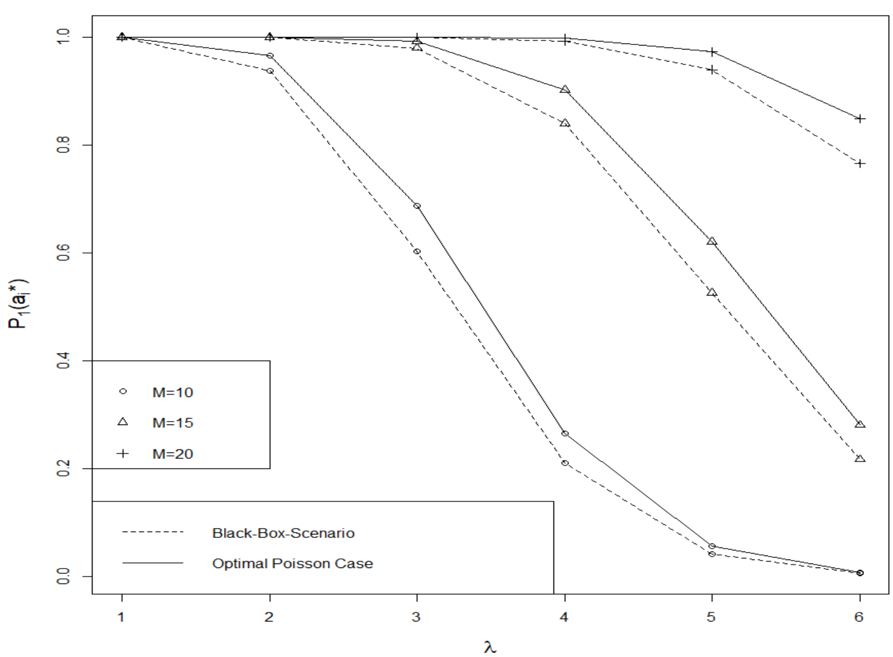

Fig.5. Comparisons of the MSPs (T=5) for the black-box and degradation scenarios for different values of *M* and λ.

We see now from Fig. 3 that the optimal inspection is scheduled *earlier* than at $T/2$. Its monotonicity pattern (as a function of M and λ) is due to the range of the considered parameters and the relevant properties of the Poisson distribution, whereas the graphs just show the optimal inspection time obtained numerically via the proposed procedure. On the other hand, the increase of the optimal inspection time with λ can be loosely explained by the fact that in this case, the 'importance' of the observed degradation is diminishing and the procedure becomes 'closer' to the black-box scenario when the switching is performed at $T/2$. Fig.4 already depicts the actual time of switching as the function of the observed degradation *m*. We see that, as expected, this time decreases with *m* in the range from approximately 3.5 (for *m*=0) to 2-2.3 (for different values of λ and m=8). When m=9, switching should be performed at inspection, which means that $a^*(9)=0$, whereas a_i^* should be taken from Fig. 3. Thus, one should obtain the optimal inspection time, then observe degradation at this time a_i^* and depending on the observed value schedule switching at time $a_i^*+a^*(m)$. Fig. 5, for the considered example, justifies superiority of the proposed strategy over the black-box switching at $T/2=2.5$.

Remark 3. Under the assumptions of Remark 1, the results obtained in this section can be used for scheduling the PM action as well. However, the implementation is distinct from the black-box case. First, the optimal inspection time is obtained by solving the optimization problem. This time does not depend on the observed value, however, the actual, PM time does, as illustrated by Figs. 3 and 4, respectively. Finally, Fig.5 shows that this strategy minimizes the probability of a failure of a system that have started operation at $t=0$ and, therefore, of the expected operational costs. This is because the PM cost is smaller than that of the corrective maintenance.

Remark 4 The assumption for the degradation process to be Poisson in general is often an approximate (excluding the case when degradation is manifested by the number of failed elements in a component, when it is "exact"). We use this case for methodological reasons. However, the gamma process to be considered as deterioration model is widely used and reported numerously in the literature.

