Shape preserving interpolation on surfaces

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introduction

- let be given an orientable, smooth parametric surface:
  $\mathbf{S}(u, v)$, $(u, v) \in \Omega \subseteq \mathbb{R}^2$,
  and an ordered set of points:
  $\mathcal{I} = \{\mathbf{I}_i = \mathbf{S}(u_i, v_i),\ (u_i, v_i) \in \Omega, i = 0, 1, ..., n\}$
  on it.

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  on it.

- we aim to set up a methodology for constructing smooth curves $\mathbf{c}(t)$, $t \in [t_0, t_n]$, that lie on $\mathbf{S}(u, v)$, interpolate the given points, $\mathbf{c}(t_i) = \mathbf{I}_i$ on it, with t_i being user-specified parameters in $[t_0, t_n]$, and are *shape preserving* in an appropriately defined sense.

a criterion for shape-preserving interpolation (spi)

- to introduce a notion of *shape-preserving interpolation on a surface*, we appeal to the composite curve Γ consisting of the *geodesic segments*
  $\gamma_i(\tau)$, $\tau \in [0,1]$, $\tau = (t - t_i)/h_i$, $h_i = t_{i+1} - t_i$,
  that connect each consecutive pair of interpolation points:
  $\gamma_i(0) = \mathbf{I}_i$, $\gamma_i(1) = \mathbf{I}_{i+1}$, $i = 0, 1, ..., n-1$.

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  that connect each consecutive pair of interpolation points:
  $\gamma_i(0) = \mathbf{I}_i$, $\gamma_i(1) = \mathbf{I}_{i+1}$, $i = 0, 1, ..., n-1$.
- the *geodesic curvature* κ_g of a regular curve $\mathbf{c}(t)$ on $\mathbf{S}(u, v)$, is defined as:

$$\kappa_g(t; \mathbf{c}) = \frac{(\dot{\mathbf{c}}(t), \ddot{\mathbf{c}}(t), \mathbf{n}(t))}{\|\dot{\mathbf{c}}(t)\|^3},$$

generalised convexity indicator

- in analogy to the notion of *convexity indicators* $P_{planar,i}$ used for shape-preserving interpolation on the plane

$$P_{planar,i} = \frac{(\mathbf{L}_{i-1}, \mathbf{L}_i, \mathbf{e}_3)}{\|\mathbf{L}_{i-1} \times_{2D} \mathbf{L}_i\|},$$

where $\mathbf{e}_3$ is the unit normal on the plane, $\mathbf{L}_i = \mathbf{I}_{i+1} - \mathbf{I}_i$.

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  where $\mathbf{e}_3$ is the unit normal on the plane, $\mathbf{L}_i = \mathbf{I}_{i+1} - \mathbf{I}_i$.
- we introduce the notion of *generalized convexity indicators* P_i at the vertices $\mathbf{I}_i$ of the composite geodesic Γ on $\mathbf{S}(u, v)$

$$P_i = \frac{(\dot{\gamma}_{i-1}(1), \dot{\gamma}_i(0), \mathbf{n}(t_i))}{\|\dot{\gamma}_{i-1}(1) \times \dot{\gamma}_i(0)\|}.$$

the proposed spi criterion

(i) *convexity*: If $P_m P_{m+1} > 0$ then

$$\kappa_g(t;\mathbf{c})P_n > 0, \quad t \in [t_m, t_{m+1}], n = m \text{ or } m+1.$$

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(ii) *minimum variation*: If $P_m P_{m+1} < 0$ then

$$\kappa_g(t;\mathbf{c})P_m \geq 0,\ t\in[t_m, t_{m,m+1}], \kappa_g(t;\mathbf{c})P_{m+1} \geq 0,\ t\in[t_{m,m+1}, t_{m+1}],$$

for some $t_{m,m+1}\in(t_m, t_{m+1})$.

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for some $t_{m,m+1} \in (t_m, t_{m+1})$.