5.2. Gamma process of degradation

In the previous subsection, the case of the counting HPP for modeling degradation in components was discussed. The NHPP case can be also described in the similar (although more cumbersome) way. For convenience, the corresponding optimization procedure that takes into account our reasoning and discussion at the end of Section 4, is deferred to the Appendix. Note that, we are considering the homogeneous gamma process that is characterized by the increasing failure rate for the survival model (11) [35, 36]. Thus, using the developed procedure, the optimization problem (16) can be solved numerically and illustrated by the following figures.

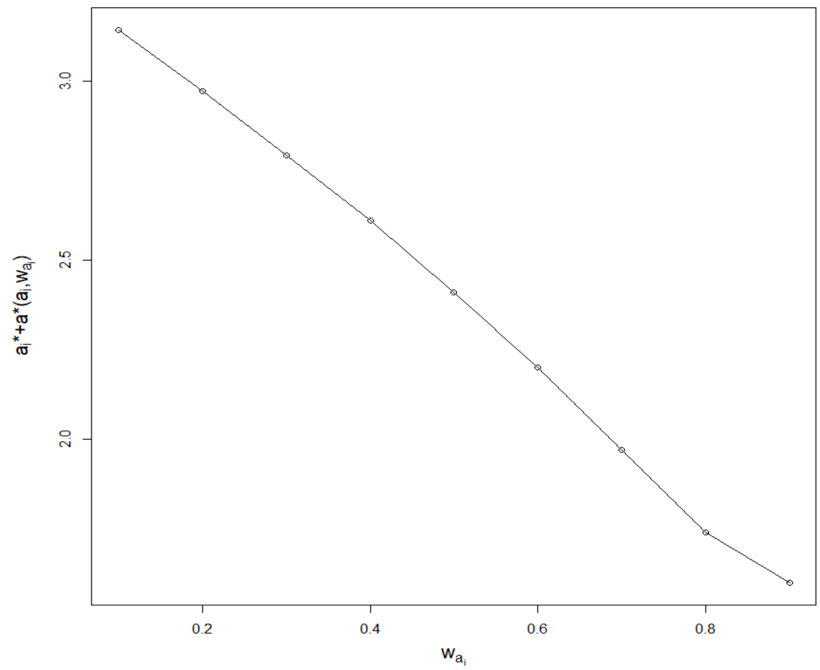

Fig.6. Optimal times of switching as a function of the observed degradation level at optimal inspection time $a_i^* = 1{,}503$ for T=5, $\alpha = 2.6$, $\lambda = 12$, $w = 1$.

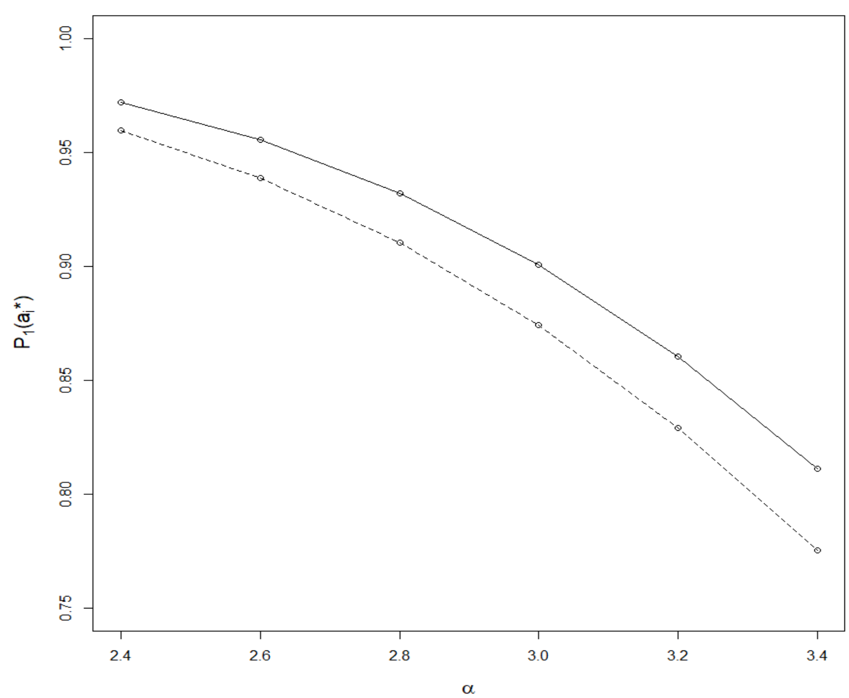

Fig. 7. Comparisons for the MSPs (T=5, $w=1$) for the black-box and degradation scenarios as functions of α and fixed $\lambda=12$.

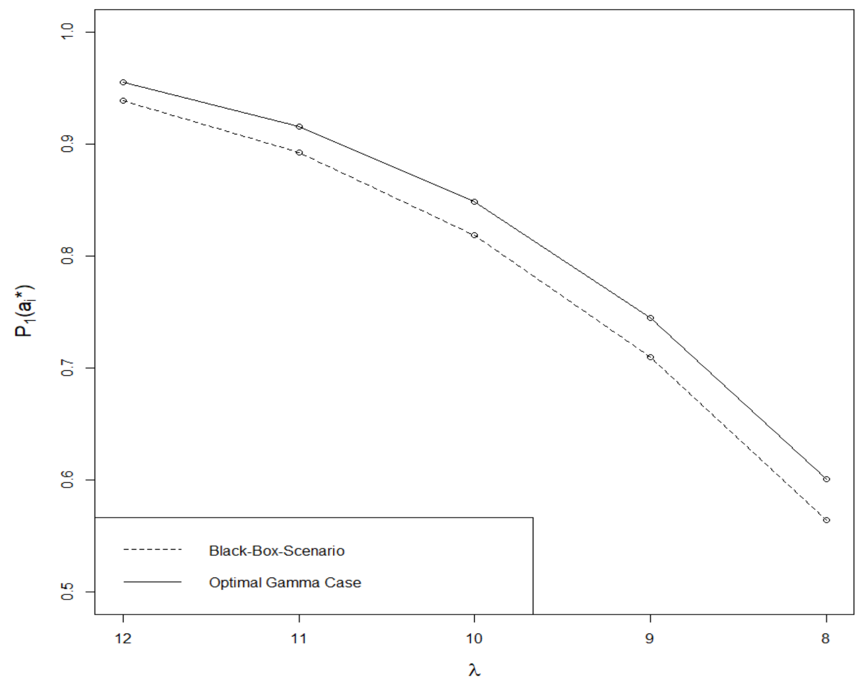

Fig. 8. Comparisons for the MSPs (T=5, $w=1$) for the black-box and degradation scenarios as functions of λ and fixed $a=2.6$.

These graphs, similar to the Poisson degradation case illustrate the efficiency of the proposed strategy as compared with the black-box one. Fig. 6 plots the scheduled optimal switching times after the optimal inspection time that was scheduled at t=1.503 (recall that the optimal black-box switching is at T/2=2.5). We see that they are decreasing when the observed level of degradation at inspection is increasing. This is supported by general considerations, as the larger values of the observed degradation imply the larger risk of failure of the operating component that is avoided by scheduling switching earlier. Figures 7 and 8 show the superiority of the proposed switching strategy over the black-box one and present the corresponding sensitivity analysis with respect to the shape parameter a and the scale parameter λ. We see that the optimal MSP $P_1(a_i^*)$ is decreasing as a is increasing and λ is decreasing. This can be easily explained as the former parameter 'controls' the intensity of jumps in the gamma process (the larger a corresponds to the larger intensity), whereas smaller values of λ correspond to the larger jump values, which obviously, also decreases the reliability function of a component.