(iii) co-geodesity: If $P_m = 0$ and $P_{m-1}P_{m+1} \neq 0$ then

$$\|\kappa_g(t;\mathbf{c})\| < \epsilon,\ t \in \eta_m, \kappa_g(t;\mathbf{c})P_{m-1} \geq 0,\ t \in [t_{m-1}, t_m] \backslash \eta_m,$$

$$\kappa_g(t;\mathbf{c})P_{m+1} \geq 0,\ t \in [t_m, t_{m+1}] \backslash \eta_m,$$

where ϵ is a user-specified small positive number in $(0, 1]$ and η_m is a closed subinterval of (t_{m-1}, t_{m+1}) that includes t_m as an interior point.

geodesic-based variable-degree splines (vd-splines)

the members $\mathbf{c}(t)$ of this family are defined by composing $\mathbf{S}(u, v)$ with a family of planar curves, $\mathbf{q}(t)$, which combine:

- the shape-preserving interpolation properties of the so-called *variable-degree polynomial splines*, with
- the pre-images, $\mathbf{g}_i(t) = \mathbf{S}^{-1}(\gamma_i(t))$, of the geodesic arcs $\gamma_i(\tau)$:

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- the pre-images, $\mathbf{g}_i(t) = \mathbf{S}^{-1}(\gamma_i(t))$, of the geodesic arcs $\gamma_i(\tau)$:

$$\mathbf{c}(t) = (\mathbf{S} \circ \mathbf{q})(t), \tag{1}$$

$$\mathbf{q}(t) = \mathbf{g}_i(\tau) + h_i^2(\ddot{\mathbf{q}}_i - \ddot{\mathbf{g}}_i(t_i))F_i(1-\tau) + h_i^2(\ddot{\mathbf{q}}_{i+1} - \ddot{\mathbf{g}}_i(t_{i+1}))F_i(\tau)$$

$$t \in [t_i, t_{i+1}],\ \ddot{\mathbf{q}}_i := d^2\mathbf{q}(t_i)/dt^2$$

$$F_i(\tau) = \frac{\tau^{k_i} - \tau}{k_i(k_i - 1)}, \quad 3 \leq k_i \in \mathbb{N},$$

$$\tau = \frac{t - t_i}{h_i} \in [0, 1],\ \ h_i = t_{i+1} - t_i,\ i = 1, ..., n-1$$

on surface interpolation problem

theorem

- Let be given a surface $\mathbf{S}(u,v)$, $(u,v) \in \Omega \subseteq \mathbb{R}^2$, along with a set $\mathcal{I} = \{\mathbf{I}_0, ..., \mathbf{I}_n\}$ of points on it, a user-specified knot sequence $\mathcal{T} = \{t_i : \ t_i < t_{i+1}, i = 0, 1, ..., n-1\}$ and a set $\mathcal{K} = \{k_1, ..., k_{n-1}\}$, $3 \leq k_i \in \mathbb{N}$, $i = 1, ..., n-1$.
- then, under appropriate boundary conditions $\mathcal{B}$, there exists a unique $C^2([t_0, t_n])$ curve $\mathbf{c}(t)$, represented as in (1), which lies on $\mathbf{S}(u,v)$ and interpolates $\mathcal{I}$ in conformity with the knot sequence $\mathcal{T}$

local asymptotic behaviour

theorem

- If k_{i-1}, k_i increase in compatibility with:

$$\lim_{k_{i-1},k_i\to\infty} \frac{k_{i-1}}{k_i} = \lambda_{i-1,i},$$

  where $\lambda_{i-1,i}$ is a non-zero positive constant,
- then, for sufficiently large degrees k_{i-1} and k_i, the sign of the geodesic curvature of $\mathbf{c}(t)$ at $t = t_i$ is equal to the sign of the quantity

$$\dot{\mathbf{g}}_{i-1}(t_i) \times_{2D} \dot{\mathbf{g}}_i(t_i)(\mathbf{S}_u(\mathbf{l}_i), \mathbf{S}_v(\mathbf{l}_i), \mathbf{n}(\mathbf{l}_i)),$$