6. Concluding remarks

In many applications, it is necessary to maintain the failure-free performance of components in the standby systems, as each sudden failure of an operating component can result in a failure of a system due to imperfect or non-instantaneous switching on failure and related adverse effects. This is especially relevant for safety-critical systems or systems, which failures result in substantial economic losses as, e.g., for important autonomous missions.

To increase the mission success probability (MSP) with the fixed mission time, the optimal strategy of switching is developed for the cold standby system of two aging components with degradation modeled by the counting Poisson process and the gamma process. The crucial assumption for justification of the proposed approach is the IFR property of the distributions that describe the time of reaching the fixed threshold for both processes.

An inspection is carried out at some optimally predetermined time and the switching is performed after that with the optimally obtained delay. These optimal times are derived via the suggested procedure for obtaining the MSP and solving (numerically) the corresponding optimization problem. The case of more than one possible switching and inspections can be also considered in a similar manner, however, it is much more cumbersome.

It is most likely that our results can be generalized to the case of the nonhomogeneous degradation processes. However, this case needs further studies as, e.g., the monotonicity properties of the failure rate for the corresponding threshold survival model defined by the nonhomogeneous gamma process are not yet described in the literature. Considering a random mission time can constitute another topic for the future research.

Acknowledgements

The authors thank the reviewers for helpful comments and advices. The work of the second author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (Grant Number: 2019R1A6A1A11051177).

References

[1]. Xing, L., Levitin, G. & Wang, W. Dynamic System Reliability: Modeling and Analysis of Dynamic and Dependent Behaviors, Wiley, ISBN: 978-1-119-50763-5, 2019.

[2]. Yang, D & Tsao, H. Reliability and availability analysis of standby systems with working vacations and retrial of failed elements. Reliability Engineering & System Safety; 182: 46-55, 2019.

[3] Levitin G, Finkelstein M, Dai Y. Optimal preventive replacement policy for homogeneous cold standby systems with reusable elements. Reliability Engineering and System Safety; 204: 107135, 2020.

[4]. Fouladirad, M., Paroissin, C., Grall, A. Sensitivity of optimal replacement policies to lifetime parameter estimates, European Journal of Operational Research, 266(3), 963-975, 2018.

[5] Yang, L., Ye, Z., Chi-Guhn, L., Su-fen, Y, Peng R., A two-phase preventive maintenance policy considering imperfect repair and postponed replacement, European Journal of Operational Research, 274(3), 966-977, 2019.

[6] Levitin, G., Xing, L. & Dai, Y., Preventive Replacements in Real-Time Standby Systems with Periodic Backups. IEEE Transactions on Reliability; 66(3): 771-782, 2017.

[7] Wang, W, Wu, Z, Xiong, J, & Xu, Y. Redundancy optimization of cold-standby systems under periodic inspection and maintenance. Reliability Engineering and System Safety; 180: 394–402, 2018.

[8] Behboudi, Z, Mohtashami, Z.R, Asadi, M. Reliability modeling of the two-init standby systems: A periodic switching approach. Applied Mathematical modeling; 92, 176-195, 2021.
[9] Levitin, G, Finkelstein M, Dai Y. Optimal shocks-driven optimal startegies with elements reuse in heterogeneous watm-standby systems.Reliability Engineering and System Safety; 210: 107517, 2021.
[10] Kim, H. Maximization of system reliability with the consideration of componets sequencing. Reliability Engineering and System Safety; 170, 64-72, 2018.
[11] Eryilmaz, S. On the mean residual life of a k-out-of-n:G system with a single cold standby component. European Journal of Operational Research 2012; 222:273–277.
[12] Eryilmaz, S, Erkan TE. Coherent system with standby components. Applied Stochastic Models in Business and Industry 2018; 34: 395-406.
[13] Ardakan MA, Talkhabi S, Juybari MN. Optimal activation order *vs.* redundancy strategies in reliability optimization problems. Reliability Engineering and System Safety; 217, 108096, 2022.
[14] Shekhar C, Kumar A, Varshneys. Load sharing redundant repairable systems with switching and reboot delay. Reliability Engineering and System Safety;193; 106656; 2020
[15] Jia X, Chen H, Cheng Z, Guo B. A comparison between two switching policies for two-unit standby system. Reliability Engineering and System Safety 2016;148: 109-118.
[16] Finkelstein, M. On systems with shared resources and optimal switching strategies. Reliability Engineering and System Safety 2009; 94: 1258-1263.
[17] Finkelstein, M. Failure rate modeling for reliability and risk. London: Springer; 2008.
[18]. Zhang, X., Zeng, J. Joint optimization of the condition-based opportunistic maintenance and spare parts provisioning policy in multiunit systems. European Journal of Operational Research, 262, 479-498, 2017.
[19] Keizer, M., Flapper, P., Teunter, R.H. Condition-based Maintenance policies for systems with multiple dependent components: A review. European Journal of Operational Research, 261, 405-420, 2017.
[20] Liu, B., Wu, S., Xie, M., Kuo,W. A condition-based maintenance policy for degrading systems with age and state-dependent operating cost. European Journal of Operational Research, 263, 879-887, 2017.
[21] Nakagawa T. Maintenance theory of reliability. London: Springer; 2005.
[22] Gertsbakh I. Reliability theory: with applications to preventive maintenance. Berlin: Springer; 2013.
[23] Cha, JH, Finkelstein, M. On optimal life extension for degrading systems. Journal of Risk and Reliability; 234: 487-495, 2020.
[24] Finkelstein M, Cha JH, Ghosh, S. On dynamic degradation-based life extension. Journal of Risk and Reliability; 235: 690-699, 2021.
[25] Yun WY, Cha, JH. Optimal design of a general warm standby system. Reliability Engineering and System Safety; 95: 880-886, 2010.
[26] Zhang Z, Yang L. Postponed maintenance scheduling integrating state variation and environmental impact. Reliability Engineering and System Safety; 202, 107065, 2020.
[27] Wang J, Qiu Q, Wang H. Joint optimization of condition-based and age-based replacement policy and inventory policy for a two-unit series system. Reliability Engineering and System Safety; 202, 107251, 2021.

[28] Yuan XX, Higo E,, Pandey MD. Estimation of the value of an inspection and maintenance program: A Bayesian gamma process model. Reliability Engineering and System Safety; 216, 106912, 2021
[29] Wu H, Li YF, Bérenguer C. Optimal inspection and maintenance for a repairable *k*-out-of-*n*: G warm standby system. Reliability Engineering and System Safety; 193, 106656, 2020.
[30] Bismut E, Straub D. Optimal adaptive inspection and maintenance planning for deteriorating structural systems. Reliability Engineering and System Safety; 215, 107891, 2021.
[31] Ardakan MA, Amini, H, JuyBary MN. Prescheduled switching time: A new strategy for systems with standby components. Reliability Engineering and System Safety. https://doi.org/10.1016/j.ress.2021.108188, 2021.
[32] Ross, S. Stochastic processes. New York: Wiley; 1996.
[33] van Noortwijk JM. A survey of the application of gamma processes in maintenance. Reliability Engineering and System Safety 2009; 4: 2-21.
[34] Çinlar E. On a generalization of gamma process. Journal of Applied Probability 1980; 17: 467–480.
[35] Liao H, Elsayed EA, Ling-Yau Chan LY (2006). Maintenance of continuously monitored degrading systems. European Journal of Operational Research 2006; 175: 821-835.
[36] Kahle W, Mercier, S, Paroissin, C. Degradation Processes in Reliability. Hoboken: Wiley; 2016.

Appendix.

The following procedure for solving the optimization problem (16) should be implemented for the gamma process of deterioration

Let us fix the time of inspection, $a_i \in [0,T]$.

Case (i). If the first component fails before a_i, then the MSP is 0.