  where the factor $(\mathbf{S}_u(\mathbf{l}_i), \mathbf{S}_v(\mathbf{l}_i), \mathbf{n}(\mathbf{l}_i))$ has constant sign for the orientable surface $\mathbf{S}(u, v)$

geodesic-based ν-splines

$$\mathbf{c}(t) = (\mathbf{S} \circ \mathbf{q})(t), \tag{2}$$

$$\mathbf{q}(t) = \mathbf{g}_i(H_3^3(\tau)) + h_i \dot{\mathbf{q}}_i H_1^3(\tau) + h_i \dot{\mathbf{q}}_{i+1} H_2^3(\tau), \;\; t \in [t_i, t_{i+1}],$$

$$\tau = \frac{t - t_i}{h_i} \in [0, 1], \;\; h_i = t_{i+1} - t_i, \; i = 0, ..., n-1, \dot{\mathbf{q}}_i := d\mathbf{q}(t_i)/dt$$

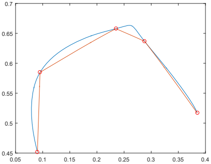
$$H_3^3(\tau) = B_2^3(\tau) + B_3^3(\tau)$$

$$H_1^3(\tau) = \frac{1}{3} B_1^3(\tau), H_2^3(\tau) = -\frac{1}{3} B_2^3(\tau)$$

G^2-continuity conditions

$$\ddot{\mathbf{q}}(t_i+) - \ddot{\mathbf{q}}(t_i-) = \nu_i \mathbf{q}_i, \; i = 1, ..., n-1$$

example-1: spi on cylinder



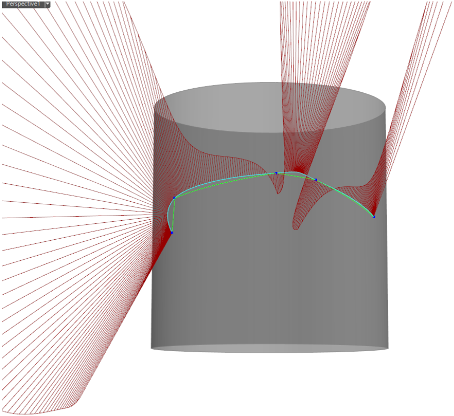
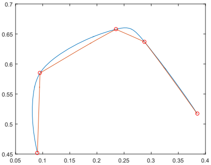


Figure 1:

initial: ν's $\rightarrow \{0, 0, 0, 0, 0\}$

example-1: spi on cylinder



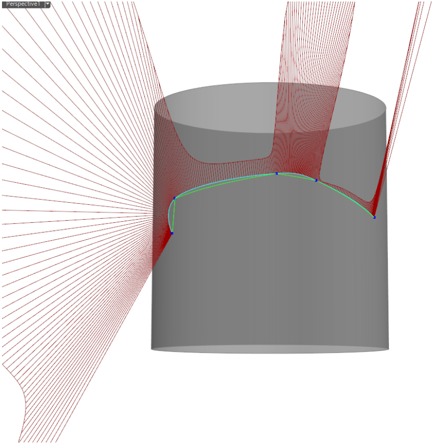
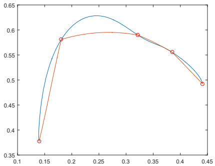


Figure 1:

spi: ν's $\rightarrow \{0, 2, 4, 4, 4\}$

example-2: spi on sphere



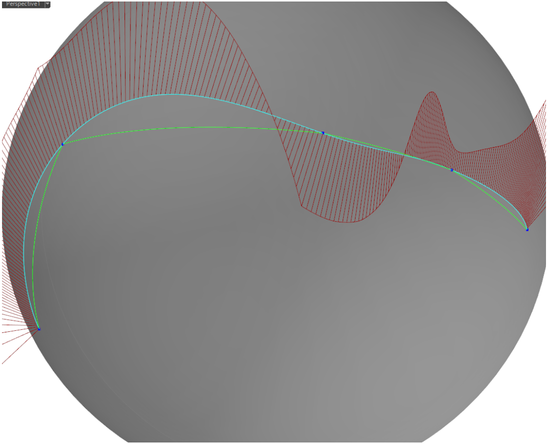
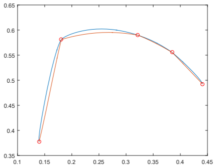