Case (ii). If the first component is operable at a_i, degradation at $W_{a_i} = w_{a_i} < w$ is observed. Thus, we have to minimize the sum of integrals

$$\int_0^a \lambda_{w_{a_i}}(u)du + \int_0^{(T-a_i)-a} \lambda_0(u)du ,$$

where, as previously, $a_i + a < T$ is the time of switching in this case and $\lambda_{w_{a_i}}(t)$ is the failure rate that corresponds to the survival model (11) with the threshold w and initial degradation w_{a_i} (see the corresponding relationships later). Then we have two possibilities (for the fixed a_i):

1. $\lambda_{w_{a_i}}(0) \geq \lambda_0(T-a_i)$.

Observe that $\int_0^a \lambda_{w_{a_i}}(u)du + \int_0^{(T-a_i)-a} \lambda_0(u)du = \int_0^{(T-a_i)} \lambda_0(u)du + (\int_0^a \lambda_{w_{a_i}}(u)du - \int_{(T-a_i)-a}^{(T-a_i)} \lambda_0(u)du)$.

In this case, for any $a>0$, $\int_0^a \lambda_{w_{a_i}}(u)du - \int_{(T-a_i)-a}^{(T-a_i)} \lambda_0(u)du > 0$ and $\int_0^a \lambda_{w_{a_i}}(u)du - \int_{(T-a_i)-a}^{(T-a_i)} \lambda_0(u)du = 0$ if $a=0$. Therefore, $a^*(a_i, w_{a_i}) = 0$ and the switching should be performed immediately.

2. If $\lambda_{w_{a_i}}(0) < \lambda_0(T-a_i)$, there exists $a>0$ such that

$$\int_0^a \lambda_{w_{a_i}}(u)du - \int_{(T-a_i)-a}^{(T-a_i)} \lambda_0(u)du < 0 \cdot$$

Therefore, the optimal $a^*(a_i, w_{a_i}) \in (0, T-a_i]$ exists, which is the solution of the following equation:

$$\lambda_{w_{a_i}}(a^*(a_i, w_{a_i})) = \lambda_0(T-(a_i + a^*(a_i, w_{a_i}))) . \tag{17}$$

Using relationships of Section 3, it can be easily shown that for the homogeneous gamma degradation process, we have

$$\lambda_0(t) = \frac{A(t,w)}{\left(\int_0^w \frac{1}{\Gamma(\alpha t)} \lambda^{\alpha t} u^{\alpha t-1} \exp(-\lambda u) du\right)\left(\int_0^\infty u^{\alpha t-1} \exp(-u)\, \mathrm{du}\right)^2},$$

where,

$$A(t,w) = \int_0^w \Bigg[-\left(\ln\lambda + \ln u\right)\; \alpha\lambda^{\alpha t} u^{\alpha t-1} \exp(-\lambda u) \left(\int_0^\infty s^{\alpha t-1} \exp(-s)\, \mathrm{ds}\right)$$

$$+\lambda^{\alpha t} u^{\alpha t-1} \exp(-\lambda u) \left(\int_0^\infty \alpha\, (\ln s)\, \mathrm{s}^{\alpha t-1} \exp(-s)\, \mathrm{ds}\right)\Bigg] \mathrm{d}u ,$$

and

$$\lambda_{w_{a_i}}(t) = \frac{A^*(t, w-w_{a_i})}{\left(\int_0^{w-w_{a_i}} \frac{1}{\Gamma(\alpha t)} \lambda^{\alpha t} u^{\alpha t-1} \exp(-\lambda u) du\right)\left(\int_0^\infty u^{\alpha t-1} \exp(-u)\, \mathrm{du}\right)^2},$$

where

$$A^*(t, w-w_{a_i}) = \int_0^{w-w_{a_i}} \Bigg[-\left(\ln\lambda + \ln u\right)\; \alpha\lambda^{\alpha t} u^{\alpha t-1} \exp(-\lambda u) \left(\int_0^\infty s^{\alpha t-1} \exp(-s)\, \mathrm{ds}\right)$$

$$+\lambda^{\alpha t} u^{\alpha t-1} \exp(-\lambda u) \left(\int_0^\infty \alpha\, (\ln s)\, \mathrm{s}^{\alpha t-1} \exp(-s)\, \mathrm{ds}\right)\Bigg] \mathrm{d}u .$$

If $W_{a_i} = w_{a_i}$, the corresponding conditional mission success probability is given by

$$\left(\int_0^{w-w_{a_i}} \frac{1}{\Gamma(\alpha\cdot a^*(a_i,w_{a_i}))}\lambda^{\alpha\cdot a^*(a_i,w_{a_i})}u^{\alpha\cdot a^*(a_i,w_{a_i})-1}\exp(-\lambda u)du\right)$$

$$\times\left(\int_0^{w}\frac{1}{\Gamma(\alpha\cdot(T-a_i-a^*(a_i,w_{a_i})))}\lambda^{\alpha\cdot(T-a_i-a^*(a_i,w_{a_i}))}u^{\alpha\cdot(T-a_i-a^*(a_i,w_{a_i}))-1}\exp(-\lambda u)du\right)$$

Indeed, the first integral defines the probability that after the inspection time a_i, the operating component did not fail before switching to the second component, whereas the second integral defines the probability that the second component does not fail in the rest of the mission time after switching.

The conditional pdf of $W_{a_i}=w_{a_i}$ given that $W_{a_i}<w$ is

$$\frac{\frac{1}{\Gamma(\alpha a_i)}\lambda^{\alpha a_i}w_{a_i}^{\alpha a_i-1}\exp(-\lambda w_{a_i})}{\left(\int_0^{w}\frac{1}{\Gamma(\alpha a_i)}\lambda^{\alpha a_i}u^{\alpha a_i-1}\exp(-\lambda u)du\right)},\ 0<w_{a_i}<w.$$

Thus, for Case (ii), the conditional probability for mission success is

$$\frac{1}{\left(\int_0^{w}\frac{1}{\Gamma(\alpha a_i)}\lambda^{\alpha a_i}u^{\alpha a_i-1}\exp(-\lambda u)du\right)}$$

$$\times\int_0^{w}\left(\int_0^{w-w_{a_i}} \frac{1}{\Gamma(\alpha\cdot a^*(a_i,w_{a_i}))}\lambda^{\alpha\cdot a^*(a_i,w_{a_i})}u^{\alpha\cdot a^*(a_i,w_{a_i})-1}\exp(-\lambda u)du\right)$$

$$\times\left(\int_0^{w}\frac{1}{\Gamma(\alpha\cdot(T-a_i-a^*(a_i,w_{a_i})))}\lambda^{\alpha\cdot(T-a_i-a^*(a_i,w_{a_i}))}u^{\alpha\cdot(T-a_i-a^*(a_i,w_{a_i}))-1}\exp(-\lambda u)du\right)$$

$$\times\frac{1}{\Gamma(\alpha a_i)}\lambda^{\alpha a_i}w_{a_i}^{\alpha a_i-1}\exp(-\lambda w_{a_i})dw_{a_i}$$

and the unconditional mission success probability is

$$P_1(a_i)=\int_0^{w}\left(\int_0^{w-w_{a_i}} \frac{1}{\Gamma(\alpha\cdot a^*(a_i,w_{a_i}))}\lambda^{\alpha\cdot a^*(a_i,w_{a_i})}u^{\alpha\cdot a^*(a_i,w_{a_i})-1}\exp(-\lambda u)du\right)$$

$$\times\left(\int_0^{w}\frac{1}{\Gamma(\alpha\cdot(T-a_i-a^*(a_i,w_{a_i})))}\lambda^{\alpha\cdot(T-a_i-a^*(a_i,w_{a_i}))}u^{\alpha\cdot(T-a_i-a^*(a_i,w_{a_i}))-1}\exp(-\lambda u)du\right)$$

$$\times\frac{1}{\Gamma(\alpha a_i)}\lambda^{\alpha a_i}w_{a_i}^{\alpha a_i-1}\exp(-\lambda w_{a_i})dw_{a_i}$$