Figure 2:

initial: ν's $\rightarrow \{0, 0, 0, 0, 0\}$

example-2: spi on sphere



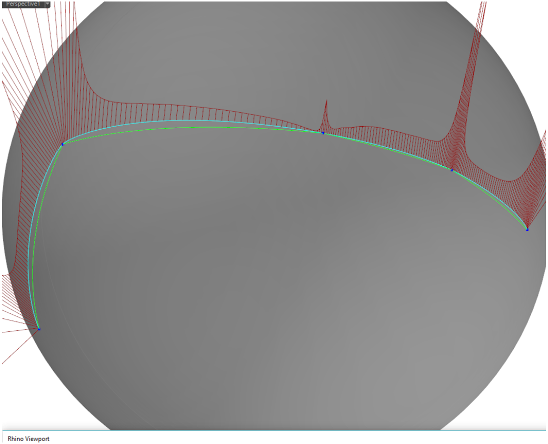
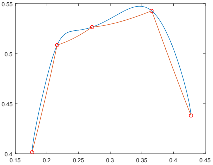


Figure 2:

spi: ν's $\rightarrow \{0, 22, 22, 22, 1\}$

example-3: spi on a free-form surface



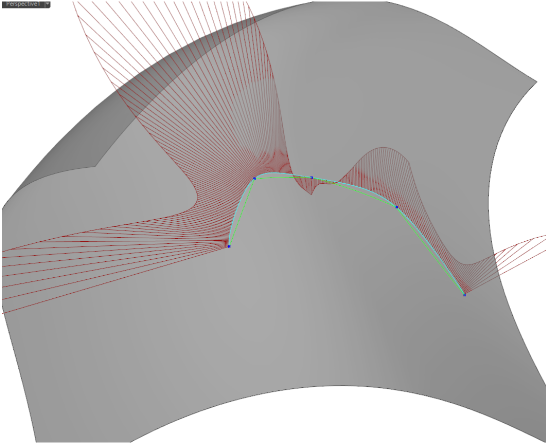
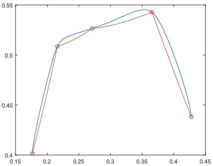


Figure 3:

initial: ν's $\rightarrow \{0, 0, 0, 0, 0\}$

example-3: spi on a free-form surface



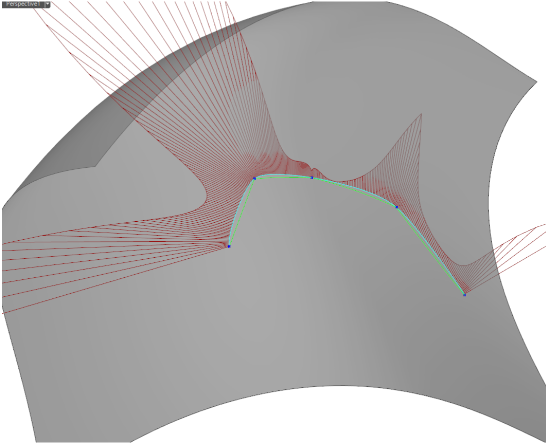


Figure 3:

spi: ν's $\rightarrow \{0, 4, 4, 4, 0\}$

future work

spi & shape optimization

- local asymptotic behaviour for geodesic-based $\nu-$splines (in the neighborhood of interpolation points)
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- local asymptotic behaviour for geodesic-based $\nu-$splines (in the neighborhood of interpolation points)
- global asymptotic behaviour for geodesic-based vd- & nu-splines (along closed parametric intervals between knots)
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spi & shape optimization

- local asymptotic behaviour for geodesic-based $\nu-$splines (in the neighborhood of interpolation points)
- global asymptotic behaviour for geodesic-based vd- & nu-splines (along closed parametric intervals between knots)
- optimal degrees or $\nu-$parameters against fairness criteria

some references

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